0/ Objectives:

Motility offers new pathways to self-organization.

Pattern formation: [Cross-Hohenberg, RMP 65, 851 (2019)]

$$\text{System} \xrightarrow{\;(1)\;} \text{Partial differential equation} \xrightarrow[\;(2)\;]{} \text{Patterns}$$

little on this.

Paradigm: (1) Navier-Stokes + external driving
(Rayleigh-Bénard convection)

(2) Turing patterns

$$\begin{cases} \partial_t A = D_A \Delta A + F(A,B) \\ \partial_t B = \underbrace{D_B \Delta B}_{\substack{\text{boring} \\ \text{diffusion}}} + \underbrace{G(A,B)}_{\substack{\text{exciting} \\ \text{non-linear chemical reaction}}} \end{cases}$$

Q: Here focus on (1) → we do not know the PDE's of active matter

+ what happens if exciting motility replaces boring diffusion.

I) Spatially varying activity

I.1) Underdamped Langevin equation

$\dot{\vec{r}} = \vec{v}$; $\dot{\vec{v}} = -\gamma \vec{v} - V'(r) + \sqrt{2\gamma k T}\, \vec{\eta}$; $\gamma(x)$, $T(x)$

Fokker-Planck eq°

$$\partial_t P(\vec{r},\vec{v}) = -\vec{\nabla}\cdot(\vec{v}\,\vec{P}) + \vec{\nabla}_v \cdot \left[\gamma \vec{v} P + V'(r) P + \gamma k T \vec{\nabla}_v P \right]$$

① $T \in \mathbb{R}$; $\gamma : \mathbb{R} \to \mathbb{R}^+$

$$P_{st}(\vec{r},\vec{v}) = Z^{-1} \exp\left[-\beta\left(\frac{\vec{v}^2}{2} + V(r) \right) \right]$$

γ does not impact steady state $\Rightarrow$ kinetic parameter

② $\gamma \in \mathbb{R}$, $V = 0$, $T(\vec{r})$

Non-uniform steady-state

Slowly varying $T(\vec{r})$; $\varphi(\vec{r}) = \int d\vec{v}\, P(\vec{r},\vec{v}) \propto \frac{1}{T(\vec{r})}$

T thermodynamic parameter

I.2) Active particles

I.2.1) Dynamics

Spatial : $m \ddot{\vec{r}} = -\gamma \dot{\vec{r}} - \vec{\nabla} V + \vec{f}_p + \sqrt{2\gamma^2 D_t}\, \vec{\eta}$

γ viscous damping

$\vec{f}_p$ self propulsion force

Large γ $\Rightarrow$ $\dot{\vec{r}} = -\mu \vec{\nabla} V + \vec{v}_p + \sqrt{2 D_t}\, \vec{\eta}$

μ: mobility, $\vec{v}_p$ self-propulsion velocity

For fixed $\vec{v}_p$:

$$\partial_t P(\vec{r}) = -\vec{\nabla}\cdot\left[\vec{v}_p P - \mu \vec{\nabla} V P - D_t \vec{\nabla} P\right]$$

Self-propelled particles defined by properties of $\vec{v}_p$

① Run & Tumble particles

$\vec{v}_p = v_0 \vec{u}$ $\quad \vec{u} \xrightarrow{\alpha} \vec{u}' \in S_n \quad ; p(\vec{u}') = \dfrac{1}{\Omega(S_n)}$

1d: $\vec{u}' = \pm e_x$ with $p = 1/2$

Master equation $\quad \partial_t P(\vec{u}) = -\alpha \vec{u} + \int \frac{d\vec{u}'}{\Omega} \alpha P(\vec{u}') \quad (d>1)$

② Rotational diffusion

2d $\quad \vec{u} = (\cos\theta, \sin\theta) \quad \dot{\theta} = \sqrt{2D_r}\,\xi \quad D_r$: inverse persistence time

$$\partial_t P(\theta) = D_r \partial_{\theta\theta} P$$

$d > 2 \quad \partial_t P = D_r \Delta_\theta P$

All together: 2d

$$\partial_t P(\vec{r},\theta) = -\vec{\nabla}\left[v_0 \vec{u}(\theta) P - \mu \vec{\nabla} V P - D_t \vec{\nabla} P\right] + D_r \Delta_\theta P - \alpha P + \frac{1}{2\pi}\Psi(\vec{r}) \quad ; \text{ with } \Psi(\vec{r}) = \int d\theta\, P(r,\theta)$$

A simpler case, RTP in 1d $R(x,t)$ and $L(x,t)$
the proba to find the particle going to the right or to the left at (x,t):

$$\begin{cases} \partial_t R(x,t) = -\partial_x [v_0 R] - \frac{\alpha}{2} R + \frac{\alpha}{2} L + \partial_{xx}[D_t R] + \partial_x[\mu V'(x) R] \\ \partial_t L(x,t) = +\partial_x [v_0 L] - \frac{\alpha}{2} L + \frac{\alpha}{2} R + \partial_{xx}[D_t L] + \partial_x[\mu V'(x) L] \end{cases}$$

From now on, focus on $V(\vec{r})=0$.

- Schnitzer, PRE 48, 2553 (1993)
- Tailleur, Cates, PRL 100, 218103 (2008), EPL 101, 20010 (2013)
- Frangipane et al, e-life 7, e36608 (2018) NEXT PAGE
- Arlt et al, Nat. Com. 9, 768 (2018); 10, 2321 (2019)

I.2.2) Spatially varying activity: the steady state

(A) The simplest case $D=0$, $v_0(r)$, $\alpha(r)$, $D_r(r)$

$$\partial_t P(r,\theta) = -\nabla[v_0(r)\vec{u}(\theta) P(r,\theta)] + \underbrace{D_r(r)\Delta_\theta P - \alpha(r) P + \frac{\alpha(r)}{2\pi}\varphi(r)}_{\textcircled{H} P(r,\theta)}$$

Up to normalization issues

$$P_s(r,\theta) = \frac{K}{v_0(r)} \quad \text{is a steady-state solution}$$

isotropic $\Rightarrow \textcircled{H} P = 0$

$v_0(r) P_s(r,\theta)\vec{u}(\theta) = K\vec{u}(\theta) \Rightarrow$ independent of $\vec{r}$

$$\boxed{\text{Self-propelled particles accumulate where they go slower}}$$

Comment: gradient of viscosity $\Rightarrow \gamma(\vec{r}) \Rightarrow \mu(\vec{r})$

$\Rightarrow v_p(\vec{r})$ even if f_p is uniform.

$\Rightarrow$ Very different from thermal equilibrium.

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(B) $D \neq 0$, slowly varying $v_0(\vec{r})$

To lighten notation, drop $D_r \Delta_u P$ for now

$$\partial_t P(r,u) = -\nabla[v u P] + D_t \Delta P - \alpha P + \frac{\alpha}{2\pi}\psi \quad (1) \; ; \; \psi(r) = \int d\vec{u}\, P(r,u)$$

$$\int (1)\, du \Rightarrow \partial_t \psi(r) = -\vec{\nabla}\cdot[v \vec{m}] + D_t \Delta P \quad (2) \; ; \; \vec{m} = \int d\vec{u}\, \vec{u}\, P(\vec{r},\vec{u})$$

$$\int (1) u_\alpha\, du \Rightarrow \partial_t m_\alpha = -\partial_\beta \left[v \int u_\alpha u_\beta P\, du\right] + D_t \Delta m_\alpha - \alpha m_\alpha$$

$$\int u_\alpha u_\beta P\, du = \int \left(u_\alpha u_\beta - \frac{\delta_{\alpha\beta}}{d}\right) P\, du + \delta_{\alpha\beta} \int \frac{1}{d} P\, du$$

$$= Q_{\alpha\beta}(r) + \delta_{\alpha\beta} \frac{\psi(r)}{d}$$

$$\partial_t m_\alpha = -\alpha m_\alpha - \partial_\alpha\left[\frac{v\psi}{d}\right] - \partial_\beta\left[v Q_{\alpha\beta}\right] + D_t \Delta m_\alpha \quad (3)$$

$$\partial_t Q_{\alpha\beta} = -\alpha Q_{\alpha\beta} - \partial_\gamma[\dots] \quad (4)$$

(2) $\Rightarrow \psi(r)$ is a slow variable $\xleftrightarrow{\ell}$ relaxation time diverge as ℓ^z, $z>1$

(3, 4) m_α, $Q_{\alpha\beta}$ fast variables, relaxation times $\sim \alpha^{-1}$

Gradient expansion

$Q_{\alpha\beta} \sim \nabla$

$$m_\alpha = -\frac{1}{\alpha}\partial_\alpha\left[\frac{\psi v}{d}\right] + \mathcal{O}(\nabla^2)$$

$$\Rightarrow \partial_t \psi = \nabla\cdot\left[\frac{v}{\alpha}\nabla\left[\frac{\psi v}{d}\right] + D_t \nabla\psi\right]$$

Restoring D_r

$$\partial_t \psi = \nabla\left[v\tau\nabla\left[\frac{\psi v}{d}\right] + D_t\nabla\psi\right] \; ; \; \tau = \left(\alpha + (d-1)D_r\right)^{-1}$$

Flux-free steady-state:

$$\left(D_t + \frac{v^2\tau}{d}\right)\nabla\psi + \psi\frac{1}{2}\nabla\left[D_t + \frac{v^2\tau}{d}\right] = 0$$

$$\nabla \ln\psi + \frac{1}{2}\nabla\ln\left[D_t + \frac{v^2\tau}{d}\right] = 0$$

$$\nabla \ln\left[\psi\sqrt{D_t + \frac{v^2\tau}{d}}\right] = 0$$

$$\psi \propto \frac{1}{\sqrt{\frac{v^2\tau}{d} + D_t}} \underset{D_t \ll \frac{v^2\tau}{\alpha}}{\simeq} \frac{\sqrt{d/\tau}}{v}\left(1 - \frac{1}{2}\frac{dD_t}{v^2\tau}\right)$$

I.2.3/ Late-time dynamics and effective equilibrium