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T II: Elektrodynamik
(Prof. E. Frey)

Problem set 1

Tutorial 1.1 *Field lines*

Field lines are integral curves tangent to the vector field, i.e., up to reparametrization they fulfill

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\vec{x}(\tau) = \vec{E}(\vec{x}(\tau))\,, \qquad \vec{x}(\tau = 0) = \vec{x}_0\,.$$

Consider the vector field $\vec{E}(\vec{x}) = (y, x, 0)$. Verify that $\vec{E}$ is irrotational, i.e., $\vec{\nabla} \times \vec{E} = 0$, and construct a scalar potential $\varphi(\vec{x})$ by evaluating the line integral

$$\varphi(\vec{x}) = -\int_{\mathcal{C}} \vec{E} \cdot \mathrm{d}\vec{l}\,,$$

for curves $\mathcal{C}$ connecting the origin with the point $\vec{x} = (x, y, z)$. Discuss the equipotential surfaces and calculate the field lines corresponding to $\vec{E}(\vec{x})$.

Tutorial 1.2 *Newton's theorem*

a) Prove the electrostatic analog of Newton's theorem:

For a spherically symmetric charge (or mass, in the case of gravity) distribution $\rho(r)$, the radial component of the electric field, $E_r = \vec{E} \cdot \vec{r}/r$, is given by

$$E_r = \frac{Q(r)}{r^2} \qquad \text{with} \qquad Q(r) = 4\pi \int_0^r \rho(R) R^2 \mathrm{d}R\,,$$

i. e. the same as if the charge in the sphere of radius R is located at the center of the sphere.

Calculate also the associated electrostatic potential.

Note that the Poisson equation in spherical coordinates reads

$$-4\pi\rho = \nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial \varphi}{\partial \vartheta}\right) + \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2 \varphi}{\partial \phi^2}\,.$$

b) As an application of Newton's theorem, consider a charge-free spherical cavity concentric with the center of a spherically symmetric charge distribution. What is the electric force on a test charge inside this hole?

Problem 1.3 *Hamilton formulation*

Consider a magnetic field in two dimensions. By virtue of the magnetic Gauss's law $\vec{\nabla} \cdot \vec{B} = \nabla_x B_x + \nabla_y B_y = 0$, a 'vector' potential $A_z(x, y)$ can be introduced, such that $B_x = \nabla_y A_z$, $B_y = -\nabla_x A_z$.

a) Argue that the field lines corresponding to $\vec{B}$ can be interpreted in terms of Hamiltonian flows in phase space of a suitable equivalent mechanical system with a single degree of freedom.

b) Summarize some basic facts on the geometry of a Hamiltonian flow in the p–q-plane.

c) Discuss the magnetic field lines corresponding to the vector potentials

$$\text{i)} \quad A_z(x, y) = \frac{y^2}{2} + [1 - \cos(x)]\,, \qquad \text{ii)} \quad A_z(x, y) = \frac{y^2}{2} - \frac{x^2}{2}\,.$$

Problem 1.4 *Penning trap*

Consider the motion of a particle that has a charge q and mass m in a constant uniform magnetic field $\vec{B} = B\hat{e}_z$ and an electric quadrupol potential ($U_0 > 0$)

$$\varphi(\vec{x}) = -\frac{U_0}{2r_0^2}(x^2 + y^2 - 2z^2)\,, \qquad \vec{x} = (x, y, z)\,.$$

a) Show that the non-relativistic equation of motion for the particle in the x–y plane for the case $U_0 = 0$ leads to oscillatory motion. Determine the cyclotron frequency ω_c characterizing the oscillation. It is favorable to introduce a complex variable $\xi := x + \mathrm{i}y$.

b) Determine the electric field $\vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x})$ and verify that $\vec{E}$ is solenoidal, i.e., $\vec{\nabla} \cdot \vec{E}(\vec{x}) = 0$.

c) Show that the magnetic field does not couple to the motion along the z-direction, and determine the characteristic frequency ω_z for the corresponding harmonic oscillations in the quadrupol field.

d) Solve the complete equations of motion in the x–y plane and show that the general solution is a superposition of two oscillatory motions with a perturbed cyclotron frequency ω_c' and the *magnetron* frequency ω_M. Provide conditions such that the orbits are stable. Discuss the case $\omega_z \ll \omega_c$ in particular.

Problem 1.5 *Hydrogen atom*

Quantum mechanics reveals that the electron in a hydrogen atom should be described in terms of a wave function $\psi(\vec{r})$ (probability amplitude) giving rise to a smeared electron cloud corresponding to a charge density, $\rho_e(\vec{r}) = -e|\psi(\vec{r})|^2$. At the center of the atom, the proton is localized at a much smaller length scale, and the contribution to the charge density may be modeled as a point charge, $e\delta(\vec{r})$. Determine the (total) electrostatic potential φ

a) for the (1s orbital, K-shell) ground state of the hydrogen atom. Here the wave function is spherically symmetric

$$\psi(\vec{r}) = \frac{1}{\sqrt{\pi a^3}}\,\mathrm{e}^{-r/a}$$

where $a = \hbar^2/2me^2 = 0.529 \times 10^{-8}$cm denotes the Bohr radius.

b) for the spherically symmetric first excited state (2s orbital, L-shell)

$$\psi(\vec{r}) = \frac{1}{\sqrt{8\pi a^3}}\left(1 - \frac{r}{2a}\right)\mathrm{e}^{-r/2a}\,.$$

Problem 1.6 *Harmonic functions*

Consider a scalar field in three-dimensional space $\varphi : U \to \mathbb{R}$ (where U is an open subset of $\mathbb{R}^3$) that is harmonic, i.e., satisfies the Laplace equation $\nabla^2\varphi(\vec{x}) = 0$.

a) Proof the mean value theorem: Let $\vec{x} \in U$ and take a ball $B_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^3 : |\vec{y} - \vec{x}| \leq r\} \subset U$ of radius r around $\vec{x}$. If φ is a harmonic function then

$$\varphi(\vec{x}) = \frac{1}{4\pi r^2} \int_{\partial B_r(\vec{x})} \varphi(\vec{y})\,\mathrm{d}f(\vec{y})\,, \qquad \partial B_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^3 : |\vec{y} - \vec{x}| = r\}\,,$$

i.e., the average over the surface of a sphere of a harmonic function reproduces its value at the center of the sphere.
To demonstrate the property argue that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{4\pi r^2} \int_{\partial B_r(\vec{x})} \varphi(\vec{y})\,\mathrm{d}f(\vec{y})\right] = \frac{1}{4\pi r^2} \int_{\partial B_r(\vec{x})} \left(\vec{\nabla}\varphi(\vec{y})\right) \cdot \vec{n}\,\mathrm{d}f(\vec{y})\,,$$

where $\vec{n}$ denotes the normal vector of the sphere. Then apply Gauss's theorem to show that the derivative actually vanishes. Complete the proof by evaluating the mean value for sufficiently small radii.

b) Apply the mean value theorem to proof the related property

$$\varphi(\vec{x}) = \frac{1}{4\pi r^3/3} \int_{B_r(\vec{x})} \varphi(\vec{y})\,\mathrm{d}f(\vec{y})\,,$$

i.e., the volume average over a sphere of a harmonic function yields the value at the center of the sphere.

c) As a corollary conclude the maximum principle: if $K \subset U$ is compact, then φ restricted to K attains its maximum and minimum on the boundary of K.

d) Prove Earnshaw's theorem:

> A charge cannot be maintained in a stable stationary equilibrium solely by an electrostatic potential.

Due date: Wednesday, 5/2/07, at 1 p.m.