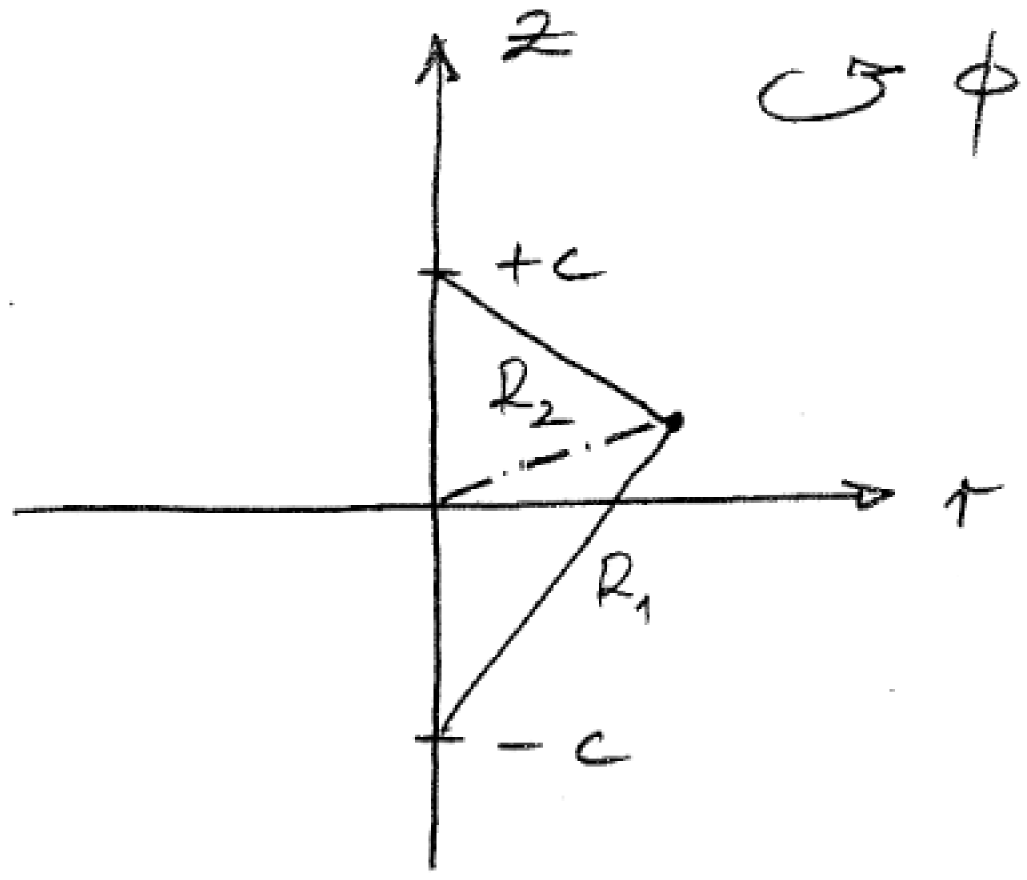
Elliptische Koordinaten

Die Brennpunkte des Rotationsellipsoids sind durch $z^2 = c^2 = a^2 - b^2$ gegeben.



(r, z, ϕ) zylindrische Koordinaten
(u, v, ϕ) elliptische Koordinaten

$$R_1 + R_2 := 2u = 2\rho c$$
$$R_1 - R_2 := 2v = 2\xi c$$

Aus der Figur entnimmt man

$$R_1^2 = (z+c)^2 + r^2, \quad R_2^2 = (z-c)^2 + r^2$$

Folglich gilt

$$u \cdot v = \frac{R_1^2 - R_2^2}{4} = c \cdot z$$

$$u^2 + v^2 = \frac{1}{2}\left(R_1^2 + R_2^2\right) = z^2 + c^2 + r^2$$

$$\Rightarrow r^2 = u^2 + v^2 - \left(\frac{uv}{c}\right)^2 - c^2$$

$$\left\{ \begin{array}{l} z = uv/c \\ r = \frac{1}{c}\sqrt{(u^2-c^2)(c^2-v^2)} \end{array} \right.$$

Umrechnung des Laplace operators auf elliptische Koordinaten (spheroidale)

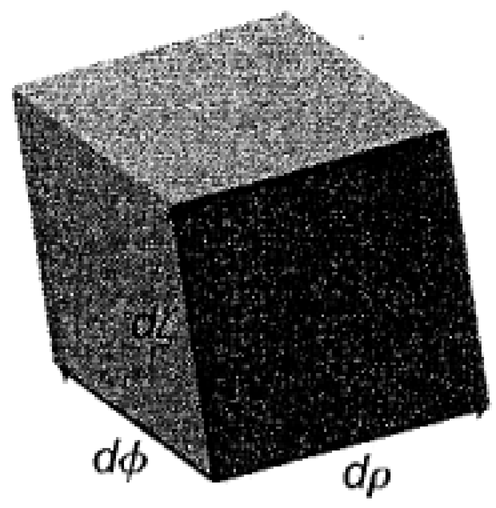


Figure 3.16 Rectangular volume in spheroidal coordinate system.

Therefore, far from the origin, the spheroidal coordinates $(\rho c, \zeta, \phi)$ are analogous to the spherical polar coordinates $(R, \cos\theta, \phi)$. At the origin, $\zeta = 0$ and $\rho = 1$.

Transforming the Laplace equation to spheroidal coordinates (or any other curvilinear coordinates) is inevitably a tedious process. We proceed in the following way. If we consider a small rectangular volume formed by the coordinates (ρ, ζ, ϕ) as shown in Figure 3.16, we find by differentiating (3.147) and (3.148) that the increments dx, dy, and dz along the edges are

$$dx = c\sqrt{\frac{1-\zeta^2}{\rho^2-1}}\rho\cos\phi\, d\rho + c\sqrt{\frac{\rho^2-1}{1-\zeta^2}}\zeta\cos\phi\, d\zeta - c\sqrt{(\rho^2-1)(1-\zeta^2)}\sin\phi\, d\phi \tag{3.153}$$

$$dy = c\sqrt{\frac{1-\zeta^2}{\rho^2-1}}\rho\sin\phi\, d\rho + c\sqrt{\frac{\rho^2-1}{1-\zeta^2}}\zeta\sin\phi\, d\zeta + c\sqrt{(\rho^2-1)(1-\zeta^2)}\cos\phi\, d\phi \tag{3.154}$$

and

$$dz = c\zeta\, d\rho + c\rho\, d\zeta \tag{3.155}$$

Summing the squares of all these terms, we find that the increment of length from one corner of the rectangular volume to the diagonal corner (the metric in the new coordinate system) is

$$dl^2 = dx^2 + dy^2 + dz^2 = h_\rho^2\, d\rho^2 + h_\zeta^2\, d\zeta^2 + h_\phi^2\, d\phi^2 \tag{3.156}$$

where

$$h_\rho = c\sqrt{\frac{\rho^2-\zeta^2}{\rho^2-1}} \tag{3.157}$$

$$h_\zeta = c\sqrt{\frac{\rho^2-\zeta^2}{1-\zeta^2}} \tag{3.158}$$

$$h_\phi = c\sqrt{(\rho^2-1)(1-\zeta^2)} \tag{3.159}$$

The increments of length corresponding to increments of the curvilinear coordinates are evidently

$$dl_\rho = h_\rho\, d\rho \tag{3.160}$$

$$dl_\zeta = h_\zeta\, d\zeta \tag{3.161}$$

$$dl_\phi = h_\phi\, d\phi \tag{3.162}$$

To express the Laplace equation in spheroidal coordinates, we recognize that the Laplace equation is equivalent, by the divergence theorem, to the statement

$$\oint_S \frac{\partial \Phi}{\partial n}\, dS = 0 \tag{3.163}$$

where $\partial\Phi/\partial n$ is the derivative normal to the surface element dS in the direction outward from the volume enclosed by S. To apply this to the infinitesimal rectangular volume shown in Figure 3.16 we note that the electric flux through the face normal to the coordinate ρ is

$$\frac{h_\zeta\, d\zeta\, h_\phi\, d\phi}{h_\rho} \frac{\partial \Phi}{\partial \rho}$$

so the net electric flux outward from the volume through this face and the opposite face is

$$\frac{\partial}{\partial \rho}\left(\frac{h_\zeta h_\phi}{h_\rho}\frac{\partial \Phi}{\partial \rho}\right) d\rho\, d\zeta\, d\phi$$

Doing the same calculation for the other faces and adding, we obtain the Laplace equation in the curvilinear coordinate system

$$\left[\frac{\partial}{\partial \rho}\left(\frac{h_\zeta h_\phi}{h_\rho}\frac{\partial \Phi}{\partial \rho}\right) + \frac{\partial}{\partial \zeta}\left(\frac{h_\rho h_\phi}{h_\zeta}\frac{\partial \Phi}{\partial \zeta}\right) + \frac{\partial}{\partial \phi}\left(\frac{h_\rho h_\zeta}{h_\phi}\frac{\partial \Phi}{\partial \phi}\right)\right] d\rho\, d\zeta\, d\phi = 0 \tag{3.164}$$

In the present case, when we substitute for h_ρ, h_ζ, and h_ϕ we find that the Laplace equation in prolate spheroidal coordinates is

$$\frac{\partial}{\partial \rho}\left[(\rho^2 - 1)\frac{\partial \Phi}{\partial \rho}\right] + \frac{\partial}{\partial \zeta}\left[(1 - \zeta^2)\frac{\partial \Phi}{\partial \zeta}\right] + \frac{\rho^2 - \zeta^2}{(\rho^2 - 1)(1 - \zeta^2)}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{3.165}$$

Quelle: C. A. Brau
"Modern Problems in Classical Electrodynamics",
Oxford Univ. Press