Solutions for String Theory 101

Lectures at the International School of Strings and Fundamental Physics

Munich July 26 - August 6 2010

Neil Lambert
Theory Division
CERN
1211 Geneva 23
Switzerland
Email: neil.lambert@cern.ch

Problem: Show that if

$$m\frac{d}{d\tau}\left(\frac{v^i}{\sqrt{1-v^2}}\right) = qF_{i0} + qF_{ij}v^j \tag{0.1}$$

is satisfied then so is

$$-m\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) = qF_{0i}v^i \tag{0.2}$$

Solution: We simply multiply (0.1) by v^i and use the anti-symmetry of F_{ij} to deduce

$$-m\frac{d}{d\tau}\left(\frac{v^i}{\sqrt{1-v^2}}\right)v^i = qF_{0i}v^i \tag{0.3}$$

Now the left hand side is

$$\begin{aligned}
-m\frac{d}{d\tau}\left(\frac{v^i}{\sqrt{1-v^2}}\right)v^i &= -mv^2\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) - \frac{m}{\sqrt{1-v^2}}v^i\frac{dv^i}{d\tau} \\
&= -mv^2\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) - \frac{1}{2}\frac{m}{\sqrt{1-v^2}}\frac{dv^2}{d\tau} \\
&= -mv^2\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) - m(1-v^2)\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) \\
&= -m\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right)
\end{aligned} \tag{0.4}$$

This agrees with the left hand side of (0.2) and since the right hand sides already agree we are done.

Problem: Show that, in static gauge $X^0 = \tau$, the Hamiltonian for a charged particle is

$$H = \sqrt{m^2 + (p^i - qA^i)(p^i - qA^i)} - qA_0 \tag{0.5}$$

Solution: In static gauge the Lagrangian is

$$L = -m\sqrt{1 - \dot{X}^i\dot{X}^i} + qA_0 + qA_i\dot{X}^i \tag{0.6}$$

so the momentum conjugate to X^i is

$$\begin{aligned}
p_i &= \frac{\partial L}{\partial \dot{X}^i} \\
&= m\frac{\dot{X}^i}{\sqrt{1-v^2}} + qA_i
\end{aligned} \tag{0.7}$$

Inverting this gives

$$\frac{\dot{X}^i}{\sqrt{1-v^2}} = (p^i - qA^i)/m \tag{0.8}$$

We square to find v^2

$$\frac{v^2}{1-v^2} = (p-qA)^2/m^2 \iff v^2 = \frac{(p-qA)^2}{m^2+(p-qA)^2} \tag{0.9}$$

and hence

$$\dot{X}^i = \frac{p^i - qA^i}{\sqrt{m^2+(p-qA)^2}} \tag{0.10}$$

Finally we calculate

$$\begin{aligned} H &= p_i\dot{X}^i - L \\ &= \frac{(p^i-qA^i)p_i}{\sqrt{m^2+(p-qA)^2}} + \frac{m^2}{\sqrt{m^2+(p-qA)^2}} - qA_0 - q\frac{(p^i-qA^i)A_i}{\sqrt{m^2+(p-qA)^2}} \\ &= \sqrt{m^2+(p-qA)^2} - qA_0 \end{aligned} \tag{0.11}$$

Problem: Find the Schödinger equation, contraint and effective action for a quantized particle in the backgroud of a classical electromagnetic field using the action

$$S_{pp} = -\int \frac{1}{2}e\left(-\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2\right) - A_\mu\dot{X}^\mu \tag{0.12}$$

Solution: Proceeding as before we first calculate

$$\begin{aligned} p_\mu &= \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} \\ &= \frac{1}{e}\dot{X}^\nu\eta_{\mu\nu} + A_\mu \end{aligned} \tag{0.13}$$

Inverting this gives

$$\dot{X}^\mu = e\eta^{\mu\nu}(p_\nu - A_\nu) \tag{0.14}$$

Thus the main effect is merely to shift $p_\mu \to p_\mu - A_\mu$. The constraint is unchanged as the new term is independent of e:

$$\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2 = 0 \tag{0.15}$$

however in terms of the momentum it becomes

$$(p_\mu - A_\mu)(p_\nu - A_\nu)\eta^{\mu\nu} + m^2 = 0 \tag{0.16}$$

In the calculation of the Hamiltonian we have two effects. The first is that we which find from the replacement $p_\mu \to p_\mu - A_\mu$ in the old Hamiltonian. The second is the addition of the $A_\mu \dot{X}^\mu$ term which leads to an addition term

$$A_\mu \dot{X}^\mu = e\eta^{\mu\nu}(p_\nu - A_\nu)A_\mu \tag{0.17}$$

The factors of A_μ from these two effects combine and we find

$$H = \frac{e}{2}\left(\eta^{\mu\nu}(p_\mu - A_\mu)(p_\nu - A_\nu) + m^2\right) \tag{0.18}$$

Next consider the quantum theory where we consider wavefunctions $\Psi(X^\mu, tau)$ and promote

$$\hat{p}_\mu \Psi = -i\frac{\partial \Psi}{\partial X^\mu}\ , \qquad \hat{X}^\mu \Psi = X^\mu \Psi \tag{0.19}$$

Thus the Schrodinger equation is

$$i\frac{\partial \Psi}{\partial \tau} = \frac{e}{2}\left(-\eta^{\mu\nu}\left(\frac{\partial}{\partial X^\mu} - iA_\mu\right)\left(\frac{\partial}{\partial X^\nu} - iA_\nu\right) + m^2\right)\Psi \tag{0.20}$$

and the constraint is

$$\left(-\eta^{\mu\nu}\left(\frac{\partial}{\partial X^\mu} - iA_\mu\right)\left(\frac{\partial}{\partial X^\nu} - iA_\nu\right) + m^2\right)\Psi = 0 \tag{0.21}$$

Thus we again find that Ψ is independent of τ. The effective action is just found by replacing $\partial_\mu \to -ip_\mu + A_\mu$ and hence we have

$$S = \frac{1}{2}\int d^D x(\partial_\mu \Psi - iA_\mu \Psi)(\partial_\nu \Psi - iA_\nu \Psi)\eta^{\mu\nu} + m^2\Psi^2 \tag{0.22}$$

You should recognize this as a Klein-Gordon scalar field coupled to a background Electromagnetic field.

Problem: Show that by solving the equation of motion for the metric $\gamma_{\alpha\beta}$ on a d-dimensional worldsheet the action

$$S_{HT} = -\frac{1}{2}\int d^d\sigma \sqrt{-\det(\gamma)}\left(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - m^2(d-2)\right) \tag{0.23}$$

one finds the action

$$S_{NG} = m^{2-d}\int d^d\sigma \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)} \tag{0.24}$$

for the remaining fields X^μ, *i.e.* calculate and solve the $\gamma_{\alpha\beta}$ equation of motion and then substitute the solution back into S_{HT} to obtain S_{NG}. Note that the action S_{HT} is often referred to as the Howe-Tucker form for the action whereas S_{NG} is the Nambu-Goto form. (Hint: You will need to use the fact that $\delta\sqrt{-\det(\gamma)}/\delta\gamma^{\alpha\beta} = -\frac{1}{2}\gamma_{\alpha\beta}\sqrt{-\det(\gamma)}$)

Solution: From S_{HT} we calculate the $\gamma_{\alpha\beta}$ equation of motion

$$-\frac{1}{2}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + \frac{1}{4}\gamma_{\alpha\beta}\left(\gamma^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} - m^2(d-2)\right) = 0 \tag{0.25}$$

This implies that

$$\gamma_{\alpha\beta} = b\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{0.26}$$

for some b. To determine b we substitute back into the equation of motion to find

$$-\frac{1}{2} + \frac{b}{4}(d/b - m^2(d-2)) = 0 \tag{0.27}$$

where we have used the fact that if $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ then

$$\gamma^{\alpha\beta} g_{\alpha\beta} = d/b \tag{0.28}$$

This tells us that $b = m^{-2}$. Substituting back into S_{HT} gives

$$\begin{aligned} S_{HT} &= -\frac{1}{2}m^{-d}\int d^d\sigma\sqrt{-\det g}\left(dm^2 - m^2(d-2)\right) \\ &= m^{2-d}\int d^d\sigma\sqrt{-\det g} \end{aligned} \tag{0.29}$$

where again $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$. This is precisely S_{NG}.

Problem: What transformation law must $\gamma_{\alpha\beta}$ have to ensure that S_{HT} is reparameterization invariant? (Hint: Use the fact that

$$\frac{\partial\sigma'^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\beta}{\partial\sigma'^\gamma} = \delta_\alpha^\beta \tag{0.30}$$

why?)

Solution: Under a reparameterization $\sigma^\alpha = \sigma^\alpha(\sigma')$ we have that

$$\frac{\partial X^\mu}{\partial\sigma^\alpha} = \frac{\partial X^\mu}{\partial\sigma'^\beta}\frac{\partial\sigma'^\beta}{\partial\sigma^\alpha} \tag{0.31}$$

Since the m^2 term is invariant it must be that

$$\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{0.32}$$

is invariant in order for the expression to make sense. Thus we are lead to postulate that

$$\gamma'_{\alpha\beta} = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\alpha}\gamma_{\gamma\delta} \qquad \Longleftrightarrow \qquad \gamma'^{\alpha\beta} = \frac{\partial\sigma'^\alpha}{\partial\sigma^\gamma}\frac{\partial\sigma'^\beta}{\partial\delta^\alpha}\gamma_{\gamma\delta} \tag{0.33}$$

since

$$\frac{\partial\sigma'^{\gamma}}{\partial\sigma^{\alpha}}\frac{\partial\sigma^{\beta}}{\partial\sigma'^{\gamma}} = \delta^{\beta}_{\alpha} \qquad and \qquad \frac{\partial\sigma^{\gamma}}{\partial\sigma'^{\alpha}}\frac{\partial\sigma'^{\beta}}{\partial\sigma^{\gamma}} = \delta^{\beta}_{\alpha} \tag{0.34}$$

It remains to check that

$$d^d\sigma\sqrt{-\det(\gamma)} \tag{0.35}$$

is invariant. However this follows from the above formula and the Jacbobian transformation rule for integration

$$d^d\sigma = \det\left(\frac{\partial\sigma^{\alpha}}{\partial\sigma'^{\beta}}\right)d^d\sigma' \tag{0.36}$$

Problem: Show that if $x^\mu, p^\mu \neq 0$ then we also have

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{0.37}$$

with the other commutators vanishing.

Solution: Recall that we have

$$\begin{aligned}
\hat{X}^\mu &= x^\mu + p^\mu\tau + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\left(\frac{a^\mu_n}{n}e^{in(\tau+\sigma)} + \frac{\tilde{a}^\mu_n}{n}e^{in(\tau-\sigma)}\right) \\
\hat{P}^\mu &= \frac{1}{2\pi\alpha'}\left(p^\mu - \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}^{\infty}a^\mu_n e^{in(\tau+\sigma)} + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}^{\infty}\tilde{a}^\mu_n e^{in(\tau-\sigma)}\right)
\end{aligned} \tag{0.38}$$

and we require

$$[\hat{X}^\mu(\tau,\sigma), \hat{P}_\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\delta^\mu_\nu \tag{0.39}$$

In the lectures we considered terms that come from two oscillators, *i.e.* terms with a factor of $e^{in\tau+im\tau}$. It should be clear that any term in the commutator with a single exponential must also vanish. Thus we see that the commutator of x^μ, w^μ, p^μ with any oscillators $a^\mu_n, \tilde{a}^\mu_n$ must vanish. Thus the remaining terms are

$$\frac{1}{2\pi\alpha'}[x^\mu + w^\mu\sigma, \alpha' p_\nu] = \frac{i}{2\pi}\delta^\mu_\nu \tag{0.40}$$

where on the right hand side we have included a left over term from the calculation of the oscilators (*i.e.* the $n = 0$ term from the Fourier decomposition of $\delta(\sigma - \sigma')$). Okay, it is clear that the term linear in σ on the left hand side must vanish. Thus we find

$$[x^\mu, p_\nu] = i\delta^\mu_\nu\ , \qquad [w^\mu, p_\nu] = 0 \tag{0.41}$$

Problem: Show that in these coordinates

$$\begin{aligned}
\hat{T}_{++} &= \partial_+\hat{X}^\mu\partial_+\hat{X}^\nu\eta_{\mu\nu} \\
\hat{T}_{--} &= \partial_-\hat{X}^\mu\partial_-\hat{X}^\nu\eta_{\mu\nu} \\
\hat{T}_{+-} &= T_{-+} = 0
\end{aligned} \tag{0.42}$$

Solution: We have

$$\hat{T}_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} \tag{0.43}$$

The new coordinates are

$$\begin{array}{l} \sigma^+ = \tau + \sigma \\ \sigma^- = \tau - \sigma \end{array} \iff \begin{array}{l} \tau = \frac{\sigma^+ + \sigma^-}{2} \\ \sigma = \frac{\sigma^+ - \sigma^-}{2} \end{array} \tag{0.44}$$

and hence it follows that $ds^2 = -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-$. From this we read off that $\eta_{++} = \eta_{--} = 0$ and $\eta_{-+} = \eta_{+-} = -\frac{1}{2}$. Hence $\eta^{++} = \eta^{--} = 0$ and $\eta^{-+} = \eta^{+-} = -2$. Thus we see that

$$\begin{aligned} \hat{T}_{++} &= \partial_+ \hat{X}^\mu \partial_+ \hat{X}^\nu \eta_{\mu\nu} \\ \hat{T}_{--} &= \partial_- \hat{X}^\mu \partial_- \hat{X}^\nu \eta_{\mu\nu} \end{aligned} \tag{0.45}$$

For the $\hat{T}_{-+}$ components we note that

$$\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = -4\partial_- X^\mu \partial_+ X^\nu \eta_{\mu\nu} \tag{0.46}$$

so that

$$\hat{T}_{-+} = \partial_- X^\mu \partial_+ X^\nu \eta_{\mu\nu} - \frac{1}{2}\frac{1}{2}4\partial_- X^\mu \partial_+ X^\nu \eta_{\mu\nu} = 0 \tag{0.47}$$

Problem: Show that

$$< 0,0;0| : L_2 :: L_{-2} : |0;0,0 >= \frac{D}{2} \tag{0.48}$$

Solution: We have

$$\begin{aligned} |2> &= L_{-2}|0;0,0> \\ &= \frac{1}{2}\sum_n \eta_{\mu\nu}\alpha^\mu_{-2-n}\alpha^\nu_n|0;0,0> \\ &= \frac{1}{2}\eta_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0;0,0> \end{aligned} \tag{0.49}$$

so

$$\begin{aligned} <2|2> &= \frac{1}{4}\eta_{\mu\nu}\eta_{\lambda\rho} <0,0:0|\alpha^\lambda_1\alpha^\rho_1\alpha^\mu_{-1}\alpha^\nu_{-1}|0;0,0> \\ &= \frac{1}{4}\eta_{\mu\nu}\eta_{\lambda\rho} <0,0:0|\alpha^\lambda_1\alpha^\mu_{-1}\alpha^\rho_1\alpha^\nu_{-1}|0;0,0> + \frac{1}{4}\eta_{\mu\nu}\eta_{\lambda\rho}\eta^{\rho\mu} <0,0:0|\alpha^\lambda_1\alpha^\nu_{-1}|0;0,0> \\ &= \frac{1}{4}\eta_{\mu\nu}\eta_{\lambda\rho}\eta^{\rho\nu} <0,0:0|\alpha^\lambda_1\alpha^\mu_{-1}|0;0,0> + \frac{1}{4}\eta_{\mu\nu}\eta_{\lambda\rho}\eta^{\rho\mu}\eta^{\lambda\nu} <0,0:0|0;0,0> \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\eta_{\mu\nu}\eta_{\lambda\rho}\eta^{\rho\nu}\eta^{\lambda\mu} < 0,0:0|0;0,0> + \frac{1}{4}\eta_{\mu\nu}\eta_{\lambda\rho}\eta^{\rho\mu}\eta^{\lambda\nu} < 0,0:0|0;0,0> \\
&= \frac{1}{2}\eta_{\mu\nu}\eta_{\lambda\rho}\eta^{\rho\nu}\eta^{\lambda\mu} \\
&= \frac{D}{2}
\end{aligned}
\tag{0.50}
$$

Problem: Show that the state $(a^0_{-1} + a^1_{-1})|0>$ has zero norm.

Solution:

$$
\begin{aligned}
< 0|(a^0_1 + a^1_1)(a^0_{-1} + a^1_{-1})|0> &= < 0|a^0_1 a^0_{-1} + a^1_1 a^1_{-1}|0> \\
&= \eta^{00} + \eta^{11} \\
&= 0
\end{aligned}
\tag{0.51}
$$

Problem: Show that the boundary conditions on an open string are

$$
\eta_{\mu\nu}\delta X^\mu \partial_\sigma X^\nu = 0 \tag{0.52}
$$

at $\sigma = 0, \pi$.

Solution: In calculating the Euler-Lagrange equations for the action one integrates by parts:

$$
\int d^2\sigma \eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta \delta X^\nu = -\int d^2\sigma \eta_{\mu\nu}\eta^{\alpha\beta}\partial_\beta\partial_\alpha X^\mu \delta X^\nu + \int d^2\sigma \eta_{\mu\nu}\eta^{\alpha\beta}\partial_\beta(\partial_\alpha X^\mu \delta X^\nu) \tag{0.53}
$$

Thus we need to discard the second term

$$
\int d^2\sigma \partial_\beta(\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha X^\mu \delta X^\nu) = \eta_{\mu\nu}\partial_\sigma X^\mu \delta X^\nu \tag{0.54}
$$

where we have used the fact that the normal to the boundary is $\sigma = \sigma^1$. Locally implies that this term should vanish at each end point separately.

Problem: Show that the constraints imply that $p^\mu G_{\mu\nu} = p^\nu G_{\mu\nu} = 0$ for the level one closed string states $|G_{\mu\nu}>= G_{\mu\nu}\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0;p>$

Solution: Consider L_1 first. We find

$$
L_1 G_{\mu\nu}\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0;p> = \frac{1}{2}\sum_n \eta_{\lambda\rho}\alpha^\lambda_{1-n}\alpha^\rho_n \alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0;p> \tag{0.55}
$$

Now if $n > 1$ then α_n^ρ can be commuted through until is annhilates $|0;p>$. Similarly if $n < 0$ α_{1-n}^λ can be commuted through until is annhilates $|0;p>$. Thus we have

$$\begin{aligned}
L_1 G_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0;p> &= \frac{1}{2}\eta_{\lambda\rho}(\alpha_0^\lambda\alpha_1^\rho + \alpha_1^\lambda\alpha_0^\rho)G_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu G_{\mu\nu}|0;p> \\
&= \eta_{\lambda\rho}\alpha_0^\lambda\alpha_1^\rho\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu G_{\mu\nu}|0;p> \\
&= \eta_{\lambda\rho}\alpha_0^\lambda\tilde{\alpha}_{-1}^\nu\alpha_1^\rho\alpha_{-1}^\mu G_{\mu\nu}|0;p> \\
&= \eta_{\lambda\rho}\alpha_0^\lambda\tilde{\alpha}_{-1}^\nu[\alpha_1^\rho,\alpha_{-1}^\mu]G_{\mu\nu}|0;p> \\
&= \eta_{\lambda\rho}\alpha_0^\lambda\tilde{\alpha}_{-1}^\nu\eta^{\rho\mu}G_{\mu\nu}|0;p> \\
&= \sqrt{\frac{\alpha'}{2}}p^\mu\tilde{\alpha}_{-1}^\nu G_{\mu\nu}|0;p>
\end{aligned} \tag{0.56}$$

Since this must vanish we find $p^\mu G_{\mu\nu} = 0$. Similarly evaluating $\tilde{L}_1 G_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0;p>= 0$ will lead to $p^\nu G_{\mu\nu} = 0$.

Problem: Show that

$$\begin{aligned}
g_{\mu\nu} &= G_{(\mu\nu)} - \frac{1}{D}\eta^{\lambda\rho}G_{\lambda\rho}\eta_{\mu\nu} \\
b_{\mu\nu} &= G_{[\mu\nu]} \\
\phi &= \eta^{\lambda\rho}G_{\lambda\rho}
\end{aligned} \tag{0.57}$$

will transform into themselves under spacetime Lorentz transformations.

Solution: Let us adopt a matrix notation. Under a Lorentz transformation a tensor G transforms as

$$G' = \Lambda G \Lambda^T \tag{0.58}$$

and Lorentz transformations satisfy $\eta = \Lambda\eta\Lambda^T$. Now

$$\begin{aligned}
b' &= \frac{1}{2}(G' - G'^T) \\
&= \frac{1}{2}(\Lambda G\Lambda^T - (\Lambda G\Lambda^T)^T) \\
&= \frac{1}{2}(\Lambda G\Lambda^T - \Lambda G^T\Lambda^T) \\
&= \Lambda b\Lambda^T
\end{aligned} \tag{0.59}$$

so indeed b transforms into itself. It also follows that the symmetric part of G transforms into itself so we need only show that $\phi = \mathrm{Tr}(\eta^{-1}G)$ is invariant. To do this we note that

$$\eta^{-1} = (\Lambda^{-1})^T\eta^{-1}\Lambda^{-1} \tag{0.60}$$

so that

$$\begin{aligned}
\phi' &= \mathrm{Tr}(\eta^{-1}G') \\
&= \mathrm{Tr}(\eta^{-1}\Lambda G\Lambda^T) \\
&= \mathrm{Tr}((\Lambda^{-1})^T\eta^{-1}\Lambda^{-1}\Lambda G\Lambda^T) \\
&= \mathrm{Tr}((\Lambda^{-1})^T\eta^{-1}G\Lambda^T) \\
&= \mathrm{Tr}(\eta^{-1}G) \\
&= \phi
\end{aligned} \tag{0.61}$$

Problem: Show that in light cone gauge

$$X^- = x^- + p^-\tau + \sqrt{\frac{\alpha'}{2}}i\left(\sum_n \frac{\alpha_n^-}{n}e^{-in\sigma^+} + \frac{\tilde{\alpha}_n^-}{n}e^{-in\sigma^-}\right) \tag{0.62}$$

where

$$\alpha_n^- = \frac{1}{2p^+}\sum_m \alpha_{n-m}^i\alpha_m^j\delta_{ij} \tag{0.63}$$

and similarly for $\tilde{\alpha}_n^-$.

Solution: We need to solve

$$\begin{aligned}
-2\alpha' p^+\dot{X}^- + \frac{1}{2}\dot{X}^i\dot{X}^j\delta_{ij} + \frac{1}{2}X'^iX'^j\delta_{ij} &= 0 \\
-2\alpha' p^+X'^- + \dot{X}^iX'^j\delta_{ij} &= 0
\end{aligned} \tag{0.64}$$

We have that

$$\begin{aligned}
\dot{X}^i &= \sqrt{\frac{\alpha'}{2}}\sum_n \left(\alpha_n^i e^{-in\sigma_+} + \tilde{\alpha}_n^i e^{-in\sigma_-}\right) \\
X'^i &= \sqrt{\frac{\alpha'}{2}}\sum_n \left(\alpha_n^i e^{-in\sigma_+} - \tilde{\alpha}_n^i e^{-in\sigma_-}\right)
\end{aligned} \tag{0.65}$$

where $\alpha_0^i = \tilde{\alpha}_0^i = \sqrt{\alpha'/2}p^i$. Thus

$$\dot{X}^iX'^j\delta_{ij} = \frac{\alpha'}{2}\delta_{ij}\sum_{nm}\alpha_n^i\alpha_m^j e^{-i(n+m)\sigma_+} - \tilde{\alpha}_n^i\tilde{\alpha}_m^j e^{-i(n+m)\sigma_-} \tag{0.66}$$

From the second equation we find that

$$X^- = F(\tau) + \frac{i}{4p^+}\delta_{ij}\sum_{m+n\neq 0}\frac{\alpha_n^i\alpha_m^j}{n+m}e^{-i(n+m)\sigma_+} + \frac{\tilde{\alpha}_n^i\tilde{\alpha}_m^j}{n+m}e^{-i(n+m)\sigma_-} \tag{0.67}$$

where $F(\tau)$ is an integration constant.

Let us now consider the first equation so we calculate

$$\begin{aligned}\dot{X}^i\dot{X}^j\delta_{ij} &= \frac{\alpha'}{2}\delta_{ij}\sum_{nm}\alpha^i_n\alpha^j_m e^{-i(n+m)\sigma_+}+\tilde{\alpha}^i_n\tilde{\alpha}^j_m e^{-i(n+m)\sigma_-}+(\alpha^i_n\tilde{\alpha}^j_m+\tilde{\alpha}^i_n\alpha^j_m)e^{-in\sigma_+-im\sigma_-}\\ X'^iX'^j\delta_{ij} &= \frac{\alpha'}{2}\delta_{ij}\sum_{nm}\alpha^i_n\alpha^j_m e^{-i(n+m)\sigma_+}+\tilde{\alpha}^i_n\tilde{\alpha}^j_m e^{-i(n+m)\sigma_-}-(\alpha^i_n\tilde{\alpha}^j_m+\tilde{\alpha}^i_n\alpha^j_m)e^{-in\sigma_+-im\sigma_-}\end{aligned} \tag{0.68}$$

and hence

$$\dot{X}^i\dot{X}^j\delta_{ij}+X'^iX'^j\delta_{ij}=\alpha'\delta_{ij}\sum_{nm}\alpha^i_n\alpha^j_m e^{-i(n+m)\sigma_+}+\tilde{\alpha}^i_n\tilde{\alpha}^j_m e^{-i(n+m)\sigma_-} \tag{0.69}$$

Substituting our solution into the first equation leads to

$$\begin{aligned}0 &= -2\alpha'p^+\dot{F}-\frac{\alpha'}{2}\delta_{ij}\sum_{n+m=0}\alpha^i_n\alpha^j_m e^{-i(n+m)\sigma_+}+\tilde{\alpha}^i_n\tilde{\alpha}^j_m e^{-i(n+m)\sigma_-}\\ &\quad+\frac{\alpha'}{2}\delta_{ij}\sum_{nm}\alpha^i_n\alpha^j_m e^{-i(n+m)\sigma_+}+\tilde{\alpha}^i_n\tilde{\alpha}^j_m e^{-i(n+m)\sigma_-}\end{aligned} \tag{0.70}$$

This implies that

$$2\alpha'p^+\dot{F}=\frac{\alpha'}{2}\delta_{ij}\sum_p\alpha^i_p\alpha^j_{-p}+\tilde{\alpha}^i_p\tilde{\alpha}^j_{-p} \tag{0.71}$$

Thus $F=\alpha'p^-\tau+x^-$ is linear with

$$-2\alpha'p^+p^-+\frac{1}{2}\delta_{ij}\sum_p\alpha^i_p\alpha^j_{-p}+\tilde{\alpha}^i_p\tilde{\alpha}^j_{-p}=0 \tag{0.72}$$

Seperating out the zero-mode piece we can rewrite this as

$$-4\alpha'p^+p^-+\alpha'p^ip^j\delta_{ij}+2(N+\tilde{N}=0 \tag{0.73}$$

where

$$N+\tilde{N}=\frac{1}{2}\delta_{ij}\sum_{n\neq0}\alpha^i_n\alpha^j_{-n}+\tilde{\alpha}^i_n\tilde{\alpha}^j_{-n} \tag{0.74}$$

can be identified with the total oscillator number of the transverse coordinates.

Lastly we summarise our expression by writing

$$X^-=x^-+\alpha'p^-\tau+i\sum_{n\neq0}\frac{\alpha^-_n}{n}e^{-in\sigma_+}+\frac{\tilde{\alpha}^-_n}{n}e^{-in\sigma_-} \tag{0.75}$$

where

$$\alpha^-_n=\frac{1}{4p^+}\sum_m\alpha^i_{n-m}\alpha^j_m\delta_{ij}\qquad \tilde{\alpha}^-_n=\frac{1}{4p^+}\sum_m\tilde{\alpha}^i_{n-m}\tilde{\alpha}^j_m\delta_{ij} \tag{0.76}$$

Problem: Show that for a periodic Fermion, where $L_0 = \sum_l ld_{-l}d_l + \frac{1}{24}$ and $\{d_n, d_m\} = \delta_{n,-m}$, one has

$$Z_1 = q^{\frac{1}{24}} \prod_{l=1}^{\infty}(1+q^l) \tag{0.77}$$

and for an anti-periodic Fermion, where $L_0 = \sum_r rb_{-r}b_r - \frac{1}{48}$, $\{b_r, b_s\} = \delta_{r,-s}$ and $r, s \in \mathbf{Z} + \frac{1}{2}$, one has

$$Z_1 = q^{-\frac{1}{48}} \prod_{l=1}^{\infty}(1+q^{l-\frac{1}{2}}) \tag{0.78}$$

Solution: Everything follows as it did for the Boson. However there are only two possible cases for each oscillator d_l, either it isn't present or it is present once. In other words because of the anti-commutivity there are just two states $|0>$ and $d_{-n}|0>$ so one has $\sum q^{ld_{-l}d_l} = 1 + q^l$ and hence

$$Z_1 = q^{\frac{1}{24}} \prod_{l=1}^{\infty}(1+q^l) \tag{0.79}$$

Similarly for the anti-Periodic Fermion only now we simply write $r = l - \frac{1}{2}$ with $l = 1, 2, 3, ...$ to find

$$Z_1 = q^{-\frac{1}{48}} \prod_{l=1}^{\infty}(1+q^{l-\frac{1}{2}}) \tag{0.80}$$

Problem: Obtain the equations of motion of

$$S_{effective} = \frac{1}{2\alpha'^{12}} \int d^{26}x\sqrt{-g}e^{-2\phi}\left(R - 4(\partial\phi)^2 + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) \tag{0.81}$$

and show that they agree with

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{4}H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} + 2D_\mu D_\nu\phi \\ D^\lambda H_{\lambda\mu\nu} &= 2D^\lambda\phi H_{\lambda\mu\nu} \\ 4D^2\phi - 4(D\phi)^2 &= R + \frac{1}{12}H^2 \end{aligned} \tag{0.82}$$

You may need to recall that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $g^{\mu\nu}\delta R_{\mu\nu} = D_\mu D_\nu \delta g^{\mu\nu} - g_{\mu\nu}D^2\delta g^{\mu\nu}$.

Solution:

The equations of motion for ϕ follows pretty much as normal and one finds

$$8D_\mu(e^{-2\phi}D^\mu\phi) - 2e^{-2\phi}\left(R - 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) = 0 \tag{0.83}$$

or

$$8D^2\phi - 8D_\mu\phi D^\mu\phi - 2R - \frac{1}{6}H_{\mu\nu\lambda}H^{\mu\nu\lambda} = 0 \tag{0.84}$$

The equation for $b_{\mu\nu}$ is also fairly standard and leads to

$$D_\mu(e^{-2\phi}D^{[\mu}b^{\nu\lambda]}) = 0 \tag{0.85}$$

or

$$D_\mu H^{\mu\nu\lambda} - 2D_\mu\phi H^{\mu\nu\lambda} = 0 \tag{0.86}$$

The important point here is that when we vary the metric we find a term like

$$\int \sqrt{-g}e^{-2\phi}g^{\mu\nu}\delta R_{\mu\nu} \tag{0.87}$$

appearing. We have that

$$g^{\mu\nu}\delta R_{\mu\nu} = D_\mu D_\nu \delta g^{\mu\nu} - g_{\mu\nu}D^2\delta g^{\mu\nu} \tag{0.88}$$

is a total derivative. But now this won't be the case. Integrating the above term by parts gives

$$\begin{aligned} \int \sqrt{-g}e^{-2\phi}g^{\mu\nu}\delta R_{\mu\nu} &= \int \sqrt{-g}e^{-2\phi}\left(D_\mu D_\nu\delta g^{\mu\nu} - g_{\mu\nu}D^2\delta g^{\mu\nu}\right) \\ &= \int \sqrt{-g}e^{-2\phi}\Big(4D_\nu\phi D_\mu\phi - 4D^\lambda\phi D_\lambda\phi g_{\mu\nu} \\ &\qquad -2D_\mu D_\nu\phi + 2D^2\phi g_{\mu\nu}\Big)\delta g^{\mu\nu} \end{aligned} \tag{0.89}$$

Thus one finds, after including all the usual terms,

$$\begin{aligned} 0 &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 4D_\mu\phi D_\nu\phi - 4D^\lambda\phi D_\lambda\phi g_{\mu\nu} - 2D_\mu D_\nu\phi + 2D^2\phi g_{\mu\nu} \\ &\quad -4D_\mu\phi D_\nu\phi + 2D_\lambda\phi D^\lambda\phi g_{\mu\nu} + \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} - \frac{1}{24}g_{\mu\nu}H_{\lambda\rho\sigma}H^{\lambda\rho\sigma} \\ &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 2D^\lambda\phi D_\lambda\phi g_{\mu\nu} - 2D_\mu D_\nu\phi + 2D^2\phi g_{\mu\nu} \\ &\quad +\frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} - \frac{1}{24}g_{\mu\nu}H_{\lambda\rho\sigma}H^{\lambda\rho\sigma} \end{aligned} \tag{0.90}$$

Next we substitute in the scalar equation, writen as

$$R = 4D^2\phi - 4D_\mu\phi D^\mu\phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} \tag{0.91}$$

and find that

$$0 = R_{\mu\nu} - 2D_\mu D_\nu\phi + \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} \tag{0.92}$$

Problem: Show that

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\eta^{\alpha\beta} + i\bar{\psi}^\mu\gamma^\alpha\partial_\alpha\psi^\nu\eta_{\mu\nu} \tag{0.93}$$

is invariant under

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu \ , \qquad \delta\psi^\mu = \gamma^\alpha\partial_\alpha X^\mu\epsilon \tag{0.94}$$

for any constant ϵ. Here $\bar{\psi} = \psi^T\gamma_0$ and γ^α are real 2×2 matrices that satisfy $\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}$. A convenient choice is $\gamma^0 = i\sigma^2$ and $\gamma^1 = \sigma^1$.

Solution: First we note that

$$\delta S = -\frac{1}{4\pi\alpha'}\int d^2\sigma 2\partial_\alpha X^\mu \partial_\beta \delta X^\nu \eta_{\mu\nu}\eta^{\alpha\beta} + i\delta\bar{\psi}^\mu\gamma^\alpha\partial_\alpha\psi^\nu\eta_{\mu\nu} + i\bar{\psi}^\mu\gamma^\alpha\partial_\alpha\delta\psi^\nu\eta_{\mu\nu} \tag{0.95}$$

Looking at the final term we can write it as

$$\begin{aligned} i\bar{\psi}^\mu\gamma^\alpha\partial_\alpha\delta\psi^\nu\eta_{\mu\nu} &= \partial_\alpha(i\bar{\psi}^\mu\gamma^\alpha\delta\psi^\nu\eta_{\mu\nu}) - i\partial_\alpha\bar{\psi}^\mu\gamma^\alpha\delta\psi^\nu\eta_{\mu\nu} \\ &\equiv -i\partial_\alpha\bar{\psi}^\mu\gamma^\alpha\delta\psi^\nu\eta_{\mu\nu} \end{aligned} \tag{0.96}$$

where we dropped a total derivative. Next we note that (using a, b for spinor indices)

$$\begin{aligned} \partial_\alpha\bar{\psi}^\mu\gamma^\alpha\delta\psi^\nu\eta_{\mu\nu} &= \partial_\alpha\psi^\mu_a(\gamma_0\gamma^\alpha)^{ab}\delta\psi^\nu_b\eta_{\mu\nu} \\ &= -\delta\psi^\nu_b(\gamma_0\gamma^\alpha)^{ab}\partial_\alpha\psi^\mu_a\eta_{\mu\nu} \\ &= -\delta\bar{\psi}^\mu\gamma^\alpha\partial_\alpha\psi^\nu\eta_{\mu\nu} \end{aligned} \tag{0.97}$$

In the second line we used the fact that spinors are anti-commuting and in the third line we used that $\gamma_0\gamma^\alpha$ is a symmetric matrix (convince yourself of this!) and swapped $\mu \leftrightarrow \nu$. Thus putting these together we find

$$\delta S = -\frac{1}{2\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial_\beta \delta X^\nu \eta_{\mu\nu}\eta^{\alpha\beta} + i\delta\bar{\psi}^\mu\gamma^\alpha\partial_\alpha\psi^\nu\eta_{\mu\nu} \tag{0.98}$$

To continue we observe that

$$\begin{aligned} \delta\bar{\psi}^\mu &= (\gamma^\alpha\partial_\alpha X^\mu\epsilon)^T\gamma_0 \\ &= \epsilon^T(\gamma^\alpha)^T\gamma_0\partial_\alpha X^\mu \\ &= \epsilon^T\gamma_0\gamma^\alpha\gamma_0\gamma_0\partial_\alpha X^\mu \\ &= -\bar{\epsilon}\gamma^\alpha\partial_\alpha X^\mu \end{aligned} \tag{0.99}$$

where in the third line we have used $(\gamma^\alpha)^T = \gamma_0\gamma^\alpha\gamma_0$ (convince yourself of this too!). We now have

$$\begin{aligned}
\delta S &= -\frac{1}{2\pi\alpha'}\int d^2\sigma i\partial_\alpha X^\mu\partial_\beta\bar\epsilon\psi^\nu\eta_{\mu\nu}\eta^{\alpha\beta} - i\bar\epsilon\gamma^\beta\partial_\beta X^\mu\gamma^\alpha\partial_\alpha\psi^\nu\eta_{\mu\nu} \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma i\partial_\alpha X^\mu\partial_\beta\bar\epsilon\psi^\nu\eta_{\mu\nu}\eta^{\alpha\beta} - i\bar\epsilon\partial_\beta X^\mu(\eta^{\alpha\beta}+\gamma^{\beta\alpha})\partial_\alpha\psi^\nu\eta_{\mu\nu} \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma - i\bar\epsilon\partial_\beta X^\mu\gamma^{\beta\alpha}\partial_\alpha\psi^\nu\eta_{\mu\nu} \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma - i\partial_\alpha\left(\bar\epsilon\partial_\beta X^\mu\gamma^{\beta\alpha}\psi^\nu\eta_{\mu\nu}\right) \\
&\equiv 0
\end{aligned} \tag{0.100}$$

Problem: Show that

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\partial_\alpha X^\mu\partial^\beta X^\nu\eta_{\mu\nu} + i\bar\psi^\mu_-\gamma^\alpha\partial_\alpha\psi^\nu_-\eta_{\mu\nu} + i\bar\lambda^A_+\gamma^\alpha\partial_\alpha\lambda^B_+\delta_{AB} \tag{0.101}$$

is invariant under

$$\begin{aligned}
\delta X^\mu &= i\bar\epsilon_+\psi^\mu_- \\
\delta\psi^\mu_- &= \gamma^\alpha\partial_\alpha X^\mu\epsilon_+ \\
\delta\lambda^A_+ &= 0
\end{aligned} \tag{0.102}$$

provided that $\gamma_{01}\epsilon_+ = \epsilon_+$.

Solution: This calculation is an exact copy of the previous problem. The important point is that the ϵ_+ generator does not involve the wrong chiral component ψ^μ_+ of ψ^μ:

$$\begin{aligned}
\frac{1}{2}(1-\gamma_{01})\delta\psi^\mu_- &= \frac{1}{2}(1-\gamma_{01})\gamma^\alpha\partial_\alpha X^\mu\epsilon_+ \\
&= \frac{1}{2}\gamma^\alpha\partial_\alpha X^\mu(1+\gamma_{01})\epsilon_+ \\
&= \gamma^\alpha\partial_\alpha X^\mu\epsilon_+ \\
&= \delta\psi^\mu_-
\end{aligned} \tag{0.103}$$

Problem: Show that the action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\partial_\alpha X^\mu\partial^\beta X^\nu\eta_{\mu\nu} + i\bar\psi^\mu_-\gamma^\alpha\partial_\alpha\psi^\nu_-\eta_{\mu\nu} + i\bar\lambda^A_+\gamma^\alpha\partial_\alpha\lambda^B_+\delta_{AB} \tag{0.104}$$

can be written as

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\partial_\alpha X^\mu\partial^\beta X^\nu\eta_{\mu\nu} + i(\psi^\mu_-)^T(\partial_\tau-\partial_\sigma)\psi^\nu_-\eta_{\mu\nu} + i(\lambda^A_+)^T(\partial_\tau+\partial_\sigma)\lambda^B_+\delta_{AB} \tag{0.105}$$

So that ψ^μ_- and λ^A_+ are indeed left and right-moving respectively.

Solution: Simply write

$$\begin{aligned}\bar{\psi}_-^\mu \gamma^\alpha \partial_\alpha \psi_-^\nu \eta_{\mu\nu} &= (\psi_-^\mu)^T \gamma_0 (\gamma^0 \partial_\tau + \gamma^1 \partial_\sigma) \psi_-^\nu \eta_{\mu\nu} \\ &= (\psi_-^\mu)^T (\partial_\tau + \gamma_{01} \partial_\sigma) \psi_-^\nu \eta_{\mu\nu} \\ &= (\psi_-^\mu)^T (\partial_\tau - \partial_\sigma) \psi_-^\nu \eta_{\mu\nu} \end{aligned} \tag{0.106}$$

and

$$\begin{aligned}\bar{\lambda}_+^A \gamma^\alpha \partial_\alpha \lambda_+^B \delta_{AB} &= (\lambda_+^A)^T \gamma_0 (\gamma^0 \partial_\tau + \gamma^1 \partial_\sigma) \lambda_+^B \delta_{AB} \\ &= (\lambda_+^A)^T (\partial_\tau + \gamma_{01} \partial_\sigma) \lambda_+^B \delta_{AB} \\ &= (\lambda_+^A)^T (\partial_\tau + \partial_\sigma) \lambda_+^B \delta_{AB} \end{aligned} \tag{0.107}$$