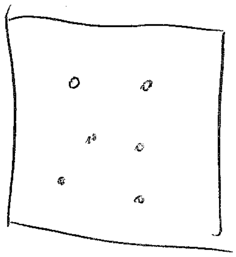
Many-body localization

Is it possible to break ergodicity in many-body systems?
Violate ETH?

Of course, there are integrable many-body systems.
For example, non-interacting fermions in a box



$$H_0 = \sum \varepsilon_i c_i^+ c_i$$, where c_i^+, c_i - single-particle eigenstates

Many I.O.M. (= Integrals Of Motion)

$$n_i = c_i^+ c_i \text{ is IOM} \quad : \quad [\hat{n}_i, H] = [\hat{n}_i, \hat{n}_j] = 0$$

Many-body eigenstates are Slater determinants

$$|\Psi\rangle = c_{i_1}^+ \ldots c_{i_N}^+ |0\rangle \qquad n_i = 0 \text{ or } 1$$

Such a system clearly violates ETH. For example, energy levels are sums of single particle energies

$$E = \sum_{i \text{ filled}} \varepsilon_i$$

Obey Poisson (not Wigner Dyson) statistics

A more non-trivial example: 1D systems solvable by Bethe ansatz

$$H = J \sum \vec{\sigma}_i \vec{\sigma}_{i+1}$$

Has extensive number of I.O.M.s $\{\hat{I}_\alpha\}$, including total spin

However, this kind of integrability is fragile

H_0- integrable (e.g. non-interacting or Bethe ansatz)

Very weak generic perturbations break this integrability

$H(\lambda) = H_0 + \lambda V$ - ergodic. Fix system size L

$H_0(\lambda)$ is ergodic for $\lambda \geqslant \lambda_*(L) \to 0$ as $L \to \infty$

Simple example: weak interactions lead to thermalization (kinetic theory etc)

This kind of integrability does not represent phase of matter

(special points in phase space)

MBL: a generic mechanism to break ergodicity

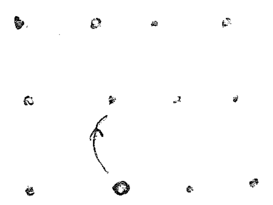
Driven by quenched disorder. Shows (new kind) of robust integrability

Three "kinds" of localization phenomena

- 1) Anderson localization in non-interacting systems
- 2) Zero-T (obscures) localization in the presence of interactions (has to do with Metal-insulator transition)
- 3) Many-body localization

Interactions and finite $T > 0$

Reminder about Anderson localization



N sites

$$H = \sum E_i \, a_i^+ a_i + \sum t_{ij} \, a_i^+ a_j$$

on-site random energy, $E_i \in [-W, W]$, independent;

short-range hopping (e.g., nearest neighbor)

[Precise distribution not important]

Single-particle eigenstates can be localized

$$\psi(\vec{r}) \sim e^{-\frac{|\vec{r}-\vec{r}_0|}{\xi}}, \quad \xi \text{ - localiz. length}$$

$\psi(r_0)$ remains finite as $N \to \infty$.

Quantified e.g. by participation ratio:

$$\text{IPR} = \sum_i |\psi(r_i)|^4 = \text{const}.$$

In $d \geq 3$, if disorder is not too strong, eigenstates can be extended / delocalized.

$$|\psi_0(r_i)| \sim \frac{1}{\sqrt{N}}$$

~same amplitudes on all sites. Dynamics is diffusive (in Anderson insulator, no transport)

$$\text{IPR} \propto \frac{1}{N}$$

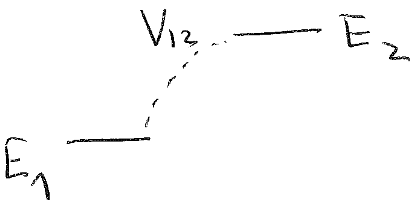
Localization is sensitive to dimensionality

$d \leq 2$: All states localized, even for weak disorder

$d \geq 3$ either loc. or extended

$d \leq 2$: localization dominates because quantum interference is enhanced (weak localization).

Qualitative picture of strong localization: absence of resonances / hybridization.



$$H = \begin{bmatrix} E_1 & V_{12} \\ V_{12} & E_2 \end{bmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix}$$

1, 2 - wave functions on given sites (in our model, nearby)

If $|V_{12}| \ll |E_1 - E_2|$,

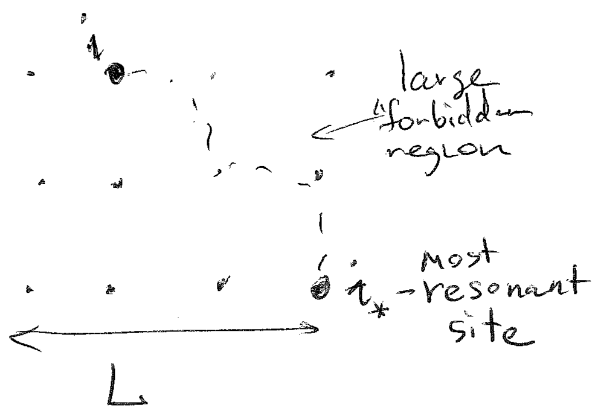
$$|\psi_1\rangle \simeq |1\rangle + \frac{V_{12}}{E_1 - E_2}|2\rangle$$

$$|\psi_2\rangle \simeq |2\rangle - \frac{V_{12}}{E_1 - E_2}|1\rangle$$

perturbative to 1st order

Disorder suppresses hybridization, particle is "stuck". This is a typical situation. But with a small probability, we can have a resonance:

$$P_{res} \sim K\frac{t}{W}, \quad K\text{-connectivity of the lattice}$$



<u>But</u>: particle can also try to hybridize with a remote site with a closer energy.

N levels,

level spacing

$$\Delta(N) \sim \frac{W}{N} \ll W \text{ (typical energy diff b/w nearby sites)}$$

But typically, i_* would be distance $L \sim N^{1/d}$ away in real space.

Competition b/w tunneling matrix element and energy separation:

<u>Naive estimate</u> t_{ii_*} arises in Lth order in pert. theory in t/W.

$$t_{ii_*} \sim t\left(\frac{t}{W}\right)^L = t\cdot\left(\frac{t}{W}\right)^{N^{1/d}} \ll \Delta(N) = \frac{W}{N}$$

exponentially small in $L = N^{1/d}$ — Only power-law small

Suggest that higher-order processes leave particle localized

This indeed, can be shown rigorously (Anderson '58 ...) Math literature

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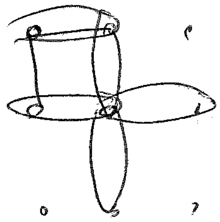
3D (and higher) delocalization happens when $t \sim W$



Percolation of resonant bonds $\Rightarrow$ delocalized states, filling finite fraction of Hilbert space.

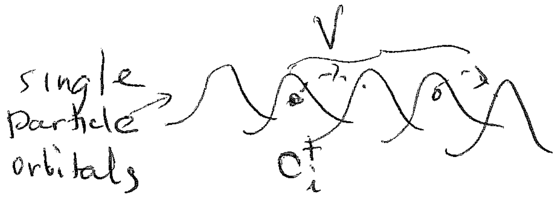
What about ETH? Naturally, Anderson-localized systems violate ETH.

Do <u>NOT</u> serve as a bath, no transport

Is localization robust to generic perturbations?
E.g. adding weak interactions?
[Recall: In the absence of disorder, even small $H_{int} \to$ ETH]

(1) $$H = \underbrace{\sum \epsilon_i c_i^\dagger c_i}_{H_0} + \underbrace{\sum V_{ijkl}}_{} c_i^\dagger c_j^\dagger c_k c_l$$

H_0 " single-particle localised ~~eigenst~~ eigenstates — weak, retains locality



Locality - only nearby pairs of particles can scatter

Sometimes, convenient to study MBL for spins.

$\sigma_i^z = \;\uparrow$ if an orbital is filled
$\quad\quad\;\; \downarrow$ - if empty

(2) $$H = \underbrace{\sum_i h_i \sigma_i^z}_{\text{"single-particle" } H_0} + \sum \sigma_i^\alpha J_{ij}^{\alpha\beta} \sigma_j^\beta$$

$\uparrow$ local generic interaction which allows

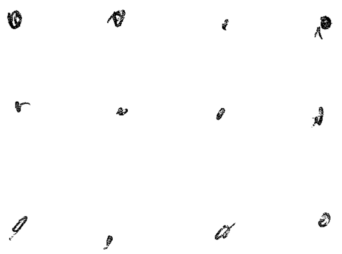
Note: no rigorous mapping b/w fermionic model (1) and spin model (2). Nevertheless, physics is similar. Moreover, for a simple 1D model of MBL, which was studied extensively, such a mapping does exist (Jordan-Wigner) (below)

Eigenstates are "up-down" states $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$

Why is this a difficult problem? (~50 years!)
Already Anderson '58 was motivated by disordered spins but it was too hard

How could we approach it? Naive approaches: trying to reduce to Anderson localization in many-body space

Let's think about eigenstates of H_0 as "sites"



For a spin model,

$\bullet = \uparrow\downarrow\uparrow\downarrow\uparrow$ (L spins)

(specify each $\sigma_i^z = \uparrow$ or $\downarrow$)

$\cancel{E(\{\sigma_i^z\}) = \sum h_i \sigma_i^z}$

$N = 2^L$ sites. Hoppings? Consider a simple model

$$\boxed{H_0 = \sum h_i \sigma_i^z + J\sum\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z\right)}$$

$h_i \in [-W;W]$ Random field XXZ spin chain. Has been used as "standard model" of MBL

On-site $E(\{\sigma^z\}) = \sum h_i\sigma_i^z + J\sum\sigma_i^z\sigma_{i+1}^z$

"Hopping" on this lattice:

If two nearby spins $i, i+1$ are opposite, amplitude to flip them is J.

"Connectivity" of our hopping problem.

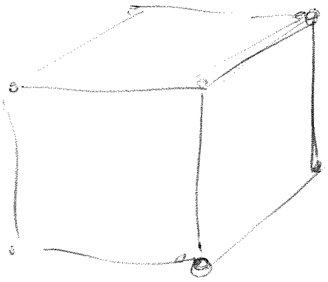
$$K \sim L/2 \quad \text{grows with system size!}$$

~~How to simplify? Assume energies are random and independent?~~
~~But then we can always~~

Could we define MBL as localization of this many-body lattice? No. $\langle\psi_i|\psi_i^0\rangle \to 0$ as $L\to\infty$ (single-particle definition)

true many-body closest to ψ_i — "non-interacting" site

Product eigenstates $\uparrow\downarrow\uparrow\ldots\downarrow$ can be thought of as vertices of a hypercube in N dimensions.



Naively, there are many resonances! (Since connectivity $K \sim L/2$, and change of energy b/w two connected sites is finite, $\leq 2W + 2J$)

So if we use "percolating resonances" logic from Anderson loc., we would expect that this model is always delocalized!

This is incorrect, because resonances are typically spatially separated and "independent"

$$\uparrow \underbrace{(\downarrow\ \uparrow)}_{RS\,1} \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \uparrow \underbrace{(\downarrow\ \uparrow)}_{RS\,2} \downarrow$$

here $h_i - h_{i+1} \leq J$ $\qquad$ $h_j - h_{j+1} \leq J$

In the hypercube picture, this information is kept in the correlations b/w ~~one~~site energies (since they came from $H_0 = \sum h_i \sigma_i^z$) Naive "percolation" estimates lose this information

So, intuitively we expect that wave function looks like this:

no resonance here $\uparrow\downarrow\uparrow\downarrow$

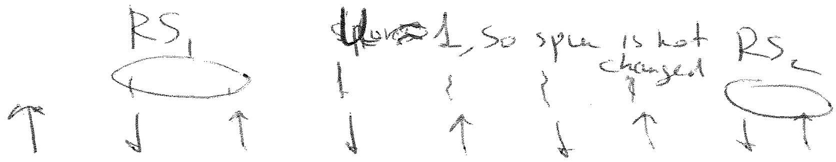
$$\Psi_* \approx \left(\frac{\uparrow\downarrow \pm \downarrow\uparrow}{\sqrt{2}}\right)_{RS_1} \otimes \left(\frac{\uparrow\downarrow \pm \downarrow\uparrow}{\sqrt{2}}\right)_{RS_2}$$ (assuming perfect resonance)

$L \sim \sqrt{D\tau}$

And not like $\sum A_{\sigma_1\sigma_2\sigma_3\sigma_4} |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes |\sigma_3\rangle \otimes |\sigma_4\rangle$

(as naive hypercube picture would suggest)

Key difference: true eigenfunction has <u>little</u> entanglement
Eigenstates obtained from product states by quasi-local unitary transformations



Eigenstates remain close to product st. but not in terms of overlaps!

$$|\psi_i\rangle = \underset{\text{quasi-local}}{\mathcal{U}}\,|\mathrm{prod}_i\rangle \qquad (1)$$

Quasi-locality means that 1) $\mathcal{U}|\mathrm{prod}_i\rangle$ have little (boundary-law) entanglement

2) If I view it as evolution operator, information does not spread:

$$\underset{\underline{\text{quasi-local}}}{\tilde{\hat{O}}} = \mathcal{U}^{+}\underset{\text{local}}{\hat{O}}\,\mathcal{U}$$

$$\tilde{O} = \sum \tilde{O}_K \Leftarrow \text{part with range } K$$

(Consider example, $\tilde{O} = \sigma_i^z + \frac{1}{2}\sigma_{i-1}^z\,\sigma_{i+1}^x + \frac{1}{4}\sigma_{i-2}^z\cdot\sigma_i^y\,\sigma_{i+2}^x$]

Lets take (1) as a <u>definition</u> of MBL

One immediate consequence is <u>integrability</u>

Quasi-local I.O.Ms.

$|prod_\alpha\rangle = |\uparrow\uparrow\ldots\downarrow\rangle$ - initial basis

$\sigma_i^z |prod_\alpha\rangle = \pm |prod_\alpha\rangle$

$|\{\sigma_i^z\}_\alpha\rangle$ - eigenstates

$|\psi_\alpha\rangle = \hat{\mathcal{U}} |\{\sigma^z\}_\alpha\rangle$ - new eigenstates

$\boxed{\tau_i^z = \mathcal{U} \sigma_i^z \mathcal{U}^+}$ - dressed σ^z operator

$\tau_i^z |\psi_\alpha\rangle = \tau_i^z \cdot \hat{\mathcal{U}} |\{\sigma_i^z\}_\alpha\rangle = \mathcal{U} \sigma_i^z \cdot |\{\sigma^z\}_\alpha\rangle = \pm \mathcal{U} |\{\sigma^z\}_\alpha\rangle =$

$= \pm |\psi_\alpha\rangle$

τ_i^z is diagonal in the eigenstate basis

$$[\tau_i^z, H] = 0 \;, \quad [\tau_i^z, \tau_j^z] = 0$$

quasi-local deformations of (non-inter.) I.O.Ms

New kind of integrability which is <u>robust</u>. <u>Phase</u>!

Expore consequences, justify robustness.

A complete set of operators:

$\sigma_1^{\alpha_1} \sigma_2^{\alpha_2} \ldots \sigma_N^{\alpha_N}$. $\quad \alpha_i = 0, x, y, z, \quad \sigma_i^0 = 1$

4^N operators

Introduce $\quad \tau_i^\alpha = \mathcal{U} \sigma_i^\alpha \mathcal{U}^+ \quad$ also a basis

$H = \sum H^{\alpha_1 \ldots \alpha_N} \cdot \mathcal{T}_{\alpha_1 \ldots \alpha_N} \quad (\mathcal{T}_{\alpha_1 \ldots \alpha_N} = \tau_1^{\alpha_1} \ldots \tau_N^{\alpha_N})$

$[H, \tau_i^z] = 0 \Rightarrow H^{\alpha_1 \ldots \alpha_N} = 0$ if $\alpha_i = x, y$ for some i

$H^{\alpha_1 \ldots \alpha_N} \neq 0$ only if all $\alpha_i = 0, z$.

$$H = \sum \tilde{h}_i \tau_i^z + \sum J_{ij} \tau_i^z \tau_j^z + \sum J_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots \quad (2)$$

Has to be quasi-local! $J_{i_1 \ldots i_K} \sim e^{-\frac{\max[i_1 - i_K]}{\varkappa}}$

Alternatively, $H = \sum h_i \sigma_i^z + J \sum \sigma_i^\alpha \sigma_{i+1}^\alpha$

We can express $\sigma_i^\alpha = \mathcal{U}^+ \tau_i^\alpha \mathcal{U}$, and obtain Hamiltonian (2). Need to know the form of $\hat{\mathcal{U}}$, which depends on disorder realization.

$\hat{\tau}_i^z$ are called LIOMs, l-bits, effective spins - all equiv.

τ_i^z is a q-bit with $T_1 \to \infty$.

We have a complete set of LIOMs (N spins $\Rightarrow$ N LIOMs)

But it experiences ~~z~~-field which depends on the state of other spins

Field on I: $H(\{\tau\}, i = I) \, \tau_I^z$

$$H(\{\tau_i\}, i \neq I) = \tilde{h}_I + \sum_{i \neq I} J_{iI} \tau_i^z + \ldots$$