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Basics of classical black holes

$D=4$ Schwarzschild bh

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\underbrace{\left(d\theta^2+\sin^2\theta\, d\phi^2\right)}_{d\Omega^2} \quad , \quad \underline{R_{\mu\nu}=0}$$

$$\left|\frac{\partial}{\partial t}\right|^2 = g_{tt} = -\left(1-\frac{2M}{r}\right)$$: null vector at $r=2M \rightarrow$ null surface generated by $\partial/\partial t$

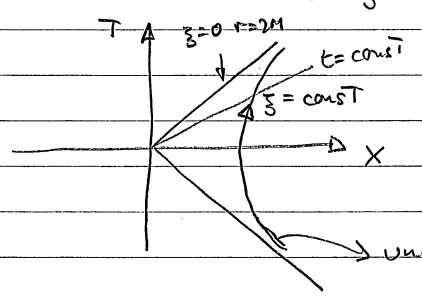
$r=2M$ is smooth:

$$r-2M \simeq \xi^2/8M \quad \Rightarrow \quad ds^2 \simeq \underbrace{-\frac{\xi^2}{16M^2}dt^2 + d\xi^2}_{} + 4M^2 d\Omega^2$$

This is reminiscent of $ds^2 = \rho^2 d\phi^2 + d\rho^2$ if $\phi \rightarrow i\frac{t}{4M}$

Change $X = \xi \cosh t/4M$

$T = \xi \sinh t/4M$ so $-\frac{\xi^2}{16M^2}dt^2 + d\xi^2 = -dT^2 + dX^2$



$X^2 - T^2 = \xi^2$ $\quad \frac{T}{X} = \tanh t/4M$

This is the Kruskal extension, near the horizon

uniformly accelerated trajectory

Beyond $r=2M$, ∂_r is timelike: $|\partial_r|^2 < 0$: geometry is time-dependent
horizon recedes away at light-speed

Singularity at $r=0$ is spacelike : in the future

- For <u>static</u> black holes, the Killing ∂_t is tangent to the horizon
  ie can always locate the Killing horizon solving for $g_{tt}=0$

($\exists$ hypersurf-orth timelike Killing)

For <u>stationary</u> bh's, eg Kerr, ∂_t is <u>not</u> tangent to horizon, and horizon does <u>not</u> correspond to $g_{tt}=0$

($\exists$ timelike Killing)

Instead $K = \partial_t + \Omega_H \partial_\phi$, $|K|^2 = 0$ at horizon

<u>Won't do much rotation.</u>

Ω_H : angular velocity

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note $R=0$

Reissner-Nordstrom $\quad R_{\mu\nu} = 2\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu} F^2\right)$

$\partial_\mu\left(\sqrt{-g}\, F^{\mu\nu}\right) = 0$

: in these units the Coulomb force is $\frac{G_1 Q_1 Q_2}{r^2}$ (geometrized units: Q has units of mass)

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2$$

electric $\quad A = -\frac{Q}{r}\, dt$, ie $F = \frac{Q}{r^2}\, dt \wedge dr$

magnetic $\quad A = -(Q\cos\theta + K)\, d\phi$

$\downarrow$ Dirac string

Note $Q = \frac{1}{4\pi}\int_{S^2} * F$

$= \frac{1}{4\pi}\int d\theta\, d\phi\, \frac{1}{2}\varepsilon_{\theta\phi tr} F^{tr} r^2 \sin\theta$

This electromagnetic field creates a repulsive gravitational effect, which acts also on neutral particles: $-g_{tt} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$

Horizons: $g_{tt} = 0 \Rightarrow r^2 - 2Mr + Q^2 = 0 \Rightarrow r = r_\pm = M \pm \sqrt{M^2 - Q^2}$

(Note $r_+ < 2M$: repulsive grav eff of electromagnetic field, smaller horizon)

Cases:

- $M < |Q|$: no horizon, $r=0$ is naked. This is the case of electron. $\frac{|Q|}{M} \sim 10^{21}$
  Singularity usually resolved by non-quantum-gravity effects
  eg QED (modifies $T^{(e)}_{\mu\nu}$ at $r \sim \frac{\hbar}{m_{el}}$) or stringy α'-effects

- $M > |Q|$: inner : $r = r_-$
  outer : $r = r_+$
  Can easily extend across them as we did for Schw
  Singularity at $r=0$ is timelike

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Can rewrite metric as

$$ds^2 = -\frac{f}{H^2}dt^2 + H^2\left(\frac{d\tilde{r}^2}{f} + \tilde{r}^2 d\Omega^2\right)$$ by making $\tilde{r} = r - r_-$

$r_0 = r_+ - r_- = 2\sqrt{M^2 - Q^2}$

$H = 1 + \frac{r_1}{\tilde{r}}$ $\quad f = 1 - \frac{r_0}{\tilde{r}}$ $\quad r_1 = r_-$

- $\underline{M = |Q|}$: <u>extremal bh</u> $\quad r_0 = 0$ $\quad$ (r_0 = "non-extremality parameter")

  $r_+ = r_- = M$ $\quad$ <u>degenerate</u> horizon

$$ds^2 = -\left(1 - \frac{M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{M}{r}} + r^2 d\Omega^2$$ horizon at $r = M$

$$= -\frac{1}{H^2}dt^2 + H^2\left(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right)$$ $\quad H = 1 + \frac{Q}{\tilde{r}}$ $\quad$ (Take $Q > 0$ w/o l.o.g.)

spatially conformally flat

Horizon at $\tilde{r} = 0$: but this is not a point, it's a S^2

(singularity at $\tilde{r} = -M$: -ve value of $\tilde{r}$!)

$$A_t = \frac{Q}{\tilde{r}\left(1 + \frac{Q}{\tilde{r}}\right)} = -\frac{1}{H} + 1$$ $\hookrightarrow$ can gauge away

- $r = M$ ($\tilde{r} = 0$) is at infinite spatial proper distance $\int_{r=M} \sqrt{g_{rr}}\, dr' = \int_{r=M} \frac{dr'}{1 - M/r'} \to \infty$

  at finite timelike / null proper distance (ex: check)

- $r = 0$: singularity is timelike

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Throat geometry

Take $t = \text{const}$ surface

$$ds^2 = \frac{dr^2}{\left(1-\frac{M}{r}\right)^2} + r^2 d\Omega^2 = H^2\left(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right)$$

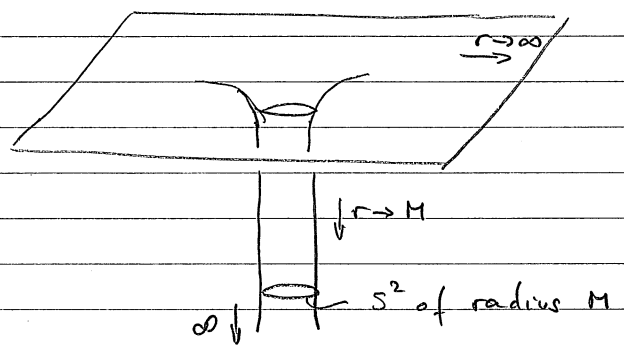
As $r \to \infty$: flat space

As $r \to M$: size of $S^2 \to$ constant (area $4\pi M^2$) $\qquad H^2\tilde{r}^2 \to \frac{Q^2}{\tilde{r}^2}\tilde{r}^2 = Q^2$

proper radial distance diverges $\left(1+\frac{Q}{\tilde{r}}\right)d\tilde{r} \simeq Q\frac{d\tilde{r}}{\tilde{r}} \simeq Q\, d(\log \tilde{r})$

$R_{prop} \simeq Q \log \tilde{r} \xrightarrow{\tilde{r}\to 0} \infty$

Eliminating one angle ($S^2 \to S^1$)



• Multi-RN extremal bh's (Majumdar-Papapetrou)

Ansatz: $\quad ds^2 = -\frac{1}{H^2}dt^2 + H^2(dx^2+dy^2+dz^2) \qquad H(x,y,z)$: "conformastatic"

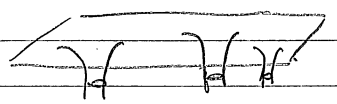
$A_t = -\frac{1}{H} + 1$

Eqs are solved iff $\boxed{\nabla^2 H = 0}$: H harmonic in (x,y,z) <u>Linear equation</u>!

Simplest: RN $\quad H = 1 + \frac{Q}{(x^2+y^2+z^2)^{1/2}} = 1 + \frac{Q}{|\vec{x}|}$

Multi-RN: $\quad H = 1 + \sum_i \frac{Q_i}{|\vec{x}-\vec{x}_i|}$: multi-center $\vec{x}_i = \vec{x}$

Throat near each $\vec{x} = \vec{x}_i$



<u>Static</u> and <u>non-singular</u> outside horizons

Cancellation of forces: $M_i = Q_i \Rightarrow \frac{M_i M_j}{r^2} = \frac{Q_i Q_j}{r^2}$

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Near the horizon of extremal RN:

go near $\tilde{r}=0$ $\quad H = 1 + \frac{Q}{\tilde{r}} \simeq \frac{Q}{\tilde{r}}$ $\quad$ (formally: scale $\tilde{r} = \varepsilon \hat{r}$, $t = \frac{1}{\varepsilon}\hat{t}$, $\varepsilon \to 0$)

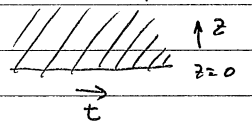
$$ds^2 \to \underbrace{-\frac{\tilde{r}^2}{Q^2}dt^2 + \frac{Q^2}{\tilde{r}^2}d\tilde{r}^2}_{AdS_2} + \underbrace{Q^2 d\Omega^2}_{S^2}$$

$\tilde{r}/Q = Q/z$

$$-\frac{\tilde{r}^2}{Q^2}dt^2 + \frac{Q^2}{\tilde{r}^2}d\tilde{r}^2 = \frac{Q^2}{z^2}(-dt^2 + dz^2)$$

$0 < z < \infty$

Poincare upper-half plane $\quad -\infty < t < \infty$



Note: $H = \frac{Q}{\tilde{r}}$ is harmonic, but not asymptotically const (flat)

$A_t = -\frac{\tilde{r}}{Q}$

$F_{t\tilde{r}} = \frac{1}{Q}$ is covariantly constant $(\nabla_\mu F_{\nu\alpha} = 0)$

Extremal RN interpolates between flat space and $AdS_2 \times S^2$ (aka Bertotti-Robinson)

4D dilatonic black holes

$$\int d^4x \sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - e^{-a\phi}F^2\right)$$

$$\partial_\mu\left(\sqrt{-g}\, e^{-a\phi} F^{\mu\nu}\right) = 0$$

$$\Box\phi + a e^{-a\phi} F^2 = 0$$

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + 2\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\right)e^{-a\phi}$$

Easy to find the solutions by guesswork w/ ansatz $\quad ds^2 = -H^\alpha f\, dt^2 + H^\beta\left(\frac{dr^2}{f} + r^2 d\Omega^2\right)$

$$A_t = \frac{\gamma}{H} \qquad e^{-a\phi} = H^\delta \qquad H = 1 + \frac{r_1}{r} \qquad f = 1 - \frac{r_0}{r}$$

Solution: $\quad \alpha = -\frac{2}{1+a^2} \quad \beta = \frac{2}{1+a^2} \quad \gamma = \frac{\sqrt{1 + r_0/r_1}}{\sqrt{1+a^2}} \quad \delta = \frac{2a^2}{1+a^2}$

$M = -\frac{\alpha r_1}{2} = \frac{r_1}{1+a^2}$ $\quad$ (from $-g_{tt} = 1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)$)

$Q = \gamma r_1 = \frac{r_1}{\sqrt{1+a^2}}$ $\quad$ (from $Q = \frac{1}{4\pi}\int_{S^2} e^{-a\phi} * F$)

If $r_0 \neq 0$ then $r = r_0$ is a regular non-degenerate horizon

Extremal limit: $r_0 = 0$. $\quad r = 0$ is singularity if $a \neq 0$

Area $\quad r^{2 - \frac{2}{1+a^2}} = r^{\frac{2a^2}{1+a^2}} \to 0$ if $a \neq 0$

PLUS

Black holes in string theory and sugra

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Higher-dimensional black holes and branes

Vacuum $R_{\mu\nu}=0$ in $D=3+n$

Generalize Schwarzschild by changing fall-off $\frac{1}{r} \to \frac{1}{r^{n}}$

$$ds^2 = -\left(1-\frac{\mu}{r^n}\right)dt^2 + \frac{dr^2}{1-\frac{\mu}{r^n}} + r^2 d\Omega^2_{(n+1)}$$

$$\Omega_{(n-1)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

Mass $M \propto \mu$

↳ from linearized analysis near ∞ : $-g_{tt} = 1 - \frac{16\pi G}{(n+1)\Omega_{(n+1)}}\frac{M}{r^n}$

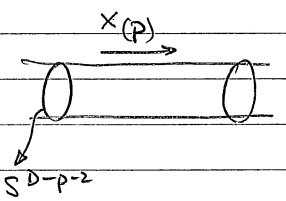
Single zero at $r = r_0 \equiv \mu^{\frac{1}{n}}$: horizon. Same causal structure as in 4D

Black branes: $ds^2_D = ds^2(\text{Schw}_{D-p}) + d\vec{x}^2_{(p)}$: automatically solves $R_{\mu\nu}=0$

↳ flat

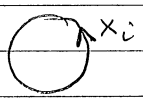
This is a neutral black p-brane

↳ spatial dimensions



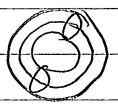
They only exist in $D>4$

When $\vec{x}_{(p)}$ directions are compact, $x_i \sim x_i + 2\pi R_i$



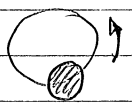
The horizon is $S^{D-p-2} \times T^p$

For simplicity $p=1$, black string



wrapped black string

but can have also



: localized black hole (no exact analytic solution)

- Can boost it:

$$ds^2 = \left(\eta_{ab} + \left(\frac{r_0}{r}\right)^n u_a u_b\right)dx^a dx^b + \frac{dr}{1-\left(\frac{r_0}{r}\right)^n} + r^2 d\Omega^2_{(n+1)}$$

u^a: velocity vector, $u^a u^b \eta_{ab} = -1$ $\quad$ η_{ab} = Minkowski in $p+1$ dimensions $a,b = 0,\dots,p$

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$u^a = (\cosh\alpha, \sinh\alpha, 0, \dots, 0)$ α: boost parameter

Boost to light-speed: $\alpha \to \infty$, $r_0 \to 0$, keep $r_0^n \sinh^2\alpha \equiv q$ finite

Then $r_0^{n/2} u^a \to q^{1/2} \ell^a$, $\ell^a \ell^b \eta_{ab} = 0$: null vector

$$ds^2 = \left(\eta_{ab} + \frac{q}{r^n} \ell_a \ell_b\right) dx^a dx^b + dr^2 + r^2 d\Omega^2_{(n+1)}$$: momentum wave

- Boosting can be seen as solution-generating (but when is the solution actually different, ie, not diffeo-equivalent?)

- Other solution-generating techniques: <u>duality symmetries in low-energy string th</u>

<u>T-duality</u>

(String vs Einstein frame OVER)

NS-NS sector of closed strings string frame

$$I = \int d^D x \sqrt{-g}\, e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_{(3)}^2\right)$$

$H_{(3)}$: antisymmetric $H_{\mu\nu\rho}$

$H_{(3)} = dB_{(2)}$

$B_{(2)}$: potential $B_{\mu\nu}$

Take a solution w/ an isometric direction x. (in coords adapted to the isometry), ϕ, $g_{\mu\nu}$, $B_{\mu\nu}$

Then a new solution (in string frame) is obtained as

$$e^{2\tilde\phi} = e^{2\phi}/g_{xx} \qquad \tilde g_{xx} = \frac{1}{g_{xx}} \qquad \tilde g_{\mu x} = \frac{B_{\mu x}}{g_{xx}} \qquad \tilde g_{\mu\nu} = g_{\mu\nu} - \frac{g_{\mu x} g_{\nu x} - B_{\mu x} B_{\nu x}}{g_{xx}}$$

$$\tilde B_{\mu x} = \frac{g_{\mu x}}{g_{xx}} \qquad \tilde B_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu x} g_{\nu x} - g_{\mu x} B_{\nu x}}{g_{xx}}$$

For instance take a black string boosted to speed of light $\ell^t = 1$ $\ell^x = 1$ ie <u>momentum wave</u> $r_q^n \equiv q$

$$ds^2 = -\left(1 - \frac{q}{r^n}\right) dt^2 + H dx^2 + 2(H-1)\, dt\, dx + dr^2 + r^2 d\Omega^2_{(n+1)}$$

$H = 1 + \frac{q}{r^n}$ $\phi = 0$ $B_{\mu\nu} = 0$ $g_{xx} = H$ $g_{tx} = H - 1$

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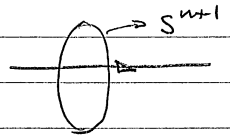
$$\Rightarrow \quad d\tilde{s}^2 = \frac{1}{H}\left(-dt^2 + dx^2\right) + dr^2 + r^2 d\Omega^2_{(n+1)}$$

$$e^{2\phi} = \frac{1}{H} \qquad B = \left(1 - \frac{1}{H}\right) dt \wedge dx$$

This is the sugra soln for a fundamental string (F-string, F1)

It has Poincaré invce along (t,x) and is an electric source for $B_{\mu\nu}$ like a fundamental string



$$\int_{S^{n+1}} e^{-2\phi} * H$$ is a conserved charge $\propto$ <u>winding</u> number.

Thus T-duality exchanges momentum $\Leftrightarrow$ winding

- Note: any H such that $\nabla^2_{(n+1)} H = 0$ harmonic yields a solution (extremal)

  There are also non-extremal F-strings, obtained by T-duality of boosted black strings.

Another duality: S-duality of Type IIB

$$I = \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{1}{12} H^2_{(3)}\right) \underbrace{- F^2_{(1)} - \frac{1}{12}\tilde{F}^2_{(3)} - \frac{1}{4\times 5!}\tilde{F}^2_{(5)}}_{\text{RR sector}} \right\}$$

Set $A_{(0)} = 0$ so $F_{(1)} = 0$

and $\tilde{F}_{(3)} = F_{(3)} = dA_{(2)}$ (no axion, for simplicity)

$$\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2} A_{(2)} \wedge H_{(3)} + \frac{1}{2} B_{(2)} \wedge F_{(3)}$$

Go to <u>Einstein frame</u> $g^{(E)}_{\mu\nu} = e^{-\phi/2} g_{\mu\nu}$

so $$I = \int d^{10}x \sqrt{-g^{(E)}} \left\{ R_E - \frac{1}{2}(\partial\phi)^2 - \frac{e^{-\phi}}{12} H^2_{(3)} - \frac{e^{\phi}}{12} F^2_{(3)} - \frac{1}{4\times 5!}\tilde{F}^2_{(5)} \right\}$$

is invariant under

$$g^{(E)}_{\mu\nu} \longleftrightarrow g^{(E)}_{\mu\nu}$$

$$\phi \longleftrightarrow -\phi$$

$$H_{(3)} \longleftrightarrow F_{(3)}$$

$$F_{(5)} \longleftrightarrow F_{(5)}$$

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S-duality of F-string gives D-string.

In general one can construct:

- D-p-branes metric in string frame

$$ds^2 = H^{-1/2} \overbrace{\eta_{\mu\nu} dx^\mu dx^\nu}^{p+1} + H^{1/2}\left(dr^2 + r^2 d\Omega^2_{(8-p)}\right) \qquad H = 1 + \frac{q}{r^{7-p}}$$

$$e^{2\phi} = H^{\frac{3-p}{2}} \qquad A_{01\cdots p} = 1 - \frac{1}{H} \quad \text{(electric)}$$

$$dA = (7-p)\, q\, \omega_{(8-p)} \quad \text{(magnetic)} \qquad \omega_{8-p} : \text{volume-form in } \Omega_{8-p}$$

$0 \leq p < 7$ ($p=-1$, $p=7$ and $p=8$ are special)

- NS5-brane : electric-magnetic dual (of $H_{(3)}$) of F-string

$$ds^2 = \underbrace{\eta_{\mu\nu} dx^\mu dx^\nu}_{1+5} + H\left(dr^2 + r^2 d\Omega^2_{(3)}\right) \qquad H = 1 + \frac{q}{r^2}$$

string frame

$$dB_{(2)} = 2q\, \omega_3 : \text{magnetic}$$

- KK monopole : pure vacuum solution

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \underbrace{\frac{1}{H}\left(dx + q\cos\theta\, d\phi\right)^2 + H\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right)}_{\text{Euclidean Taub-NUT}} \qquad H = 1 + \frac{q}{r}$$

Dirac: $x \sim x + 4\pi q$

Won't study this one much more

Also M2, M5 branes

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Core structure near $r=0$: consider Transverse space

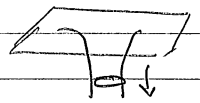
$$H^{\alpha}\left(dr^2 + r^2 d\Omega^2_{(n+1)}\right) \qquad H = 1 + \frac{q}{r^n}$$

$r \to 0 \quad H \to \frac{q}{r^n}$ $\qquad q^{\alpha}\left(\frac{dr^2}{r^{\alpha n}} + r^{2-\alpha n}\, d\Omega^2_{(n+1)}\right)$

if $\alpha n > 2$: $r=0$ at ∞ distance S^{n+1} blows up → singular $\quad D_p$

$\alpha n = 2$: $r=0$ at ∞ distance, S^{n+1} finite size $\quad p\ 3$

$\alpha n < 2$: $r=0$ at finite distance S^{n+1} shrinks to 0 → singular $\quad p=3 \quad p\ 3$

$p=3$ is non-dilatonic  Also NS5 (in string frame)

Non-extremal D3

$$ds^2 = H^{-1/2}\left(-f\,dt^2 + d\vec{x}^2_{(3)}\right) + H^{1/2}\left(\frac{dr^2}{f} + r^2 d\Omega^2_{(5)}\right) \qquad f = 1 - \left(\frac{r_0}{r}\right)^4$$

$$\left(\text{also } ds^2 = H^{-1/2}\left(\eta_{\mu\nu} + \left(\frac{r_0}{r}\right)^4 u_\mu u_\nu\right) dx^\mu dx^\nu + \text{''}\right) \qquad H = 1 + \left(\frac{R}{r}\right)^4$$

$\mu,\nu = 0,\dots,3$

Near-horizon limit (decoupling) $\quad H \to \left(\frac{R}{r}\right)^4$ $\qquad$ properly: $r \to \varepsilon r$, $r_0 \to \varepsilon r_0$, $\varepsilon \to 0$

$$ds^2 \to \frac{r^2}{R^2}\left(-f\,dt^2 + d\vec{x}^2_3\right) + \frac{R^2}{r^2}\left(\frac{dr^2}{f} + r^2 d\Omega^2_{(5)}\right)$$

$$= \frac{r^2}{R^2}\left(-f\,dt^2 + d\vec{x}^2_3\right) + \frac{R^2}{r^2}\frac{dr^2}{f} + R^2 d\Omega^2_{(5)}$$

If $f=1$ (extremal) Then, change $\frac{r}{R} = \frac{R}{z}$

$$ds^2 = R^2 \frac{\overbrace{\eta_{\mu\nu}dx^\mu dx^\nu + dz^2}^{3+1}}{z^2} + R^2 d\Omega^2_{(5)} \quad : AdS_5 \times S^5$$

If $f = 1 - \left(\frac{r_0}{r}\right)^4$: planar bh in $AdS_5 \times S^5$

$\quad$ ‖

$\quad$ black brane

- Although S^5 has finite size, area of extremal D3 brane horizon is zero:

$$\text{Area} = \int dx_1 dx_2 dx_3\, d\Omega_{(5)} \sqrt{g_{11} g_{22} g_{33} g_{\theta_i\theta_i} \cdots} = 0 \quad \text{since} \quad \int dx_1 dx_2 dx_3 \sqrt{g_{11} g_{22} g_{33}} = 0$$

Horizon shrinks to zero longitudinally (due to brane tension)

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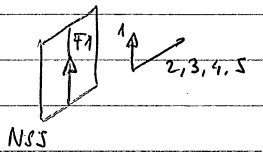
Branes within branes & brane intersections (Towards an extremal bh w/ finite horizon area)

Eg F1-NS5



	1	2	3	4	5	6	7	8	9
F1	x								
NS5	x	x	x	x	x				

$$ds^2 = \frac{1}{H_1}\left(-dt^2+dx^2\right) + dx_2^2+dx_3^2+dx_4^2+dx_5^2 + H_5\left(dr^2+r^2 d\Omega^2_{(3)}\right) \qquad \text{(string frame)}$$

$$e^{2\phi} = \frac{H_5}{H_1} \qquad H_1 = 1+\frac{q_1}{r^2} \qquad H_5 = 1+\frac{q_5}{r^2}$$

(F1 is "smeared" along NS5 directions)

Note: dilaton is regular at $r=0$ $e^{2\phi} \to \frac{q_5}{q_1}$. Curvature remains regular, but (mild) orbifold singularity when $x \sim x+2\pi R$

But area of horizon $r=0$ vanishes since $\int dx \sqrt{g_{xx}} = 0$ due to string tension.

Add momentum wave (will create pressure) : NS5-F1-P

$$ds^2 = \frac{H_P}{H_1}\left(dx + \left(1-\frac{1}{H_P}\right)dt\right)^2 - \frac{1}{H_1 H_P}dt^2 + dx^2_{2,3,4,5} + H_5\left(dr^2+r^2 d\Omega^2_{(3)}\right)$$

$$H_i = 1+\frac{q_i}{r^2}$$

$$e^{2\phi} = \frac{H_5}{H_1}$$

To measure area go to Einstein frame $g^{(E)}_{\mu\nu} = e^{-\phi/2} g_{\mu\nu} = \left(\frac{H_1}{H_5}\right)^{1/4} g_{\mu\nu}$

$$\text{Area} = \Omega_{(3)}\sqrt{\left(\frac{H_1}{H_5}\right)^{3/4}}\sqrt{\frac{H_P}{H_1}H_5^3}\; r^3 \Big|_{r\to 0} = \Omega_{(3)} r^3 \sqrt{H_1 H_5 H_P}\Big|_{r\to 0} = \Omega_3\sqrt{q_1 q_5 q_P} \neq 0 \qquad \Omega_{(3)} = 2\pi^2$$

Can S-dualize to D1-D5-P: same Einstein-frame metric

Then compactify on T^4 (could be K3 Torus), and go to the 6D D1D5-P string in Einstein frame

$$ds^2_6 = \frac{H_P}{\sqrt{H_1 H_5}}\left(dx + \left(1-\frac{1}{H_P}\right)dt\right)^2 - \frac{1}{H_P\sqrt{H_1 H_5}}dt^2 + \sqrt{H_1 H_5}\left(dr^2+r^2 d\Omega^2_{(3)}\right)$$

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Near-horizon of D1-D5-P black hole

↓

"Decoupling" limit $\quad q_1, q_5 \gg q_P, r^2 \quad H_{1,5} \to \frac{q_{1,5}}{r^2}$

$$ds_6^2 \to \frac{1}{\sqrt{q_1 q_5}} \underbrace{r^2\left[H_P\left(dx + \left(1-\frac{1}{H_P}\right)dt\right)^2 - \frac{1}{H_P}dt^2\right] + \sqrt{q_1 q_5}\left(\frac{dr^2}{r^2}\right.}_{\text{constant -ve curvature 3D spacetime, radius } \sqrt{q_1 q_5}} \underbrace{\left. + d\Omega_{(3)}^2\right)}_{S^3 \text{ of radius } \sqrt{q_1 q_5}}$$

$\Rightarrow$ locally equiv to AdS_3

Define $\ell^2 = \sqrt{q_1 q_5}$; identify $x \sim x + 2\pi R$

and change $x = R\phi$; $\phi \sim \phi + 2\pi$

$$r^2 + q_P = \frac{\ell^2}{R^2}\rho^2 \qquad q_P = \frac{\ell^2}{R^2}\rho_+^2 \qquad t \to \frac{R}{\ell}t$$

Then

$$ds_6^2 = \underbrace{-\frac{(\rho^2-\rho_+^2)^2}{\ell^2\rho^2}dt^2 + \frac{\ell^2\rho^2}{(\rho^2-\rho_+^2)^2}d\rho^2 + \rho^2\left(d\phi + \frac{\rho_+^2}{\ell^2\rho^2}dt\right)^2}_{\text{extremal BTZ black hole}} + \ell^2 d\Omega_{(3)}^2$$

↳ double zero at $\rho = \rho_+$ $\quad$ is a solution of $\int d^3x\sqrt{-g}\left(R + \frac{2}{\ell^2}\right)$

Nonextremal D1-D5-P

$$ds_6^2 = \frac{H_P}{\sqrt{H_1 H_5}}\left(dx + \frac{r_0^2\sinh\alpha_P}{2r^2 H_P}dt\right)^2 - \frac{f}{H_P\sqrt{H_1 H_5}}dt^2 + \sqrt{H_1 H_5}\left(\frac{dr^2}{f} + r^2 d\Omega_{(3)}^2\right)$$

$$H_i = 1 + \frac{r_0^2\sinh^2\alpha_i}{r^2} \qquad f = 1 - \frac{r_0^2}{r^2}$$

$$i = 1, 5, P$$

When $r^2 \ll r_0^2\sinh^2\alpha_{1,5}$

Then the decoupling limit yields <u>non-extremal BTZ</u> $\times S^3$

$$\rho_+^2 = \frac{R^2}{\ell^2}r_0^2\cosh^2\alpha_P \qquad \rho_-^2 = \frac{R^2}{\ell^2}r_0^2\sinh^2\alpha_P$$

$$ds_{BTZ}^2 = -\frac{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}{\ell^2\rho^2}dt^2 + \frac{\ell^2\rho^2}{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}d\rho^2 + \rho^2\left(d\phi + \frac{\rho_+\rho_-}{\ell^2\rho^2}dt\right)^2$$

String frame vs Einstein frame

$$I = \int d^{10}x \sqrt{-g}\; e^{-2\phi} \left(R + 4(\partial\phi)^2 + \dots \right)$$

: $g_{\mu\nu}$ is the metric that appears in Polyakov's ~~Embedded~~ action,

$$\int d^2\sigma \sqrt{-\gamma}\; g_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu$$

String worldsheets extremize the area measured with this metric

We can perform a field-redefinition to disentangle graviton and dilaton

$$g^{(E)}_{\mu\nu} = e^{-\phi/2} g_{\mu\nu}$$

so that

$$I = \int d^{10}x \sqrt{-g_{(E)}} \left(R_{(E)} - \tfrac{1}{2}(\partial\phi)^2 + \dots \right)$$

$\Downarrow$ no $e^{-2\phi}$ in front of R_E

Remarks:
- gravitational eqs are more easily derived in Einstein frame (= "graviton frame")
- bh area thm is valid in Einstein frame : entropy measured in Einstein frame
- energy conditions in Einstein frame
- Mass, angular mom " " "
  Tension ...
- Things that belong properly to the gravitational field should be computed in Einstein frame

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Black hole Thermodynamics

Laws of bh mechanics (Bardeen + Carter + Hawking)

0th : κ uniform over Killing horizon (equilibrium)

1st : $dM = \frac{\kappa}{8\pi G} dA_H + \Omega dJ$ ..

2nd : $\Delta A_H \geq 0$

Analogous to laws of Thermo, but proofs couldn't be more different

Suggests that area is entropy, κ is Temperature

On dimensional grounds $S_{BH} = c \frac{A_H}{G\hbar}$ $T_H = \frac{\hbar \kappa}{8\pi c}$ c = constant

But if S is an entropy then area law will not be strict but only statistically valid. A bh should be able to emit particles w/ suppression factor

$\Gamma \sim e^{-|\delta S|}$

$$\left.\begin{aligned} T\delta S &= \delta M - \Omega \delta J \\ \delta M &= -\omega \\ \delta J &= -m \end{aligned}\right\} \Rightarrow -|\delta S| = -\frac{\omega - m\Omega}{T}$$

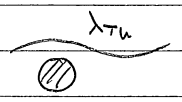
$\Gamma \sim e^{-\frac{\omega - m\Omega}{T}}$: Thermal spectrum

Observe that if we keep the area fixed then $S_{BH} = c\frac{A_H}{G\hbar} \xrightarrow{\hbar \to 0} \infty$: nothing can escape from a classical bh.

Hawking (1974) : indeed $T_H = \frac{\hbar \kappa}{2\pi}$ ie $c = \frac{1}{4}$ and $S_{BH} = \frac{A_H}{4G\hbar}$

For Schwarzschild : $\kappa = \frac{1}{4GM}$ so $T = \frac{\hbar}{8\pi GM}$

Note $\lambda_{Thermal} \sim \frac{\hbar}{T} \sim GM \sim R_{schw}$



: BH is a small radiator

- T_H increases as M decreases

→ bh will produce any particle in its evaporation

→ Specific heat is negative

$C = \frac{dE}{dT} = \frac{dM}{dT_H} = -8\pi M^2 < 0$: Thermodynamically unstable

bh heats up by radiating energy

cools down " absorbing "

(Typical in gravitating systems)

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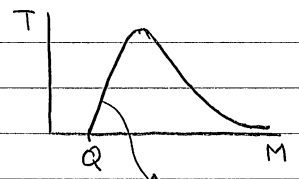
$\Rightarrow$ can't keep Schw bh in stable Thermal equilibrium w/ a bath of radiation

(but can stabilize it by putting it in a sufficiently small box; more later)

- Charged bhs: Reissner - Nordstrom

$T_H = \frac{\sqrt{M^2-Q^2}}{2\pi r_+^2}$ $\qquad r_+ = M + \sqrt{M^2-Q^2}$ $\qquad$ Recall $M \geq Q$, $M=Q$ is extremal



fixed Q

$\frac{dM}{dT} > 0$ so $C_Q = \left(\frac{\partial M}{\partial T}\right)_Q > 0$ close to extremality

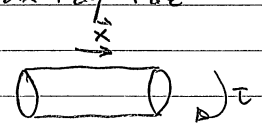
- <u>Euclidean bh Thermodynamics</u>

<u>Imaginary Time formalism for QFT at finite Temperature:</u>

Kubo-Martin-Schwinger: The correlation functions of a QFT at Temperature $T = 1/\beta$ are periodic in imaginary Time $t = -i\tau$, w/ period $\tau \sim \tau + \beta$.

i.e. QFT at finite Temp is formulated w/ periodic boundary conditions in imaginary Time

Define the Theory in $ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2$ with $\tau \sim \tau + \beta$



Partition function $Z = \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi \; e^{-I_E[\phi]}$: computed over field configurations periodic in imaginary Time

Once Z is computed, apply Thermo: $Z = e^{-\beta F}$ $\qquad$ F: free energy

$E = \frac{\partial}{\partial\beta}(\beta F) = -\frac{\partial}{\partial\beta}\log Z$ $\qquad$ $F = E - TS$

$S = \left(\beta\frac{\partial}{\partial\beta} - 1\right)(\beta F) = -\left(\beta\frac{\partial}{\partial\beta} - 1\right)\log Z$

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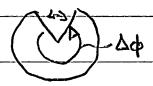
What does this have to do with bh's?

Near a (non-degenerate) horizon the geometry is

$$ds^2 \simeq -\kappa^2\xi^2 dt^2 + d\xi^2 + R^2 d\Omega$$

$$t \to -i\tau \qquad ds^2 \simeq \underbrace{\kappa^2\xi^2 d\tau^2 + d\xi^2}_{\text{Like polar coords } \rho^2 d\phi^2 + d\rho^2 \text{ with } \rho \to \xi,\ \phi \to \kappa\tau} + R^2 d\Omega$$

If origin $\xi = 0$ is to be smooth w/out conical singus



(ie solving Einstein's eqs w/out external sources)

Then $\kappa\tau \sim \kappa\tau + 2\pi$

ie $\tau \sim \tau + \frac{2\pi}{\kappa}$: periodic in imaginary time:

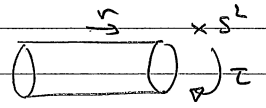
finite temperature $\beta = \frac{2\pi}{\kappa}$, ie $\underline{\underline{T = \frac{\kappa}{2\pi}}}$: Hawking's result!

Eg Euclidean Schwarzschild is

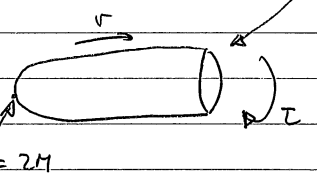
$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2_{(2)}$$

Near $r = 2M$ is 2D plane $\times S^2$

Near ∞ is cylinder



$\Rightarrow$ It's a "cigar" geometry



$\times S^2$

Topology is $\mathbb{R}^2 \times S^2$ unlike finite-temp Minkowski $S^1 \times \mathbb{R}^3$

$\therefore$ The euclidean action I_E of a classical regular euclidean solution yields its partition function and free energy $\beta F = I_E[g_{ce}]$

$$I_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^D x \sqrt{g}\, R - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{D-1}x \sqrt{h}\, K$$

Einstein-Hilbert

Gibbons-Hawking

Needed to have well-defined variational problem with $g_{\mu\nu}$ fixed at $\partial\mathcal{M}$

$h_{\mu\nu}$: metric induced at the boundary

K: Trace of extrinsic curvature of $h_{\mu\nu}$

$\sqrt{h}K = n^\mu \partial_\mu \sqrt{h}$, n^μ: outward normal to $\partial\mathcal{M}$

Analogous to scalar field action $\int_{\mathcal{M}} -\phi\partial^2\phi + \int_{\partial\mathcal{M}} \phi\, n\cdot\partial\phi$

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Using this we can show in general that the gravitational entropy is $\boxed{S_{BH} = \frac{A_H}{4G}}$
↳ E-H + G-H gravitational action

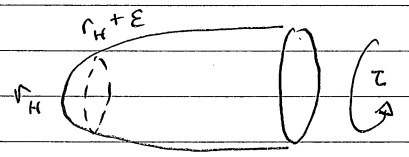
First note that for any field theory in flat space, if $\frac{\partial}{\partial \tau}$ is an isometry of the solution (ie. static) then

$$I_E = \int d\tau\, d^3x\, \mathcal{L} = \int d\tau \int d^3x\, \mathcal{L} = \beta H \qquad \text{where } H \text{ is the Hamiltonian}$$

so $S = (\beta \partial_\beta - 1) I_E = 0$: the classical contribution to the entropy is zero. Entropy in field theory arises at one-loop

This applies to any Euclidean spacetime where $\frac{\partial}{\partial \tau}$ is Killing and acts without fixed-points, ie Hamiltonian evolution is well-defined, and $I_E \propto \beta \Rightarrow S=0$

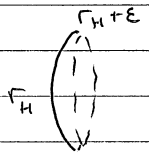
When there's a bh (for simplicity, static; can be easily extended to stationary) then $\frac{\partial}{\partial \tau}$ acts w/ a fixed point at the horizon, and the evolution generated by the Hamiltonian degenerates there



Split the integration of the action between $r \in [r_H, r_H+\varepsilon]$ and $r \in [r_H+\varepsilon, \infty)$

The contribution of to the entropy is zero
(to the action: $I = \beta H \Rightarrow F = E$)

For the "cap"



The bulk term $\int d^Dx \sqrt{g} R = 0$ on-shell

The boundary term:

$$I_E^{(\varepsilon)} = -\frac{1}{8\pi G} \int_{\partial M} d^{D-1}x \sqrt{h}\, K = -\frac{1}{8\pi G} n^\mu \partial_\mu \int_{r \approx r_H} d^{D-1}x \sqrt{h}$$

$$ds^2 \simeq \kappa^2 x^2 d\tau^2 + dx^2 + d\Sigma^2 \longrightarrow \text{horizon geometry} \quad \int d\Sigma = A_H$$

$$\sqrt{h} = \kappa x\, \mathrm{vol}(\Sigma) \qquad n^\mu \partial_\mu = \partial_x \qquad \kappa = \frac{2\pi}{\beta}$$

$$I_E^{(\varepsilon)} = -\frac{1}{8\pi G} \partial_x \int d\tau \int \sqrt{h}\, \mathrm{vol}(\Sigma) = -\frac{1}{8\pi G} \beta A_H\, \partial_x(\kappa x) = -\frac{1}{4G} A_H$$

$$S = (\beta \partial_\beta - 1) I_E = \underline{\underline{\frac{A_H}{4G}}}$$

Note that this is a classical contribution to the entropy $S = \frac{A_H}{4G\hbar}$. At one-loop $S \sim \hbar^0$
$O(\frac{1}{\hbar})$ counts states running on a loop

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In an exercise we compute the full I_E for the Schwarzschild solution, to find

$$\beta F = \frac{1}{2}\beta M = \frac{\beta^2}{16\pi G}$$

$$\Rightarrow E = \frac{\partial}{\partial\beta}(\beta F) = \frac{\beta}{8\pi G} = M$$

$$S = \frac{E-F}{T} = 4\pi G M^2 = \frac{A_H}{4G} \qquad \left(A_H = 4\pi(2GM)^2\right)$$

This can be applied to other black holes

- <u>Schwarzschild-Anti-de Sitter black holes</u>

For definiteness I'll discuss only 5D

$$ds^2 = -\left(1-\frac{\mu}{r^2}+\frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1-\frac{\mu}{r^2}+\frac{r^2}{\ell^2}} + r^2 d\Omega^2_{(3)} \qquad R_{\mu\nu} = -\frac{4}{\ell^2} g_{\mu\nu}$$

If $\mu=0$ this is AdS_5 in <u>global coordinates</u> (not in Poincaré)

Horizon at $r=r_+$ $\quad r_+^4 + \ell^2 r_+^2 - \mu\ell^2 = 0$ ie $\mu = \frac{r_+^2(r_+^2+\ell^2)}{\ell^2}$ (use r_+ as parameter)

AdS has the effect of a box, or a potential well $\left(r_+^2 = \mu - \frac{r_+^4}{\ell^2} < \mu : \text{smaller bh}\right)$

There's a length scale ℓ so we can distinguish between <u>small and large bhs</u>

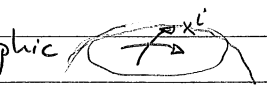
Small: $r_H \ll \ell$: can neglect cosmo constant, similar to Schw

Large bhs: $r_H \gg \ell$: do not have counterparts in the absence of Λ

Very large limit: $r \to \lambda r$ $\qquad \lambda \to \infty$

$\mu \to \lambda^4 \mu$ (so that r_H scales like λ)

$t \to \frac{1}{\lambda} t$

$r^2 d\Omega^2_{(3)} \to \lambda^2 r^2 d\Omega^2_{(3)} \to \frac{r^2}{\ell^2} d\vec{x}^2_{(3)}$ (stereographic 

$$\Rightarrow ds^2 \to -\left(\frac{r^2}{\ell^2}-\frac{\mu}{r^2}\right)dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2}-\frac{\mu}{r^2}} + \frac{r^2}{\ell^2}d\vec{x}^2_{(3)}$$

: "planar" bh in AdS_5

Same as we found from non-extremal D3-brane!

$\mu=0$ is AdS_5 in Poincaré coords

Now μ can be absorbed by a rescaling of coords: no meaning of "large" bh

underlying conformal invce

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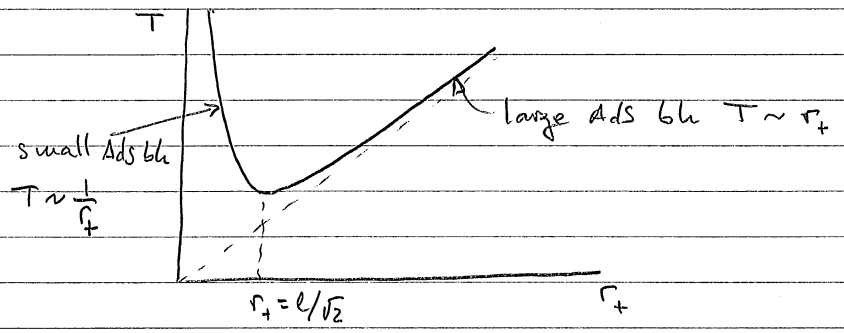
Back To bh in global AdS

We can compute using previous results The surface gravity and horizon Temperature

$$T = \frac{2r_+^2 + \ell^2}{2\pi \ell^2 r_+}$$



There are Two bhs w/ different size for any Temperature $> T_{min}$

The large one has higher entropy, clearly

$$S_{BH} = \frac{A_H}{4G} = \frac{\pi^2}{2G} r_+^3$$

but also it has positive specific heat

$$C = T\frac{\partial S}{\partial T} = \frac{3\pi^2}{2G} T r_+^2 \frac{\partial r_+}{\partial T}$$

$\frac{\partial r_+}{\partial T} > 0 \Rightarrow C > 0$: large bh

Can be in stable local Thermodynamic equilibrium w/ a bath of radiation

- Can compute The Euclidean action: w/ background subtraction wrT AdS_5 or w/ counterterm subtraction

$$I = I_{EH} + I_{GH} + I_{ct}$$

$$I_{ct} = \frac{1}{8\pi G}\int_{\partial M} d^4x \sqrt{h}\left(\frac{3}{\ell} + \frac{\ell}{4}\mathcal{R}\right)$$

$\mathcal{R}$: Ricci of $h_{\mu\nu}$

(Note: bulk Term I_{EH} is not vanishing)

Result is: $I = \beta \frac{\pi}{8G\ell^2}\left(-r_+^4 + r_+^2\ell^2\right) \overset{?}{=} \beta F$ (if counterterm subtraction, Then also $= \beta \frac{3\pi}{32G}\ell^2$, $\propto \beta$: Casimir energy of AdS_5)

$$E = \frac{\partial}{\partial\beta} I = \frac{3\pi}{8G}\mu$$

Note That at high T for large bhs, $r_+ \to \infty$ $T \sim \frac{r_+^2}{\ell^2}$

$F \sim T^4$, $E \sim T^4$, $S \sim T^3$ like a CFT_4

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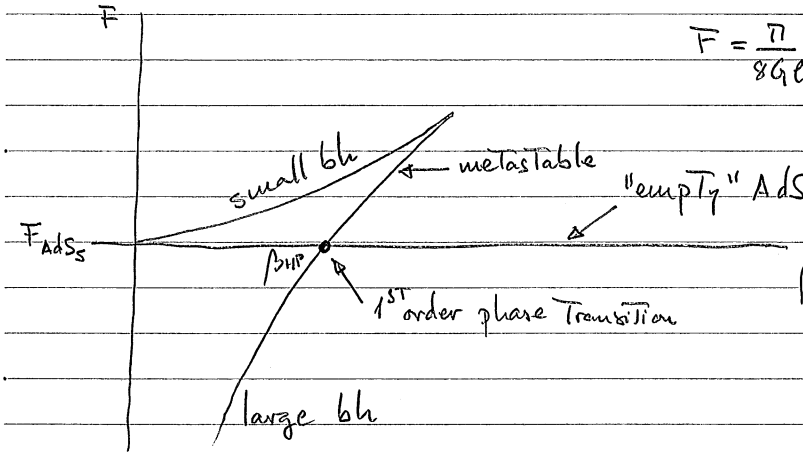
Phase Transitions in AdS (Hawking + Page)

Canonical ensemble: fix T



$$F = \frac{\pi}{8G\ell^2}\left(-r_+^4 + r_+^2\ell^2\right)$$

(+ E_0 if Casimir energy included: doesn't play a role here)

"empty" AdS_5 : would be filled by a thermal gas of gravitons w/ negligible backreaction and small F (one-loop)

"Swallowtail", 1st order phase Transition Hawking-Page

Small r_+:

$$F \sim r_+^2 \sim \beta^2$$

Large r_+:

$$F \sim -r_+^4 \sim -\beta^{-4}$$

β_{HP} occurs at $F_{BH} = F_{AdS} = 0$ ie $r_+ = \ell$: when the bh becomes AdS-size

1st order phase Transition: bh becomes globally preferred phase

$$\Delta S = S_{BH}(r_+ = \ell) = \frac{\pi^2}{2G}\ell^3$$

- In AdS_d $\quad ds^2 = -\left(1 - \frac{\mu}{r^{d-3}} + \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1 - \frac{\mu}{r^{d-3}} + \frac{r^2}{\ell^2}} + r^2 d\Omega^2_{(d-2)}$

it's qualitatively the same

For large bh's $\quad 1 - \frac{\mu}{r^{d-3}} + \frac{r^2}{\ell^2} \longrightarrow -\frac{\mu}{r^{d-3}} + \frac{r^2}{\ell^2} \qquad r_+^{d-1} \sim \mu\ell^2$

$$T \sim \frac{r_+}{\ell^2} \qquad E \sim \mu \sim T^{d-1}$$

$$F \sim T^{d-1} \quad : \text{like CFT in } d-1$$

$$S \sim T^{d-2}$$

No.
Date

In The Poincare patch, if The boundary is $\underset{\text{Eucl. Time}}{S^1} \times \underset{\text{compactified spatial direction}}{S^1} \times \mathbb{R}^2$

There can be another Transition.

With This boundary we have (besides AdS_5 in Poincare patch) which has an orbifold singularity if a spatial direction is a circle)

$$ds^2 = \left(\frac{r^2}{\ell^2} - \frac{\mu}{r^2}\right) d\tau^2 + \frac{r^2}{\ell^2}(d\theta^2 + dy^2 + dz^2) + \frac{dr^2}{\frac{r^2}{\ell^2} - \frac{\mu}{r^2}} \quad : \text{AdS black brane}$$

$$\theta \sim \theta + \Delta\theta$$

but also

$$ds^2 = \frac{r^2}{\ell^2} d\tau^2 + \left(\frac{r^2}{\ell^2} - \frac{\tilde{\mu}}{r^2}\right) d\theta^2 + \frac{r^2}{\ell^2}(dy^2 + dz^2) + \frac{dr^2}{\frac{r^2}{\ell^2} - \frac{\tilde{\mu}}{r^2}} \quad : \text{"AdS soliton"}$$

(allowed if fermions are antiperiodic ie ~~susy~~)

In The first one μ is fixed by The Temperature : $\tau \sim \tau + \beta$ $\quad \beta = \frac{\pi \ell^2}{r_+}$ $\quad r_+ = (\mu \ell^2)^{1/4}$ $\quad \mu = \left(\frac{\pi \ell}{\beta}\right)^4 \ell^2$

In The second one $\tilde{\mu}$ is fixed by $\Delta\theta$: $\theta \sim \theta + \Delta\theta$ $\quad \Delta\theta = \frac{\pi \ell^2}{\tilde{r}_+}$ $\quad \tilde{r}_+ = (\tilde{\mu} \ell^2)^{1/4}$ $\quad \tilde{\mu} = \left(\frac{\pi \ell}{\Delta\theta}\right)^4 \ell^2$

The Two solutions are possible with The same boundary conditions (fixed by β and $\Delta\theta$) and Therefore There can be phase Transitions among Them, depending on which one has The lowest F (free energy)

- F for The black brane is obtained from limit of previous results in large bh limit

$$\beta \to \frac{\pi \ell^2}{r_+}, \quad \ell^2 \Omega_{(3)} \to V_2 \Delta\theta \qquad V_2 = \int dy\, dz \ : \text{extensive} \qquad \Omega_{(3)} = 2\pi^2$$

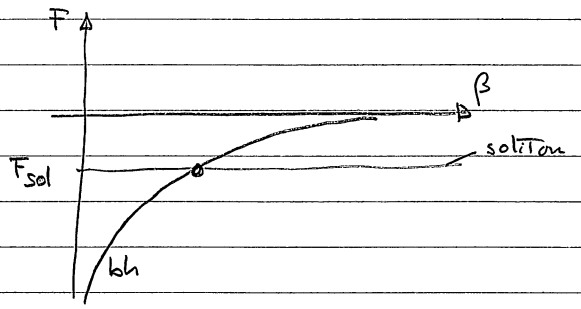
$$\beta F_{bh} \to -\frac{\pi \ell^2}{8G}\left(\frac{\pi \ell}{\beta}\right)^4 = -\frac{\ell^2 \Omega_{(3)}}{16\pi G}\left(\frac{\pi \ell}{\beta}\right)^4 \to \frac{V_2 \Delta\theta}{16\pi G}\left(\frac{\pi \ell}{\beta}\right)^4$$

- For The soliton, just change $\beta \leftrightarrow \Delta\theta$ $\quad \beta F_{sol} = \frac{-V_2}{16\pi G}\beta\left(\frac{\pi \ell}{\Delta\theta}\right)^4$: note $\beta F \propto \beta$ so $S = 0$ no horizon

Keep $\Delta\theta$ fixed, vary β:

Then $F_{BH} > F_{sol}$ at $\beta < \Delta\theta$: hi-Temp

$F_{BH} < F_{sol}$ at $\Delta\theta < \beta$: low-Temp

$F_{BH} = F_{sol}$ at $\Delta\theta = \beta$: 1st order phase Transition

$\Delta S = S_{BH}$