Introduction to Gauge/Gravity Duality

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(Don't forget to pick up examples sheet outside before you leave for lunch!)

Introduction: What is gauge/gravity duality?

Some quantum field theories are equivalent to (quantum) gravity theories.
In a particular limit, the quantum theory becomes classical and the QFT becomes strongly coupled.

- gauge/gravity duality originates from string theory
- further remarks about the limit mentioned above:
  The limit taking quantum gravity to classical gravity is a saddle point approximation. On QFT side, where we have an $SU(N)$ gauge theory, this same approximation corresponds to taking the limit $N \to \infty$. In this limit, only planar Feynman diagrams contribute.
- Gauge/gravity duality generalizes the AdS/CFT correspondence
  AdS $\hat{=}$ Anti-de Sitter spacetime
  CFT $\hat{=}$ conformal field theory

Conjecture (Maldacena 1997):

There is an equivalence between

① 4d QFT with conformal symmetry (e.g. supersymmetric SU(N) Yang-Mills theory)

② superstring theory on a particular 10-dimensional curved background involving Anti-de Sitter spacetime

The conjecture states that these two theories are equivalent including observables, states, correlation functions and the dynamics.

Purpose of lectures:

1) Explain the ingredients of AdS/CFT
   a) conformal FT in $d=4$
   b) geometry of AdS spacetime
2) Explain the AdS/CFT correspondence and its origin from superstring theory
3) Introduce some generalizations of AdS/CFT (in particular finite temperature QFT, with potential application to real-world physics)

Conformal field theories in $d=4$

1) Conformal coordinate transformations leave angles invariant (locally), i.e.

$$dx_\mu dx^\mu = \Omega^{-2}(x)\, dx'_\mu dx'^\mu$$

(here we consider d-dimensional Euclidean space $\mathbb{R}^d$)

Infinitesimally ($x'_\mu = x_\mu + v_\mu$): $\Omega(x) = 1 - \sigma(x)$ ③

$$\sigma(x) = \frac{1}{d}\, \partial \cdot v$$

$\Rightarrow$ conformal Killing equation

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\,\sigma(x)\,\eta_{\mu\nu} \quad , \quad \eta_{\mu\nu} = \delta_{\mu\nu}$$

Note $\partial_\mu v_\nu + \partial_\nu v_\mu$ is the flat space restriction of a diffeomorphism of a metric

$$\delta_v g^{\mu\nu} = \left(-\nabla^\mu v^\nu - \nabla^\nu v^\mu\right)$$

Solutions of conformal Killing equations

$$\underbrace{v_\mu = a_\mu}_{\text{translation}} + \underbrace{\omega_{\mu\nu} x^\nu}_{\text{rotation}} + \underbrace{\lambda x_\mu}_{\text{scaling transf.}} + \underbrace{b_\mu x^2 - 2(b\cdot x)x_\mu}_{\text{special conformal transf.}}$$

$SO(d+1,1)$ Minkowski $SO(d,2)$

$$\left(\omega_{\mu\nu} = -\omega_{\nu\mu} \quad , \quad \sigma(x) = \lambda - 2(b\cdot x)\right)$$

In $d=2$ the conformal Killing equation reduces to the Cauchy-Riemann eq.

$$\partial_1 v_1 = \partial_2 v_2 \quad , \quad \partial_1 v_2 = -\partial_2 v_1$$

$\Rightarrow$ infinitely many solutions

$\Rightarrow$ infinitely many conserved quantities

In $d > 2$ conformal group remains ~~finite~~ (remains finite)

Here we study in $d = 4$ (sometimes $d = 3$) for physical applications

CFT: Fields transform covariantly under conformal transformations.

In a CFT, a conformally covariant operator $\mathcal{O}$ (so-called primary Op.) transform as

$$\delta_v \mathcal{O} = -\left(\mathcal{L}_v \mathcal{O}\right)(x) \quad , \quad \mathcal{L}_v = v\cdot\partial + \underbrace{\Delta}_{\text{scaling dimension}}\,\sigma(x) - \frac{1}{2}\,\partial_{[\gamma} v_{\delta]}\,\underbrace{S_{\gamma\delta}}_{\text{generator of group } O(d)}$$

$[\ldots]$ $\hat{=}$ antisymmetrization

For scalar we have

$$\delta_v \varphi(x) = -\left(v \cdot \partial + \Delta \partial(x)\right) \varphi(x)$$

Conformal correlation functions

in QFT

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = \langle T\, \mathcal{O}(x)\, \mathcal{O}(y) \rangle$$

In CFT, the symmetry is so strong that it determines the functions of 2- and 3-point correlations fct. up to a small number of parameters

Examples

$$\langle \varphi(x)\, \varphi(y) \rangle = \frac{C}{(x-y)^{2\Delta}}$$

scaling dim. Δ

C ← determined by field content of theory

$(x-y)^2 = (x-y)_\mu (x-y)^\mu$

$$\langle \varphi_1(x)\, \varphi_2(y)\, \varphi_3(z) \rangle = \frac{K}{(x-y)^{\Delta_1+\Delta_2-\Delta_3}\,(y-z)^{\Delta_2+\Delta_3-\Delta_1}\,(z-x)^{\Delta_1+\Delta_3-\Delta_2}}$$

$\Delta_1 \quad \Delta_2 \quad \Delta_3$

Anti-de Sitter space

The $(p+2)$ dimensional AdS space (AdS_{p+2}) is represented by the hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = L^2 \qquad (*)$$

L: AdS radius

which is embedded in $(p+3)$-dimensional space.

The metric is

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2$$

By construction, the space has isometry $SO(2,p+1)$. The space is homogen and isotropic.

(*) is solved by

$$X_0 = L\cosh\rho\cos\tau$$
$$X_{p+2} = L\cosh\rho\sin\tau$$
$$X_i = L\sinh\rho\,\Omega_i \qquad \left(i=1,\dots,p+1,\ \sum_i \Omega_i^2 = 1\right)$$

$$\Rightarrow ds^2 = L^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2\right)$$

with

$0 \leq \rho$ and $0 \leq \tau \leq 2\pi$. This covers the entire hyperboloid once $\Rightarrow$ (τ, ρ, Ω_i) are called <u>global</u> coordinates.

τ ~~wraps~~ an S^1; to obtain a <u>causal</u> spacetime, the S^1 needs to be ~~an~~ unwrapped (i.e. take $-\infty < \tau < +\infty$ without identifications)

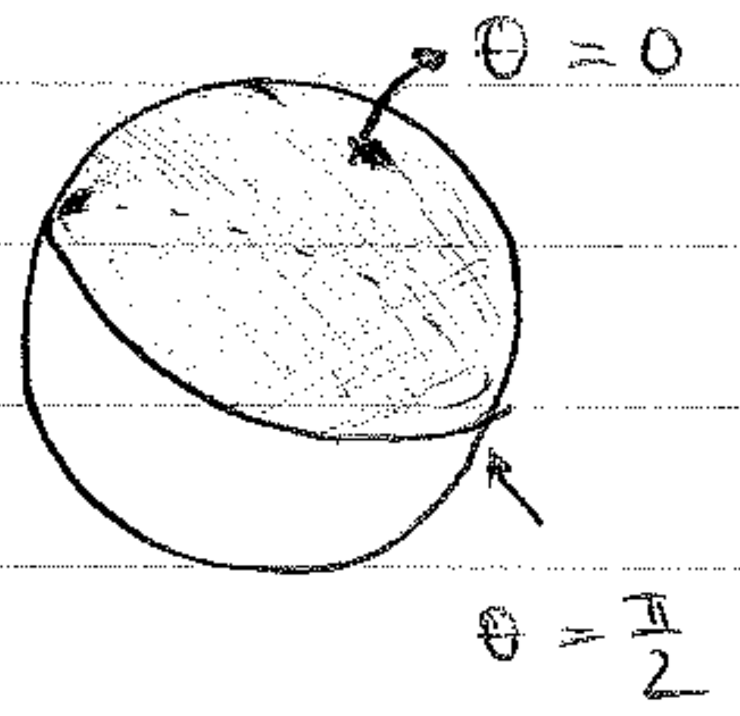
$\rightsquigarrow$ <u>universal covering</u> of hyperboloid

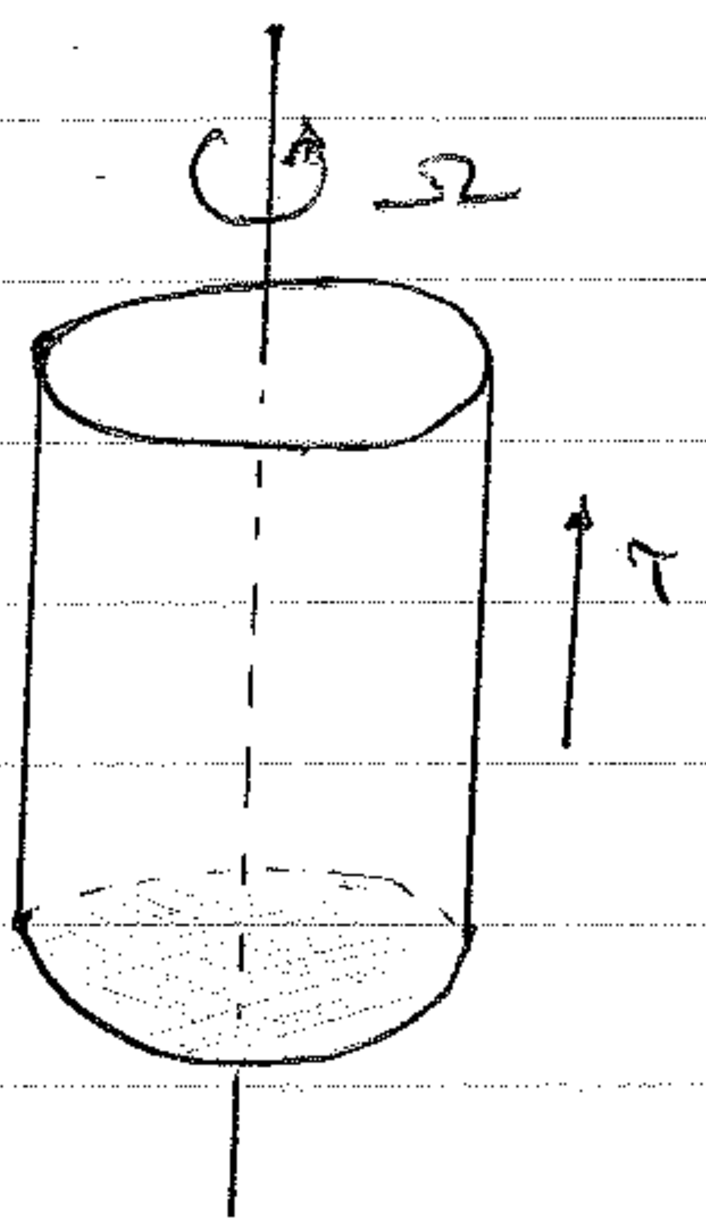
To study the causal structure, it is convenient to introduce a new coordinate θ with $\tan\theta = \sinh\rho$ $\left(0 \leq \theta < \frac{\pi}{2}\right)$

$$\Rightarrow ds^2 = \frac{L^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2\right)$$

A conformal rescaling gives $ds'^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2$

<u>AdS_3</u>





- This is the metric of one-half of the Einstein static Universe $\mathbb{R} \times S^p$
- one half because $\left(0 \leq \theta \leq \frac{\pi}{2}\right)$
- Equator at $\theta = \frac{\pi}{2}$ is a boundary of the space with the topology S^p

The boundary also extends in τ-direction (6)

> The boundary of conformally compactified AdS_{p+2} is <u>identical</u> to the conformal compactification of flat space!

Another set of useful coordinates → POINCARÉ coordinates

$$X_0 = \frac{1}{2y}\left(1 + y^2\left(L^2 + \vec{x}^2 - t^2\right)\right)$$

$$X_i = L y x_i \qquad (i = 1, \dots, p)$$

$$X_{p+1} = \frac{1}{2y}\left(1 - y^2\left(L^2 - \vec{x}^2 + t^2\right)\right)$$

$$X_{p+2} = L y t$$

$(y, t, \vec{x})$ $\quad (0 < y, \; \vec{x} \in \mathbb{R}^p)$

$$ds^2 = L^2\left(\frac{dy^2}{y^2} + y^2\left(-dt^2 + d\vec{x}^2\right)\right)$$

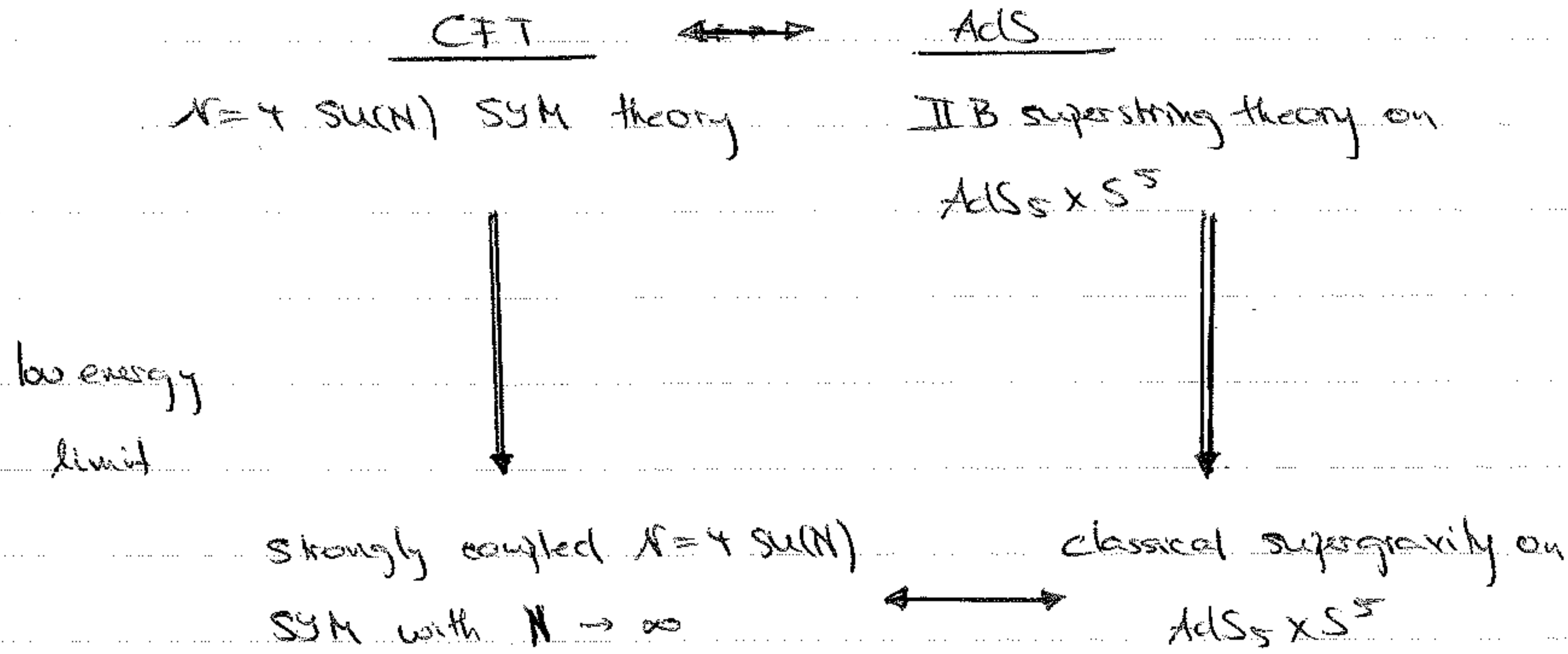
Poincaré coordinates covers only half of the hyperboloid

Curvature: AdS space has a constant negative curvature

$$R_{\mu\nu\sigma\varsigma} = -\frac{1}{L^2}\left(g_{\mu\sigma} g_{\nu\varsigma} - g_{\mu\varsigma} g_{\nu\sigma}\right)$$

Lecture 2: AdS/CFT

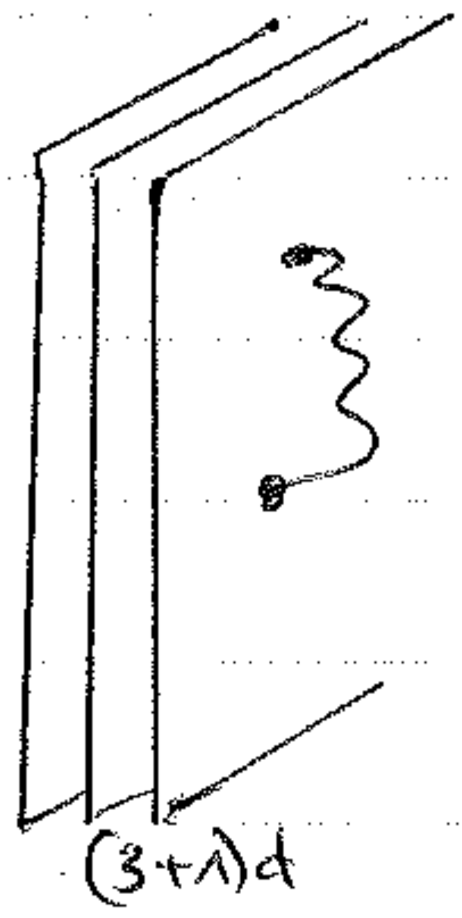
Last time:



The equivalence arises from considering N coincident D3-branes within superstring theory. D3 branes have two different interpretations.

① open string perspective

D3 branes (D stands for Dirichlet boundary conditions) are (3+1) dimensional hyperplanes in flat $d = 9+1$ spacetime on which open strings can end.



At low energies, when the string coupling $g_s \to 0$ and the inverse string tension $\alpha' \equiv l_s^2 \to 0$, the dynamics of open strings are described by $SU(N)$ $\mathcal{N}=4$ SYM theory living on the world volume of the D3 branes with $g_{YM}^2 = 2\pi g_s$.

In the Maldacena limit, where $\alpha' \to 0$ with $u = \frac{r}{\alpha'}$ fixed, for r being any distance, the open string decouple from the closed string excitations in the surrounding 10d flat spacetime. ⑧

Low-energy theory : $\mathcal{N}=4$ SYM + 10d supergravity

② <u>Closed string perspective</u> (reliable for $g_s N \gg 1$)

D-branes are solitonic solutions of the low-energy of superstrings, i.e. supergravity. They are sources for the gravitational fields which curve the surrounding spacetime. The curvature is proportional to $\frac{1}{(g_s N)^2}$. The N D3-branes are massive charged objects sourcing the various fields of supergravity.

<u>Ansatz</u> for D3-brane solutions of the equations of motion of IIB supergravity in (9+1)-dim is

metric $$ds^2 = H(r)^{-1/2} \eta_{\mu\nu} \underbrace{dx^\mu dx^\nu}_{3+1} + H(r)^{1/2} \underbrace{d\vec{y}\cdot d\vec{y}}_{\text{in 6 dim}}$$

$$r^2 = \sum_{i=4}^{9} y_i^2$$

dilaton $\exp(2\varphi(r)) = g_s^2$

4-form $C_{(4)} = (1 - H(r)^{-1}) \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$

This ansatz has the symmetry $SO(3,1) \times SO(6)$.

The EoM of 10d sugra imply that $H(r)$ is a harmonic function, and in particular

$$H(r) = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N \alpha'^2 .$$

Write

$$d\vec{y}\cdot d\vec{y} = dr^2 + r^2 d\Omega_5^2 \text{, the metric reads}$$

$$ds^2 = H(r)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H(r)^{1/2} (dr^2 + r^2 d\Omega_5^2).$$

There are two asymptotic regions : $r \to \infty$ and $r \to 0$. For $r \to \infty$, $H(r) \to 1$ and we recover 10d flat space. The region for $r \ll 1$ is called the NEAR-HORIZON region.

There are both closed strings in flat space and closed strings in the near-horizon region. In the Maldacena limit $\alpha' \to 0$ with r/α' fixed, the strings in the two regions decouple from each other. (9)

Note that in this limit

$$\frac{L^4}{r^4} = 4\pi g_s N \frac{\alpha'^2}{r^4} = 4\pi g_s N \underbrace{\left(\frac{\alpha'^4}{r^4}\right)}_{\text{const}} \underbrace{(\alpha')^{-2}}_{\to\infty} \to \infty ,$$

i.e. we zoom into the near-horizon region where $H(r) \sim \frac{L^4}{r^4}$. Then the D3 brane metric becomes

$$ds^2 = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 .$$

This is the metric of $AdS_5 \times S^5$.

Now! Both from the open and from the closed string perspective, we have two decoupled low-energy theories in the low-energy limit.

① open string: type IIB sugra on $\mathbb{R}^{9,1}$ and $\mathcal{N}=4$ SYM theory in $d=3+1$

$\updownarrow$ the same $\qquad\qquad\qquad\qquad \updownarrow$

② closed string: type IIB sugra on $\mathbb{R}^{9,1}$ and sugra on $AdS_5 \times S^5$

Maldacena's conjecture

First check of correspondence: symmetries agree!

$\mathcal{N}=4$ $\qquad \underbrace{SO(4,2)}_{\text{conformal}} \times \underbrace{SU(4)_R}_{\cong SO(6)}$

$AdS_5 \times S^5$ $\qquad SO(4,2) \times SO(6)$

Different versions (different strengths) of conjecture

① strong form : holds for all values of N and g_s
$\mathcal{N}=4$ SU(N) SYM is equivalent to IIB superstring theory for all g_s and N ($\alpha' \neq 0$)

② 't Hooft limit : keep $\lambda = g_{YM}^2 N = 2\pi g_s N$ fixed, $N \to \infty$
QFT : only planar diagrams
AdS : classical strings theory : $g_s = 0$, $\alpha' \neq 0$

③ In addition, take $\lambda = g_{YM}^2 N$ to be large $\Rightarrow g_s = 0, \alpha' = 0$
$\Rightarrow$ AdS : classical sugra
CFT : strong coupled theory

Field-operator map

1-1 map between gauge invariant operators in CFT and fields in the supergravity theory.
To calculate correlation functions in $AdS_5 \times S^5$, it is convenient to work in Euclidean AdS_5.

$$H = \{ (z_0, \vec{z}), z_0 > 0, \vec{z} \in \mathbb{R}^4 \} \text{ with metric}$$

$$ds^2 = z_0^{-2} (dz_0^2 + d\vec{z}^2) \quad ; \text{ boundary at } z_0 \to 0, \quad \partial H = \mathbb{R}^4$$

($z_0 \to 0$) is a coordinate singularity and not curvature singularity. It is natural to assume that SU(N) $\mathcal{N}=4$ SYM theory lives on this boundary of AdS_5. On the field theory side, look at typical operators in "simple" representations of the symmetry group.

On the AdS side we decompose all fields in to Kaluza-Klein-towers on S^5.

For scalar field

$$\underbrace{\varphi(z,y)}_{\text{on } AdS_5 \text{ on } S^5} = \sum_{\Delta=0}^{\infty} \varphi_\Delta(z)\, Y_\Delta(y)$$, i.e. expand in spherical harmonics on S^5

The 10d equation of motion gives

$$(\Box_5 + m_\Delta^2)\,\varphi_\Delta(z) = 0 \quad \text{with} \quad m_\Delta^2 = \Delta(\Delta-4)$$ & Δ is the scale dim. of φ_Δ.

The two independent solutions are characterized by their asymptotics,

$$\varphi_\Delta(z_0,\vec{z}) \sim \begin{cases} z_0^{\Delta} & \text{normalizable} \\ z_0^{4-\Delta} & \text{non-normalizable} \end{cases}$$

$$\varphi_\Delta(z_0,\vec{z}) \underset{z_0\to 0}{\simeq} \underbrace{\langle O \rangle}_{\text{VEV}} z_0^{\Delta} + \underbrace{\bar{\varphi}_\Delta(\vec{z})}_{\text{source}}\, z_0^{4-\Delta}$$

The boundary values of supergravity field φ is the <u>source</u> for an operator $\mathcal{O}$ in the dual quantum field theory!

A mapping between correlation fcts. in the CFT and the dynamics of SUGRA is given as follows:

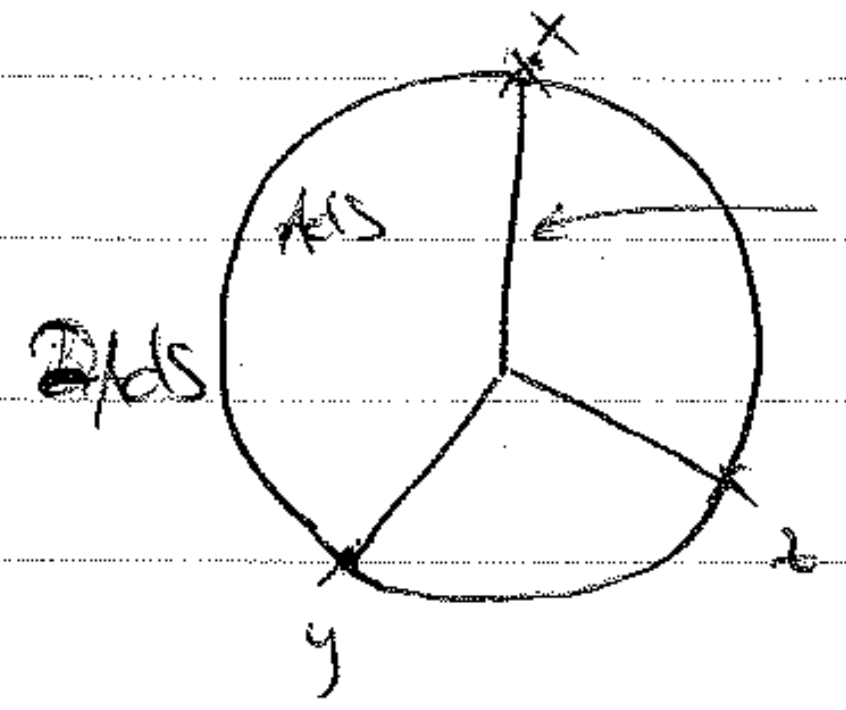
1) in CFT: generating functional $W[\bar{\varphi}_\Delta]$ for all correlations of operators $\mathcal{O}_\Delta$

$$\exp\{-W[\bar{\varphi}_\Delta]\} = \langle \exp[-\int_{\partial H} \bar{\varphi}_\Delta \mathcal{O}_\Delta] \rangle$$

→ <u>Formula for AdS/CFT conjecture</u>

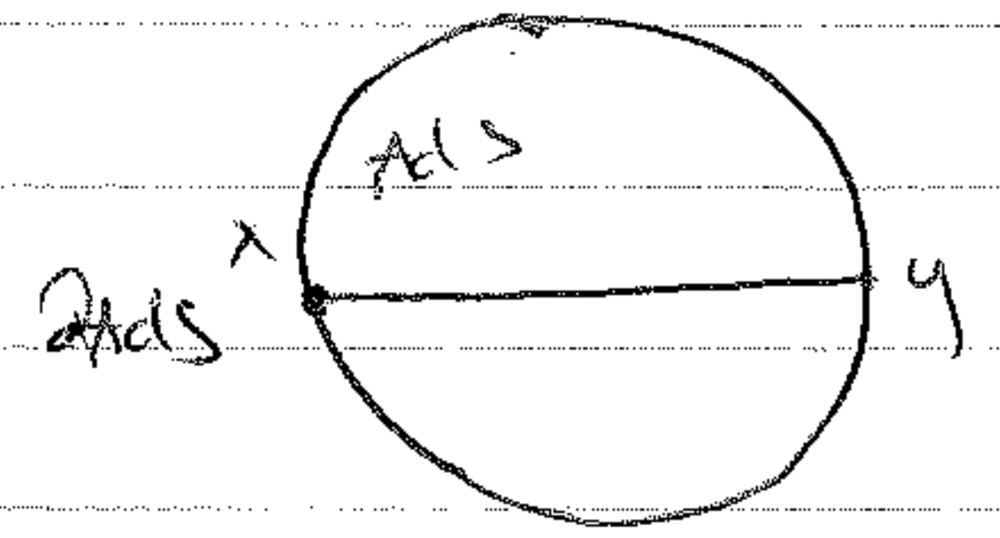
$$W[\bar{\varphi}_\Delta] \overset{!}{=} \underbrace{S_\varphi[\varphi_\Delta]}_{\text{classical sugra action on } AdS_5\times S^5}\Big|_{\lim_{z_0\to 0}\varphi_\Delta(z_0=0) = \bar{\varphi}_\Delta(\vec{z})\, z_0^{4-\Delta}}$$

Witten diagrams for computing correlation fct.



propagators $\langle \mathcal{O}(x)\,\mathcal{O}(y)\,\mathcal{O}(z)\rangle$

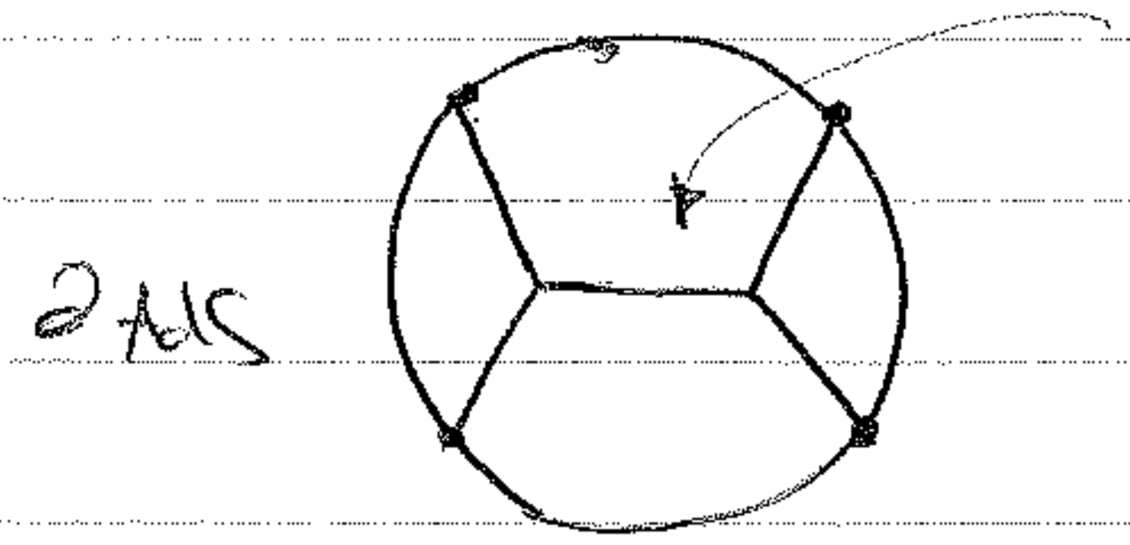
(bulk-to-boundary propagators)



$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle$

(boundary-to-boundary propagator)

(bulk-to-bulk propagator)



Lecture 3:

last time:

strongly coupled CFT $\leftrightarrow$ weakly coupled gravity on AdS

The AdS_5 space has $SO(4,2)$ symmetry and it has a boundary. It is natural to assume that the CFT (which has also $SO(4,2)$ sym.) lives on the boundary of AdS_5.

Then the boundary value $\bar{\varphi}_\Delta$ of supergravity fields may be ~~de~~-viewed as sources for the field theory operators.

$$\frac{\partial W}{\partial\bar{\varphi}_\Delta(x)\,\partial\bar{\varphi}_\Delta(y)} \equiv \langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle$$

Asymptotic behavior of the SUGRA fields near the boundary:

$$\varphi_\Delta(z_0, \vec{z}) \sim \langle \mathcal{O}_\Delta \rangle z_0^\Delta + \bar{\varphi}_\Delta z_0^{4-\Delta}$$

$$\boxed{e^{-W[\bar{\varphi}_\Delta]} = \left\langle \exp\left(- \int_{\partial H} d^4 z \, \bar{\varphi}_\Delta \mathcal{O}_\Delta \right) \right\rangle}$$

Gravity dual of finite temperature field theory

- Maldacena November 1997 → AdS/CFT
- Gubser, Klebanov, Polyakov Feb. 1998
  → correlation fct, field-operator map
- Witten, April 1998 → finite temperature

see e.g. MAGOO review, May 99

AdS Schwarzschild Black hole

Show relation to thermal field theory using the Euclidean signature metric

$$ds^2 = \frac{r^2}{L^2}\left(f(r)\, d\tau^2 + d\vec{x}^2 \right) + \frac{L^2}{r^2} \frac{1}{f(r)} dr^2 + L^2 d\Omega_5^2$$

$$f(r) = 1 - \frac{r_H^4}{r^4} \qquad r_H \text{ : location of the Schwarzschild horizon}$$

Euclidean metric defined only outside the horizon. We want to show that regularity at the horizon is obtained only if τ is periodic. Period $\sim \beta = \frac{1}{T}$ then identified with inverse temperature. Hawking temperature of black hole becomes the temperature of the field theory on the boundary.

Consider behaviour of the metric near $r \simeq r_H$

$$ds^2 = \frac{4 r_H}{L^2} (r - r_H)\, d\tau^2 + \frac{L^2}{4 r_H} \frac{1}{(r - r_H)} dr^2$$

may be rewritten as (14)

$$ds^2 = \frac{L^2}{r_H}\left(\underbrace{d\varrho^2 + \varrho^2 \frac{4r_H^2}{L^4} d\tau}\right), \quad \varrho^2 = r - r_H$$

this corresponds to the metric of a plane in ~~polar~~ polar coordinates $ds^2 = d\varrho^2 + \varrho^2 d\theta^2$ if we identify $\theta = \frac{\tau 2 r_H}{L^2}$.

Otherwise there is a conical singularity!

$\rightarrow$ deficit angle

Periodicity of θ translates into periodicity of τ with period

$$\frac{2r_H}{L^2}\beta = 2\pi$$

$$T = 1/\beta \quad \Rightarrow \quad \boxed{r_H = T\pi L^2}$$

Hawking temperature of AdS Schwarzschild BH

On the boundary, we also have compactified Euclidean time in $d=4$ $\Rightarrow$ finite temperature field theory!

Simplest possibility of defining a finite temperature field theory: Wick-rotate to Euclidean signature, compactify imaginary time $\rightarrow$ inverse temperature!

Partition fct

$$Z_E = \mathrm{Tr}\, e^{-\beta H} = \sum_{\substack{\text{all } \beta\text{-periodic} \\ \text{states}}} \langle \varphi_\beta | e^{-\beta H} | \varphi_\beta \rangle$$

$$\overset{\text{field theory}}{=} \int_{\text{all } \beta\text{-periodic states}} D\varphi \, e^{-S_E[\varphi]}$$

Goal: Thermodynamics in finite temperature field theory (15)

further generalization : finite density

Application : 1) quark-gluon-plasma strongly coupled !

2) condensed matter systems; superfluids & superconductors

In thermal field theory $Z = \int \mathcal{D}\varphi \, e^{-S_E(\varphi)} \propto e^{-S_E[\varphi^*]}$

saddle point approximation

In AdS/CFT we can also do it for the metric

$$Z \underset{\text{AdS/CFT}}{\overset{!}{=}} Z_{grav} = e^{-S_E[g^*]}$$

Evaluate gravitation action at saddle point $\Rightarrow$ free energy

$$S_E[g] = -\frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{g} \left[R + \frac{d(d-1)}{L^2} \right] +$$

$$+ \frac{1}{2\kappa^2} \int_{r\to 0} d^d x \left(-2K + \frac{2(d-1)}{L^2} \right) \sqrt{\gamma}$$

extrinsic curvature $K = \gamma^{\mu\nu} \nabla_\mu n_\nu$

$\gamma^{\mu\nu}$ = induced metric at $r \to 0$

saddle point $\hat{=}$ solution to the equation of motion

$\rightarrow$ AdS Schwarzschild metric !

Write this metric as

$$ds^2 = \frac{L^2}{r^2} \left(-f(r) dt^2 + \frac{dr^2}{f(r)} + d\vec{x} d\vec{x} \right), \quad f(r) = 1 - \frac{r^d}{r_H^d}$$

(here the boundary is at $r \to 0$)

Evaluate action on solution:

$$S_E = - \frac{L^{d+1}}{2\kappa^2 r_H^d} \frac{V_{d-1}}{T} = \frac{-(4\pi)^d L^{d-1} V_{d-1} T^{d-1}}{2\kappa^2 d^d}$$

$L^4 = 4\pi g_s N \alpha'^2$

For staying in the classical gravity regime, spacetime needs to be highly curved, $\frac{L^{d-1}}{\kappa^2} \ll 1 \rightarrow N \rightarrow \infty$ (16)

$$F = -T \ln Z = T S_E[g^*] = -\frac{(4\pi)^d L^{d-1} V_{d-1} T^d}{2\kappa^2 d^d}$$

free energy

entropy: $$S = -\frac{\partial F}{\partial T} = \frac{(4\pi)^d L^{d-1} V_{d-1} T^{d-1}}{2\kappa^2 d^{d-1}}$$

Consistency check: The expression for the entropy is equal to the area of the event horizon divided by $4G_N = \frac{\kappa^2}{2\pi}$. This area/entropy relation is universal for event horizon.

Finite density and chemical potential in a quantum field theory

Consider a QFT with a $U(1)$ gauge symmetry and a scalar and a fermion charged under this symmetry

$$\mathcal{L} = (D_\mu \varphi)^* D^\mu \varphi + i \bar{\Psi} \not{D} \Psi + \frac{1}{g^2} F^{\mu\nu} F_{\mu\nu}$$

$$D_\mu = \partial_\mu + i A_\mu$$

Let's give the time-component of A_μ a non-vanishing expectation value $\langle A_0 \rangle = \mu \Rightarrow A_0 = \langle A_0 \rangle + \delta A_0$

This ~~generat~~ generates a potential of the form

$\mu \rightarrow$ chemical potential

$$V = -\underbrace{\mu^2 \varphi^* \varphi}_{\substack{\text{upside-down}\\ \text{mass term}\\ \rightarrow \text{generates instability}}} - \mu \underbrace{\Psi^\dagger \Psi}_{}$$

$\Psi^\dagger \Psi = \hat{N}$ number operator

Gibbs energy (free energy) $\Omega = E - TS - \mu N$

How do we realise this in gauge/gravity duality?
⇒ consider charge black hole : Reissner - Nordström - black hole

On gravity side : metric $g_{\mu\nu}$, $U(1)$ gauge field A_μ

<u>EoM (Einstein's Eq.)</u>

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \frac{d(d-1)}{2L^2} g_{\mu\nu} = \frac{\kappa^2}{g^2}\left(F_{\mu\lambda} F_\nu^{\ \lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}\right)$$

Solution : Reissner Nordström AdS black hole (Minkowski signature)

- Metric:

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)\, dt^2 + \frac{dr^2}{f(r)} + d\vec{x}\cdot d\vec{x}\right)$$

$$f(r) = 1 - \left(1 + \frac{r_H^2 \mu}{\gamma^2}\right)\left(\frac{r}{r_H}\right)^d + \frac{r_H^2 \mu^2}{\gamma}\left(\frac{r}{r_H}\right)^{2(d-1)}$$

$$\text{with} \quad \gamma = \frac{(d-1) L^2 g^2}{(d-2)\kappa^2}$$

- Gauge field $A_t(r) = \mu \left(1 - \left(\frac{r}{r_H}\right)^{d-2}\right)$

Near the boundary $A_t(r)$ will give the VEV and the source which correspond to the chemical potential and the density.

→ We generalized gauge/gravity duality to finite temperature and density

→ applications to strongly coupled systems!
(quark-gluon plasma, superconductors)