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T VI: Soft Matter and Biological Physics
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Problem set 1

Problem 1.1 *Local limit theorem*
Use *Sterling*'s approximation for the factorial, $n! = \sqrt{2\pi n}(n/\mathrm{e})^n(1+R(n))$, where the remainder fulfills $R(n) \to 0$ as $n \to \infty$, to show that the binomial distribution

$$w(k,N) = \binom{N}{k} p^k q^{N-k}\,, \qquad 0 \le p \le 1\,, \quad p+q=1\,,$$

approaches a Gaussian distribution in the limit of $N \to \infty$ provided the number of successes is typical, i.e. $|k - Np| = o(Npq)^{2/3}$.

Problem 1.2 *random walker*
A particle performs a random walk on a d-dimensional hypercubic lattice Λ^d of lattice constant a. The steps are assumed to be independent and identically distributed (*iid*), and steps are performed at fixed time intervals ϵ. Furthermore, we consider only nearest neighbor hopping without bias, i.e. all allowed steps are equi-probable.

1. Use the partition theorem to argue that the probability distribution for the random walker $P(\vec{x},t)$ satisfies the following master equation

$$P(\vec{x}, t+\epsilon) = \frac{1}{2d} \sum_{\vec{x}'\,\text{neighbor of}\,\vec{x}} P(\vec{x}', t)\,.$$

2. Defining the probability density $\rho(\vec{x},t) = P(\vec{x},t)/a^d$ perform a suitable *continuum limit*, $a \to 0$, $\epsilon \to 0$ and show that the density fulfills a diffusion equation

$$\partial_t \rho(\vec{x},t) = D\nabla^2 \rho(\vec{x},t)\,, \qquad \nabla^2 = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\,.$$

   Relate the diffusion constant D to the micro-parameters a, ϵ.

3. Consider the discrete lattice again. The characteristic function corresponding to $P(\vec{x},t)$ is the discrete Fourier transform of the probability distribution

$$\tilde{P}(\vec{k},t) = \sum_{\vec{x}\in\Lambda^d} \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} P(\vec{x},t)\,.$$

   Specify the range of allowed wave vectors $\vec{k}$ and determine the equation of motion for $P(\vec{k},t)$. Show that the Fourier modes are decoupled and solve for $\tilde{P}(\vec{k},t)$ for the initial condition specified by $P(\vec{x},t=0) = \delta_{\vec{x},\vec{x}_0}$. How should the inverse Fourier transform $\tilde{P}(\vec{k},t) \mapsto P(\vec{x},t)$ be defined, reflecting that positions live on a lattice $\vec{x} \in \Lambda^d$?

Problem 1.3 *Characteristic Functions*

For a probability density $p(x) \geq 0$ (a.e.), $p \in L^1(\mathbb{R})$ the corresponding characteristic function is defined as

$$C(\xi) \equiv \langle \mathrm{e}^{\mathrm{i}\xi x} \rangle = \int \mathrm{e}^{\mathrm{i}\xi x} p(x) \mathrm{d}x \,.$$

Demonstrate the following properties:
(a) $C(0) = 1$.
(b) $|C(\xi)| \leq C(0)$.
(c) $C(\xi)$ is continuous on the real axis, even if $p(x)$ is highly irregular.
(d) $C(-\xi) = C(\xi)^*$
(e) $C(\xi)$ is positive semi-definite, i.e. for an arbitrary set of N real numbers $\xi_1, \xi_2, \ldots, \xi_N$ and N arbitrary complex numbers $a_1, a_2, \ldots, a_N$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i^* a_j C(\xi_i - \xi_j) \geq 0 \,.$$

Problem 1.4 *Markov process*

Determine the solution of the d-dimensional diffusion equation

$$\partial_t \rho(\vec{x}, t) = D \nabla^2 \rho(\vec{x}, t) \,, \qquad \nabla^2 = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \,.$$

for the initial condition $\rho(\vec{x}, t = 0) = \delta(\vec{x})$, i.e. the distribution is concentrated at the origin. Show explicitly that the solution satisfies the Chapman-Kolmogorov relation

$$\rho(\vec{x}, t) = \int \mathrm{d}^d \vec{y} \; \rho(\vec{y}, s) \rho(\vec{x} - \vec{y}, t - s) \,.$$

In particular, diffusion is a *Markov* process.