Heterotic Branes in M & F Theories

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A. Lukas, B. Ovrut, K.S.S. & D. Waldram, hep-th/9806051
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Outline

- Heterotic M-Theory in 5D and its 3-brane solution
- Oxidation up to an intersection of 3 M5 branes
- Elliptically fibered CY3 and oxidation to 6D
- T-duality and the F-Theory limit
- The 3-brane from an F-Theory perspective

Heterotic M-Theory in 5D

• Dimensional reduction of 11D supergravity on a Calabi-Yau 3-fold with 4-form flux turned on produces a 5D 8-supercharge (minimal 5D) supergravity coupled to a matter sector determined by the cohomology of the CY3. Overall, there are $h^{1,1}$ vector fields in 5D, one of which must belong to the minimal 5D supergravity multiplet. So there are $(h^{1,1} - 1)$ vectors belonging to 5D vector supermultiplets. All $h^{1,1}$ of the 5D vectors emerge from the 11D 3-form gauge field upon compactification in which two indices of the 3-form correspond to $(1, 1)$ CY3 indices.

Cadavid, Ceresole, D'Auria & Ferrara 1995; Lukas, Ovrut, K.S.S & Waldram 1998

• Each of the $(h^{1,1} - 1)$ 5D vector multiplets also contains a single real scalar field, emerging from the 11D metric as a Kaluza-Klein scalar. In addition, there is a set of 4 scalar degrees of freedom belonging to a 5D "universal hypermultiplet" containing the last of the $h^{1,1}$ KK scalars arising from the 11D metric, plus a complex scalar arising from the 11D 3-form gauge field upon expansion on the complex $(3, 0)$ form of the CY3, plus one more axionic scalar arising upon dualizing the 5D 3-form components of the 11D 3-form gauge field.

Truncation to the 5D effective theory

• Dimensional reduction on a Calabi-Yau manifold does not, in general, produce a consistent truncation to a lower-dimensional Kaluza-Klein theory. Nonetheless, CY manifolds can produce a kind of "intermediate consistency" upon dimensional reduction in the sense that integration over the KK massive modes does produce corrections to the lower-dimensional theory, but these corrections can be purely of higher-derivative character. CY3 reductions of M-theory fall into this class.

Duff, Ferrara, Pope & K.S.S 1990

• Of the $(1,1)$ type 5D KK scalar fields, $(h^{1,1}-1)$ correspond to "shape moduli" $b^{\Lambda} = \mathcal{V}^{-\frac{1}{3}} a^{\Lambda}$ of the CY3 manifold, while the remaining scalar of this type corresponds to the overall volume $\mathcal{V}$ of the CY3. One may consider a 5D bosonic-field effective action for just the 5D metric plus the CY3 shape and volume moduli. Fluxes turned on for the 4-form field strength in the CY directions are characterized by $h^{2,2} = h^{1,1}$ moduli, $G_{ABCD} = \alpha_{\Lambda} \nu^{\Lambda}_{ABCD}$.

3-brane solutions in 5D

• The $h^{2,2}$ fluxes contribute a potential for the 5D modulus fields, with the consequence that the 5D effective theory does not admit a maximally symmetric solution. Instead it favors 3-brane solutions, which in 5D are of codimension one. Such solutions are characterized by linear harmonic functions, and in order to prevent the harmonic function from reaching zero, one puts in kinks corresponding to positive and negative tension Hořava-Witten end-branes. Including sources at the end-brane locations $M_4^{(1)}$ and $M_4^{(2)}$, the resulting 5D effective action is

$$-\int_{M_5}\sqrt{-g}\left[R+G_{\Lambda\Sigma}\partial_\alpha b^\Lambda\partial^\alpha b^\Sigma+\frac{1}{2}\mathcal{V}^{-2}\partial_\alpha\mathcal{V}\partial^\alpha\mathcal{V}+\frac{1}{2}\mathcal{V}^{-2}G^{\Lambda\Sigma}\alpha_\Lambda\alpha_\Sigma\right]$$
$$+2\sqrt{2}\int_{M_4^{(1)}}\sqrt{-g}\,\mathcal{V}^{-1}\alpha_\Lambda b^\Lambda-2\sqrt{2}\int_{M_4^{(2)}}\sqrt{-g}\,\mathcal{V}^{-1}\alpha_\Lambda b^\Lambda$$

where the b^Λ shape modulus fields need further to be constrained by the volume-fixing condition $d_{\Lambda\Sigma\Theta}b^\Lambda b^\Sigma b^\Theta=6$; $\Lambda=1,\ldots,h^{1,1}$.

• The 5D effective action has a class of 3-brane solutions given by

$$
\begin{aligned}
ds_5^2 &= a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2 \\
a &= \tilde{k}\mathcal{V}^{1/6} , \qquad b = k\mathcal{V}^{2/3} \\
b^\Lambda &= \mathcal{V}^{-1/6} f^\Lambda , \qquad \mathcal{V} = \left(\frac{1}{6} d_{\Lambda\Sigma\Theta} f^\Lambda f^\Sigma f^\Theta\right)^2 \\
d_{\Lambda\Sigma\Theta} f^\Sigma f^\Theta &= H_\Lambda , \qquad H_\Lambda = 2\sqrt{2} k \alpha_\Lambda |y| + k_\Lambda
\end{aligned}
$$

determined by a set of linear harmonic functions H_Λ, as is natural for a codimension-one brane solution.

• The general 5D 3-brane solution above involves all the CY3 shape and volume modulus scalar fields. A specialization of this general solution involves only the CY3 volume modulus, or "breathing mode". Let

$$
d_{\Lambda\Sigma\Theta}\bar{\alpha}^\Sigma\bar{\alpha}^\Theta = \frac{2}{3}\alpha_\Lambda , \quad \alpha = 9\left(\frac{1}{6} d_{\Lambda\Sigma\Theta}\bar{\alpha}^\Lambda\bar{\alpha}^\Sigma\bar{\alpha}^\Theta\right)^{2/3} , \quad k_\Lambda = 6kc_0\frac{\alpha_\Lambda}{\alpha}
$$

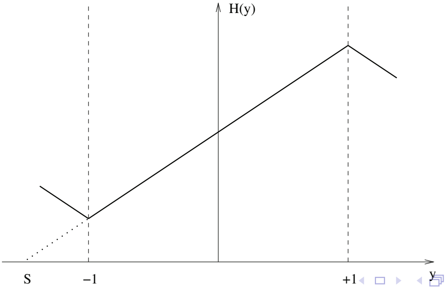
Universal brane solution

• Then one has the "universal solution"

$$\begin{aligned} a &= a_0 H^{1/2} \ , \qquad b = b_0 H^2 \ , \qquad b^\Lambda = 3\alpha^{-1/2}\bar{\alpha}^\Lambda \\ \mathcal{V} &= b_0 H^3 \ , \qquad H = \frac{\sqrt{2}}{3}\alpha|y| + c_0 \end{aligned}$$

depending only on the CY3 breathing mode $\mathcal{V}(y)$.

• The 'kinks' in the harmonic function correspond to the end-brane source actions on $M_4^{(1)}$ and $M_4^{(2)}$. They are needed to keep the harmonic function $H(y)$ from hitting zero and giving rise to a singularity:

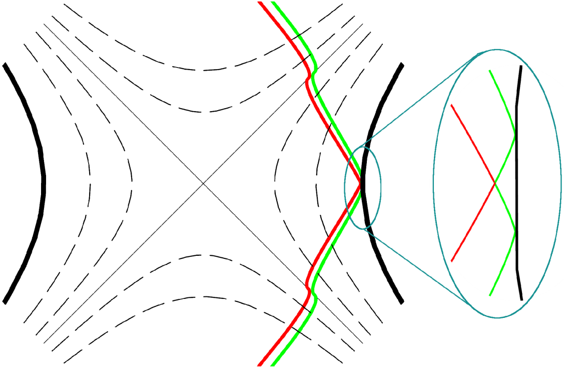


• Such 3-brane solutions, including also extra branes in the middle of the 5D bulk, have played a key rôle in the development of cyclic cosmological models. Provided the overall volume of the CY3 remains small, the lack of a completely consistent truncation from 11D to 5D can be ignored, owing to the mass gap between the zero modes and the first non-zero Kaluza-Klein level.

• One question that has been raised is whether such 3-branes can avoid falling into the singularity when the static solution is promoted to a time-dependent one. Time dependence can be introduced into the solution simply by changing the transverse-space harmonic function to $H = \frac{\sqrt{2}}{3}\alpha|y| + kt$. Then there is a time at which $H = 0$ regardless of where the end-brane sources are put in the static solution.

Chen, Chong, Gibbons, Lu & Pope 2005

• A detailed analysis of the brane solution evolution shows, however, that the inclusion of simple matter on the end-branes (such as scalar-field kinetic energy) causes the encounter with the singularity to separate into a sequence of bounces. Lehners, McFadden & Turok 2007



Exact oxidation to 3 intersecting M5 branes in 11D

- For the universal brane solution, a special situation obtains: this solution can be *exactly* oxidized to a solution of 11D supergravity.

Chen, Chong, Gibbons, Lu & Pope 2005; Lehners, Liu, Lu & K.S.S., work in progress

- The bosonic equations of motion of 11D supergravity are given by

$$
\begin{gathered}
dF_{(4)} = 0\,, \qquad d{*}F_4 = \tfrac{1}{2} F_{(4)} \wedge F_{(4)} \\
R_{MN} = \tfrac{1}{12}\left(F^2_{MN} - \tfrac{1}{12} F^2 g_{MN}\right)
\end{gathered}
$$

- These equations have a solution for 3 intersecting M5 branes

$$
\begin{aligned}
ds^2_{11} &= (H_1 H_2 H_3)^{-\frac{1}{3}} dx^\mu dx_\mu \\
&\quad + H_1^{-\frac{1}{3}} (H_2 H_3)^{\frac{2}{3}} (dz_1^2 + dz_2^2) + H_2^{-\frac{1}{3}} (H_1 H_3)^{\frac{2}{3}} (dz_3^2 + dz_4^2) \\
&\quad + H_3^{-\frac{1}{3}} (H_1 H_2)^{\frac{2}{3}} (dz_5^2 + dz_6^2) + (H_1 H_2 H_3)^{\frac{2}{3}} dy^2\,, \\
{*}F_{(4)} &= d^4x \wedge (dz^1 \wedge dz^2 dH_1^{-1} + dz^3 \wedge dz^4 dH_2^{-1} + dz^5 \wedge dz^6 dH_3^{-1})
\end{aligned}
$$

where the H_i harmonic functions depend only on the overall-transverse coordinate y but not on the 6 relative-transverse coordinates $z_1, \ldots, z_6$.

• Taking the case of three coincident M5 branes, $H_1 = H_2 = H_3 \equiv H = 1 + ky$, one can generalize the three-intersecting-M5 brane solution to a solution describing three M5 branes wrapping three different 2-cycles of a CY3:

$$\begin{aligned} ds^2_{11} &= H^{-1} dx^\mu dx_\mu + H ds^2_{CY} + H^2 dy^2 \\ F_{(4)} &= k I_{(4)} \ , \qquad *F_{(4)} = dx^4 \wedge I_{(2)} \wedge dH^{-1} \end{aligned}$$

where $I_{(4)} = *_6 I_{(2)}$. The Bianchi identity and equation of motion for the 4-form field strength are satisfied provided $I_{(2)}$ and $I_{(4)}$ are harmonic.

• The difficult point in establishing consistency of this solution without turning on an infinite number of CY3 KK modes arises from the Einstein equation. One must make sure that the quadratic terms in $F_{(4)}$ in the stress tensor on the RHS of the Einstein equation do not generate sources for higher KK modes. For this, one requires that $(I_{(4)})^2_{ab} = 12\delta_{ab}$ and that the harmonic 4-form $I_{(4)}$ be symmetric in the six a directions in some vielbein basis. This gives a unique solution $I_{(4)} = e^{1234} + e^{3456} + e^{1256}$.

• The dual of $I_{(4)}$ in the CY space is then the Kähler 2-form,

$$*_6 I_{(4)} = J_{(2)} = e^{12} + e^{34} + e^{56}$$

Can one separate the 5-branes?

In the above construction of the heterotic M-theory 3-brane ground state in terms of M-theory 5-branes, the 3 harmonic functions H_n are taken to be identical, corresponding to coincident 5-brane centers in the orbifold direction. For 5-branes wrapped on toroidal cycles, the centers could equally well be separated, but in the case of 5-branes wrapping CY3 cycles, any attempt at separation leads directly to a loss of the consistent embedding in $D = 11$ M-theory.

• However, one can certainly embed 5-branes at differing orbifold locations while also taking into account the resulting Kaluza-Klein corrections to the CY metric and 3-form background. The techniques for doing this were developed for non-standard 5-brane embeddings in heterotic M-theory. Lukas, Ovrut & Waldram 1998 This leads to a calculable expansion in terms of $\epsilon_S = (\frac{\kappa}{4\pi})^{2/3}\frac{2\pi\rho}{v^{2/3}}$ where v is the CY volume and $\pi\rho$ is the orbifold size.

Elliptically fibered CY3

F. Bonetti & T. Grimm 2011; T. Grimm, T. Pugh & K.S.S, work in progress

• The exact intersecting 11D M5-brane solution $\leftrightarrow$ 3-brane in 5D discussed above was obtained for an arbitrary CY3 compactification manifold.

• Now we will specialize the CY3 to an elliptically fibered space and will perform a T-duality transformation on the T^2 fiber in order to reinterpret the above 3-brane solution as 6D F-theory solution. For this purpose, we will need to consider again the full 5D effective theory, but now specialized to the case of an elliptically fibered CY3.

• Let the 2-form divisor ω_0 be associated to the elliptic fiber, let ω_α be associated to the 4-dimensional base and let ω_i be associated to singularity resolution blowups. Then 5D vector multiplets naturally can be labeled in a similar way, with vectors $A^\Lambda = (A^0, A^\alpha, A^i)$ and scalars $L^\Lambda = (L^0, L^\alpha, L^i)$.

• The CY3 intersection numbers $d_{\Lambda\Sigma\Theta}$ then become constrained:

$$
\begin{aligned}
&d_{000} = \eta_{\alpha\beta} K^\alpha K^\beta \,, && d_{00\alpha} = \eta_{\alpha\beta} K^\beta \,, && d_{0\alpha\beta} = \eta_{\alpha\beta} \,, \\
&d_{\alpha\beta\gamma} = 0 \,, && d_{0i\Lambda} = 0 \,, && d_{\alpha\beta i} = 0 \,, \\
&d_{\alpha ij} = -C_{ij} \eta_{\alpha\beta} C^\beta \,, && d_{ijk} = \text{ unconstrained}
\end{aligned}
$$

• The fields then arrange themselves into multiplets of 5D supersymmetry. For example, the 5D metric g_{mn} together with the vector A^0_m form the bosonic part of the 5D supergravity multiplet. The remaining $h^{1,1}(\hat{Y}) - 1$ vectors combine with the constrained scalars b^Λ to form $n^5_{\mathcal{V}} = h^{1,1}(\hat{Y}) - 1$ vector multiplets. Note also that the $4(h^{1,2}(\hat{Y}) + 1)$ scalars given by $q^u = (\mathcal{V}, \Phi, z^k, \bar{z}^{\bar{k}}, \xi^K, \tilde{\xi}_K)$ belong to $n^5_H = h^{1,2}(\hat{Y}) + 1$ hypermultiplets.

- The resulting 5D action is then given by

$$S^{(5)}_{(M)} = \int_{\mathcal{M}_5} \Big[\frac{1}{2} R * 1 - \frac{1}{2} G_{\Lambda\Sigma}(M) dM^\Lambda \wedge * dM^\Sigma - \frac{1}{2} h_{uv} dq^u \wedge * dq^v$$
$$- \frac{1}{2} G_{\Lambda\Sigma}(M) F^\Lambda \wedge * F^\Sigma - \frac{1}{12} \mathcal{N}_{\Lambda\Sigma\Theta} A^\Lambda \wedge F^\Sigma \wedge F^\Theta \Big]$$

where

$$G_{\Lambda\Sigma}(M) = -\frac{1}{2}\mathcal{N}_{\Lambda\Sigma} + \frac{1}{2}\mathcal{N}_\lambda \mathcal{N}_\Sigma \, , \qquad \mathcal{N}_\Lambda = \partial_{M^\Lambda} \mathcal{N}_{(M)}|_{\mathcal{N}_{(M)}=1} \, ,$$
$$\mathcal{N}_{\Lambda\Sigma} = \partial_{M^\Lambda}\partial_{M^\Sigma} \mathcal{N}_{(M)}|_{\mathcal{N}_{(M)}=1} \, , \quad \mathcal{N}_{\Lambda\Sigma\Theta} = \partial_{M^\Lambda}\partial_{M^\Sigma}\partial_{M^\Theta} \mathcal{N}_{(M)}|_{\mathcal{N}_{(M)}=1}$$

and

$$\mathcal{N}_{(M)} = \eta_{\alpha\beta} M^0 M^\alpha M^\beta - 4 \eta_{\alpha\beta} C^\alpha C_{ij} M^\beta M^i M^j + \frac{1}{192} \eta_{\alpha\beta} K^\alpha K^\beta M^0 M^0 M^0$$
$$+ \frac{1}{2} \eta_{\alpha\beta} C^\alpha C_{ij} M^0 K^\beta M^i M^j + \frac{4}{3} d_{ijk} M^i M^j M^k$$

is a function of the scalar fields M which is constrained to take the value 1.

Oxidizing to 6D

• The T^2 fiber of the elliptically fibered CY3 allows for a reinterpretation of the above 5D theory in $D = 6$, by comparison with a general (1,0) 6D supergravity with gauged matter upon making a standard Kaluza-Klein reduction to 5D on a circle. The 6D theory will be described by a "pseudo-action" whose equations of motion need to be complemented by a 3-form self-duality condition imposed by hand after variation of the pseudo-action.

• The 6D theory is taken to contain n_T^6 (1,0) 6D tensor multiplets, n_H^6 hypermultiplets and to have a group G gauged by the vectors of the 6D vector multiplets. The reduction from the 6D to the 5D theory will work in the Coulomb branch of the 6D nonabelian theory, in which adjoint representation scalar fields take VeVs that trigger a breakdown of the gauge group G to its Cartan subalgebra. F. Bonetti & T. Grimm 2011

• 6D (1,0) supergravity contains fields $(e_M^A, \psi_M, B^+_{MN})$, where the 2-form gauge field B^+_{MN} is required to have a 6D self-dual 3-form field strength and the gravitino ψ_M is chiral.

• 6D (1,0) tensor multiplets contain fields (B^-_{MN}, χ, σ), where B^-_{MN} has an anti-self-dual field strength, while the spinor χ is antichiral; 6D (1,0) vector multiplets contain fields (A_M, λ), where λ is a chiral spinor.

• Self-duality or anti-self-duality constraints can either be handled by a non-Lorentz-covariant procedure Schwarz; Henneaux & Teitelboim or by covariant methods including additional gauge symmetry Pasti, Sorokin & Tonin or simply by imposing the self-duality/anti-self-duality constraint by hand on solutions of the field equations derived from a pseudo-action.

• 6D (1,0) theories are generically anomalous; construction of anomaly-free models imposes various constraints on the matter content. For suitable matter content, the anomaly polynomial factorizes and the remaining anomalies can be removed by a generalization of the Green-Schwarz mechanism. Green & Schwarz 1984; Sagnotti 1992; Schwarz 1996

T-duality transition to F-Theory

• In order to pass over to the F-Theory realization of the above system, consider M-Theory on a product manifold

$$\mathcal{M}_{11} = T^2 \times B_2 \times \mathbb{R}^{1,4} ,$$

where B_2 is a Kähler manifold of complex dimension 2.

• One can write the T^2 fiber metric as

$$ds^2_{T^2} = \frac{v^0}{\mathrm{Im}\,\tau}\left[(dx_A + \mathrm{Re}\,\tau\, dx_B)^2 + (\mathrm{Im}\,\tau)^2 dx_B^2\right]$$

where v^0 is the volume of the torus and τ is the complex structure parameter. The period 1 coordinates x_A and x_B parametrize the A and B cycles of the torus. Compactification along the A-cycle reduces M-theory to type 10D Type IIA theory.

• Application of T-duality on the B cycle in this 11D $\to$ 10D reduction instead yields Type IIB theory on

$$\mathcal{M}_{10} = S^1 \times B_2 \times \mathbb{R}^{1,4} \ ,$$

where the circle S^1 corresponds to the B' cycle, *i.e.* the T-dual of the B cycle.

• The F-Theory limit is one in which the volume of the T^2 fiber and the resolution blowups vanish, but in which the overall CY3 volume remains finite. In the limit of vanishing T^2 volume v^0, the size of the T-dualized compact B'-cycle S^1 becomes infinite, thus leading effectively to Type IIB on $\mathcal{M}'_{10} = B_2 \times \mathbb{R}^{1,5}$, since the B' cycle has become decompactified.

• On an elliptically-fibered CY3, the same T-duality can be carried out fiberwise, with the T^2 moduli depending holomorphically on the complex coordinates of the base manifold B_2. Since the complex structure modulus τ yields the dilaton-axion system of Type IIB theory, one will end up with a non-trivial dilaton-axion profile varying along the base B_2.

Recovering the 3-brane solution in F-Theory

- The bosonic sector of the 6D theory has a pseudo-action of the form

$$S^{(6)} = \int_{\mathcal{M}_6} \Big[\frac{1}{2}\hat{R}\hat{*}1 - \frac{1}{4}g_{\alpha\beta}\hat{G}^{\alpha}_{(3)} \wedge \hat{*}\hat{G}^{\beta}_{(3)} - \frac{1}{2}g_{\alpha\beta}dj^{\alpha} \wedge \hat{*}dj^{\beta} - \frac{1}{2}h_{UV}\hat{D}q^{U} \wedge \hat{*}\hat{D}q^{V}$$

$$- 2\Omega_{\alpha\beta}j^{\alpha}b^{\beta}C_{IJ}\hat{F}^{I}_{(2)} \wedge \hat{*}\hat{F}^{J}_{(2)} - \Omega_{\alpha\beta}b^{\alpha}C_{IJ}\hat{B}^{\beta}_{(2)} \wedge \hat{F}^{I}_{(2)} \wedge \hat{F}^{J}_{(2)} - V^{(6)}\hat{*}\hat{1}\Big]$$

including a potential $V^{(6)} = -\frac{1}{4}\frac{1}{\Omega_{\alpha\beta}j^{\alpha}b^{\beta}}C^{-1IJ}{A_U}^{A}{}_{B}{A_V}^{B}{}_{A}k^{U}_{I}k^{V}_{J}$, where k^{U}_{I} and ${A_U}^{A}{}_{B}$ are functions of the hypermultiplet scalars. $A = 1, 2$ is an $SU(2)$ R-symmetry index of the 6D theory; $\alpha = 0, \dots n^{6}_{T}$ is an index in the fundamental representation of $SO(n^{6}_{T}, 1)$ for the tensor multiplets; $I = 1, \dots, \dim(G)$ and $U = 1, \dots, 4n^{6}_{H}$ counts the n^{6}_{H} hypermultiplets.

- It is natural that the 6D theory involve a potential of scalar fields, because the 5D theory from which it is lifted has such a potential, as a consequence of the 4-form fluxes that were turned on in the CY3 directions.

• As in 5D, the presence of the scalar potential has the consequence that the theory does not admit a 6D maximally symmetric solution. Instead, as in 5D, it has a preference for brane solutions.

• Accordingly, one tries a 3-brane ansatz of codimension two:

$$d\hat{s}^2 = e^{2W(y^c)}\eta_{\mu\nu}dx^\mu dx^\nu + g_{ab}(y^c)dy^a dy^b \ .$$

• Of the 6D hypermultiplet scalars q^U, all but two can consistently be set equal to constants; the remaining two, $\mathcal{V}$ and Φ, are affected by the runaway potential $V^{(6)}$; one then finds that the Einstein equation in the T^2 directions requires $D_a\Phi = 2\epsilon_{ab}\partial^b\mathcal{V}$, leaving just the scalar $\mathcal{V}$ to determine the solution. Using the Killing spinor equation to ensure an unbroken supersymmetry then leads to a 6D solution with quadratic structure

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + r^2(dz^2 + d\phi^2) \\ \mathcal{V} &= -2AV_0z^2 + Bz + C \ , \qquad r^2 = -2A^2V_0z^2 + ABz + AC \end{aligned}$$

for some integration constants A, B, C.

• The 6D brane solution may now be compared to the 3-brane of 5D heterotic M-Theory. For this, one must recall the structure of the CY3 divisors and the associated restrictions on intersection numbers $d_{\Lambda\Sigma\Theta}$. It is also necessary to take into account certain scalings of the 5D scalar fields needed in the F-theory limit: $M^0 \to \epsilon M^0\,,\ M^\alpha \to \epsilon^{-\frac{1}{2}} M^\alpha\,,\ M^i \to \epsilon^{\frac{1}{4}} M^i$.

• One then finds agreement between the quadratic-structure 6D solution and the general 5D 3-brane solution provided

$$k_0 f^0 = \pm k_0 \left(\frac{k_\alpha b^\alpha \Omega^{\beta\gamma} k_\beta k_\gamma}{8 k_0 k_\alpha b^\alpha - C^{-1ij} H_i H_j} \right)^{\frac{1}{2}}$$

$$H_i f^i = \mp C^{-1ij} H_i H_j \left(\frac{\Omega^{\alpha\beta} k_\alpha k_\beta}{16 k_\alpha b^\alpha (8 k_0 k_\beta b^\beta - C^{-1ij} H_i H_j)} \right)^{\frac{1}{2}}$$

$$k_\alpha f^\alpha = \pm (16 k_0 k_\alpha b^\alpha - C^{-1ij} H_i H_j) \left(\frac{\Omega^{\alpha\beta} k_\alpha k_\beta}{64 k_\alpha b^\alpha (8 k_0 k_\beta b^\beta - C^{-1ij} H_i H_j)} \right)^{\frac{1}{2}}$$

$$\mathcal{V} = \frac{\Omega^{\beta\gamma} k_\beta k_\gamma}{256 k_\alpha b^\alpha} (-C^{-1ij} k^2 \theta_i \theta_j z^2 - 2 C^{-1ij} k k_i \theta_j z + 8 k_0 k_\alpha b^\alpha - C^{-1ij} k_i k_j)$$

• The 6D solution has singularities where $\mathcal{V} = 0$. Choosing the constant C, one can place one of these at $z = 0$ and write $\mathcal{V} = 2AV_0z(p - z)$. The geodesic distance between the singularities is then $\frac{1}{8}p^2\sqrt{2AV_0}\pi$.

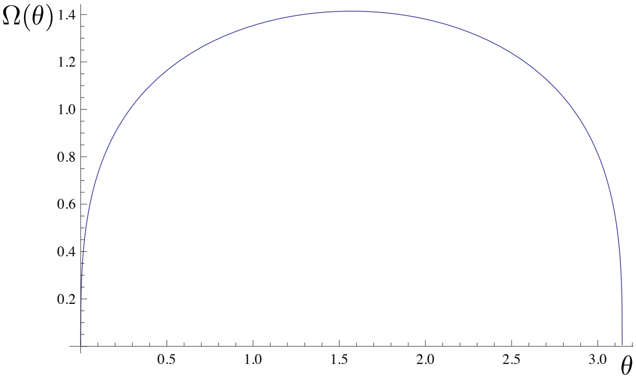
• In order to visualize the 6D solution and its singularity structure better, one can make a change of the 2D coordinates to give

$$ds^2 = 2A^2V_0z(p - z)(dz^2 + d\phi^2) = d\theta^2 + \Omega^2(\theta)d\phi^2$$

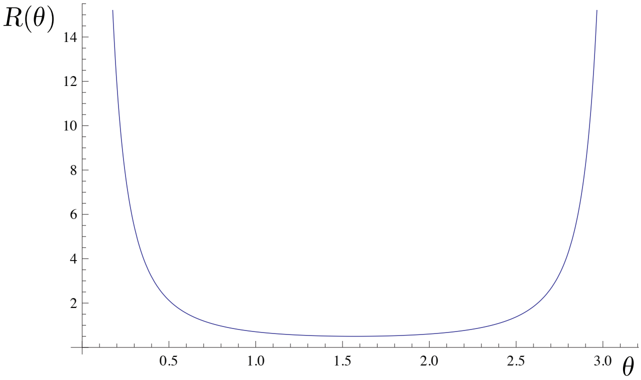
where

$$\begin{aligned}
\Omega^2 &= 2A^2V_0z(p - z) \\
\theta &= \frac{A\sqrt{V_0}}{2\sqrt{2}}\left((2z - p)\sqrt{z(p - z)} + p^2\tan^{-1}\left(\frac{\sqrt{z}}{\sqrt{p - z}}\right)\right)
\end{aligned}$$

- Then one may sketch $\Omega(\theta)$ for $A = (2V_0)^{-\frac{1}{2}}$ and $p = \sqrt{8}$:



- One may also sketch the corresponding 2D Ricci scalar:



• To understand better the nature of the 6D solution's singularities, include a 4D brane source term in the 6D action:

$$S_{brane} = Q \int_{\mathcal{M}_b} (\frac{1}{\mathcal{V}} \tilde{*}_4 1 + \tilde{s}^* C_4)$$

This should be located slightly away from $z = 0$ in order to move the resulting delta function term into the interior of the $z \in [0, p]$ interval.

• One then finds $\mathcal{V}(z) = -2V_0 A z^2 + 2Qz$, which yields the charge identification $Q = AV_0 p$.

Codimension-one lifts to codimension-two

• A natural question arises: how did a 5D codimension-one brane solution lift to a codimension-two solution in 6D?

• Normally, one would think that a 5D codimension-one solution would oxidize to a 6D solution that was evenly smeared along the A cycle of the CY3 T^2. The return to 5D would then be an instance of "vertical dimensional reduction" in the transverse space to the 3-brane's worldvolume. H. Lu, C.N. Pope & K.S.S. 1996

• However, in the present case, the lifted F-theory solution appears to be a genuinely codimension-two solution. What happens is that in the F-theory limit, the A and B cycles of the T^2 torus shrink to zero size while the T-dualized B' cycle opens up. The source remains a 3-brane, however, because although the B' cycle generically blows up, the fibration pinches at the singular point corresponding to the brane source.

Some open questions

• The 6D solution, with its nontrivial dilaton-axion sector, might seem to be associated to Type IIB 7-branes. However, this does not seem to be right because the Φ and V scalars of the 3-brane system do not derive from the IIB dilaton-axion system. So the proper F-theory origin of this 3-brane solution remains obscure.

• Is there a direct relation between the 6D F-theory brane solution and the interpretation of the 5D 3-brane as an intersection of three M5 branes?

• Is there added freedom in the 6D realization of these 3-brane solutions that could be of use in cosmological applications?