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T II: Elektrodynamik

(Prof. E. Frey)

Problem set 6

Tutorial 6.1 *Sound waves in a fluid*

The macroscopic properties of a fluid are characterized in terms of a few fields, e.g., the mass density $\rho(\vec{r},t)$, the mass current density $\vec{j}(\vec{r},t)$, the fluid velocity $\vec{v}(\vec{r},t)$, and the pressure $p(\vec{r},t)$. Euler's equations specify the field equations; the first set encodes the conservation of mass and momentum,

$$\partial_t \rho + \nabla_k j_k = 0\,, \quad \partial_t j_k + \nabla_l \Pi_{kl} = 0\,. \qquad (*)$$

The mass current density is connected to the fluid velocity by $\vec{j}(\vec{r},t) = \rho(\vec{r},t)\vec{v}(\vec{r},t)$, and Π_{kl} denotes the momentum current tensor

$$\Pi_{kl} = \rho v_k v_l - \sigma_{kl} = \rho v_k v_l + p\delta_{kl}\,, \qquad (**)$$

which closes the equations. The term $\rho v_k v_l$ is the contribution to momentum current by the inertia of the flow (the terms responsible for turbulence). The quantity $\sigma_{kl} = -p\delta_{kl} + \sigma'_{kl}$ is known as stress tensor, p denotes the pressure; last σ'_{kl} encompasses the (bulk and shear) viscous forces, i.e. dissipative processes, which are neglected in Euler's equations, $\sigma'_{kl} = 0$.

a) Demonstrate that

$$\rho(\vec{r},t) = \rho_0 = const, \quad \vec{v}(\vec{r},t) = 0, \quad \text{and} \quad p(\vec{r},t) = p_0 = const$$

constitutes a solution of the field equations. Show that the linearized field equations for small perturbations $\delta\rho = \rho - \rho_0$, $\vec{v}$, and $\delta p = p - p_0$ to this reference state read

$$\partial_t \delta\rho + \rho_0 \nabla_k v_k = 0\,, \qquad \rho_0 \partial_t v_k = -\nabla_k \delta p\,.$$

Introduce the *isothermal compressibility* κ_T that reflects the pressure increase due to compression at constant temperature to linear order, $\delta p = \delta\rho/\rho_0\kappa_T$.

b) Derive a local conservation law, $\partial_t u + \operatorname{div}\vec{S} = 0$, for the energy density

$$u(\vec{r},t) = \frac{\rho_0}{2}\vec{v}(\vec{r},t)^2 + \frac{A}{2}\delta\rho(\vec{r},t)^2$$

for suitably chosen A relying on the approximations introduced so far. Determine the energy current density $\vec{S}(\vec{x},t)$.

c) Show that the linearized field equations allow for monochromatic longitudinal waves in $\vec{v}$ and scalar waves in $\delta\rho(\vec{r},t)$.

Tutorial 6.2 *Polaritons*

Consider the constitutive equation of the Lorentz-Drude model,

$$\partial_t^2 \vec{P}(\vec{x},t) + \frac{1}{\tau}\partial_t \vec{P}(\vec{x},t) + \omega_0^2 \vec{P}(\vec{x},t) = \frac{\omega_p^2}{4\pi}\vec{E}(\vec{x},t)\,,$$

with the relaxation time τ, characteristic frequency ω_0 and the plasma frequency ω_p.

a) Perform a spatio-temporal Fourier transform and determine the complex susceptibility $\chi(\omega)$, with $\vec{P}(\vec{k},\omega) = \chi(\omega)\vec{E}(\vec{k},\omega)$, as well as the dielectric function $\varepsilon(\omega) = 1 + 4\pi\chi(\omega)$.

b) Argue that the longitudinal modes follow from the zero of the dielectric function, $\varepsilon(\omega_*) = 0$, and determine the complex frequency ω_* in the case of weak damping.

c) Ignoring the damping, $\tau \to \infty$, determine the dispersion relation of the transverse modes.

d) Explain without calculation, in what frequency regime the damping is most important.

Problem 6.3 *Doppler effect*

Consider again the ideal fluid of Tutorial 6.1.

a) Convince yourself that a fluid moving at constant velocity $\vec{v}(\vec{r},t) = \vec{v}_0$ with constant density and pressure constitutes a valid solution of the field equations $(*)$ and $(**)$. Linearize the field equations around this new reference state relying again on the thermodynamic relation $\delta p = \delta\rho/\rho_0\kappa_T$.

b) Demonstrate the existence of monochromatic plane density waves and discuss the dispersion relation $\omega = \omega(\vec{k})$ with respect to the direction of the wave propagation $\vec{k}$ to the fluid velocity $\vec{v}_0$ and interpret your result.

Problem 6.4 *Elastic waves*

Small deformations of an isotropic elastic medium are described in terms of the vector field of displacements $\vec{u}(\vec{x},t)$, the velocities $\vec{v}(\vec{x},t)$, and the symmetric stress tensor field $\sigma_{ij}(\vec{x},t)$. The linearized field equations read

$$\partial_t u_i(\vec{x},t) = v_i(\vec{x},t) \quad \text{and} \quad \varrho_0 \partial_t v_i(\vec{x},t) = \nabla_j \sigma_{ij}(\vec{x},t)\,.$$

The field equations are closed with Hooke's law as constitutive equation,

$$\sigma_{ij} = K\delta_{ij}\,\mathrm{div}\,u + \mu\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\,\mathrm{div}\,u\right).$$

Here $\varrho_0 > 0$ is the mass density, $\mu > 0$ denotes the shear modulus and $K > 0$ the bulk modulus; the inverse of K gives the compressibility.

a) Perform a spatio-temporal Fourier transform and show that the medium supports longitudinal as well as transverse waves. Derive the corresponding dispersion relations and compare the different sound velocities.

b) Argue that in the limit of zero shear modulus, $\mu = 0$, one recovers the hydrodynamics of an ideal fluid introduced in Tutorial 6.1.

Problem 6.5 *Debye-Hückel and Thomas-Fermi theory*

As a natural extension of Drude's theory of metals, consider the constitutive equation

$$\partial_t \vec{j}^{(\mathrm{ind})}(\vec{r},t) + \frac{1}{\tau}\vec{j}^{(\mathrm{ind})}(\vec{r},t) + c_0^2 \,\mathrm{grad}\, \rho^{(\mathrm{ind})}(\vec{r},t) = \frac{\omega_p^2}{4\pi}\vec{E}(\vec{r},t)\,.$$

Here the new term $c_0^2 \,\mathrm{grad}\, \rho^{(\mathrm{ind})}(\vec{r},t)$ describes a restoring force similar to Euler's theory of fluids. The material is characterized by the velocity c_0, the relaxation time τ, and the plasma frequency ω_p. Derived quantities are the screening length $\lambda_0 = c_0/\omega_p$ and the conductivity $\sigma = \omega_p^2\tau/4\pi$.

a) Formulate a continuity equation for the scalar field

$$u_{\mathrm{M}}(\vec{r},t) = \frac{2\pi}{\omega_p^2}\left[\vec{j}^{(\mathrm{ind})}(\vec{r},t)^2 + c_0^2\rho^{(\mathrm{ind})}(\vec{r},t)^2\right]\,,$$

and specify the corresponding source term. Derive a conservation law for the total energy density consisting of the energy density of matter and the electromagnetic fields.

b) Consider an external charge Q located at the origin $\vec{r}_0 = 0$ of the medium. Discuss the induced charge density $\rho^{(\mathrm{ind})}(\vec{r})$, the electric field $\vec{E}(\vec{r})$, and the displacement field $\vec{D}(\vec{r})$ for the static case. Introduce an appropriate limit to recover the properties of an ideal conductor.

Hint: Since the question involves a calculation from Problem 4.3, you may refer to this result.

c) Neglecting dissipation, $\tau \to \infty$, the material supports longitudinal and transverse waves. Discuss the monochromatic plane waves; in particular, derive the dispersion relation of the longitudinal plasma oscillations and the transverse modes.

Due date: Tuesday, 6/5/07, at 9 a.m.