Supersymmetric Hidden Sectors for Heterotic Standard Models

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New Developments in Gravity, Cosmology and Strings

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Munich, Germany

March, 2013

The Compactification Vacuum

with Pantev

Calabi-Yau Threefold:

Consider the fiber product $\tilde{X} = B_1 \times_{\mathbb{P}^1} B_2$ where B_1, B_2 are both dP_9 surfaces. In a region of their moduli space such manifolds admit a fixed point free $\mathbb{Z}_3 \times \mathbb{Z}_3$ isometry. Then

$$X = \frac{\tilde{X}}{\mathbb{Z}_3 \times \mathbb{Z}_3}$$

is a smooth Calabi-Yau threefold torus-fibered over dP_9 with fundamental group

$$\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$$

Its Hodge data is

$$h^{1,1} = h^{1,2} = 3$$

ignore complex structure

Relevant here is the Dolbeault cohomology group

$$H^{1,1}(X,\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{\omega_1, \omega_2, \omega_3\}$$

where $\omega_i = \omega_{ia\bar{b}} dz^a d\bar{z}^{\bar{b}}$ are dimensionless (1,1)-forms on X with the properties

$$\omega_3 \wedge \omega_3 = 0, \quad \omega_1 \wedge \omega_3 = 3\omega_1 \wedge \omega_1, \quad \omega_2 \wedge \omega_3 = 3\omega_2 \wedge \omega_2$$

Defining the intersection numbers as

$$d_{ijk} = \frac{1}{v} \int_X \omega_i \wedge \omega_j \wedge \omega_k, \quad i,j,k = 1,2,3$$

where v is a reference volume $\Rightarrow$

$$d_{ijk} = \begin{pmatrix} (0,\frac{1}{3},0) & (\frac{1}{3},\frac{1}{3},1) & (0,1,0) \\ (\frac{1}{3},\frac{1}{3},1) & (\frac{1}{3},0,0) & (1,0,0) \\ (0,1,0) & (1,0,0) & (0,0,0) \end{pmatrix}$$

The {ij}-th entry is the triplet $(d_{\{ij\}k} | k = 1,2,3)$.

Noting that the structure group of TX is SU(3), we find that

$$c_1(TX) = c_3(TX) = 0$$

self − mirror

and

$$c_2(TX) = \frac{1}{v^{2/3}}(12\omega_1 \wedge \omega_1 + 12\omega_2 \wedge \omega_2)$$

Choosing the SU(3) generators to be hermitian ⇒

$$c_2(TX) = -\frac{1}{16\pi^2} trR \wedge R$$

where R is the curvature two-form. Note that $H^{2,0} = H^{0,2} = 0$ on a Calabi-Yau threefold ⇒ $H^{1,1}(X, \mathbb{C}) = H^2(X, \mathbb{R})$.

Furthermore, each $\omega_i, i = 1, 2, 3$ is dual to an effective curve

⇒ the Kahler cone is the positive quadrant

$$\mathcal{K} = H^2_+(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$$

⇒ The Kahler form can be expanded as

$$\omega = a^i \omega_i, \quad a^i > 0, \ i = 1, 2, 3$$

The a^i are the (1,1) Kahler moduli. Define the dimensionless volume modulus

$$V = \frac{1}{v}\int_X \sqrt{{}^6 g} = \frac{1}{6v}\int_X \omega \wedge \omega \wedge \omega = \frac{1}{6} d_{ijk} a^i a^j a^k$$

It is useful to consider the scaled “shape” moduli

$$b^i = V^{-1/3} a^i, \quad i = 1, 2, 3$$

They satisfy the constraint

$$d_{ijk} b^i b^j b^k = 6$$

and, hence, represent only two degrees of freedom.

Note that all moduli a^i, V, b^i are functions of the four coordinates $x^\mu, \mu = 0, \ldots 3$ of Minkowski space M_4.

Observable Sector Vector Bundle:

Consider a holomorphic vector bundle $\tilde{V}^{(1)}$ on $\tilde{X}$ with structure group $SU(4) \subset E_8$ constructed as the extension

$$0 \to V_1 \to \tilde{V}^{(1)} \to V_2 \to 0$$

of two rank 2 bundles V_1, V_2 that is equivariant under $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Take the observable sector vector bundle $V^{(1)}$ on X to be

$$V^{(1)} = \frac{\tilde{V}^{(1)}}{\mathbb{Z}_3 \times \mathbb{Z}_3}$$

The SU(4) structure group ⇒

$$E_8 \longrightarrow Spin(10)$$

The asociated Chern classes are $c_1(V^{(1)}) = 0$,

$$c_2(V^{(1)}) = \frac{1}{v^{2/3}}(\omega_1 \wedge \omega_1 + 4\omega_2 \wedge \omega_2 + 4\omega_1 \wedge \omega_2)$$

and

$$c_3(V^{(1)}) = 3 \;\Rightarrow\; \text{three matter families}$$

Choosing E_8 generators $\Rightarrow$

$$c_2(V^{(1)}) = -\frac{1}{16\pi^2} tr_{E_8} F^{(1)} \wedge F^{(1)}$$

To preserve N=1 supersymmetry in four-dimensions, $V^{(1)}$ must be

- $\text{slope} - \text{stable}$
- vanishing slope

where the slope is defined as

$$\mu(\mathcal{F}) = \frac{1}{rank(\mathcal{F})v^{2/3}} \int_X c_1(\mathcal{F}) \wedge \omega \wedge \omega$$

Clearly

$$\mu(V^{(1)}) = 0$$

It will be slope-stable if seven "maximally destabilizing" line sub-bundles have negative slope. This translates into the the following seven conditions.

$$-3(a^1 - a^2)(a^1 + a^2 + 6a^3) - 18a^1a^2 < 0$$
$$3(a^1 - a^2)(a^1 + a^2 + 6a^3) - 18a^1a^2 < 0$$
$$6(a^1 - a^2)(a^1 + a^2 + 6a^3) < 0$$
$$-6(a^1 - a^2)(a^1 + a^2 + 6a^3) - 18a^1a^2 < 0$$
$$-3(5a^1 - 2a^2)(a^1 + a^2 + 6a^3) + 9a^1a^2 < 0$$
$$-3(4a^1 - a^2)(a^1 + a^2 + 6a^3) + 9a^1a^2 < 0$$
$$3(a^1 - 4a^2)(a^1 + a^2 + 6a^3) + 9a^1a^2 < 0$$

The subspace $\mathcal{K}^s \subset \mathcal{K}$ satisfying these conditions is given by

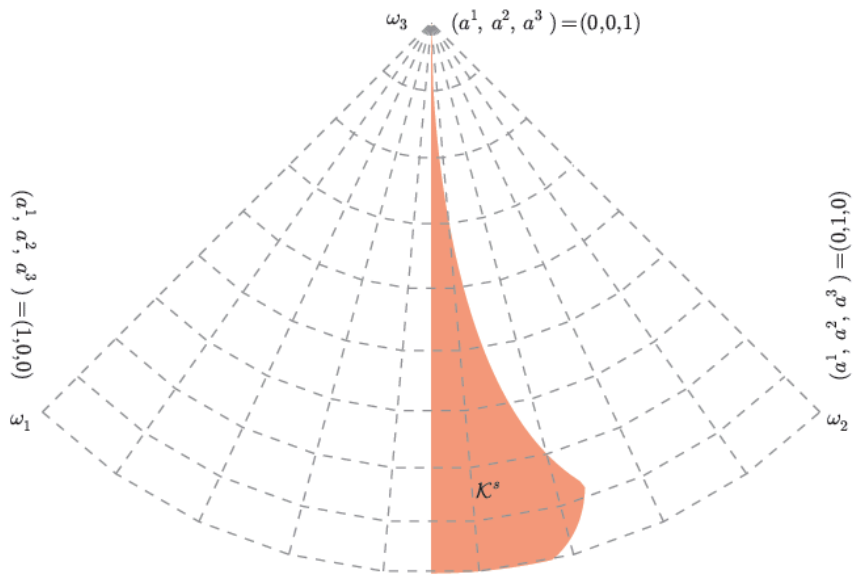

Figure 1: *Map projection of the unit sphere intersecting the Kähler cone, that is, the positive octant in $H^2(X, \mathbb{R}) \simeq \mathbb{R}^3$. The visible sector bundle $V^{(1)}$ is stable inside the red teardrop-shaped region $\mathcal{K}^s$. Every point in the projection represents a ray in the Kähler cone. For example, $(a^1, a^2, a^3) = (0, 1, 0)$ generates the ray in the ω_2 direction.*

In addition to $V^{(1)}$ turn on two flat Wilson lines, each generating a different $\mathbb{Z}_3$ factor of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ homotopy. $\Rightarrow$

$$Spin(10) \longrightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$$

Hidden Sector Vector Bundle:

We will consider the bundle

entirely as the sum of holomorphic line bundles classified by the elements of

$$H^2(X, \mathbb{Z}) = \{a\omega_1 + b\omega_2 + c\omega_3 | a, b, c \in \mathbb{Z}, a + b = 0 \text{ mod } 3\}$$

Denote the line bundle associated with

$$\mathcal{O}_X(a, b, c)$$

It is not necessary for a,b,c to be even integers since the bundle is always "spin".

Choose the hidden sector bundle to be

$$V^{(2)} = \oplus_{r=1}^{R} L_r, \quad L_r = \mathcal{O}_X(l_r^1, l_r^2, l_r^3)$$

where

$$l_r^1 + l_r^2 = 0 \text{ mod } 3, \quad r = 1, \dots, R$$

The structure group is $U(1)^R$ where each factor group has a specific embedding into the hidden E_8 .

Since $V^{(2)}$ is a sum of line bundles ⇒

$$c_1(V^{(2)}) = \sum_{r=1}^{R} c_1(L_r), \quad c_1(L_r) = \frac{1}{v^{2/3}}(l_r^1\omega_1 + l_r^2\omega_2 + l_r^3\omega_3)$$

and

$$c_2(V^{(2)}) = c_3(V^{(2)}) = 0$$

However, the relevant quantity is

$$ch_2(V^{(2)}) = \sum_{r=1}^{R} ch_2(L_r) = \sum_{r=1}^{R} \frac{1}{2} c_1(L_r) \wedge c_1(L_r)$$

Specifically, we will need

$$\frac{1}{16\pi^2} tr_{E_8} F^{(2)} \wedge F^{(2)} = \sum_{r=1}^{R} a_r c_1(L_r) \wedge c_1(L_r)$$

where

$$a_r = \frac{1}{4} tr_{E_8} Q_r^2$$

Blumenhagen, Honecker, Weigand

Q_r is the generator of the i-th U(1) factor embedded into the 248 representation of the hidden E_8

Wrapped Five-Branes:

The vacuum can also contain five-branes wrapped on two-cycles $\mathcal{C}_2^{(n)}, \ n = 1, \ldots, M$ in X. ⇒ Each five-brane is described by a (2,2)-form $W^{(n)}$ Poincare dual to $\mathcal{C}_2^{(n)}$. To preserve N=1 supersymmetry, each $W^{(n)}$ must be an effective class.

The Vacuum Constraint Conditions

Anomaly Cancellation:

$$-\frac{1}{16\pi^2}\,\mathrm{tr}\,R\wedge R+\frac{1}{16\pi^2}\,\mathrm{tr}_{E_8}\,F^{(1)}\wedge F^{(1)}+\frac{1}{16\pi^2}\,\mathrm{tr}_{E_8}\,F^{(2)}\wedge F^{(2)}-\sum_{m=1}^{M}W^{(m)}=0$$

or equivalently

$$c_2(TX)-c_2(V^{(1)})+\sum_{r=1}^{R}a_r c_1(L_r)\wedge c_1(L_r)-W=0,\quad W=\textstyle\sum_{m=1}^{M}W^{(m)}$$

This can be expanded in the basis of $H^4(X,\mathbb{R})$ dual to $(\omega_1,\omega_2,\omega_3)$. The coefficient of the i-th vector in this basis is found by wedging each term with ω_i and integrating over X.

We find

$$\frac{1}{v^{1/3}}\int_X\left(c_2(TX)-c_2(V^{(1)})\right)\wedge\omega_{1,2,3}=\left(\tfrac{4}{3},\tfrac{7}{3},-4\right)$$

$$\frac{1}{v^{1/3}}\int_X c_1(L_r)\wedge c_1(L_r)\wedge\omega_i=d_{ijk}\ell_r^j\ell_r^k,\quad i=1,2,3$$

$$W_i=\left(\tfrac{4}{3},\tfrac{7}{3},-4\right)_i+\sum_{r=1}^{R}a_r d_{ijk}\ell_r^j\ell_r^k\geq 0,\quad i=1,2,3$$

$\Rightarrow$ the anomaly condition becomes

- $$W_i = (\tfrac{4}{3}, \tfrac{7}{3}, -4)_i + \sum_{r=1}^{R} a_r d_{ijk} \ell_r^j \ell_r^k \geq 0, \quad i = 1,2,3$$

Supersymmetric Hidden Sector Bundle:

Each U(1) factor in the structure group of $V^{(2)}$ leads to an an anomalous U(1) gauge group in the d=4 effective theory and an associated D-term. Let L_r be any of the sub-line bundles. The Fayet-Iliopoulos term is

Anderson, Gray, Lukas, Ovrut $\mathcal{O}(\kappa^{4/3})$
Blumenhagen, Honecker, Weigand

$$FI^{U(1)_r} \propto \underbrace{\mu(L_r)}_{\text{tree level}} - \frac{g_s^2 l_s^4}{v^{2/3}} \int_X c_1(L_r) \wedge \left(\sum_{s=1}^{R} a_s c_1(L_s) \wedge c_1(L_s) + \underbrace{\frac{1}{2} c_2(TX)}_{\text{one-loop}} - \sum_{m=1}^{M} (\tfrac{1}{2} + \lambda_m)^2 W^{(m)} \right)$$

where

$$g_s = e^{\phi_{10}}, \; l_s = 2\pi\sqrt{\alpha'}, \; \lambda_m \in [-\frac{1}{2}, \frac{1}{2}]$$

are the string coupling/length and m-th five-brane modulus.

Assuming the vev's of all $U(1)^R$ charged zero-modes vanish

$\Rightarrow$ the hidden sector is N=1 supersymmetric iff each

$FI^{U(1)^r} = 0$. $\Rightarrow$

$$\int_X c_1(L_r) \wedge \omega \wedge \omega - g_s^2 l_s^4 \int_X c_1(L_r) \wedge \left(\sum_{s=1}^{R} a_s c_1(L_s) \wedge c_1(L_s) + \frac{1}{2} c_2(TX) - \sum_{m=1}^{M} (\tfrac{1}{2} + \lambda_m)^2 W^{(m)} \right) = 0$$

for $r = 1, \ldots, R$. Using

$$\frac{1}{v^{1/3}} \int_X \frac{1}{2} c_2(TX) \wedge \omega_i = (2, 2, 0)_i$$

$\Rightarrow$ the hidden sector supersymmetry condition becomes

- $$d_{ijk} l_r^i a^j a^k - \frac{g_s^2 l_s^4}{v^{2/3}} \left(d_{ijk} \ell_r^i \sum_{s=1}^{R} a_s \ell_s^j \ell_s^k + \ell_r^i (2, 2, 0)_i - \sum_{m=1}^{M} (\tfrac{1}{2} + \lambda_m)^2 \ell_r^i W_i^{(m)} \right) = 0$$

for $r = 1, \ldots, R$.

Gauge Threshold Corrections:

The gauge couplings of the non-anomalous components of the observable and hidden sector gauge interactions have been computed to the string one-loop level. Including five-branes these are

Lukas, Ovrut, Waldram $\mathcal{O}(\kappa^{4/3})$

Blumenhagen, Honecker, Weigand

$$\frac{4\pi}{g^{(1)2}} = \underbrace{\frac{1}{6v}\int_X \omega\wedge\omega\wedge\omega}_{\text{tree level}} - \underbrace{\frac{g_s^2 l_s^4}{2v}\int_X \omega\wedge \left(-c_2(V^{(1)}) + \frac{1}{2}c_2(TX) - \sum_{m=1}^{M}(\tfrac{1}{2}-\lambda_m)^2 W^{(m)}\right)}_{\text{one-loop}}$$

and

$$\frac{4\pi}{g^{(2)2}} = \frac{1}{6v}\int_X \omega\wedge\omega\wedge\omega - \frac{g_s^2 l_s^4}{2v}\int_X \omega\wedge \left(\sum_{r=1}^{R} a_r c_1(L_r)\wedge c_1(L_r) + \frac{1}{2}c_2(TX) - \sum_{m=1}^{M}(\tfrac{1}{2}+\lambda_m)^2 W^{(m)}\right)$$

respectively. Clearly $g^{(1)2}, g^{(2)2}$ must be positive. $\Rightarrow$

$$\frac{1}{3}\int_X \omega\wedge\omega\wedge\omega - g_s^2 l_s^4 \int_X \omega\wedge\Big(-c_2(V^{(1)}) + \frac{1}{2}c_2(TX) - \sum_{m=1}^{M}(\tfrac{1}{2}-\lambda_m)^2 W^{(m)}\Big) \; > \; 0$$

$$\frac{1}{3}\int_X \omega\wedge\omega\wedge\omega - g_s^2 l_s^4 \int_X \omega\wedge\Big(\sum_{r=1}^{R} a_r c_1(L_r)\wedge c_1(L_r) + \frac{1}{2}c_2(TX) - \sum_{m=1}^{M}(\tfrac{1}{2}+\lambda_m)^2 W^{(m)}\Big) \; > \; 0$$

Re-writing these in terms of the moduli gives

- $d_{ijk}a^i a^j a^k - 3\frac{g_s^2 l_s^4}{v^{2/3}}\left(-(\tfrac{8}{3}a^1 + \tfrac{5}{3}a^2 + 4a^3) + 2(a^1+a^2) - \sum_{m=1}^{M}(\tfrac{1}{2}-\lambda_m)^2 a^i\, W_i^{(m)}\right) > 0$

and

- $d_{ijk}a^i a^j a^k - 3\frac{g_s^2 l_s^4}{v^{2/3}}\left(d_{ijk}a^i \sum_{r=1}^{R} a_r \ell_r^j \ell_r^k + 2(a^1+a^2) - \sum_{m=1}^{M}(\tfrac{1}{2}+\lambda_m)^2 a^i\, W_i^{(m)}\right) > 0$

for the observable and hidden gauge couplings respectively.

Example: Constraints For A Single Line Bundle

Consider the case where the hidden sector consists of a single line bundle

$$V^{(2)} = L, \quad L = \mathcal{O}_X(l^1, l^2, l^3)$$

with

$$l^1, l^2, l^3 \in \mathbb{Z}, \quad l^1 + l^2 = 0 \bmod 3$$

The explicit embedding of L into E_8 is as follows. Recall

$$SU(2) \times E_7 \subset E_8$$

is a maximal subgroup. With respect to $SU(2) \times E_7$

$$\underline{\mathbf{248}} \longrightarrow (\underline{\mathbf{1}}, \underline{\mathbf{133}}) \oplus (\underline{\mathbf{2}}, \underline{\mathbf{56}}) \oplus (\underline{\mathbf{3}}, \underline{\mathbf{1}})$$

We embed the generator Q of the U(1) structure group of L so that under

$$SU(2) \rightarrow U(1)$$

the two-dimensional SU(2) representation decomposes as

$$\underline{\mathbf{2}} \rightarrow \underline{\mathbf{1}} \oplus -\underline{\mathbf{1}}$$

It follows that the U(1) structure group breaks

$$E_8 \to U(1) \times E_7$$

such that

$$\underline{\mathbf{248}} \longrightarrow (0, \underline{\mathbf{133}}) \oplus \Big((1, \underline{\mathbf{56}}) \oplus (-1, \underline{\mathbf{56}})\Big) \oplus \Big((2, \underline{\mathbf{1}}) \oplus (0, \underline{\mathbf{1}}) \oplus (-2, \underline{\mathbf{1}})\Big)$$

The generator Q can be read off from this expression.
It follows that

$$a = \frac{1}{4} tr_{E_8} Q^2 = 1$$

For the single line bundle with this embedding--and assuming there is only a single five-brane with modulus λ--the anomaly, hidden supersymmetry and positive squared gauge coupling constraints become

$$W_i = \left(\tfrac{4}{3}, \tfrac{7}{3}, -4\right)_i + d_{ijk}\ell^j\ell^k \ \geq 0, \quad i = 1, 2, 3$$

$$d_{ijk}\ell^i a^j a^k - \frac{g_s^2 l_s^4}{v^{2/3}}\Big(d_{ijk}\ell^i\ell^j\ell^k + \ell^i(2,2,0)_i - \left(\tfrac{1}{2}+\lambda\right)^2\ell^i W_i\Big) \ = 0,$$

$$d_{ijk}a^i a^j a^k - 3\frac{g_s^2 l_s^4}{v^{2/3}}\Big(- \left(\tfrac{8}{3}a^1 + \tfrac{5}{3}a^2 + 4a^3\right) + +2(a^1+a^2) - \left(\tfrac{1}{2}-\lambda\right)^2 a^i W_i\Big) \ > 0,$$

$$d_{ijk}a^i a^j a^k - 3\frac{g_s^2 l_s^4}{v^{2/3}}\Big(d_{ijk}a^i\ell^j\ell^k + +2(a^1+a^2) - \left(\tfrac{1}{2}+\lambda\right)^2 a^i W_i\Big) \ > 0.$$

respectively. We must solve these along with the conditions for the slope-stability of the observable sector E_8 bundle. Note that these equations, as well as the conditions for slope-stability, are homogeneous with respect to the rescaling

$$\left(a^1,\ a^2,\ a^3,\ \frac{g_s^2 l_s^4}{v^{2/3}}\right) \mapsto \left(\mu a^1,\ \mu a^2,\ \mu a^3,\ \mu^2\frac{g_s^2 l_s^4}{v^{2/3}}\right), \quad \mu > 0.$$

$\Rightarrow$ one can set $\frac{g_s^2 l_s^4}{v^{2/3}} = 1$.

Let us try to solve this using

$$V^{(2)} = L = \mathcal{O}_X(1,2,3) \quad \Rightarrow \quad (l^1, l^2, l^3) = (1,2,3)$$

This gives

$$W = (16, 10, 0) \;\; \Rightarrow \;\; \text{effective}$$

FI=0 $\Rightarrow$

$$a^3 = \frac{-2(a^1)^2 - (a^2)^2 - 24a^1a^2 - 108\lambda^2 - 108\lambda + 117}{6\,(2a^1 + a^2)}$$

Inserting this leads to three polynomial inequalities in a^1, a^2, λ .

$\Rightarrow$ Scan through the range $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$ and plot the region of validity in the $a^1 - a^2$ plane.

For example, choosing

$$\lambda = 0.496$$

$\Rightarrow$

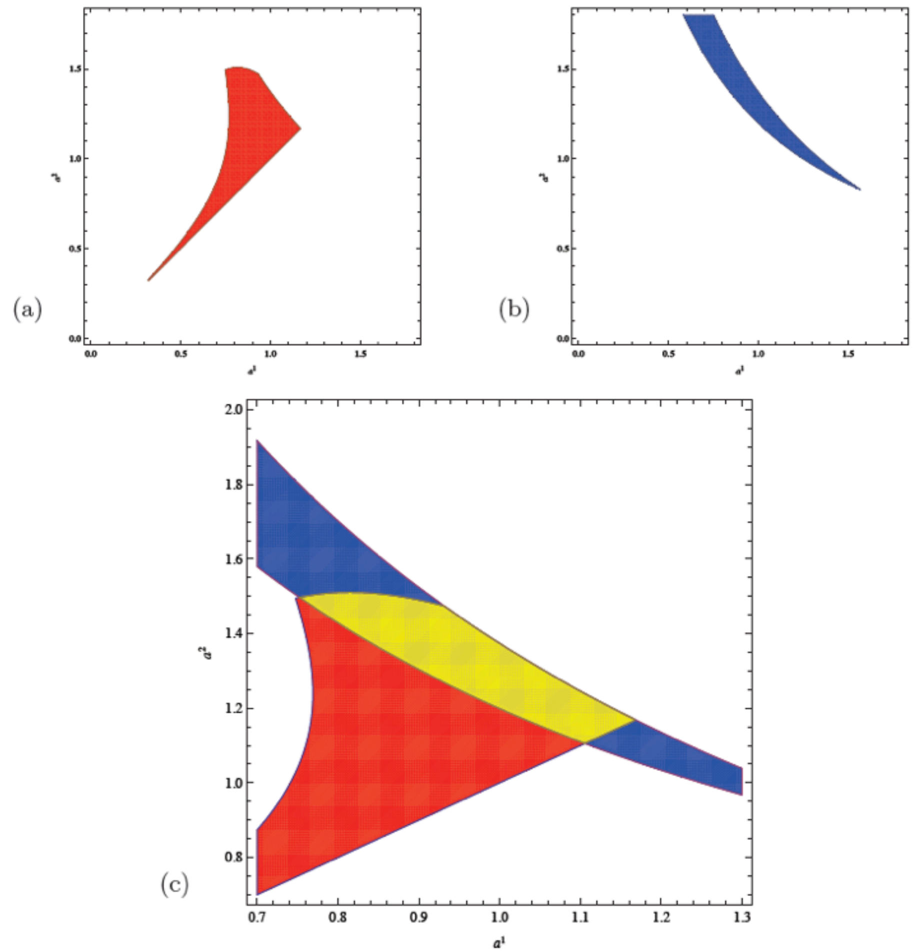

Figure 2: *The two-dimensional slice through the Kähler cone where the FI-term of the hidden line bundle $L = \mathcal{O}_X(1,2,3)$ with five-brane position $\lambda = 0.496$ vanishes. The slice is parametrized by (a^1, a^2) with a^3 given by (61). In red, the visible sector stability condition, see sub-figures a) and c). In blue, the region where the both the visible and hidden sector gauge couplings are positive, see sub-figures b) and c). Their intersection is drawn in yellow, see sub-figure c).*

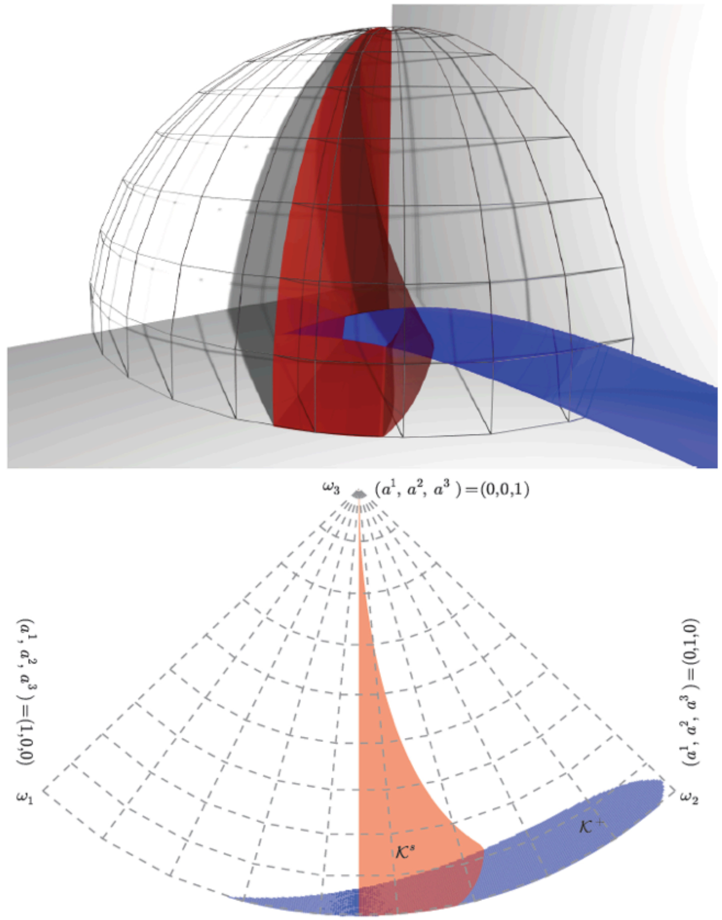

Figure 3: *The Kähler cone, in 3 dimensions (top) and the projection in radial directions (bottom). The blue region* $\mathcal{K}^+$ *is our hidden sector solution for* $L = \mathcal{O}_X(1,2,3)$ *at* $\lambda = 0.496$. *It shows the Kähler moduli* $\omega = a^1\omega_1 + a^2\omega_2 + a^3\omega_3$ *simultaneously satisfying the* $FI = 0$ *condition and the positivity of the visible and hidden sector gauge couplings. The red region* $\mathcal{K}^s$ *is the stability region of the visible sector bundle from Figure 1. The intersection* $\mathcal{K}^s \cap \mathcal{K}^+$ *is where all physical constraints are satisfied.*

Spectrum For A Single Line Bundle

For the above embedding of L, $E_8 \to U(1) \times E_7$ and

$$\underline{\mathbf{248}} \longrightarrow (0, \underline{\mathbf{133}}) \oplus \Big((1, \underline{\mathbf{56}}) \oplus (-1, \underline{\mathbf{56}}) \Big) \oplus \Big((2, \underline{\mathbf{1}}) \oplus (0, \underline{\mathbf{1}}) \oplus (-2, \underline{\mathbf{1}}) \Big)$$

The multiplicity of each representation is given by the associated sheaf cohomology. We find

$U(1) \times E_7$ representation	Multiplicity	Multiplicity if L is ample
$(0, \underline{\mathbf{133}})$	$h^0(X, \mathcal{O}_X)$	$h^0(X, \mathcal{O}_X)$
$(1, \underline{\mathbf{56}})$	$h^0(X, L) + h^2(X, L)$	$h^0(X, L)$
$(-1, \underline{\mathbf{56}})$	$h^0(X, L^*) + h^2(X, L^*)$	0
$(2, \underline{\mathbf{1}})$	$h^0(X, L^2) + h^2(X, L^2)$	$h^0(X, L^2)$
$(-2, \underline{\mathbf{1}})$	$h^0(X, L^{2*}) + h^2(X, L^{2*})$	0
$(0, \underline{\mathbf{1}})$	$h^0(X, \mathcal{O}_X)$	$h^0(X, \mathcal{O}_X)$

Table 1: *Matter spectrum for the $E_8 \to U(1) \times E_7$ breaking pattern with a line bundle L. The multiplicity counts the number of left-chiral $N = 1$ multiplets with the given gauge charge.*

On a Calabi-Yau threefold one always has

$$h^0(X, \mathcal{O}_X) = h^3(X, \mathcal{O}_X) = 1$$

with the remaining two 0. Our chosen line bundle

$$V^{(2)} = L = \mathcal{O}_X(1,2,3) \quad \Rightarrow \quad (l^1, l^2, l^3) = (1,2,3)$$

is "ample" ⇒ The multiplicities are then reduced to the to the right-hand column of the Table. It follows from the the Atiyah-Singer index theorem that

$$h^0(X, \mathcal{O}_X(\ell_1, \ell_2, \ell_3)) = \int_X \left(\frac{1}{12} c_1(V^{(2)}) \wedge c_2(TX) + \mathrm{ch}_3(V^{(2)}) \right)$$
$$= \frac{1}{3}\ell_1^i + \frac{1}{3}\ell_2^i + \frac{1}{6} \sum_{r,s,t=1}^{3} \kappa_{rst} \ell_r^i \ell_s^i \ell_t^i$$

and, hence

$$h^0(X, L) = 8 \quad \text{for} \quad L = \mathcal{O}_X(1,2,3)$$

$$h^0(X, L^2) = 58 \quad \text{for} \quad L^2 = \mathcal{O}_X(2,4,6)$$

We conclude that the complete $U(1) \times E_7$ matter spectrum of the hidden sector is

$$\underline{1} \times (0, \underline{\mathbf{133}}) + \underline{8} \times (1, \underline{\mathbf{56}}) + \underline{58} \times (2, \underline{\mathbf{1}}) + \underline{1} \times (0, \underline{\mathbf{1}})$$