Holography with and without gravity

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Plan for the lectures

1. Stringless motivation of holographic duality and basic dictionary
2. A good problem for holography: systems with Fermi surfaces
3. Phases of matter which are *characterized by* their edge states
   (a kind of holography without gravity)

Some references for the first lecture:

JM, *Holographic duality with a view toward many-body physics*, 0909.0518

Hartnoll, *Horizons, holography and condensed matter,* 1106.4324.

Maldacena, *The gauge/gravity duality*, 1106.6073

Polchinski, *Introduction to Gauge/Gravity Duality*, 1010.6134

Hartnoll, *Quantum Critical Dynamics from Black Holes*, 0909.3553

Horowitz, Polchinski *Gauge/gravity duality*, gr-qc/0602037

Preface

In these lectures we're going to think about (holographic perspectives on) physical systems with *extensive* degrees of freedom.

This includes QFT and also lattice models, classical fluids, and many other interesting systems...

0.1 Introductory remarks

I will begin with some comments about my goals for this course.

The main goal is to make a study of coarse-graining in quantum systems with extensive degrees of freedom. For silly historical reasons, this is called the renormalization group (RG) in QFT. By 'extensive degrees of freedom' I mean that we are going to study models which, if we like, we can sprinkle over vast tracts of land, like *sod* (see Fig. 1). And also like sod, each little patch of degrees of freedom only interacts with its neighboring patches: this property of sod and of QFT is called *locality*.

Figure 1: Sod.

Figure: The beginning of my QFT lecture notes (at `http://physics.ucsd.edu/~mcgreevy/w13/`).

What is a QFT?

Sometimes such systems can be understood directly in terms of some weakly interacting particle picture (aka normal modes).

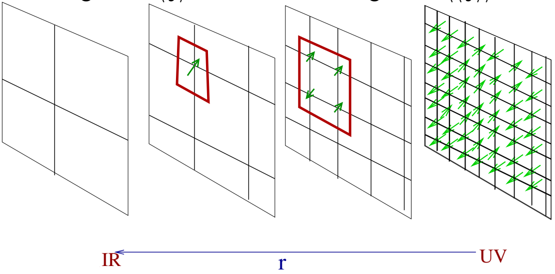
(eg: $D = 3 + 1$ QED in its coulomb phase)

There are many interesting systems for which such nearly-gaussian variables are not available.

Renormalization group

Wilson: the right way to approach such systems is scale-by-scale.

$$eg: \quad S_i = \pm 1 \quad H = \sum_{\text{neighbors}, \langle ij \rangle} J_{ij} S_i S_j + \sum_{\text{next neighbors}, \langle\langle ij \rangle\rangle} K_{ij} S_i S_j + \ldots$$



Idea: measure the system with coarser and coarser rulers.

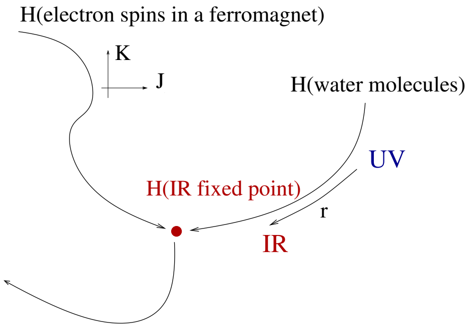
Let 'block spin' = average value of spins in block.

Define a Hamiltonian $H(r)$ for block spins preserving IR observables.

$\longrightarrow$ a renormalization group (RG) flow on the space of hamiltonians: $H(r)$

Observation: $\frac{dg(r)}{dr} \equiv \beta_g = \beta_g(g(r))$ is local in scale.

RG fixed points give universal physics

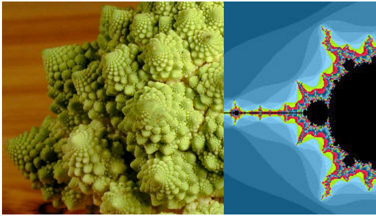


Universality: fixed points are rare.
Many microscopic theories will flow to the same fixed-point.

$\Longrightarrow$ same critical exponents.

The fixed point theory is scale-invariant (self-similar):
if you change your resolution you get the same picture back.

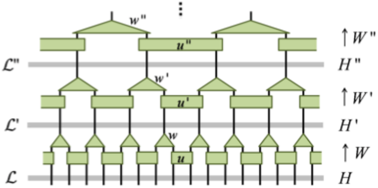
Often the fixed point theory is also 'Conformally invariant'.
This is the 'C' in AdS/CFT.

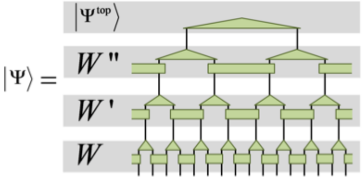


Quantum real space RG

For classical many body systems, this is the whole story.

Quantumly, in order for the coarse-grained description to be useful
— not 2^k states on each site —
we must also keep track of states
— or better, density matrices —
and throw away the unimportant ones:
density matrix RG [White, 80s], entanglement RG [Vidal 00s]





[figures: Evenbly-Vidal] [connection to holographic duality: Swingle 0905]

These coarse-graining procedures are hard to do.
Wouldn't it be great if this picture solved some diff'l equation?

A theory of gravity is not like this.

Black hole thermodynamics

Gravity $\implies$ black holes. (regions of no escape)

There are close parallels between black hole (BH) mechanics and the Laws of Thermodynamics. [70s]

Consistent laws of thermo require BH has entropy: $(k_B = 1)$

$$S_{\rm BH} = \frac{\text{area of horizon}}{4\ell_p^2}. \qquad \ell_p \equiv \sqrt{\frac{G_N \hbar^2}{c^3}}.$$

'Generalized 2d Law': $S_{total} \equiv S_{\rm ordinary\ stuff} + S_{\rm BH}$
$\Delta S_{total} \geq 0$ in processes which *happen*. [Bekenstein]

Holographic principle

Recall: In an ordinary d-dim'l system without gravity
(a chunk of stuff, the vacuum...) DoFs at each point $\Longrightarrow$
max entropy in some region of space $\sim$ volume L^d

Holographic Principle: In a gravitating system,
max entropy in a region of space V $=$
entropy of the biggest black hole that fits.

$$S_{max} = S_{BH} = \frac{1}{4\pi G_N} \times \text{horizon area}$$

$\propto$ area of ∂V in planck units. ['t Hooft, Susskind 1990s]

Why: suppose the contrary, a configuration with
$S > S_{\rm BH} = \frac{A}{4G_N}$ but $E < E_{\rm BH}$ (biggest BH fittable in V)
Then: throw in junk (increases S and E) until you make a BH.
S decreased, violating 2d law.

Punchline: Gravity in $d+1$ dimensions

has the same number of degrees of freedom as

a QFT in *fewer* (d) dimensions.

[Blundell[2]]

Questions:

- Who is the QFT on the boundary?
- From its point of view, what is the extra dimension?
- Where do I put the boundary?

Synthesis

1. RG: ordinary systems with extensive degrees of freedom (QFT) should be understood in terms of a picture with an extra dimension.
2. Holographic Principle: systems with gravity have the same number of degrees of freedom as ordinary systems in one fewer dimension.

Combining these hints, we might conjecture:

$$\begin{array}{c}\text{gravity}\\ \text{in a space with an extra dim}\\ \text{whose coord is the energy scale}\end{array} \quad \stackrel{?}{=} \quad \text{QFT}$$

To make this more precise, we consider a simple case (AdS/CFT) [Maldacena, 1997] in more detail:

Focus on *conformal* QFT (CFT). This is not much of a restriction: Many continuum QFTs can be constructed as perturbations of a UV CFT by a relevant operator.

AdS/CFT

A relativistic field theory, scale invariant ($\beta_g = 0$ for all nonzero g)

$$x^\mu \to \lambda x^\mu \quad \mu = 0...d-1, \quad u \to \lambda^{-1} u$$

u is the energy scale, RG coordinate

with d-dim'l Poincaré symmetry: Minkowski $ds^2 = -dt^2 + d\vec{x}^2$

Most gen'l $d+1$ dim'l metric w/ Poincaré plus scale inv.

$$AdS_{d+1}: \quad ds^2 = \frac{u^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + L^2 \frac{du^2}{u^2} \qquad L \equiv \text{`}AdS \text{ radius'}$$

If we rescale space and time and move in the radial dir,

the geometry looks the same (isometry).

copies of Minkowski space of varying 'size'.

(Note: this metric also has conformal symmetry $SO(d,2)$

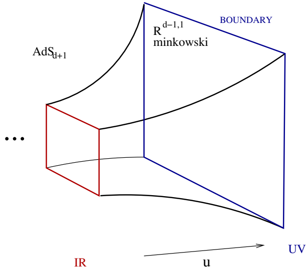
$\exists$ gravity dual $\Longrightarrow$ "Polchinski's Theorem" for any d.)

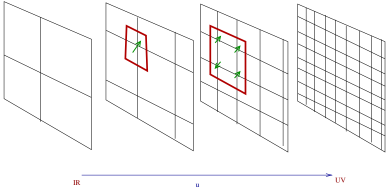
another useful coordinate:

$$z \equiv \frac{L^2}{u} \qquad ds^2 = L^2 \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2}$$

$[u]$ = energy, $[z]$ = length ($c = \hbar = 1$ units).

Geometry of *AdS* continued





The extra ('radial') dimension is the resolution scale.

(The bulk picture is a hologram.)

preliminary conjecture:

$$\text{gravity on } AdS_{d+1} \text{ space} \stackrel{?}{=} \mathrm{CFT}_d$$

crucial refinement:

in a gravity theory the metric fluctuates.

$\longrightarrow$ what does 'gravity in AdS' mean ?!?

Geometry of AdS continued

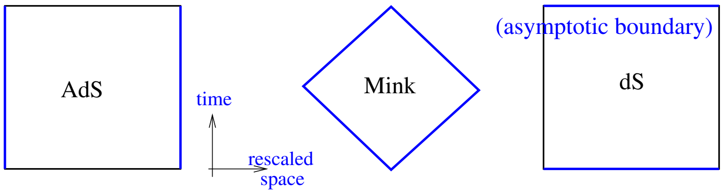
AdS has a boundary (where $u \to \infty, z \to 0$, 'size' of Mink blows up).

massless particles reach it in finite time.

$\implies$ must specify boundary conditions there.

the fact that the geometry is AdS near there is one of these boundary conditions.

different from Minkowski space or (worse) de Sitter:



so: some $\text{CFT}_d \overset{?}{=}$ gravity on *asymptotically* AdS_{d+1} space

(we will discuss the meaning of this '$=$' much more)

Preview of dictionary

"bulk" $\leftrightsquigarrow$ "boundary"

fields in AdS_{d+1} $\leftrightsquigarrow$ operators in CFT

(Note: operators in CFT don't make particles.)

mass $\leftrightsquigarrow$ scaling dimension

$$m^2 L^2 = \Delta(\Delta - d)$$

a simple bulk theory with a small # of light fields	$\leftrightsquigarrow$	CFT with a small # of ops of small Δ (like rational CFT)

What to calculate

some observables of a QFT (Euclidean for now):
vacuum correlation functions of local operators:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle$$

standard trick: make a generating functional $Z[J]$ for these correlators by perturbing the action of the QFT:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \sum_A J_A(x)\mathcal{O}_A(x) \equiv \mathcal{L}(x) + \mathcal{L}_J(x)$$

$$Z[J] = \langle e^{-\int \mathcal{L}_J}\rangle_{CFT}$$

$J_A(x)$: arbitrary functions (sources)

$$\langle \prod_n \mathcal{O}_n(x_n)\rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln Z \Big|_{J=0}$$

Hint: $\mathcal{L}_J$ is a UV perturbation – near the boundary, $z \to 0$

Holographic duality made quantitative

[Witten; Gubser-Klebanov-Polyakov (GKPW)]

$$Z_{QFT}[\text{sources}] = Z_{\text{quantum gravity}}\left[\text{boundary conditions at } u \to \infty\right]$$
$$\approx e^{-S_{\text{bulk}}[\text{boundary conditions at } u\to\infty]}\Big|_{\text{saddle of } S_{\text{bulk}}}$$
$$J = \phi_0 \qquad \text{"}\phi \overset{u\to\infty}{\to} \phi_0\text{"}$$

What's S_{bulk}? AdS solves the EOM for

$$S_{\text{bulk}} = \frac{1}{\# G_N}\int d^{d+1}x\sqrt{g}\left(-2\Lambda + \mathcal{R} + \ldots\right)$$

($\ldots$ = fields which vanish in groundstate, more irrelevant couplings.)

expansion organized by decreasing relevance

$\Lambda = -\frac{d(d-1)}{2L^2}$ note tuning!

$\mathcal{R} \sim \partial^2 g \implies G_N \sim \ell_p^{d-1}$

gravity is classical if $L \gg \ell_p$.

This is what comes from string theory (when we can tell)

at low E and for $\frac{1}{L} \ll \frac{1}{\sqrt{\alpha'}} \equiv \frac{1}{\ell_s}$ ($\frac{1}{\alpha'}$ = string tension)

(One basic role of string theory here: fill in the dots.)

Conservation of evil

large AdS radius $L \leftrightsquigarrow$ strong coupling of QFT

(avoids an immediate disproof – obviously a perturbative QFT isn't usefully an extra-dimensional theory of gravity.)

a special case of a

Useful principle (Conservation of evil):
different weakly-coupled descriptions
have non-overlapping regimes of validity.

strong/weak duality: hard to check, very powerful

Info goes both ways: once we believe the duality, this is our best definition of string theory.

Holographic counting of degrees of freedom

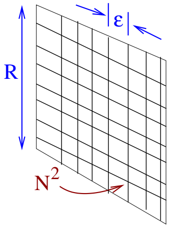
[Susskind-Witten]

$$S_{max} = \frac{\text{area of boundary}}{4G_N} \stackrel{?}{=} \# \text{ of dofs of QFT}$$

$$\text{yes :} \qquad \infty = \infty$$

need to regulate two divergences: dofs at every point in space (UV) ($\#$ dofs $\equiv N^2$) ,
spread over $\mathbb{R}^{d-1}$ (IR).
counting in QFT$_d$:

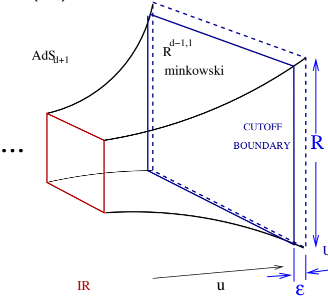
$$S_{max} \sim \left(\frac{R}{\epsilon}\right)^{d-1} N^2$$



counting in AdS$_{d+1}$: at fixed time: $ds^2_{AdS} = L^2 \frac{dz^2 + d\vec{x}^2}{z^2}$

$$A = \int_{bdy,\ z\ fixed} \sqrt{g} d^{d-1}x = \int_{\mathbb{R}^{d-1}} d^{d-1}x \left(\frac{L}{z}\right)^{d-1} |_{z\to 0}$$

$$A = \int_0^R d^{d-1}x \frac{L^{d-1}}{z^{d-1}} |_{z=\epsilon} = \left(\frac{RL}{\epsilon}\right)^{d-1}$$



The holographic principle

then says that the maximum entropy in the bulk is

$$\frac{A}{4G_N} \sim \frac{L^{d-1}}{4G_N} \left(\frac{R}{\epsilon}\right)^{d-1} .$$

$$\boxed{\frac{L^{d-1}}{G_N} = N^2}$$

lessons:

1. parametric dependence on R checks out.
2. gravity is classical if QFT has lots of dofs/pt: $N^2 \gg 1$

$$Z_{QFT}[\text{sources}] \approx e^{-N^2 I_{\text{bulk}}[\text{boundary conditions at } r\to\infty]}|_{\text{extremum of } I_{\text{bulk}}}$$

classical gravity (sharp saddle) $\leftrightsquigarrow$ many dofs per point, $N^2 \gg 1$

A word about large N^2

Most prominent example: 't Hooft limit of $N \times N$ matrix fields X.
Physical operators are $\mathcal{O}_k = \mathrm{tr}\, X^k$

This accomplishes several related things:

- $$\langle \mathcal{O}\mathcal{O} \rangle \sim \langle \mathcal{O} \rangle \langle \mathcal{O} \rangle + o\left(N^{-2}\right)$$

is the statement that *something* (the excitations created by $\mathcal{O}$) behaves classically.

- provides notion of single-particle states in bulk.
- makes saddle well-peaked $Z \sim e^{-N^2 I}$

comments:

- This is different from a *vector-like* large-N limit, where the fields are vectors with N components. In that case, QFT techniques are more useful (fewer diagrams) and holographic duality (higher spins) is less useful.
- this is just the best-understood class of examples.

in other examples, the # of dofs goes like $N^b, b \neq 2$.

I'll always write N^2 as a proxy for this large number.

Recap

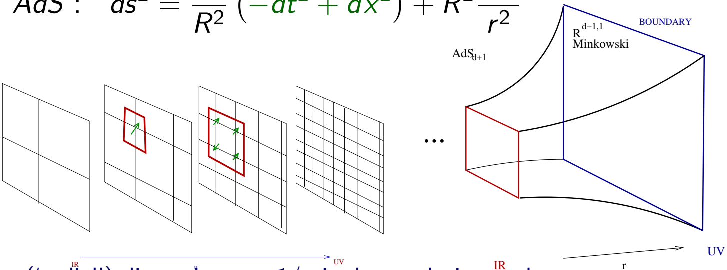
gravity in spacetimes$_{d+1}$
with timelike asymptotic boundaries $\quad \leftrightsquigarrow \quad QFT_d$

important special case:

gravity in AdS_{d+1} = d-dimensional conformal field theory (CFT)

isometries of $AdS_{d+1} \quad \leftrightsquigarrow \quad$ conformal symmetry

$$AdS: \quad ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + R^2 \frac{dr^2}{r^2}$$



The extra ('radial') dimension $r = 1/z$ is the resolution scale.

fields in bulk $\quad \leftrightsquigarrow \quad$ (possibly-) running couplings

$$Z_{QFT}[\text{sources}, \phi_0] \approx e^{-N^2 I_{\text{bulk}}[\text{boundary conditions at } r \to \infty]}\Big|_{\text{saddle of } I_{\text{bulk}}}$$

More of the dictionary

really a ϕ_a for every $\mathcal{O}^a$ in CFT. how to match?

1. organize into reps of conformal group
2. single-trace operators correspond to 'elementary fields' in the bulk.

states from multitrace ops $(\mathrm{tr}\ X^k)^2|0\rangle$ — 2-particle states of ϕ.

3. simple egs fixed by symmetry:

• gauge fields in bulk A_μ – global currents J^μ in bdy

$S_{QFT} \ni \int A_\mu J^\mu$ (massless A $\leftrightsquigarrow$ conserved J)

• def of QFT stress tensor: response to change in metric on boundary $S_{QFT} \ni \int \delta g_{\mu\nu} T^{\mu\nu}$

$$\begin{array}{ccc} \text{energy momentum tensor: } T^{\mu\nu} & & \text{graviton: } g_{ab} \\ \text{global current: } J^\mu & & \text{Maxwell field: } A_a \\ \text{scalar operator: } \mathcal{O}_B & \leftrightsquigarrow & \text{scalar field: } \phi \\ \text{fermionic operator: } \mathcal{O}_F & & \text{fermionic field: } \psi\,. \end{array}$$

boundary conditions on bulk fields $\leftrightsquigarrow$ couplings in field theory

e.g.: boundary value of bulk metric $\lim_{r\to\infty} g_{\mu\nu}$

$=$ source for stress-energy tensor $T^{\mu\nu}$

different couplings in bulk action $\leftrightsquigarrow$ different field theories

Next: a few technical slides from which we can confirm our interpretation

$$u = \text{RG scale}$$

and see the machinery at work.

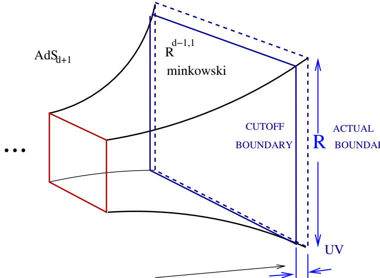
How to calculate

$$Z_{QFT}[\text{sources}] \approx e^{-N^2 I_{\text{bulk}}[\text{boundary conditions at } z\to 0]}\Big|_{\text{extremum of } I_{\text{bulk}}}$$

more explicitly:

$$\begin{aligned} Z_{QFT}[\text{sources}, \phi_0] &\equiv \left\langle e^{-\int \phi_0 \mathcal{O}} \right\rangle_{CFT} \\ &\approx e^{-N^2 I_{\text{bulk}}[\phi | \phi(z=\epsilon) \stackrel{?}{=} \phi_0]}\Big|_{\phi \text{ solves EOM of } I_{\text{bulk}}} \end{aligned}$$

As when counting dofs, we anticipate UV divergences at the boundary $z \to 0$,
cut off the bulk at $z = \epsilon$
and set bc's there.



Example: scalar probe

Simple example: scalar field in the bulk. Natural (covariant) action:

$$\Delta S[\phi] = -\frac{\mathfrak{K}}{2}\int d^{d+1}x\sqrt{g}\left[g^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2 + b\phi^3 + \dots\right]$$

$\mathfrak{K}$, a normalization constant: assume the theory of ϕ is weakly coupled, $\mathfrak{K} \propto N^2$.

$\left(\sqrt{g} = \sqrt{|\det g|} = \left(\frac{L}{z}\right)^{d+1}, \quad g^{AB} = \delta^{AB}z^2\right)$

We will study fluctuations around the solution $\phi = 0$, AdS.

(Recall: $\langle \mathcal{O}\mathcal{O}\rangle = \left(\frac{\delta}{\delta\phi_0}\right)^2 \ln Z|_{\underbrace{\phi_0 = 0}}$)

$\longrightarrow$ ignore interactions of ϕ for now.

Integrate by parts:

$$S = -\frac{\mathfrak{K}}{2}\int_{\partial AdS} d^d x\,\sqrt{g}\,g^{zB}\phi\,\partial_B\phi - \frac{\mathfrak{K}}{2}\int\sqrt{g}\,\phi\left(-\Box + m^2\right)\phi + o(\phi^3)$$

From this expression we learn:

- the EOM for small fluctuations of ϕ is $(-\Box + m^2)\underline{\phi} = 0$
  (An underline will indicate fields which solve the equations of motion.)
- If $\underline{\phi}$ solves the equation of motion, the on-shell action
  $S[\underline{\phi}], \quad Z \equiv e^{-S[\underline{\phi}]}$
  is just given by the boundary term.

next: relate bulk masses and operator dimensions

$$\Delta(\Delta - d) = m^2 L^2$$

by studying the AdS wave equation near the boundary.

Wave equation in AdS

translational invariance in d dimensions, $x^\mu \to x^\mu + a^\mu$,

$$\text{Fourier}: \ \phi(z, x^\mu) = e^{ik_\mu x^\mu} f_k(z), \qquad k_\mu x^\mu \equiv -\omega t + \vec{k}\cdot\vec{x}$$

$$\begin{aligned} 0 &= \left(g^{\mu\nu} k_\mu k_\nu - \frac{1}{\sqrt{g}}\partial_z(\sqrt{g} g^{zz}\partial_z) + m^2\right) f_k(z) \\ &= \frac{1}{L^2}[z^2k^2 - z^{d+1}\partial_z(z^{-d+1}\partial_z) + m^2L^2] f_k(z), \end{aligned} \qquad (1)$$

we used $g^{AB} = (z/L)^2\delta^{AB}$, $\quad \sqrt{g} = \sqrt{|\det g|} = \left(\frac{L}{z}\right)^{d+1}$.

Near boundary ($z \to 0$), power law solns, (spoiled by the z^2k^2 term).

Try $f_k = z^\Delta$ in (1):

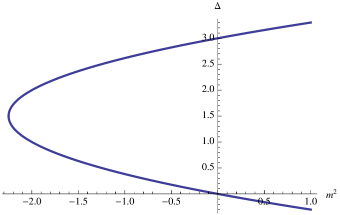
$$\begin{aligned} 0 &= k^2 z^{2+\Delta} - z^{d+1}\partial_z(\Delta z^{-d+\Delta}) + m^2L^2z^\Delta \\ &= (k^2z^2 - \Delta(\Delta - d) + m^2L^2)z^\Delta, \end{aligned}$$

and for $z \to 0$ we get:

$$\Delta(\Delta - d) = m^2L^2 \qquad (2)$$

The two roots of (2) are $\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2L^2}$.

Comments



$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}.$$

- The solution proportional to z^{Δ_-} is bigger near $z \to 0$. $\to$ usually the source ('non-normalizable')
- $\Delta_+ > 0 \ \ \forall \ \ m$: z^{Δ_+} always decays near the boundary
- $\Delta_+ + \Delta_- = d$.

We want to impose boundary conditions that allow solutions. Leading $z \to 0$ behavior of generic solution: $\phi \sim z^{\Delta_-}$, we impose

$$\phi(x,z)|_{z=\epsilon} \stackrel{!}{=} \phi_0(x,\epsilon) = \epsilon^{\Delta_-} \phi_0^{Ren}(x),$$

where ϕ_0^{Ren} is a renormalized source field.

Wavefunction renormalization of $\mathcal{O}$ (Heuristic but useful)

Suppose: ($g_{\mu\nu} \stackrel{z \approx \epsilon}{=} \frac{dz^2}{z^2} + \gamma_{\mu\nu} dx^\mu dx^\nu$ defines the boundary metric γ .)

$$
\begin{aligned}
S_{bdy} &\ni \int_{z=\epsilon} d^d x \sqrt{\gamma}\, \phi_0(x,\epsilon)\mathcal{O}(x,\epsilon) \\
&= \int d^d x \left(\frac{L}{\epsilon}\right)^d (\epsilon^{\Delta_-}\phi_0^{Ren}(x))\mathcal{O}(x,\epsilon),
\end{aligned}
$$

where we have used $\sqrt{\gamma} = (L/\epsilon)^d$.

Demanding that this be finite as $\epsilon \to 0$:

$$
\begin{aligned}
\mathcal{O}(x,\epsilon) &\sim \epsilon^{d-\Delta_-}\mathcal{O}^{Ren}(x) \\
&= \epsilon^{\Delta_+}\mathcal{O}^{Ren}(x),
\end{aligned}
$$

(we used $\Delta_+ + \Delta_- = d$)

The scaling dimension of $\mathcal{O}^{Ren}$ is $\Delta_+ \equiv \Delta$.

- To confirm: $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$
- For small m^2, $\exists$ 'alternative quantization': another CFT from the same bulk theory with Neumann boundary conditions on ϕ.

Relevantness

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}$$

• If $m^2 > 0$: $\Delta \equiv \Delta_+ > d$, so $\mathcal{O}_\Delta$ is an irrelevant operator.

$$\Delta S = \int d^d x \; (\mathrm{mass})^{d-\Delta} \mathcal{O}_\Delta,$$

the effects of such an operator go away in the IR, at energies $E < \mathrm{mass}$.

$\phi \sim z^{\Delta_-}\phi_0$ *is* this coupling.

It grows in the UV (small z). If ϕ_0 is a finite perturbation, it will back-react on the metric and destroy the asymptotic AdS-ness of the geometry: extra data about the UV will be required.

• $m^2 = 0 :\leftrightarrow \Delta = d$ means that $\mathcal{O}$ is marginal.

• If $m^2 < 0$: $\Delta < d$, so $\mathcal{O}$ is a relevant operator. Note that in AdS, $m^2 < 0$ is ok (*i.e.* not unstable) if m^2 is not too negative.

(Note: $\Delta(m)$ depends on the spin of the bulk field.)

Vacuum of CFT, euclidean case

Return to the scalar wave equation in momentum space:

$$0 = [z^{d+1}\partial_z(z^{-d+1}\partial_z) - m^2L^2 - z^2k^2]f_k(z)$$

If $k^2 > 0$ (spacelike or Euclidean) the general solution is
(a_K, a_I, integration consts):

$$f_k(z) = a_K z^{d/2}K_\nu(kz) + a_I z^{d/2}I_\nu(kz), \quad \nu = \Delta - \frac{d}{2} = \sqrt{(d/2)^2 + m^2L^2}.$$

In the interior of AdS ($z \to \infty$), the Bessel functions behave as

$$K_\nu(kz) \overset{z\to\infty}{\approx} e^{-kz} \qquad I_\nu(kz) \overset{z\to\infty}{\approx} e^{kz} \ .$$

regularity in the interior uniquely fixes $\underline{f}_k \propto K_\nu$.
Plugging this into the action S gives $\langle\mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$
($\star$: some details here. See slides of 'sample calculations.')
note: $\exists$ nonlinear uniqueness statement, 'Graham-Lee theorem'

Real-time

In Euclidean signature (or Lorentzian signature with spacelike k^2) regularity in the IR uniquely determined the correct solution.

In Lorentzian signature with timelike k^2 ($\omega^2 > \vec{k}^2$),
$\exists$ many solutions with the same UV behavior ($z \to 0$), different IR behavior:

$$z^{d/2} K_\nu(\pm iqz) \overset{z\to\infty}{\approx} e^{\pm iqz} \qquad q \equiv \sqrt{\omega^2 - \vec{k}^2}$$

these modes oscillate near the Poincaré horizon.
this ambiguity reflects the multiplicity of real-time Green's f'ns.

Important example: retarded Green's function, describes causal response of the system to a perturbation.

Linear response: nothing fancy, just QM

The retarded Green's function for two observables $\mathcal{O}_A$ and $\mathcal{O}_B$ is

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) = -i\int d^{d-1}x dt\ e^{i\omega t - ik\cdot x}\theta(t)\langle[\mathcal{O}_A(t,x), \mathcal{O}_B(0,0)]\rangle$$

$\theta(t) = 1$ for $t > 0$, else zero.

(We care about this because it determines what $\langle\mathcal{O}_A\rangle$ does if we kick the system via $\mathcal{O}_B$.)

the source is a time dependent perturbation to the Hamiltonian:

$$\delta H(t) = \int d^{d-1}x\phi_{B(0)}(t,x)\mathcal{O}_B(x)\,.$$

$$\begin{aligned}\langle\mathcal{O}_A\rangle(t,x) &\equiv \mathrm{Tr}\ \rho(t)\,\mathcal{O}_A(x) \\ &= \mathrm{Tr}\ \rho_0\, U^{-1}(t)\mathcal{O}_A(t,x)U(t)\end{aligned}$$

in interaction picture: $U(t) = Te^{-i\int^t \delta H(t')dt'}$ (*e.g.* $\rho_0 = e^{-\beta H_0}$)

Linear response, cont'd

linearize in small perturbation:

$$\begin{aligned}
\delta\langle\mathcal{O}_A\rangle(t,x) &= -i\mathrm{Tr}\ \rho_0 \int^t dt'[\mathcal{O}_A(t,x),\delta H(t')] \\
&= -i\int^t d^{d-1}x' dt' \langle[\mathcal{O}_A(t,x),\mathcal{O}_B(t',x')]\rangle \phi_{B(0)}(t',x') \\
&= \int dx' G_R(x,x')\phi_B(x')
\end{aligned}$$

fourier transform:

$$\delta\langle\mathcal{O}_A\rangle(\omega,k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\delta\phi_{B(0)}(\omega,k)$$

Linear response, an example

perturbation: an external electric field, $E_x = i\omega A_x$
couples via $\delta H = A_x J^x$ where J is the electric current ($\mathcal{O}_B = J_x$)
response: the electric current ($\mathcal{O}_A = J_x$)

$$\delta\langle\mathcal{O}_A\rangle(\omega, k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k)\delta\phi_{B(0)}(\omega, k)$$

it's safe to assume $\langle J\rangle_{E=0} = 0$:

$$\langle\mathcal{O}_J\rangle(\omega, k) = G^R_{JJ}(\omega, k)A_x = G^R_{JJ}(\omega, k)\frac{E_x}{i\omega}$$

Ohm's law: $J = \sigma E$

$$\implies \textit{Kubo formula}: \qquad \boxed{\sigma(\omega, k) = \frac{G^R_{JJ}(\omega, k)}{i\omega}}$$

Holographic real-time prescription

Claim [Son-Starinets 2002]: to compute G_R, take the solution which at $z \to \infty$ describes stuff falling into the horizon

- Both the retarded response and stuff falling through the horizon describe things that *happen*, rather than *unhappen*.
- You can check that this prescription gives the correct analytic structure of $G_R(\omega)$ ([Son-Starinets] and all the hundreds of papers that have used this prescription).
- It has been derived from a holographic version of the Schwinger-Keldysh prescription [Herzog-Son, Maldacena, Skenderis-van Rees].

The fact that stuff goes past the horizon and doesn't come out is what breaks time-reversal invariance in the holographic computation of G^R.

For a scalar in empty AdS the ingoing choice is $\phi(t, z) \sim e^{-i\omega t + iqz}$: as t grows, the wavefront moves to larger z.

(the solution which computes causal response is $z^{d/2} K_{+\nu}(iqz)$.)

The same prescription, adapted to the black hole horizon, works in the finite temperature case.

What to do with the solution

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the desired IR bc, behaves near the boundary as

$$\underline{\phi}(z,x) \approx \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x) \left(1 + \mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x) \left(1 + \mathcal{O}(z^2)\right);$$

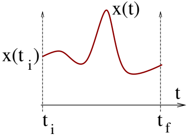
this formula defines the coefficient ϕ_1 of the subleading behavior of the solution.

All the information about G is in ϕ_0, ϕ_1.

recall: $Z[\phi_0] \equiv e^{-W[\phi_0]} \simeq e^{-S_{\mathrm{bulk}}[\underline{\phi}]}\big|_{\phi \overset{z\to 0}{\to} z^{\Delta_-}\phi_0}$

confession: this is a euclidean eqn. next: a nice general trick. [Iqbal-Liu]

classical mechanics interlude: consider a particle in 1d with action $S[x] = \int_{t_i}^{t_f} dtL$. The variation of the action with respect to the initial value of the coordinate is the initial momentum:



$$\Pi(t_i) = \frac{\delta S}{\delta x(t_i)}, \quad \Pi(t) \equiv \frac{\partial L}{\partial \dot{x}} \quad . \tag{3}$$

Thinking of the radial direction of AdS as time, a mild generalization of (3): [Iqbal-Liu]

$$\langle \mathcal{O}(x) \rangle = \frac{\delta W[\phi_0]}{\delta \phi_0(x)} = \lim_{z \to 0} \left(\frac{z}{L}\right)^{\Delta_-} \Pi(z,x)|_{\text{finite}},$$

where $\Pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)}$ is the bulk field-momentum with z treated as time.

two minor subtleties:

(1) the factor of z_-^{Δ} arises because of our renormalization of ϕ: $\phi \sim z^{\Delta_-}\phi_0$, so $\frac{\partial}{\partial \phi_0} = z^{-\Delta_-} \frac{\partial}{\partial \phi(z=\epsilon)}$.

(2) Π itself has a term proportional to the source ϕ_0

Linear response from holography

With these caveats, away from the support of the source:

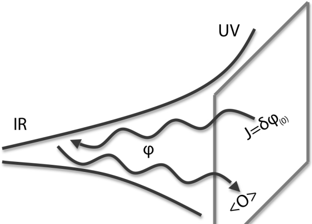
$$\langle \mathcal{O}(x) \rangle = \mathfrak{K} \frac{2\Delta - d}{L} \phi_1(x).$$

linearize in the size of the perturbing source:

$$\langle \mathcal{O}(x) \rangle = G_R \cdot \phi_0$$

summary: The leading behavior of the solution encodes the source *i.e.* the perturbation of the *action* of the QFT.
The coefficient of the subleading falloff encodes the response [Balasubramanian et al, 1996].

$$\boxed{G \propto \frac{\phi_1}{\phi_0}}$$



[figure: Hartnoll, 0909.3553]

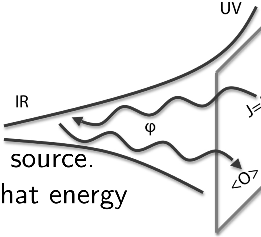
(Quasi)normal modes

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the desired IR bc, behaves near the boundary as

$$\underline{\phi}(z,x) \overset{z\to 0}{\approx} \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x)\left(1+\mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x)\left(1+\mathcal{O}(z^2)\right);$$

$$\boxed{G \propto \frac{\phi_1}{\phi_0}}$$



[figure: Hartnoll, 0909.3553]

G has poles when $\phi_1 \neq 0, \phi_0 = 0$: response without source.

this means that the system has an actual mode at that energy

(if $\omega \in \mathbb{C}$, 'quasinormal mode')

Important conceptual point: the Hilbert spaces are the same.

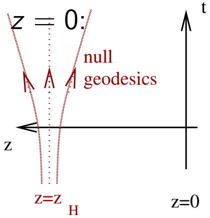
Next: thermal equilibrium

Finite temperature

AdS was scale invariant. sol'n dual to *vacuum* of CFT.
saddle point for CFT in an ensemble with a scale (some relevant perturbation) is a geometry which approaches AdS near the bdy:

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right) \qquad f = 1 - \frac{z^d}{z_H^d}$$

When the emblackening factor $f \overset{z\to 0}{\to} 1$ this is the Poincaré AdS metric.
[exercise: check that this solves the same EOM as AdS.]
It has a horizon at $z = z_H$, where the emblackening factor $f \propto z - z_H$
Events at $z > z_H$ can't influence the boundary near $z = 0$:



Physics of horizons

Claim: geometries with horizons describe thermally mixed states.

Why: Near the horizon ($z \sim z_H$),

$$ds^2 \sim -\kappa^2 \rho^2 dt^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \quad \rho^2 \equiv \frac{2}{\kappa z_H^2}(z - z_H) + o(z - z_H)^2$$

$\kappa \equiv \frac{4}{|f'(z_H)|} = d/2z_H$ is called the 'surface gravity'

Continue this geometry to euclidean time, $t \to i\tau$:

$$ds^2 \sim \kappa^2 \rho^2 d\tau^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2$$

which looks like $\mathbb{R}^{d-1} \times \mathbb{R}^2_{\rho,\kappa\tau}$ with polar coordinates $\rho, \kappa\tau$.

There is a deficit angle in this plane unless we identify

$$\kappa\tau \simeq \kappa\tau + 2\pi.$$

A deficit angle would mean nonzero Ricci scalar curvature, which would mean that the geometry is *not* a saddle point of our bulk path integral.

So: $T = \kappa/(2\pi) = 1/(\pi z_H)$.

(Note: this is the temperature of the Hawking radiation.)

Static BH describes thermal equilibrium

This identification on τ also applies at the boundary. If

$$ds^2_{bulk} \overset{z\to 0}{\approx} \frac{dz^2}{z^2} + \frac{L^2}{z^2} g^{(0)}_{\mu\nu} dx^\mu dx^\nu$$

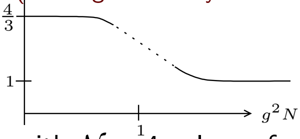
then, up to a factor, the boundary metric is $g^{(0)}_{\mu\nu}$.
This includes making the euclidean time periodic.
how to think about this:

$$Z_{CFT}(T) \approx e^{-S^{\mathrm{eucl}}_{\mathrm{bulk}}[\underline{g}]}$$

$\underline{g}$ is the saddle with the correct periodicity of eucl time at the bdy.

$$Z_{CFT}(T) = e^{-\beta F}$$

(warning: boundary terms in action are important – see below)



$$-\frac{F}{V} = \frac{L^2}{16\pi G_N} \frac{1}{z_H^4} = \frac{\pi^2}{8} N^2 T^4.$$

with $\mathcal{N} = 4$ values of parameters, $F(\lambda = \infty) = \frac{3}{4} F(\lambda = 0)$.

QFT thermodynamics from black holes cont'd

The Bekenstein-Hawking entropy is

$$S = \frac{A}{4G_N} = \frac{L^{d-1}}{4G_N}\frac{V}{z_H^{d-1}} = \frac{N^2}{2\pi}(\pi T)^{d-1}V = \frac{\pi^2}{2}N^2 V T^{d-1} \quad .$$

The Bekenstein-Hawking entropy *density* is

$$s_{BH} = \frac{S_{BH}}{V} = \frac{a_{BH}}{4G_N}.$$

where $a_{BH} \equiv \frac{A}{V}$ is the 'area density' of the black hole.
checks:

- $$\underset{\text{horizon}}{S_{BH}} = \underset{\text{integral over all spacetime}}{-\frac{\partial F}{\partial T}}$$

(That is, the first law of thermo holds.)

- $c_V > 0$ for AdS BH. (unlike schwarzchild in asymptotically flat space!)
- uniqueness of stationary BH ('no hair') $\leftrightsquigarrow$ few state variables in eq thermo

Next: deviations from thermal equilibrium

Derviations from equilibrium

bulk picture: dynamics of gravitational collapse.

dissipation: energy falls into BH [Horowitz-Hubeny, 99]

• small-amplitude perturbations: quasinormal modes of BH

An important virtue of holographic duality over other large-N approximations:

dissipation is built in at leading order in this approximation.

Puzzle: this is a closed system (and can even be finite volume).

Where does the energy *go*?

A: into the rest of the large-N matrix.

• hydrodynamic perturbations: perturb BH horizon by local boost $u^\mu(x)$, slowly varying.

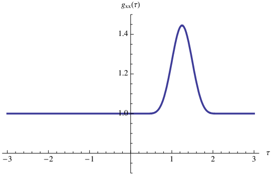
[Janik-Peschanski,Bhattacharyya et al...]: In an expansion in derivatives of $T(x), u^\mu(x)$,

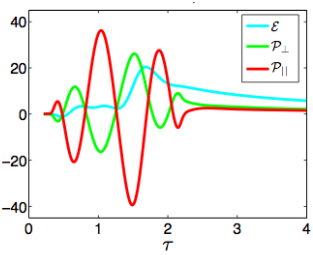
sol'ns of Einsten eqns of this form	$\leftrightarrow$	soln's of Navier-Stokes eqns with particular transport coeffs

disappointment(?): holographic duality doesn't average over turbulent flows.

Derviations from equilibrium, cont'd

- far-from equilibrium processes: [Chesler-Yaffe 08] (PDEs!)

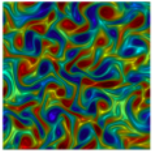
input: 

output: 

black hole forms from vacuum initial conditions.

brutally brief summary: all relaxation timescales $\tau_{th} \sim T^{-1}$.

More recently: 'turbulent' bulk solutions [Adams-Chesler-Liu 13] (not obviously in hydro regime)



End of first lecture (I hope).

Appendices

A useful visualization: 'Witten diagrams'

e.g. consider 3-point function, $\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle = \left(\frac{\delta}{\delta\phi_0}\right)^3 \ln Z|_{\phi_0=0}$.
cubic coupling matters:

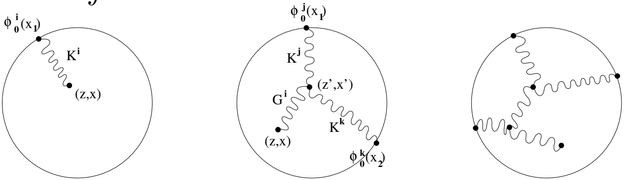
$$(\Box - m^2)\phi(z,x) = b\phi^2 \quad \text{and perms.}$$

Solve perturbatively in ϕ_0: (K, G are Green's f'ns for $\Box - m_i^2$)

$$\underline{\phi}(z,x) = \int d^d x_1 K^{\Delta}(z,x;x_1)\phi_0^{(}x_1)$$

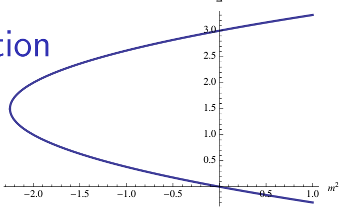
$$+ \quad b\int d^d x' dz' \sqrt{g}\, G^{\Delta}(z,x;z',x')$$

$$\times \quad \int d^d x_1 \int d^d x_2 K^{\Delta}(z',x';x_1)\phi_0(x_1) K^{\Delta}(z',x';x_2)\phi_0(x_2) + o(b^2\phi_0^3)$$



external legs $\leftrightarrow$ sources ϕ_0, vertices $\leftrightarrow$ bulk interactions

Comment on alternative quantization



$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}.$$

Lowest Δ_+ is $d/2$. Unitarity bound is $\frac{d-2}{2}$.

Idea [Klebanov-Witten 99]: If under $\delta\phi$, $\delta S \neq \infty$, maybe we can let $\delta\phi$ fluctuate.

$$\text{With} \quad S_{\text{above}} = \int_{\text{bulk}} \left[(\partial\phi)^2 + \phi^2\right]$$

$\delta\phi \sim \delta\phi_1 z^\Delta$ gives $\delta S_{\text{above}} = \infty$ for $\Delta > d/2$.

$$\text{With} \quad S = S_{\text{above}} + \# \int_{\text{bdy}} \sqrt{\gamma}\phi n \cdot \partial\phi$$

the fluctuation with $\phi \sim z^\Delta$ is normalizible for $\Delta < \frac{d-2}{2}$.

Result: can treat $\phi_1 z^{\Delta_-}$ as source, $\phi_0 z^{\Delta_+}$ as response:

$G_{\text{alt}} = \frac{\phi_0}{\phi_1} = G_{\text{usual}}^{-1}$.

Interpretation: alternative quantization is a CFT with a relevant double trace operator $\Delta\left(\mathcal{O}^2\right) = 2\Delta_-$

Perturbation (by $\Delta S_{\text{alt}} = \int_{\text{bdy}} \sqrt{\gamma}\phi^2$)leads back to ordinary quantization.

An example of a theory with a known gravity dual

$\mathcal{N} = 4$ SYM is a family of superconformal FTs.
The $\mathcal{N} = 4$ SYM action is schematically

$$\mathcal{L}_{\mathsf{SYM}} \sim \mathrm{tr}\ \left(F^2 + (D\Phi)^2 + i\bar{\Psi}\Gamma \cdot D\Psi + g^2[\Phi, \Phi]^2 + ig\bar{\Psi}[\Phi, \Psi] \right)$$

this gauge theory comes with 2 parameters:
a coupling constant $\lambda = g^2 N$ (with $\beta_\lambda \equiv 0$)
an integer, the number of colors N.

$$\boxed{\mathcal{N} = 4\ \mathrm{SYM}_{N,\lambda}} = \boxed{\text{IIB strings in } AdS_5 \times S^5 \text{ of size } \lambda, \hbar = 1/N}$$

[Maldacena 1997]

- large N makes gravity classical

(improves saddle point, suppresses splitting and joining of strings)

- strong coupling (large λ) makes the geometry big.

(improves bulk deriv. expansion)

'IIB strings in ...' specifies a list of bulk fields and interactions.
$\exists$ *infinitely many* other examples of dual pairs [*e.g.* Hanany, Vegh et al...]

Confidence-building measures

Why do we believe this enough to try to use it to do physics?

- 1. **Many** detailed checks in special examples
  examples: relativistic gauge theories (fields are $N \times N$ matrices), with extra symmetries (conformal invariance, supersymmetry)
  checks: 'BPS quantities,' integrable techniques, some numerics
- 2. Sensible answers for physics questions
  rediscoveries of known physical phenomena: *e.g.* color confinement, chiral symmetry breaking, thermo, hydro, thermal screening, entanglement entropy, chiral anomalies, superconductivity, ...
  Gravity limit, when valid, says who are the correct variables.
  Answers questions about thermodynamics, transport, RG flow, ...
  in terms of geometric objects.
- 3. Applications to quark-gluon plasma (QGP)
  benchmark for viscosity, hard probes of medium, approach to equilibrium

Appendix:

Counting powers of N^2

large N counting

consider a matrix field theory
Φ_a^b is a matrix field. $a, b = 1..N$. other labels (*e.g.* spatial position, spin) are suppressed.

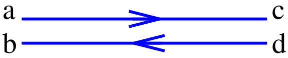
$$\mathcal{L} \sim \frac{1}{g^2} \mathrm{Tr}\ \left((\partial \Phi)^2 + \Phi^2 + \Phi^3 + \Phi^4 + \dots \right)$$

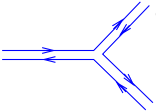
here *e.g.* $(\Phi^2)_a^c = \Phi_a^b \Phi_b^c$,
the interactions are invariant under the $U(N)$ symmetry
$\Phi \to U^{-1} \Phi U$

't Hooft counting

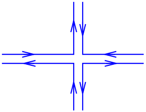
double-line notation:

$$\langle \tilde{\Phi}^a_b \tilde{\Phi}^d_c \rangle \propto g^2 \delta^a_c \delta^d_b \equiv g^2$$





$\propto g^{-2}$



$\propto g^{-2}$

$$\text{diagram} \sim \left(\frac{\lambda}{N}\right)^{\text{no. of prop.}} \left(\frac{N}{\lambda}\right)^{\text{no. of int. vert.}} N^{\text{no. of index loops}}$$

't Hooft limit: take $N \to \infty, g \to 0$ holding $\lambda \equiv g^2 N$ fixed.

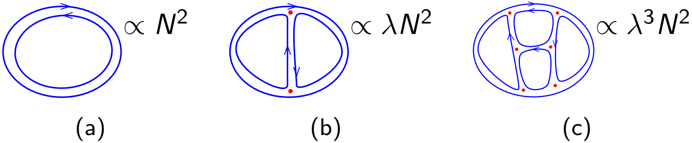


Figure: planar graphs that contribute to the vac→vac ampl.

topology of graphs

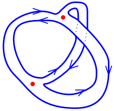
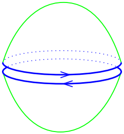
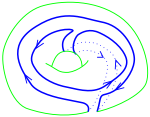
 $\propto g^2 N = \lambda N^0$

Figure: Non-planar (but still oriented!) graph that contributes to the vacuum→vacuum amplitude.

 $\sim S^2$  $\sim T^2$

If $E = \#$of propagators, $V = \#$ of vertices, and $F = \#$ of index loops,
a diagram contributes $N^{F-E+V} \lambda^{E-V}$.
$F - E + V = \chi(\text{surface}) = 2 - 2h - b$
($h =$ number of handles, $b =$ number of boundaries)

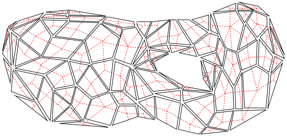
topology of graphs, cont'd

the effective action (the sum over connected vacuum-to-vacuum diagrams) has the expansion:

$$\ln Z = \sum_{h=0}^{\infty} N^{2-2h} \sum_{\ell=0}^{\infty} c_{\ell,h} \lambda^{\ell} = \sum_{h=0}^{\infty} N^{2-2h} \mathfrak{F}_h(\lambda)$$

['t hooft]: $1/N$ as a small parameter, string expansion. $1/N$ suppresses splitting and joining of strings.

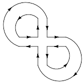
e.g.



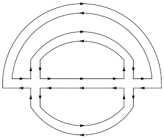
concrete point: at large N, $\ln Z \sim N^2$.

N counting for correlation functions

$$\mathcal{O}(x) = c(k, N)\mathrm{Tr}(\Phi_1(x)...\Phi_k(x))$$



Disconnected diagram contributing to the correlation function
$\langle \mathrm{Tr}(\Phi^4)\mathrm{Tr}(\Phi^4)\rangle \sim N^2$



Connected diagram contributing to the correlation function $\langle \mathrm{Tr}(\Phi^4)\mathrm{Tr}(\Phi^4)\rangle$
goes like N^0