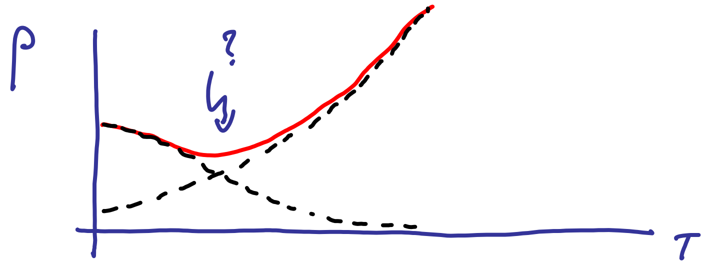
Natan Andrei : Lectures on Nonequilibrium Bethe Ansatz   28/8/06

Equilibrium Review: cond-mat/9408101

Nonequilibrium : P. Metha, N. Andrei, PRL, 2005

Introduction:



Kondo, 1964: s-d-model for dilute random impurities

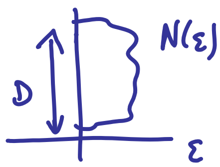
dimensionless

Description of magnetic impurities in metal:

$$H_0 = \sum_{ka} \varepsilon_k c^\dagger_{ka} c_{ka}, \qquad H_I = J \sum_{kk'} \overbrace{\left(c^\dagger_{ka} \vec{\sigma}_{ab} c_{k'b}\right)}^{\vec{\sigma}_{el}(0)} \cdot \vec{S}$$

$\vec{S}$: $1/2, 1, 3/2, \ldots$

$a = \pm 1/2$



D = bandwidth $\simeq$ 10.000 K = huge

Kondo found $\rho = \rho_0 + J^3 \ln T/D$, logarithm was surprising!

Impurity Susceptibility : $\chi^i = \frac{\mu^2}{T}\left[1 - J + J^2 \ln T/D - J^3 \ln^2 T/D\right]$

$\chi = \left.\frac{\partial M_i}{\partial h}\right|_{h=0} (T)$

Curie law, diverges for $T \to 0$

Logarithms show up again: infrared instability.

Stimulated new theoretical approaches :

- resummation methods
- poor man's scaling (Anderson)
- numerical renormalization (Wilson)
- Bethe Ansatz (Andrei)
- CFT (Affleck Ludwig)
- Bosonization
- Flow-equation RG (Kehrein)

resummation

$$\chi_i = \frac{\mu^2}{T}\left[1 - \frac{J}{1+J\ln T/D}\right] = \frac{\mu^2}{T}\left[1 - \frac{1}{\ln T/T_K}\right]$$

$$T_K = D e^{-\frac{1}{J}}$$: low-energy scale that parametrizes theory.

Summing subleading logs does not help!

For $J > 0$ (AF coupling): IF – unstable

For $J < 0$ (FM coupling): IF – stable

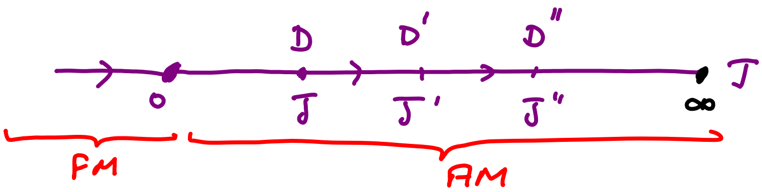
<u>Anderson scaling:</u> Start with $H(D, J) \overset{\text{renormalize}}{\rightsquigarrow} H(D', J')$ with $D' < D$

so T can become smaller before problems arise.

$$J \rightsquigarrow J' = J + \underset{\uparrow\ \text{positive!}}{J^2 \#}$$

$\boxed{J \text{ grows as } D \text{ is reduced}}$

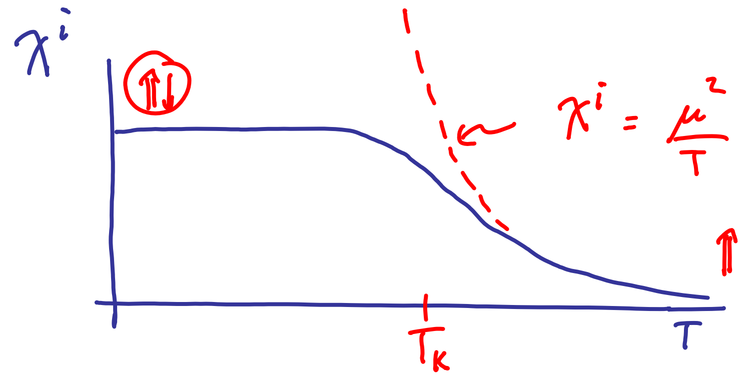
<u>Anderson, Wilson:</u>



For $s = 1/2$: $J'(D' \to 0) \to \infty$.

$(\Uparrow\Downarrow)$ = singlet!

impurity disappears!



$$\chi^i = \frac{\mu^2}{T}\left[1 - \frac{1}{\ln T/T_K}\right]$$

Response to magnetic field becomes finite as spins get bound into singlet.

Analogous to confinement in QCD: quarks look free when probed at high momenta, but are confined at small momenta.

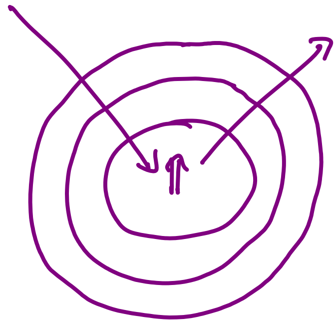
1980: Solving Kondo Model Exactly by Bethe Ansatz (N. Andrei; P. Wiegmann, A. Tsvelik)

$$H = \sum_{ka} \varepsilon_k c^\dagger_{ka} c_{ka} + J \sum_{kk'} \left(c^\dagger_{ka} \vec{\sigma}_{aa'} c_{k'a'} \right) \cdot \vec{\sigma}_0$$

Bandwidth large, only low-energy properties are of interest:

Assume $T, h \ll D \to \infty$, universal results, independent of details of bandwidth.

Field-theoretic description:



$c_{\vec{k},a}$ good for translational invariance

$\Big\downarrow$ change to spherical basis

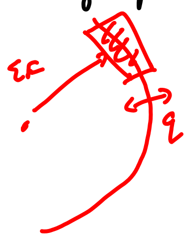
$c_{klm,a}$ only $l = l_0$ couples to impurity

l_0: angular momentum of local level

Typically one takes $l_0 = 0$, $l = 0$

For $l_0 > 0$, $|m| \le l_0$ is possible, "multi-channel model"

Linearize free spectrum: $\varepsilon_k = \varepsilon_F + q\, v_F$, for $|q| < D$



$$c_{\vec{k},a} \to c_{k,a} \to \underset{|q| < D}{c_{q,a}} \xrightarrow{\text{Fourier transform}} \psi_a(x) \quad \text{with } x \in \mathbb{R}$$

$x < 0$: incoming

$x > 0$: outgoing

Then:

$$\boxed{\hat{H} = -i \int_{-\infty}^{\infty} dx\, \psi^\dagger_a(x)\, \partial_x\, \psi_a(x) + J\, \psi^\dagger_a(0)\, \vec{\sigma}_{ab}\, \psi_b(0) \cdot \vec{\sigma}_0}$$

(only right-movers, chiral fields)

units: $v_F = 1$

DOS: $\nu = \frac{1}{\pi}$, J dimensionless.

Strictly speaking $|x| < 1/D \to \infty$ as $D \to 0$

Cutoffs will be introduced while diagonalizing H, in a way that preserves symmetries.

We are interested in finding eigenstates in Fock space, with

$$\hat{H} |F\rangle = E |F\rangle$$

, $|F\rangle$ is many-body state

Build up $|F\rangle$ one electron at a time.

Electron number: $\hat{N}_e = \int dx\, \psi^\dagger(x)\psi(x)$, $\quad [\hat{H}, \hat{N}_e] = 0$

We are interested in thermodynamic limit ① : $\quad N_e \to \infty$, $L \to \infty$, $\quad$ with $\quad D = \frac{N_e}{L} = $ fixed.

② Scaling limit: $D \to \infty$ will restore universality.

<u>$N_e = 1$</u> :

general form of $|F\rangle_1 = \int dx\, F_{aa_0}(x)\, \psi_a^\dagger(x)|0\rangle \otimes |a_0\rangle$

where $|0\rangle$ is "drained Fermi sea" : $\quad \psi_a(x)|0\rangle = 0$

$|a_0\rangle$ is impurity state.

Go to "1st quantization" :

$$H_0|F\rangle = -i\int dy\, \psi_b^\dagger(y)\,\partial_y\, \underbrace{\psi_b(y) \int dx\, F_{aa_0}(x)\, \psi_a^\dagger(x)}_{\delta_{ab}\,\delta(x-y)}|0\rangle \otimes |a_0\rangle$$

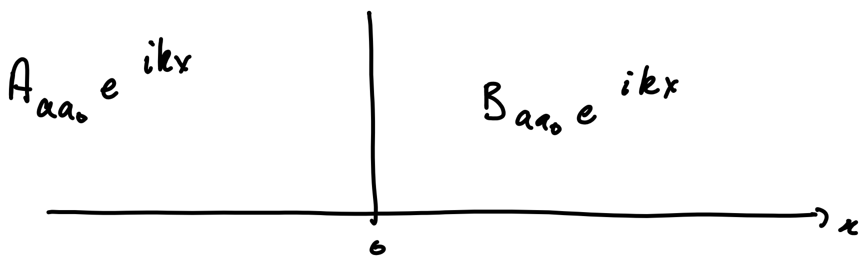
$$= -i\int dx\, \partial_x F_{aa_0}(x)\, \psi_a^\dagger(x)|0\rangle \otimes |a_0\rangle$$

$\Rightarrow N_e = 1 \qquad h(x) = -i\partial_x + J\,\delta(x)\,\vec{\sigma}\cdot\vec{\sigma}_0$

N_e : $\qquad h(x) = -i\sum_j \partial_{x_j} + J\,\delta(x_j)\,\vec{\sigma}\cdot\vec{\sigma}_0$

Schrödinger Eq: $\quad \left[-i\partial_x + J\,\delta(x)\,\vec{\sigma}_{ab}\cdot(\vec{\sigma}_0)_{a_0b_0}\right] F_{bb_0}(x) = E\, F_{aa_0}(x)$

$E = k$:



only e^{ikx}, due to ∂_x (linearized spectrum)

$$F = e^{ikx}\left[A_{aa_0}\,\theta(-x) + B_{aa_0}\,\theta(x)\right]$$

Condition between A, B:

$$B_{aa_0} = \sum_{bb_0} S_{aa_0}^{bb_0} \, A_{bb_0}$$

S-matrix.

$$\delta(x)\,\theta(\pm x) = \tfrac{1}{2}\delta(x)$$

$$hF = kF + -i\left[-A_{aa_0}\,\delta(x) + B_{aa_0}\,\delta(x)\right]$$

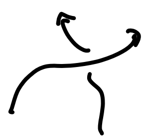
$$+ J\,\sigma_{ab}\,(\sigma_0)_{a_0b_0}\,\delta(x)\,e^{ikx}\left[\underbrace{A_{bb_0}\,\theta(-x) + B_{bb_0}\,\theta(x)}_{\frac{1}{2}[A+B]}\right]$$

$\Rightarrow$ F is eigenstate if $\quad i[A-B] + \frac{1}{2} J\, \sigma\cdot\sigma_0\,(A+B) = 0.$

$$\Rightarrow \quad \left(i + \tfrac{1}{2}J\,\sigma_{ab}\,(\sigma_0)_{a_0b_0}\right)A_{bb_0} = \left(i - \tfrac{1}{2}J\,\sigma_{ab}\cdot(\sigma_0)_{a_0b_0}\right)B_{bb_0}$$

$$\Rightarrow \quad S_{aa_0}^{bb_0} = \frac{1}{i - \frac{1}{2}J\,\vec{\sigma}\cdot\vec{\sigma}_0}\left[i + \tfrac{1}{2}J\,\vec{\sigma}\cdot\vec{\sigma}_0\right]$$

Introduce: $\quad P_{aa_0}^{bb_0} = \frac{1}{2}\left(1 + \sigma\cdot\sigma_0\right)_{aa_0}^{bb_0} \overset{\text{claim}}{=} \delta_a^{b_0}\,\delta_{a_0}^{b}$



= exchange operator

$$\Rightarrow \quad (PA)_{ab} = A_{ba}$$

Proof: $\quad \frac{1}{2}\left(1 + \sigma^z_{(1)}\cdot\sigma^z_{(2)} + \sigma^+_{(1)}\sigma^-_{(2)} + \sigma^-_{(1)}\sigma^+_{(2)}\right)$

$$A_{\uparrow\uparrow} = A_{\uparrow\uparrow}$$
$$A_{\downarrow\downarrow} = A_{\downarrow\downarrow}$$
$$A_{\uparrow\downarrow} = A_{\downarrow\uparrow}$$
$$A_{\downarrow\uparrow} = A_{\uparrow\downarrow}$$

$$\Rightarrow \quad P^2 = 1$$

$$S_{aa_0}^{bb_0} = \frac{1}{i - \frac{1}{2}J\left(\vec{\sigma}\cdot\vec{\sigma}_0 + 1\right) + \frac{1}{2}J}\left[i + \tfrac{1}{2}J\,\vec{\sigma}\cdot\vec{\sigma}_0\right]$$

$$= \frac{1}{i + \frac{1}{2}J - JP}\;\frac{i + \frac{1}{2}J + JP}{(i + \frac{1}{2}J) + JP}\quad[\qquad]$$

$$= \frac{1}{(i+\frac{1}{2}J)^2 - J^2}\left[\left(i+\tfrac{1}{2}J + JP\right)\left(i - \tfrac{1}{2}J + JP\right)\right]$$

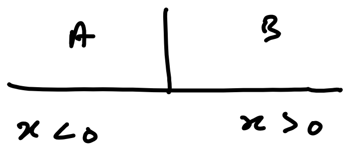
$$= a + ib\,P$$

$$\Rightarrow \quad S^{10} = \frac{1 - icP}{1 - ic} \qquad \text{acts on } A, \text{ gives } B,$$

$$c = \frac{J}{1 - \frac{3}{4}J^2} \sim J .$$

So, $N_e = 1$ problem has been solved!

$\underline{N_e = 2}$:



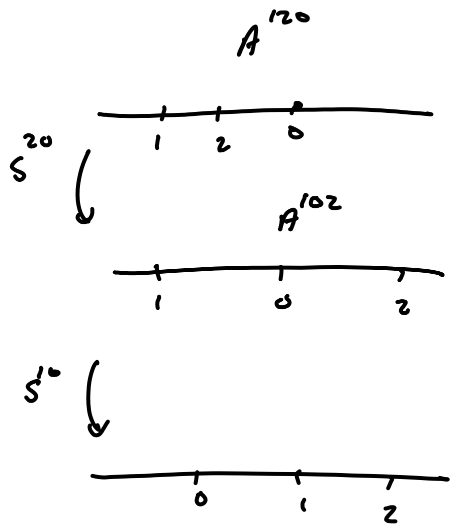
for $N_e = 1$

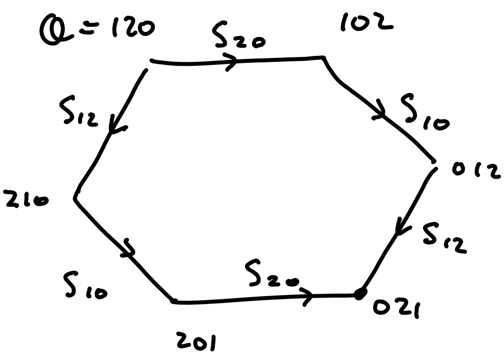
For $N_e = 2$; $3! = 6$ regions; labeled by Q. for 6 orderings of x_1, x_2, 0.

$$F(x_1, x_2)_{a_1 a_2 a_0} = \sum_Q A^Q_{a_1 a_2 a_0}\, \theta(x_Q)$$

$$S^{20} A^{120} = A^{102}$$

$$S^{10} A^{102} = A^{012}$$





two ways of reading 021 from 120 :

$$\boxed{S^{12} S^{10} S^{20} = S^{20} S^{10} S^{12}}$$

Yang-Baxter equation.

If S satisfy YB, 2-electron wave-function can be constructed.

Deep assumption is made, (which usually fails):

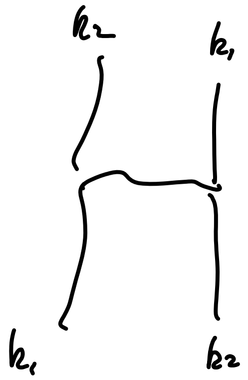
$$F = e^{ik_1x_1+ik_2x_2}\sum_Q A_{a_1a_2a_0}\,\theta(x_Q)\qquad,\quad E = k_1+k_2$$

Assumption was made that the same k's work in all Q's.

more generally, we should use: k_1+p_Q, k_2-p_Q,

where p_Q is not the same in each Q.

$\boxed{\text{Bethe Ansatz is:}\quad p_Q = 0 \quad \text{for all } Q}$

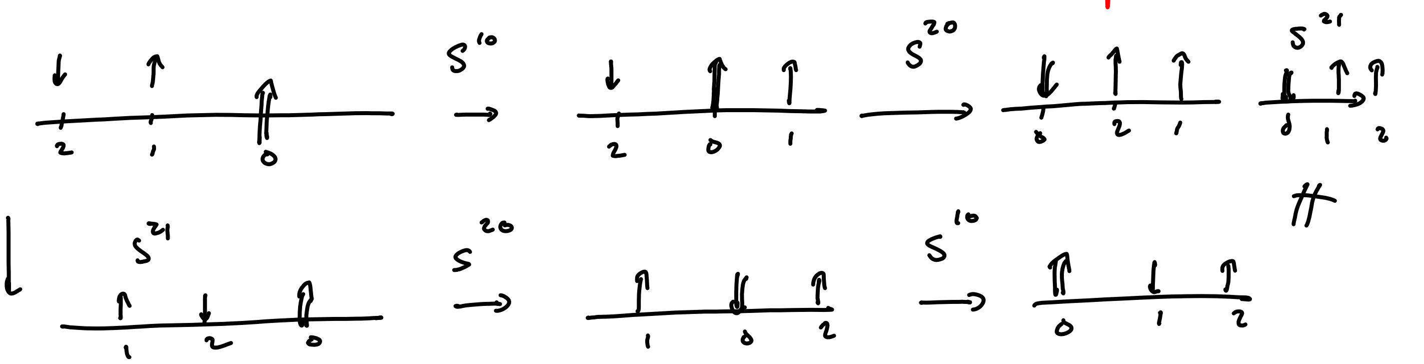


Essence of integrability: only momenta are interchanged, not shifted!!

$S^{12} = 1$?

does not work since. $S^{10}S^{20} \neq S^{20}S^{10}$

one possible outcome.



Different order of swapping leaves us in different state, since scatterer has internal degrees of freedom $\Rightarrow$ Noncommutativity!

So $S^{12}=1$ does not work. Reverse question: does S^{12} exist such that YB holds.

Claim: $S^{12} = P^{12}$! $\Rightarrow$ $P^{12}S^{10}S^{20} = S^{20}S^{10}P^{12}$, YB holds.

But is it reasonable to use $S^{12} = P^{12}$? Does it introduce interactions among electrons ?

(We may be solving the wrong model!) Answer : No!

Consider 2 free electrons: $\qquad h = -i(\partial_1 + \partial_2)$

a solution: $\qquad F_{a_1 a_2} = e^{i(k_1 x_1 + k_2 x_2)} A_{a_1 a_2}$, $E = k_1 + k_2$

another solution: $F_{a_1 a_2}(x_1, x_2) = \sum_q e^{i(k_1+q)x_1 + i(k_2 - q)x_2} A_q$,

$$= e^{i(k_1 x_1 + k_2 x_2)} \sum_q e^{iq(x_1 - x_2)} A_q$$

$$= e^{i(k_1 x_1 + k_2 x_2)} f(x_1 - x_2) \leftarrow \text{infini}$$

$\Rightarrow$ the level E is infinitely degenerate.

antisymmetrize $\rightarrow$ $\qquad$ any S can be used!

$$F_{a_1 a_2} = \mathcal{A}\, e^{i(k_1 x_1 + k_2 x_2)} \Big(A_{a_1 a_2} \theta(x_1 - x_2) + \underbrace{SA}_{A_{a_2 a_1}} \theta(x_2 - x_1) \Big).$$

no continuity concerns, since ∂_x is linear!

To perturb degenerate levels, choose special basis, such that

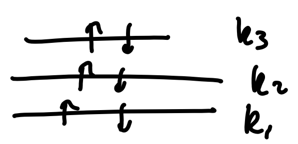
$$\langle i | V | j \rangle = V \delta_{ij} , \quad \text{else} \quad \frac{\langle i | V | j \rangle}{E_i - E_j} \text{ blows up}$$

So, the choice $S^{12} = P^{12}$ corresponds to choice of basis in the degenerate space of $k_1 + k_2 = E$ two-electron states.

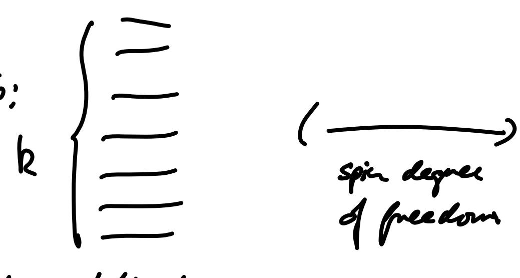
$S^{12} = P^{12}$ corresponds to charge-spin decoupling :

$$F_{a_1 a_2} = \left(e^{i(k_1 x_2 + k_2 x_1)} - e^{i(k_1 x_1 + k_2 x_2)} \right) \Big(A_{a_1 a_2} \theta(x_1 - x_2) + A_{a_2 a_1} \theta(x_2 - x_1) \Big).$$

usual basis:



Bethe basis:



but two electrons with same k would give 0.

charge degree of freedom