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T VI: Soft Matter and Biological Physics
(Prof. E. Frey)

Problem set 5

Problem 5.1 *classical elasticity of a bending rod*
The total energy of a rod under tension is given in terms of the functional

$$\mathcal{A} = \int_0^L \mathrm{d}s \left[\frac{\kappa}{2} \left(\frac{\mathrm{d}\vec{t}(s)}{\mathrm{d}s} \right)^2 - \vec{F} \cdot \vec{t}(s) \right]$$

The material parameter characterizing the rod is the bending rigidity κ. The configuration of the rod is given in terms of a curve $\vec{r}(s)$, specified by its (normalized) tangent $\vec{t}(s) = \mathrm{d}\vec{r}/\mathrm{d}s$, and parameterized by its arc length s. The actual curve is obtained by *minimizing* the total energy $\mathcal{A}$. Appropriate boundary conditions have to be imposed in order to apply the calculus of variations. In this problem we choose both ends clamped

$$\vec{t}(s=0) = \vec{t}_a \,, \qquad \vec{t}(s=L) = \vec{t}_b \,.$$

1. Introduce spherical coordinates to fulfill the constraint, $|\vec{t}(s)| = 1$, i.e. the tangent is normalized. Show that the variational problem maps to the principle of least action for a classical pendulum. Determine the equations of motion and find possible first integrals.

2. To simplify specialize to a *weakly bending rod*, i.e. the tangent vector $\vec{t} = (\vec{t}_\perp, t_\parallel)$ is essentially aligned with some axis. Show that the energy functional is then approximately given by

$$\mathcal{A} = -F_\parallel L + \int_0^L \mathrm{d}s \left[\frac{\kappa}{2} \left(\frac{\mathrm{d}\vec{t}_\perp(s)}{\mathrm{d}s} \right)^2 - \vec{F}_\perp \cdot \vec{t}_\perp(s) + \frac{1}{2} F_\parallel \, \vec{t}_\perp(s)^2 \right] \qquad (*)$$

   with corresponding boundary conditions $\vec{t}_\perp(0) = \vec{t}_{a,\perp}, \vec{t}_\perp(L) = \vec{t}_{b,\perp}$. The problem to calculate the minimal $\mathcal{A}$ is then equivalent to determine the classical action of a harmonic oscillator. Calculate the classical trajectory $\vec{t}_\perp(s)$ as the solution of the corresponding Euler-Lagrange equations and determine the minimal elastic energy $\mathcal{A}$.

3. For vanishing perpendicular forces $\vec{F}_\perp = 0$ and clamped ends $\vec{t}_{\perp,a} = \vec{t}_{\perp,b} = 0$ the Euler-Lagrange equation has a trivial solution $\vec{t}_\perp(s) \equiv 0$. For strong negative forces $F_\parallel < 0$ this solution does not correspond to a minimum of the functional given by Eq. $(*)$. Estimate the critical force of the instability by a scaling argument. Introduce discrete Fourier modes that fulfill the boundary condition

$$\vec{t}_\perp(s) = \sum_{n=1}^{\infty} \vec{t}_{n,\perp} \sin\left(\frac{\pi n}{L} s \right)$$

   and determine the force $F_{\parallel,c}$ beyond which it is favorable for the rod to buckle.