Holography with and without gravity
SAMPLE CALCULATIONS

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August 2013

Outline

1. Green's functions of scalar operators in CFT from AdS
2. Thermodynamics of neutral holographic plasma from AdS-Schwarzchild black hole
3. Shear viscosity of neutral holographic plasma
4. Fermion response function in AdS Reissner-Nördstrom

Sample calculation 1:

Green's functions of scalar operators in CFT from AdS

(in Euclidean spacetime)

Correlation functions of scalar operators from *AdS*

The solution with $f_k(z=\epsilon)=1$ ('the regulated bulk-to-boundary propagator'), is

$$\underline{f}_k(z)=\frac{z^{d/2}K_\nu(kz)}{\epsilon^{d/2}K_\nu(k\epsilon)} \qquad \left(\int dk\ e^{ikx} f_k(\epsilon)=\delta^d(x)\right)$$

The general position space solution can be obtained by Fourier decomposition:

$$\underline{\phi}^{[\phi_0]}(x)=\int d^dk e^{ikx}\underline{f}_k(z)\phi_0(k,\epsilon)\ .$$

'on-shell action' (*i.e.* the action evaluated on the saddle-point solution):

$$S[\underline{\phi}]=-\frac{\mathfrak{K}}{2}\int d^dx\sqrt{\gamma}\underline{\phi}n\cdot\partial\underline{\phi}$$

$$=-\frac{\mathfrak{K}L^{d-1}}{2}\int d^dk\phi_0(k,\epsilon)\mathcal{F}_\epsilon(k)\phi_0(-k,\epsilon)$$

$$\mathcal{F}_\epsilon(k)=z^{-d}\underline{f}_{-k}(z)z\partial_z\underline{f}_k(z)|_{z=\epsilon}+(k\leftrightarrow -k)$$

$$\langle\mathcal{O}(k_1)\mathcal{O}(k_2)\rangle_c^\epsilon=-\frac{\delta}{\delta\phi_0(k_1)}\frac{\delta}{\delta\phi_0(k_2)}S=(2\pi)^d\delta^d(k_1+k_2)\mathcal{F}_\epsilon(k_1)\ .$$

$$K_\nu(u) = u^{-\nu}(a_0 + a_1 u^2 + a_2 u^4 + \cdots) \qquad \text{(leading term)}$$
$$+ u^\nu \ln u (b_0 + b_1 u^2 + b_2 u^4 + \cdots) \qquad \text{(subleading term)}$$

(red logs are only present for $\nu \in \mathbb{Z}$)

$$\begin{aligned}
\mathcal{F}_\epsilon(k) &= 2\epsilon^{-d+1}\partial_z \left(\frac{(kz)^{-\nu+d/2}(a_0 + \cdots) + (kz)^{\nu+d/2}\ln kz(b_0 + \cdots)}{(k\epsilon)^{-\nu+d/2}(a_0 + \cdots) + (k\epsilon)^{\nu+d/2}\ln k\epsilon(b_0 + \cdots)} \right. \\
&= 2\epsilon^{-d}\left[\left\{ \frac{d}{2} - \nu(1 + c_2(\epsilon^2 k^2) + c_4(\epsilon^4 k^4) + \cdots) \right\} \right. \\
&\qquad \left. + \left\{ \nu \frac{2b_0}{a_0}(\epsilon k)^{2\nu} \ln(\epsilon k)(1 + d_2(\epsilon k)^2 + \cdots) \right\} \right] \\
&\equiv \mathrm{(I)} + \mathrm{(II)}
\end{aligned}$$

(I): Laurent series in ϵ with coefficients $k^{\text{even integer}}$

(*i.e.* analytic in k at $k = 0$). $\equiv$ contact terms $\equiv$ short distance goo:

$$\int d^d k e^{-ikx}(\epsilon k)^{2m}\epsilon^{-d} = \epsilon^{2m-d}\Box_x^m \delta^d(x) \qquad (m \in \mathbb{Z}_+)$$

The ϵ^{2m-d} agrees w/ ϵ is a UV cutoff for the QFT.

Checking that $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$

The interesting bit of $\mathcal{F}(k)$, which gives the $x_1 \neq x_2$ behavior of the correlator, is non-analytic in k:

$$(\mathrm{II}) = -2\nu \cdot \frac{b_0}{a_0} k^{2\nu} \ln(k\epsilon) \cdot \epsilon^{2\nu-d}(1+\mathcal{O}(\epsilon^2)),$$

input of Bessel: $\frac{b_0}{a_0} = \frac{(-1)^{\nu-1}}{2^{2\nu}\nu\Gamma(\nu)^2}$ for $\nu \in \mathbb{Z}$

FT of leading term: $\int d^d k e^{-ikx}(\mathrm{II}) = \frac{2\nu\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\Delta_+ - d/2)} \frac{1}{x^{2\Delta_+}} \epsilon^{2\nu-d}$.

- *AdS* radius appears only in overall normalization, in the combination $\mathfrak{K}L^{d-1}$.
- Multiplicative renormalization removes the $\epsilon^{2\nu-d}$.
- Holographic Renormalization: add to S_{bulk} the local, intrinsic boundary term

$$\Delta S = S_{\mathrm{c.t.}} = \frac{\mathfrak{K}}{2}\int_{\mathrm{bdy}} d^d x \left(-\Delta_- L^{d-1}\epsilon^{2\Delta_- - d}\left(\phi_0^{\mathrm{Ren}}(x)\right)^2\right)$$

$$= -\Delta_- \frac{\mathfrak{K}}{2L}\int_{\partial AdS, z=\epsilon} \sqrt{\gamma}\,\phi^2(z,x)$$

Affect neither bulk EOM nor $G_2(x_1 \neq x_2)$, cancels divergences.

Sample calculation 2:

Thermodynamics of neutral holographic plasma from AdS-Schwarzchild black hole

Thermodynamics from gravity: boundary terms

$$Z_{CFT} \equiv e^{-\beta F} = e^{-S_{\text{bulk}}[\underline{g}]}$$

$\underline{g}$ is the euclidean saddle-point metric(s).

$$S_{\text{bulk}} = S_{EH} + S_{GH} + S_{ct} \ .$$

$$S_{EH} = -\frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{L^2} \right)$$

Two kinds of boundary terms:

def of γ: $$ds^2 \overset{z \to 0}{\approx} L^2 \frac{dz^2}{z^2} + \gamma_{\mu\nu} dx^\mu dx^\nu .$$

$$S_{ct} = \int_{\partial M} d^d x \sqrt{\gamma} \frac{2(d-1)}{L} + \dots$$

local, *intrinsic* boundary counter-term (no normal derivatives).
just like for scalar correlators. $\cdots \propto$ intrinsic curvature of bdry metric.

Gibbons-Hawking term

S_{GH}: 'Gibbons-Hawking' term is an *extrinsic* boundary term

like $\int_{\partial AdS} \phi n \cdot \partial\phi$ for scalar.

IBP in the Einstein-Hilbert term to get the EOM :

$$\delta S_{EH} = EOM + \int_{\partial AdS} \gamma^{\mu\nu} n \cdot \partial \delta \gamma_{\mu\nu},$$

but we want a Dirichlet condition on the metric: $\delta\gamma_{\mu\nu} = 0$

δS_{GH} cancels the $\partial\delta\gamma_{\mu\nu}$ bits.

$$S_{GH} = -2 \int_{\partial M} d^d x \sqrt{\gamma} \Theta$$

Θ: extrinsic curvature of the boundary

$$\Theta \equiv \gamma^{\mu\nu} \nabla_\mu n_\nu = \frac{n^z}{2} \gamma^{\mu\nu} \partial_z \gamma_{\mu\nu}.$$

n^A is an outward-pointing unit normal to the boundary $z = \epsilon$.

Stress tensor expectation value

$$\text{GKPW}: \quad \langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{\gamma}} \frac{\delta}{\delta \gamma_{\mu\nu}} S_{\text{bulk}}[\underline{g}].$$

CFT: $T^\mu_\mu = 0$ modulo scale anomaly

In thermal eqbm: $T^t_t = -\mathcal{E}, \quad T^x_x = P \qquad \mathcal{E} = d \; P$

Sample calculation 3:

η/s from holography

Example: η/s

Shear viscosity is a transport coefficient like conductivity.

source: T^x_y response: T^x_y.

$$\eta = \lim_{\omega\to 0} \frac{1}{i\omega} G^R_{T^x_y T^x_y}(k=0,\omega)$$

$$\langle T^x_y \rangle = i\omega\eta\gamma^x_y \qquad \to \qquad \text{must study fluctuations of metric}$$

[compute following Iqbal-Liu 08] Assume a bulk metric of the form

$$ds^2 = g_{tt}(z)dt^2 + g_{zz}(z)dz^2 + g_{ij}(z)dx^i dx^j$$

such that

1. g_{AB} depend only on z
2. asymptotically AdS near $z \to 0$
3. Rindler horizon at $z = z_H$

$$g_{tt} \overset{z\to z_H}{\to} -2\kappa(z_H - z) \qquad g_{zz} \overset{z\to z_H}{\to} \frac{1}{2\kappa(z_H - z)}.$$

Shear fluctuations of the metric

$$\text{Consider} \quad S = S_{\text{gravity}} - \frac{1}{2}\int d^{d+1}x\sqrt{g}\,\frac{1}{q(z)}g^{AB}\partial_A\phi\partial_B\phi$$

Claim: fluctuations of $\phi \equiv h^x_y$ in Einstein gravity are governed by this action with $\frac{1}{q(z)} = \frac{1}{16\pi G_N}$. [lots of work by Son, Starinets, Policastro, Kovtun, Buchel, J. Liu...]

$$\text{Recall:} \quad \langle \mathcal{O}(x^\mu)\rangle_{QFT} = \lim_{z\to 0}\Pi_\phi(z, x^\mu) \qquad \text{(m=0)}$$

$$\Longrightarrow \qquad \eta = \lim_{\omega\to 0}\lim_{z\to 0}\lim_{k\to 0}\left(\frac{\Pi(z,k_\mu)}{i\omega\phi(z,k_\mu)}\right)$$

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_z\phi\right)} = \frac{\sqrt{g}}{q(z)}g^{zz}\partial_z\phi.$$

Compute this in two steps:

- Find behavior near horizon.
- Use wave equation to evolve to boundary.

$$0 = \frac{\delta S_\phi}{\delta\phi(k^\mu, z)} \propto [g^{ij}k_ik_j + g^{tt}\omega^2 - \frac{1}{\sqrt{g}}\partial_z\left(g^{zz}\sqrt{g}\partial_z\right)]\phi(k^\mu, z)$$

We can safely set $\vec{k} = 0$.

Near horizon

Assumption (3) $\implies$ $z = z_H$ is a regular singular point of the wave equation.
Try $\phi(k,z) = (z - z_H)^{\alpha}$.

$$\phi(k,z) \simeq (z - z_H)^{\pm \frac{i\omega}{4\pi T}} \qquad \text{in/out.}$$

$$\implies \text{At horizon:} \quad \Pi(z_H, k) = \left[\frac{1}{q(z)} \sqrt{\frac{|g|}{g_{zz}|g_{tt}|}} i\omega\phi(z,k) \right]_{z=z_H} .$$

Propagate to boundary

EOM: $\partial_z \Pi \propto k_\mu k_\nu g^{\mu\nu} \phi \overset{\omega \to 0, \vec{k} \to 0}{\longrightarrow} 0.$

def of Π: $\partial_z(\phi\omega) = \frac{q}{\sqrt{g} g^{zz}} \omega \Pi \overset{\omega \to 0, \omega\phi \text{ fixed}}{\longrightarrow} 0.$

$$\Longrightarrow \quad \frac{\Pi}{\omega\phi}\Big|_{z=0} = \frac{\Pi}{\omega\phi}\Big|_{z=z_H}$$ 'membrane paradigm'

$$\Longrightarrow \quad \eta = \frac{1}{q(z_H)} \sqrt{\frac{|g|}{g_{zz}|g_{tt}|}} \ .$$

Entropy density: $s = \frac{a}{4G_N} = \frac{1}{4G_N} \sqrt{\frac{|g|}{g_{zz}|g_{tt}|}}$

$$\Longrightarrow \quad \boxed{\frac{\eta}{s} = \frac{1}{4\pi}}.$$

Sample Calculation 4:

Fermion response function in AdS Reissner-Nördstrom

Computing G_R

Translation invariance in $\vec{x}, t \implies$ ODE in r.
Rotation invariance: $k_i = \delta_i^1 k$
Near the boundary, solutions behave as $(\Gamma^{\underline{r}} = -\sigma^3 \otimes 1)$

$$\psi \overset{r\to\infty}{\approx} a_\alpha r^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_\alpha r^{-m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Matrix of Green's functions, has two independent eigenvalues:

$$G_\alpha(\omega, \vec{k}) = \frac{b_\alpha}{a_\alpha}, \quad \alpha = 1, 2$$

To compute G_R: solve Dirac equation in BH geometry,
impose infalling boundary conditions at horizon [Son-Starinets, Iqbal-Liu].
Like retarded response, falling into the BH is something that *happens*.

Dirac equation

$$\Gamma^a e_a{}^M \left(\partial_M + \frac{1}{4}\omega_{abM}\Gamma^{ab} - iqA_M\right)\psi - m\psi = 0$$

$$\Phi_\alpha \equiv (-gg^{rr})^{-1/4}\Pi_\alpha^{\hat{k}}\psi, \quad \psi = e^{-i\omega t + ik_i x^i}\psi_{\omega,k},$$

$$\boxed{\left(\partial_r + M\sigma^3\right)\Phi_\alpha = \left((-1)^\alpha K\sigma^1 + Wi\sigma^2\right)\Phi_\alpha, \quad \alpha = 1, 2}$$

with

$$M \equiv m\sqrt{g_{rr}} = \frac{m}{r\sqrt{f}}, \quad K \equiv k\sqrt{\frac{g_{rr}}{g_{ii}}} = \frac{k}{r^2\sqrt{f}}, \quad W \equiv u\sqrt{\frac{g_{rr}}{g_{ii}}} = \frac{u}{r^2\sqrt{f}}.$$

$$u \equiv \sqrt{\frac{-g^{tt}}{g^{ii}}}\left(\omega + \mu_q\left(1 - \left(\frac{r_0}{r}\right)^{d-2}\right)\right)$$

Eqn depends on q and μ only through $\mu_q \equiv \mu q$

$\rightarrow$ ω is measured from the effective chemical potential, μ_q.