Department für Physik
Arnold-Sommerfeld-Zentrum
für Theoretische Physik
Ludwig-Maximillians-Universität

SoSe 05

Set 11
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T IV: Thermodynamik und Statistik
(Prof. E. Frey)

Problem set 11

Problem 11.1 *Bose gas in two dimensions*
Show that for an ideal Bose gas in two dimensions no Bose-Einstein condensation occurs.

Problem 11.2 *Pauli para-magnetism*
Consider the effect of spin on the magnetic susceptibility on a non-interacting electron gas. The hamiltonian is given by

$$\mathcal{H} = \sum_{s\mathbf{k}} \epsilon_{s\mathbf{k}} n_{s\mathbf{k}} \,, \qquad s = \uparrow, \downarrow \,, \quad n_{s\mathbf{k}} = 0, 1.$$

The electrons are fermions with a dispersion relation

$$\epsilon_{\uparrow\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - gH \,, \quad \epsilon_{\downarrow\mathbf{k}} = \frac{\hbar^2 k^2}{2m} + gH$$

Find the free energy F, the magnetization $M = -(\partial F/\partial H)_T$ and the susceptibility $\chi_T = (\partial M/\partial H)_T$ of the system for small fields H. Since the Fermi energy (chemical potential) is large compared to the temperatures of interest, restrict the discussion to $T = 0$. Then the problem reduces to finding the ground state energy compatible with the Pauli principle.

Problem 11.3 *Bose gas in a harmonic trap*
Recently it has been possible to cool simple gases to very low temperatures using optical traps. As a simplified model consider the case of $N \gg 1$ non-interacting bosons confined by a spherical harmonic trap. The energy eigenvalues for each particle are given by

$$\epsilon_{\mathbf{n}} = \hbar\omega(n_x + n_y + n_z) \,, \qquad n_x, n_y, n_z \in \mathbb{N}_0$$

Here the energies are shifted such that the ground state corresponds to $\epsilon = 0$, which can be achieved by a corresponding shift of the chemical potential. Using the grand canonical ensemble for non-interacting bosons

$$\frac{\Phi}{k_B T} = \sum_{\mathbf{n}} \ln(1 - z e^{-\beta\epsilon_{\mathbf{n}}}) \,, \qquad z = e^{\beta\mu}$$

relate the particle number N to the fugacity z. For sufficiently low temperatures the ground state is populated macroscopically, and deserves special treatment. Separate the sum over all states into ground state and remaining ones. The sum over non-condensate states can be replaced by an integral as in continuum BEC.

Determine the fugacity where the condensate occurs. Show that the critical temperature where the phase transition occurs is given by

$$k_B T_c = \hbar\omega \left(\frac{N}{g_3(1)} \right)^{1/3} \gg \hbar\omega$$

with the Bose function $g_3(z) = \sum_{j=1}^{\infty} z^j/j^3$. Evaluate the condensate fraction for $T \leq T_c$.

Calculate the energy and specific heat in the low temperature phase $T \leq T_c$.

Problem 11.4 *Rotons*

In superfluid ^{4}He the dispersion relation $\omega_{\mathbf{k}}$ of collective density fluctuations exhibits a minimum at some finite wave number k_0. The elementary excitations to wave numbers close to k_0 can approximately regarded as non-interacting bosons called rotons, and there the dispersion relation is given by

$$\hbar\omega_{\mathbf{k}} = \Delta + \frac{\hbar^2 (k - k_0)^2}{2m^*}, \qquad \Delta > 0\,, m^* > 0$$

Use the effective hamiltonian

$$\mathcal{H} = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} n_{\mathbf{k}}\,, \qquad n_{\mathbf{k}} = 0, 1, ...$$

and calculate the mean number of rotons $N_{rot} = \sum_{\mathbf{k}} \bar{n}_{\mathbf{k}}$ in leading order at low temperatures. Convince yourself that the mean occupation numbers $\bar{n}_{\mathbf{k}} = [\exp(\beta\hbar\omega_{\mathbf{k}}) - 1]^{-1}$ are small so that one can replace them by the Boltzmann value $\bar{n}_{\mathbf{k}} \approx e^{-\beta\hbar\omega_{\mathbf{k}}}$. Evaluate the mean energy $E = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \bar{n}_{\mathbf{k}}$ and find the roton contribution to the specific heat.

***Problem 11.5** *relativistic electrons*

Calculate the energy and pressure of a relativistic degenerate Fermi gas, i.e. temperature $T = 0$. Discuss the cross-over from the non-relativistic to the ultra-relativistic case.