Quasi-Local Mass in General Relativity

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References

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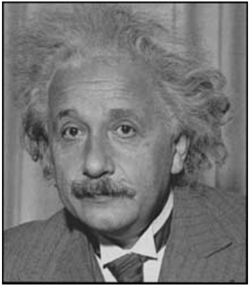
In general relativity, the Einstein equation is obtained by taking the variation of

$$\frac{1}{16\pi}\int R + \int L$$

where R is the scalar curvature of the spacetime and L is the Lagrangian of matter coupled to gravity. The gravitational interaction is described by means of a spacetime Lorentzian metric g_{ij} which has indefinite signature $(-,+,+,+)$.

For instance, the metric of the Minkowski spacetime $\mathbb{R}^{3,1}$ which is the vacuum with zero matter is

$$ds^2 = g_{ij}\, dx^i dx^j = -dt^2 + (dx)^2 + (dy)^2 + (dz)^2 \ .$$



Einstein (1879-1955)

Minkowski (1864-1909)

The variational equation has the form

$$R_{ij} - \frac{1}{2} R\, g_{ij} = T_{ij}$$

where R_{ij} is the Ricci tensor, and
T_{ij} is called the matter energy-momentum tensor.

In classical relativity, the matter tensor satisfies the weak energy condition

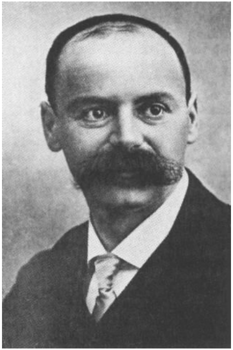
$$\sum T_{ij}\, l^i\, l^j \geq 0$$

for any four-vector l^i that is time-like

$$\sum g_{ij}\, l^i\, l^j < 0\ .$$

Karl Schwarzschild (1915)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$



Karl Schwarzschild (1873-1916)

Roy Kerr (1963)

$$ds^2 = -\frac{\Delta}{U}\left(dt - a\,\sin^2\theta\,d\phi\right)^2 + U\left(\frac{dr^2}{\Delta} + d\theta^2\right)$$
$$+\frac{\sin^2\,\theta}{U}\left(a\,dt - (r^2 + a^2)\,d\phi^2\right)^2$$

$$U = r^2 + a^2\,\cos^2\theta$$
$$\Delta = r^2 - 2\,M\,r + a^2$$
$$-\infty < t < \infty\,,\quad M + \sqrt{M^2 - a^2} < r < \infty\,,$$
$$0 < \theta < \pi,\quad 0 < \phi < 2\pi\ .$$

Both solve the Einstein equation with $T_{ij} = 0$ (vacuum). The first one is spherically symmetric and the second one is axially symmetric.

They have a null hypersurface

$$r = M + \sqrt{M^2 - a^2}$$

which is the event horizon of the black hole. This is the spacetime boundary of the black hole where any event occurring inside can not be detected by an outside observer.

The vector field $\frac{\partial}{\partial t}$ is a Killing vector field; it preserves the metric.

The Killing field $\frac{\partial}{\partial t}$ is time-like (i.e. $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) < 0$) when

$$r^2 - 2\,M\,r + a^2\,\cos^2\theta > 0$$

but space-like (i.e. $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) > 0$) when

$$r^2 - 2\,M\,r + a^2\,\cos^2\theta < 0\ .$$

This last region is called the ergosphere.

It is a bounded region outside the event horizon except at $\theta = 0$ and π.

We can consider the dynamics of a scalar field $\Phi(t, r, \theta, \phi)$ in the Kerr spacetime. Its propagation is described by a scalar wave equation.

Since the spacetime has a Killing vector field $\frac{\partial}{\partial t}$, the Lagrangian $\int |\nabla\Phi|^2$ associated to the wave equation defines a local energy density. It has the form

$$\mathcal{E} = \left(\frac{\left(r^2 + a^2\right)^2}{\Delta} - a^2 \sin^2\theta \right) |\partial_t \Phi|^2 + \Delta\, |\partial_r \Phi| + \sin^2\theta\, |\partial_{\cos\theta}\Phi|^2 + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta} \right) |\partial_\phi \Phi|^2$$

This density is positive except within the ergosphere, where it is negative.

In the Kerr geometry, no first-order or higher-order positive conserved energy density exists for the scalar wave equation as was observed by Finster, Kamran, Smoller and Yau (2005)

The energy method for the scalar wave equation breaks down due to the negativity of the energy density within the ergosphere, unless the angular momentum is small relative to the mass. (In this case, Dafermos and Rodnianski proved the solution is bounded in t if the initial data has compact support outside the event horizon.)

The decay of the solution of the scalar wave equation is a special case of the decay of solutions to the Teukolsky equation (1974) which describes the linear stability of the Kerr black hole.

Frolov and Novikov (1998):

Linear stability of the Kerr black hole is one of the few truly outstanding problems that remain in the field of black hole under gravitational wave perturbations.

The problem of linear stability of Kerr is still open and it is not even clear whether linear stability of the Kerr holds true if the angular momentum is not small relative to the mass.

The Teukolsky equation is separable by the ansatz

$$e^{-i\omega t - ik\phi}\, R(r)\, \Theta(\theta)$$

Chandrasekhar called this property of Kerr geometry

"having the aura of the miraculous."

Finster, Kamran, Smoller, Yau (2002)

The propagation of waves described by a Dirac equation in Kerr space decays in time like $t^{-5/6}$.

For the scalar wave equation, the wave with fixed angular momentum mode k also decays.

In principle, we can sum up the modes to conclude the decay of the scalar wave equation. This can be done for the Schwarzschild geometry. However, the negativity of the energy density in the ergosphere of the Kerr geometry causes problems.

The ergosphere has many strange properties including the energy extraction process proposed by Penrose. The wave analogue was due to Zeldovich (1972) and Starobinsky (1973). It is called superradiance. A complete rigorous treatment for the latter case was finally achieved recently by Finster, Kamran, Smoller and Yau.

Now we see that in the ergosphere, the energy density of the scalar wave can be negative.

The problem comes from the fact that the gravitational field itself must have energy. After all, the potential energy of a pair of gravitating particles depends on their separation distance. Hence the total energy depends on the gravitational field configuration.

In special relativity, the tensor T_{ij} can be used to define the energy-momentum vector for a domain in space.

Take any Killing vector v^j which preserves the metric. Then,

$$J_i = T_{ij} v^j$$

defines a divergence free current.

The Hodge dual of J is a closed three form. When we integrate this three form over a compact space-like hypersurface, we obtain the energy-momentum vector P associated to this hypersurface.

Since T_{ij} accounts only for matter, the energy-momentum vector defined in this way does not account for the gravitational energy. A more serious problem is that a general spacetime does not admit any Killing field.

If the spacetime admits a time-like Killing field, Noether's theorem applied to the Lagrangian provides a current associated to this Killing field. This current can be used to define a mass called the Komar mass.

Emmy Noether (1882-1935)

Noether's theorem

Let L be the Lagrangian which can be considered as a function defined on the tangent bundle of a manifold M.

Suppose we have a one parameter family of diffeomorphisms $h_t : M \to M$ so that

$$L\left((h_t)_* v\right) = L(v)$$

Then the Euler-Lagrange equation associated to L admits a first integral $I : TM \to \mathbb{R}$ given by

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{dh_t(q)}{dt}\Big|_{t=0} \quad .$$

When the spacetime is asymptotically flat, there is a space-like hypersurface which outside a compact set is diffeomorphic to $\mathbb{R}^3$ minus a ball and the metric g_{ij} has the form

$$g_{ij} = \delta_{ij} + \mathcal{O}\left(\frac{1}{r}\right)$$

and the second fundamental form p_{ij} of this hypersurface satisfies:

$$p_{ij} = \mathcal{O}\left(\frac{1}{r^2}\right) \ .$$

In particular, for asymptotically-flat spacetime admitting Lorentzian symmetry at infinity, the approximate symmetry at infinity can be used to define the total mass (or energy)

$$M_{ADM} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S^2(r)} \left(\partial_i g_{ij} - \partial_j \operatorname{tr} g\right) v^j$$

and the total linear momentum

$$P_i = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S^2(r)} (p_{ij} - p_{kk}\delta_{ij}) v^j \ .$$

The four-vector $\begin{pmatrix} M \\ P_i \end{pmatrix}$ is called the ADM (Arnowitt-Deser-Misner) energy-momentum four-vector. It was proved to be a time-like vector except for Minkowski spacetime which of course is trivially null.

For describing null infinity, it is standard to use Bondi–Sachs coordinates with the metric

$$ds^2 = -UVdw^2 - 2Udwdr + \sigma_{ab}(dx^a + W^a dw)(dx^b + W^b dw)$$

$$W^a = \mathcal{O}(r^{-2}) \ , \ U = 1 - \frac{X^2 + Y^2}{2r^2} + \mathcal{O}(r^{-2})$$

$$V = 1 - \frac{2m}{r} + \mathcal{O}(r^{-1}) \text{ and } det(\sigma_{ab}) = r^4 \sin\theta \ .$$

The Bondi-Sachs energy-momentum vector is

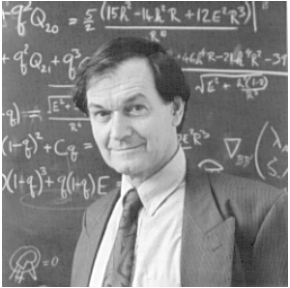
$$\frac{1}{8\pi}(\int_{S^2} 2m\, dS^2, \int_{S^2} 2mX_1\, dS^2, \int_{S^2} 2mX_2\, dS^2, \int_{S^2} 2mX_3\, dS^2)$$

where $(X_1, X_2, X_3) = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$.

The total energy in general relativity cannot be obtained by integrating any local density along a hypersurface. The reason is that the density would depend on the first order differentiation of the metric g_{ij}. But there is a coordinate system where such quantities are zero at that point.

Nonetheless, it is still possible to have a quasi-local mass, where a total energy-momentum four-vector is assigned to any space-like sphere bounding a compact portion of a space-like hypersurface.

In 1982, Penrose listed the search for a definition of such quasi-local mass as his number one problem in classical general relativity [in S.-T. Yau, *Seminar on Differential Geometry* (1982)].



Penrose

There are many reasons to search for such a concept. Many important statements in general relativity make sense only with the presence of a good definition of quasi-local mass. For example, it allows us to talk about the binding energy of two bodies rotating around each other.

More importantly, a good definition of quasi-local mass should help us to control the dynamics of the gravitational field. Hopefully, this may be used to generalize the energy method in hyperbolic equations where difficulties were encountered even in the study of linearized stability of the Kerr metric.

There were many attempts to give the definition of quasi-local mass. We shall use an approach which seems to be most promising.

Recall that for a Lorenztian manifold M with boundary ∂M, the action in general relativity should be

$$I(g, \Phi) = \int_M \left(\frac{1}{16\pi} R + L(g, \Phi) \right) + \frac{1}{8\pi} \int_{\partial M} K$$

where K is the trace of the second fundamental form of ∂M.

The last term is needed to give rise to the right variational equation if we fix the metric and the matter field on the boundary.

If we demand that a certain background (g_0, Φ_0) is a static solution to the field equation, we replace I by

$$I(g, \Phi) - I(g_0, \Phi_0) \ .$$

Hence, for flat spacetime background, we use

$$\int_M \left(\frac{1}{16\pi} R + L(g, \Phi) \right) + \frac{1}{8\pi} \int_{\partial M} (K - K_0) \ .$$

Suppose we take a family of space-like surface Σ_t and a time-like vector field t such that $t^\mu \nabla_\mu t = 1$. We can write

$$t^\mu = N\, n^\mu + N^\mu$$

where n^μ is the normal to Σ_t,

N is called the lapse function,

N^μ is called the shift vector.

In this notation,

$$I(g, \Phi) = \int N\, dt \left[\frac{1}{16\pi} \int_{\Sigma_t} \left(R + p_{\mu\nu} p^{\mu\nu} - p^2 + 16\pi L \right) + \frac{1}{8\pi} \int_{S_t^2} {}^2K \right]$$

where $p_{\mu\nu}$ is the second fundamental form of Σ_t and p is its trace, ${}^2K$ is the mean curvature of $\partial\Sigma_t = S_t$.

If one introduces the canonical momenta $k^{\mu\nu}$, k conjugate to ${}^3g_{\mu\nu}, \Phi$, we can rewrite the action to be

$$\int dt \int_{\Sigma_t} \left(k^{\mu\nu} \dot{g}_{\mu\nu} + k\dot{\Phi} - N\mathcal{H} - N^{\mu}\mathcal{H}_{\mu} \right) + \frac{1}{8\pi} \int_{S_t} \left(N\,{}^2K - N^{\mu} p_{\mu\nu} r^{\nu} \right)$$

where $\mathcal{H}$ is the Hamiltonian constraint

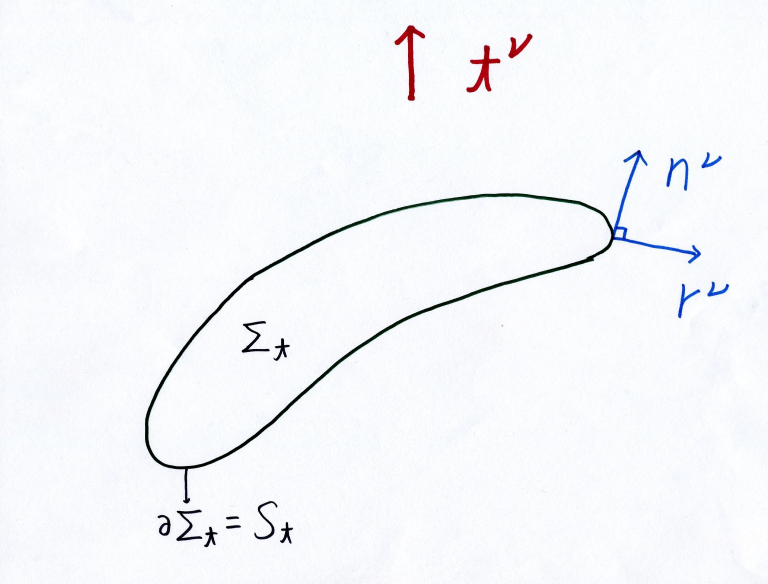
$$T_{00} - \frac{1}{2}\left(R - p_{\mu\nu}p^{\mu\nu} + p^2 \right)$$

and $\mathcal{H}_{\mu}$ is the momentum constraint

$$T_{0\mu} - p_{\mu\nu,\nu} + p_{,\mu} \ .$$

Note that $\mathcal{H} = 0$ and $\mathcal{H}_{\mu} = 0$ when the equation of motion is satisfied.

r^{ν} is spacelike unit normal to S_t and tangent to Σ_t.



t^ν
n^ν
r^ν
Σ_t
$\partial\Sigma_t = S_t$

The Hamiltonian is then derived to be

$$H = \int_{\Sigma_t} (N\,\mathcal{H} + N^{\mu}\,\mathcal{H}_{\mu}) - \frac{1}{8\pi}\int_{S_t} \left(N\,{}^{2}K - N^{\mu} p_{\mu\nu} r^{\nu}\right) \ .$$

If we take the background so that $p_{\mu\nu} = 0$, we see that the Hamiltonian relative to the background is given by

$$\int_{\Sigma_t} (N\,\mathcal{H} + N^{\mu}\,\mathcal{H}_{\mu}) - \frac{1}{8\pi}\int_{S_t} \left(N({}^{2}K - {}^{2}K_0) - N^{\mu} p_{\mu\nu} r^{\nu}\right) \ .$$

Hence associated to each time-like vector field t, we have the physical Hamiltonian

$$-\frac{1}{8\pi}\int_{S_t}\left(N({}^2K - {}^2K_0) - N^\mu p_{\mu\nu} r^\nu\right) .$$

This expression was derived by Brown-York and Hawking-Horowitz.

They proposed to simply choose $N = 1, N^\mu = 0$ for the definition of quasi-local mass. In general, the definition does not give positivity except in the time symmetric case ($p_{\mu\nu} = 0$) which was proved by Shi-Tam.

The definition of Brown-York is gauge dependent. Liu-Yau defined a gauge independent mass to be

$$-\frac{1}{8\pi}\int_S \left(\sqrt{({}^2K)^2 - (tr_S p)^2} - {}^2K_0\right)$$

and proved that it is positive whenever the mean curvature vector of S is space-like and the Gauss curvature is positive.

The proof combined arguments of Schoen-Yau and Witten. We needed to handle metrics where the mean curvature may jump along the boundary. The discontinuity of the Dirac spinor required nontrivial analysis.

$\sqrt{({}^2K)^2 - (tr_S p)^2}$ is the Lorentian norm of the mean curvature vector

$$H = -{}^2K\, r^\nu + (tr_S p)n^\nu.$$

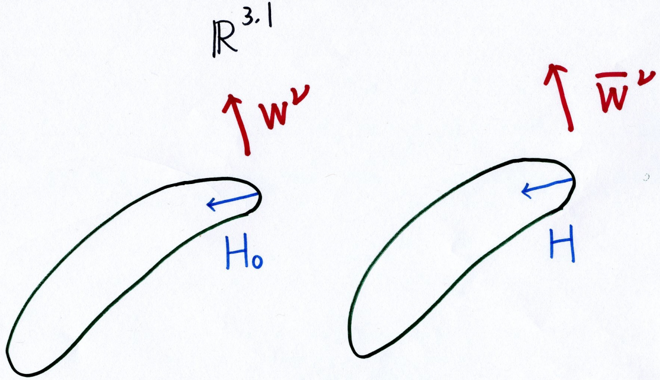
Let me now describe the work that I did with Mu-Tao Wang.

Given a surface S, we assume that its mean curvature is positive. We embed S isometrically into $\mathbb{R}^{3,1}$.

Given any constant unit future time-like vector w (observer) in $\mathbb{R}^{3,1}$, we can define a future directed time-like vector field $\overline{w}$ along S by requiring

$$\langle H_0, w \rangle \ = \ \langle H, \overline{w} \rangle$$

where H_0 is the mean curvature vector of S in $\mathbb{R}^{3,1}$
and H is the mean curvature vector of S in spacetime.



$$\langle H_0, w \rangle = \langle H, \bar{w} \rangle$$

$$w^\nu = N n^\nu + N^\nu \qquad \bar{w}^\nu = N \bar{n}^\nu + N^\nu$$

Note that given any surface S in $\mathbb{R}^{3,1}$ and a constant future time-like unit vector w^ν, there exists a canonical gauge n^μ (future time-like unit normal along S) such that

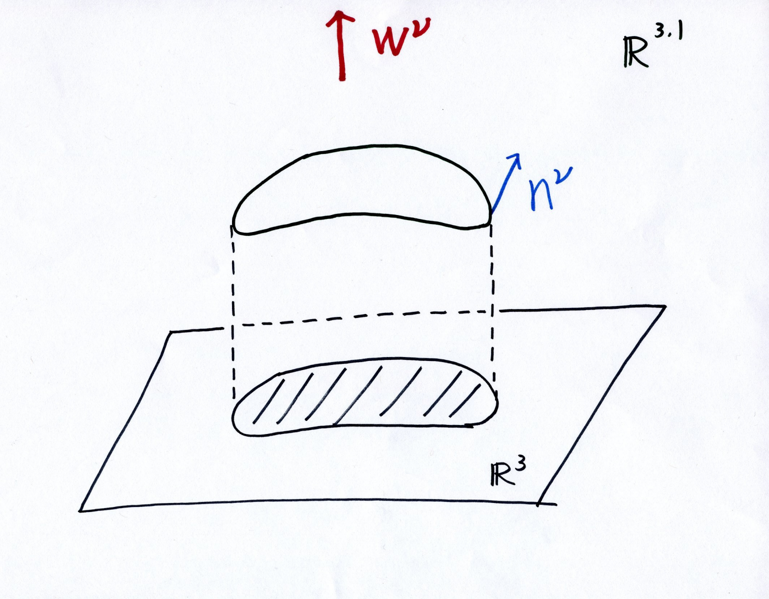
$$\int_S N^{\,2} K_0 + N^\mu (p_0)_{\mu\nu} r^\nu$$

is equal to the total mean curvature of $\hat{S}$, the projection of S onto the orthogonal complement of w^μ ,

$w^\mu = N\, n^\mu + N^\mu$,

r^μ is the space-like unit normal orthogonal to n^μ ,

p_0 is the second fundamental form calculated by the three surface defined by S and r^μ.



w^ν
$\mathbb{R}^{3,1}$
n^ν
$\mathbb{R}^3$

From the matching condition and the correspondence $(w^\mu, n^\mu) \to (\overline{w}^\mu, \overline{n}^\mu)$, we can define a similar quantity from the data in spacetime

$$\int_S N^2\overline{K} + N^\mu (\overline{p})_{\mu\nu} \overline{r}^\nu \ .$$

We write $E(w)$ to be

$$8\pi E(w) = \int_S N^2\overline{K} + N^\mu (\overline{p})_{\mu\nu} \overline{r}^\nu - \int_S N^2 K_0 + N^\mu (p_0)_{\mu\nu} r^\nu$$

and define the quasi-local mass to be

$$\inf E(w)$$

where the infimum is taken among all isometric embeddings into $\mathbb{R}^{3,1}$ and timelike unit constant vector $w \in \mathbb{R}^{3,1}$.

The Euler-Lagrange equation for minimizing $E(w)$ is

$$
\begin{aligned}
& div_S\left(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H| - \nabla\theta - V\right) \\
& - (\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} = 0
\end{aligned}
$$

where $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$, V is the tangent vector on Σ that is dual to the connection one-form $\langle \nabla^N_{(\cdot)}\frac{J}{|H|}, \frac{H}{|H|}\rangle$ and $\hat{\sigma}$, $\widehat{H}$ and $\hat{h}$ are the induced metric, mean curvature and second fundamental form of $\hat{S}$ in $\mathbb{R}^3$.

In general, the above equation should have an unique solution τ.

We prove that $E(w)$ is non-negative among admissible isometric embedding into Minkowski space.

In the proof, we use the techniques and results of the positivity of the Liu-Yau quasi-local mass. First we prove an inequality about total mean curvature for solutions of Jang's equation. Then we prove positivity of $E(W)$ by comparing the defined mass to a similar quantity defined on the graph of the solution to Jang's equation whose boundary condition is the given time function.

In summary, given a closed space-like 2-surface in spacetime whose mean curvature vector is space-like, we associate an energy-momentum four-vector to it that depends only on the first fundamental form, the mean curvature vector and the connection of the normal bundle with the properties

1. It is Lorentzian invariant;

2. It is trivial for surfaces sitting in Minkowski spacetime and future time-like for surfaces in spacetime which satisfies the local energy condition.

Spherical symmetric spacetime are foliated by the orbits of $SU(2)$. We can define a function on the spacetime by associating to its orbit the area $4\pi r^2$.

The mean curvature vector of the orbit is

$$-\frac{2}{r}\,\nabla r$$

where ∇ is with respect to the quotient Lorentzian $(1,1)$ metric.

If this vector is space-like, the quasi-local mass of this orbit sphere is

$$M = r\left(1 - |\nabla r|\right) \ .$$

Note that in 1964, Misner and Sharp defined a mass

$$m = \frac{r}{2}\left(1 - |\nabla r|^2\right)$$

which is the same as the Hawking mass (1968)

$$\sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_S |H|^2\right) \ .$$

The relation with our mass is

$$m = M - \frac{M^2}{2r} \ .$$

From the formula of quasi-local mass, which we proved to be positive, we derived a corollary that the mass m (Hawking mass) is also positive. (This was proved by Christodoulou (1995) under extra assumptions.)

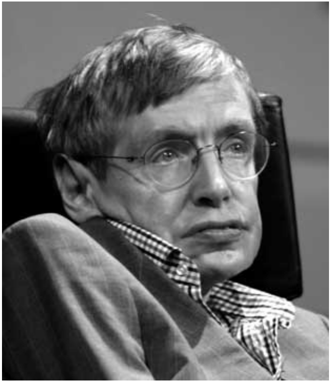
Note that

$$\frac{1}{2} M \le m \le M \ .$$

On the apparent horizon $M = 2m$,
and at space-like infinity $M = m$.

Hence our quasi-local mass is equivalent to the standard definition in the case of spherically symmetric spacetime.

In the spatial direction, the Hawking mass is monotonically increasing along the inverse mean curvature flow (Geroch) and this is important in Huisken-Ilmanen's work.



Hawking

In the future time-like or null direction, the quasi-local mass is expected to decrease up to a constant depending on the initial surface if we choose the equation of motion for the 2-surfaces carefully.

The quasi-local mass is not monotonically increasing in this sense. However, the spherical symmetric case indicates that such property may still hold, up to a constant depending on the initial surface.

In the case when $p_{ij} \equiv 0$, there is also a definition of quasi-local mass by Bartnik which is obtained by minimizing the ADM mass among all asymptotically flat extension of the data which does not contain an apparent horizon and which extends the original data.

Our quasi-local mass also satisfies the following important properties:

3. When we consider a sequence of spheres on an asymptotically flat space-like hypersurface, in the limit, the quasi-local mass (energy-momentum) is the same as the well-understood ADM mass (energy-momentum);

4. When we take the limit along a null cone, we obtain the Bondi mass(energy-momentum).

These properties of the quasi-local mass is likely to characterize the definition of quasi-local mass, i.e. any quasi-local mass that satisfies all the above four properties may be equivalent to the one that we have defined.

Strictly speaking, we associate each closed surface not a four-vector, but a function defined on the light cone of the Minkowski spacetime. Note that if this function is linear, the function can be identified as a four-vector.

It is a remarkable fact that for the sequence of spheres converging to spatial infinity, this function becomes linear, and the four-vector is defined and is the ADM four-vector that is commonly used in asymptotically flat spacetime. For a sequence of spheres converging to null infinity in Bondi coordinate, the four vector is the Bondi-Sachs four-vector.

It is a delicate problem to compute the limit of our quasi-local mass at null infinity and spatial infinity. The main difficulties are the following:

(i) The function associated to a closed surface is in non-linear in general;

(ii) One has to solve the Euler-Lagrange equation for energy minimization.

For (i), the following observation is useful:

For a family of surfaces Σ_r and a family of isometric embeddings X_r of Σ_r into $\mathbb{R}^{3,1}$, the limit of quasi-local mass is a linear function under the following general assumption that the mean curvature vectors are comparable in the sense

$$\lim_{r\to\infty} \frac{|H_0|}{|H|} = 1$$

where H is the the spacelike mean curvature vector of Σ_r in N and H_0 is that in the image of X_r in $\mathbb{R}^{3,1}$.

Under the comparable assumption of mean curvature, the limit of our quasi-local mass with respect to a constant future time-like vector $T_0 \in \mathbb{R}^{3,1}$ is given by

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \Big[-\langle T_0, \frac{J_0}{|H_0|}\rangle(|H_0| - |H|) - \langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|}\rangle + \langle \nabla^{N}_{\nabla\tau} \frac{J}{|H|}, \frac{H}{|H|}\rangle \Big] \, d\Sigma_r$$

where $\tau = -\langle T_0, X_r\rangle$ is the time function with respect to T_0.

This expression is linear in T_0 and defines an energy-momentum four-vector at infinity.

At the spatial infinity of an asymptotically flat spacetime, the limit of our quasi-local mass is

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} (|H_0| - |H|)\, d\Sigma_r = M_{ADM}$$

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|} \rangle d\Sigma_r = P_i$$

where $\begin{pmatrix} M \\ P_i \end{pmatrix}$ is the ADM energy-momentum four-vector, assuming the embeddings X_r into $\mathbb{R}^3$ inside $\mathbb{R}^{3,1}$.

At the null infinity, the limit of quasi-local mass was found by Chen-Wang-Yau to recover the Bondi-Sachs energy-momentum four-vector.

On a null cone $w = c$ as r goes to infinity, the limit of the quasi-local mass is

$$\lim_{r\to\infty} \frac{1}{8\pi}\int_{\Sigma_r} (|H_0| - |H|) d\Sigma_r = \frac{1}{8\pi}\int_{S^2} 2m\, dS^2$$

$$\lim_{r\to\infty} \frac{1}{8\pi}\int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|}\rangle d\Sigma_r = \frac{1}{8\pi}\int_{S^2} 2mX_i\, dS^2$$

where $(X_1, X_2, X_3) = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$.

The following two properties are important for solving the Euler-Lagrange equation for energy minimization:

(a) The limit of quasi-local mass is stable under $O(1)$ perturbation of the embedding;

(b) The four-vector obtained is equivariant with respect to Lorentzian transformations acting on X_r.

We observe that momentum is an obstruction to solving the Euler-Lagrange equation near a boosted totally geodesics slice in $\mathbb{R}^{3,1}$. Using (b), we find a solution by boosting the isometric embedding according to the energy-momentum at infinity. Then the limit of quasi-local mass is computed using (a) and (b).

Thank you for your attention!