The Basic Structure of Scattering Amplitudes

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Sofja Kowaleskaja Award

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- LMU, Munich, 10-14 9.2012 :: Lecture II

(these slides are supposed to be integrated with the blackboard notes)

References:

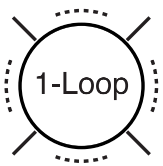
- Ossola, Papadopoulos, Pittau,
Reducing full one-Loop Amplitudes to scalar integrals at the integrand-level, hep-ph/0609007.
- Ossola, Reiter, Tramontano, & P.M.
Scattering AMplitudes from Unitarity-based Reduction At the Integrand-level, 1006.0710 [hep-ph]
- Mirabella, Peraro, & P.M.
Integrand Reduction of One-Loop Scattering Amplitudes from Laurent series expansion, arXiv:1203.0291 [hep-ph]
- Ellis, Kunszt, Zanderighi and Melnikov,
One-Loop Calculation in QFT, 1105.4319 [hep-ph]. (very nice review!!!)

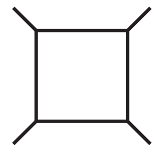
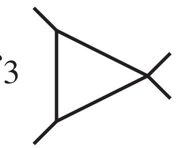
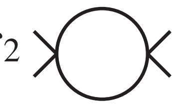
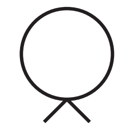
Motivation

- QFT and Scattering Amplitudes from a new perspective
- The ***singularity structures*** from complex deformation of the kinematics
- Amplitudes decomposition from factorization
- The central role of Cauchy's Residue Theorem (and its multivariate generalization)
- Reduction to Master Integrals by Integrand Decomposition
- Identify a unique Mathematical framework for any Multi-Loop Amplitude
- Based on ***one*** property of Scattering Amplitudes: the quadratic ***Feynman denominator***

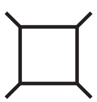
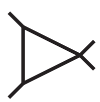
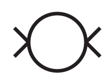
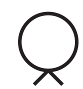
One-Loop Scattering Amplitudes

- *n*-particle Scattering: $1+2 \to 3+4+\ldots+n$
- Reduction to a Scalar-Integral Basis   Passarino-Veltman



$$= \sum_{10^2-10^3} \int d^D\ell \, \frac{\ell^\mu \ell^\nu \ell^\rho \ldots}{D_1 D_2 \ldots D_n} = c_4$$  $$+ c_3$$  $$+ c_2$$  $$+ c_1$$ 

- Known: Master Integrals

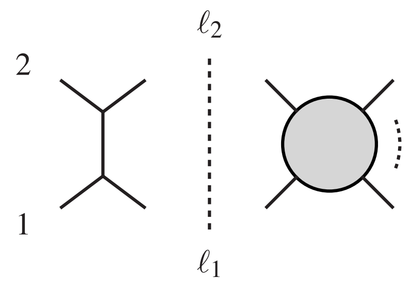
 $= \int d^D\ell \, \frac{1}{D_1 D_2 D_3 D_4}$,  $= \int d^D\ell \, \frac{1}{D_1 D_2 D_3}$,  $= \int d^D\ell \, \frac{1}{D_1 D_2}$,  $= \int d^D\ell \, \frac{1}{D_1}$

- Unknowns: c_i are rational functions of external kinematic invariants

Cutting Rules

- Discontinuity of Feynman Integrals Landau & Cutkosky

Cut Integral in the P_{12}^2-channel

$\Delta(\Gamma_1) =$ 

$$d^4\Phi \quad = \quad d^4\ell_1\, d^4\ell_2\;\; \delta^{(4)}\left(\ell_1+\ell_2-P_{12}\right)\;\delta^{(+)}\left(\ell_1^2\right)\,\delta^{(+)}\left(\ell_2^2\right)$$

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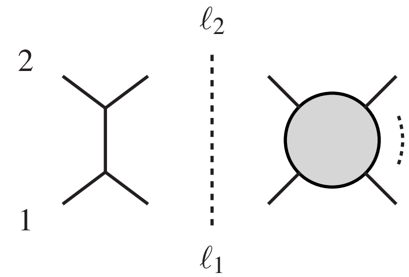
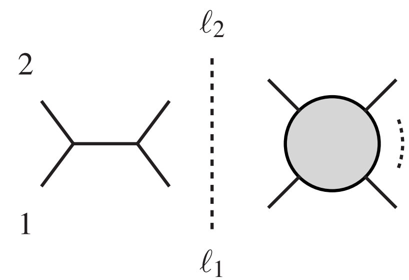


$\Delta(\Gamma_2) =$



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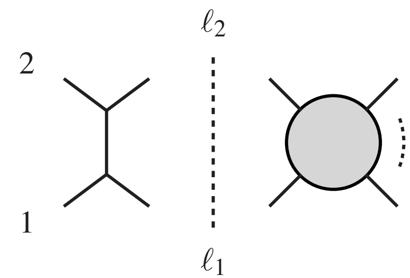
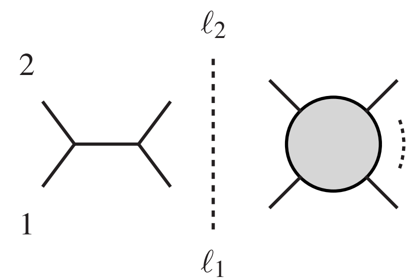
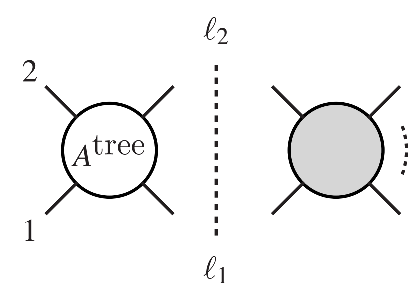


$$\Delta(\Gamma_2) =$$



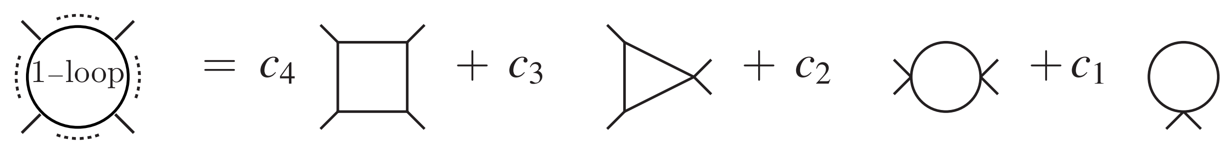
$$\Delta(\Gamma_1) + \Delta(\Gamma_2) =$$



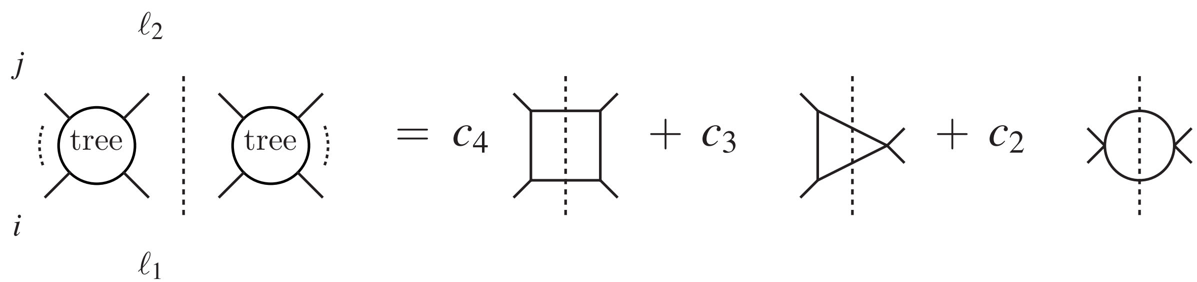
Unitarity & Cutting Rules

- Optical Theorem from Unitarity $S \equiv 1 + iT \;:\quad S^\dagger S = 1 \quad \Rightarrow \quad 2\mathrm{Im}T = -i(T - T^\dagger) = T^\dagger T$

- One-loop Amplitude:

$A_n^{\text{1-loop}} =$ 

- Discontinuity of Feynman Amplitudes Cutkosky-Veltman; Bern, Dixon, Dunbar & Kosower

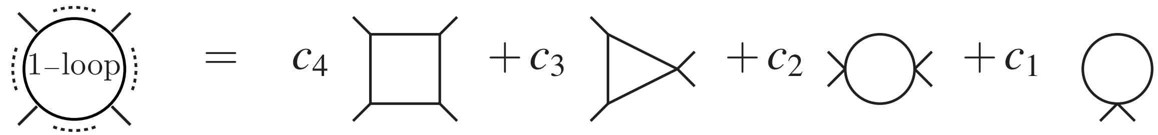
$2\mathrm{Im}\left\{A_n^{\text{1-loop}}\right\} =$ 

on − shell condition : $\dfrac{1}{(\ell_i^2 - m_i^2 + i0)} \rightarrow \delta\left(\ell_i^2 - m_i^2\right) \qquad (i = 1, 2)$

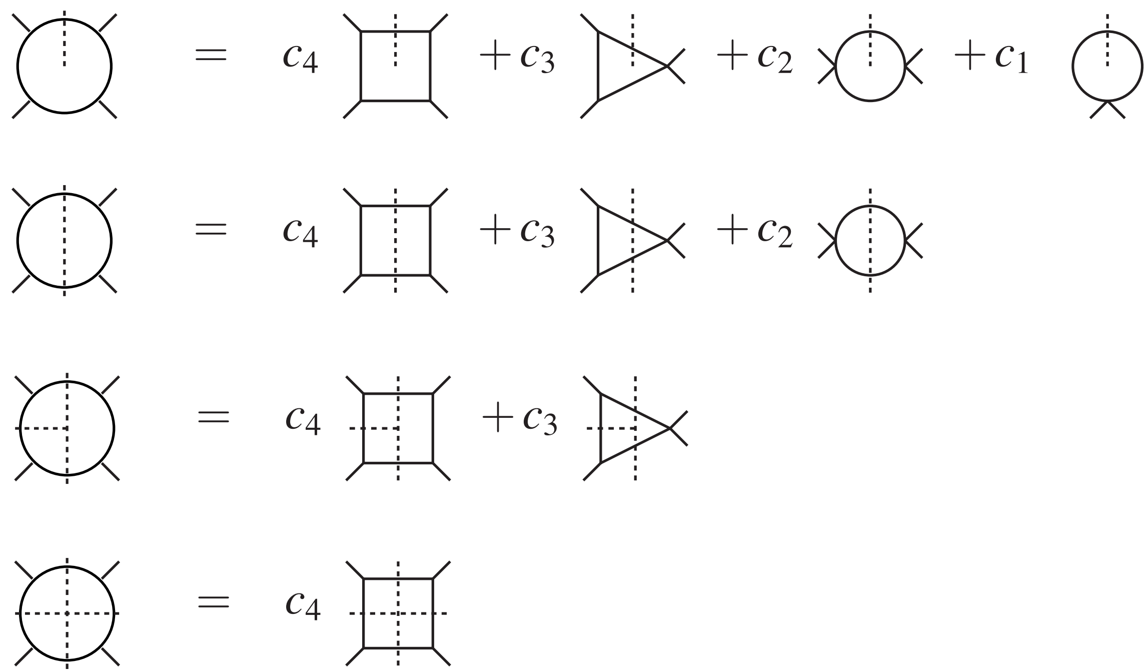
The Strategy: Generalised Unitarity

- One-loop Amplitude:

$A_n^{\text{1-loop}} =$



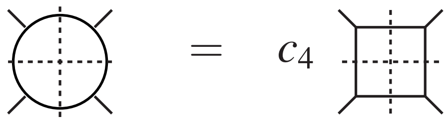
Replacing the original amplitude with simpler integrals fulfilling the same algebraic decomposition



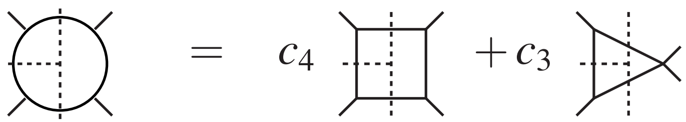
The more you cut, the more you loose, the simpler it gets

The Strategy: Generalised Unitarity

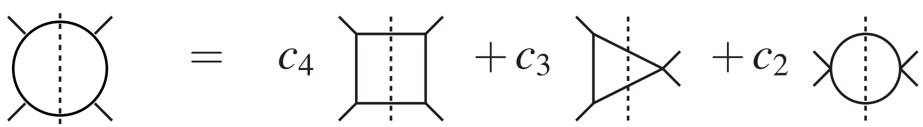
- Multiple-cuts as *optical filters*



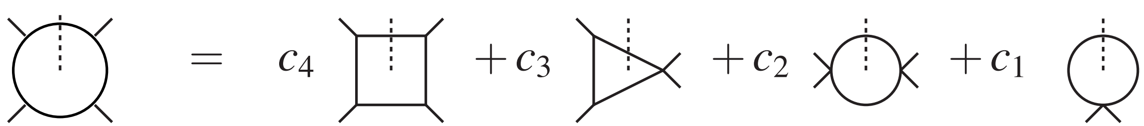
Britto, Cachazo, Feng



Bern, Dixon, Dunbar, Kosower
P.M.
Forde
Bjerrum-Bohr, Dunbar, Perkins



Bern, Dixon, Dunbar, Kosower
Brandhuber, McNamara, Spence, Travaglini
Britto, Buchbinder, Cachazo, Feng, $\oplus$ P.M.
Anastasiou, Britto, Feng, Kunszt, P.M.
Forde; Badger



Glover, Williams
Britto, Feng
Britto, Mirabella

Cut-Conditions

- Loop momentum decomposition

$$q^2 = p^2 = \varepsilon^{\pm 2} = 0 = \varepsilon^{\pm} \cdot p = \varepsilon^{\pm} \cdot q \,, \qquad \ell_\mu = x_1\, p_\mu + x_2\, q_\mu + x_3\, \varepsilon^+_\mu + x_4\, \varepsilon^-_\mu$$

- under Multiple On-shellness Conditions :
- the loop-momentum becomes complex ;
- some of its components (if not all) are frozen;
- the left over free components are *integration*-variable

- On-shell condition

$$\delta\left(\ell_i^2 - m_i^2\right)$$

- Closer look at the Integrand Structure

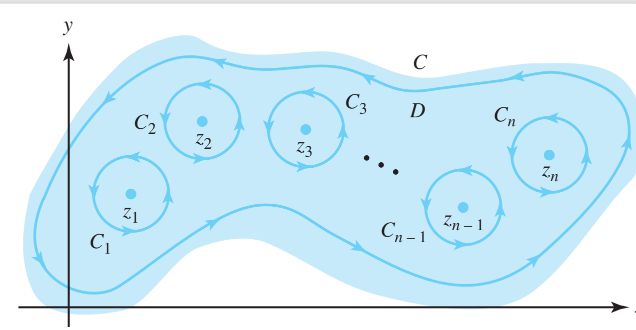
Numerator and denominator of the n-particle cut-integrand are mutivariate-polynomials in $(4-n)$ complex-variables:

$$\mathrm{Cut}_n = \oint dx_1 \ldots dx_{4-n} \; \frac{P(x_1, \ldots, x_{4-n})}{Q(x_1, \ldots, x_{4-n})}$$

▷ Contour Integrals of Rational Functions $\sim$ Integrals by *partial fractioning*

- ResidueTheorem

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{i=1}^{n} \mathrm{Res}(f, z_i) \,.$$



Unitarity-based Methods

After Integration

- One-Loop Integral basis :: MI's :: Li2(x), log(x)^2, log(x), O(x)
- log(x) ~ 1 ; Li2(x) ~ log(x) ; Li2(x) ~ 1
- Amplitude decomposition from ***matching cuts***

Unitarity-based Methods

After Integration

- One-Loop Integral basis :: MI's :: Li2(x), log(x)^2, log(x), O(x)
- log(x) ~ 1 ;   Li2(x) ~ log(x) ;   Li2(x) ~ 1
- Amplitude decomposition from ***matching cuts***

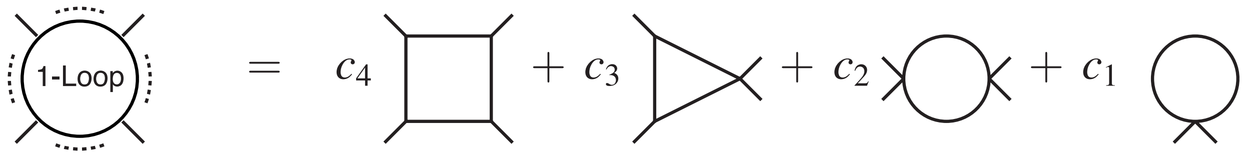
Integrand-Reduction Methods

Before Integration

- Residues are ***polynomials*** in ***irreducible scalar products*** (ISP's)
- ISP's generate MI's
- Amplitude decomposition from ***polynomial fitting*** on the cuts

At the Integrand Level

- Reduction to a Scalar-Integral Basis Passarino-Veltman



$$\int d^4q\, A(q) \;=\; c_4 \int \frac{d^4q}{D_0D_1D_2D_3} + c_3 \int \frac{d^4q}{D_0D_1D_2} + c_2 \int \frac{d^4q}{D_0D_1} + c_1 \int \frac{d^4q}{D_0}$$

- Unknowns: c_i are rational functions of external kinematic invariants

- At the Integrand-level

$$A(q) \;\neq\; \frac{c_4}{D_0D_1D_2D_3} + \frac{c_3}{D_0D_1D_2} + \frac{c_2}{D_0D_1} + \frac{c_1}{D_0}$$

$$= \frac{c_4 + f_4(q)}{D_0D_1D_2D_3} + \frac{c_3 + f_3(q)}{D_0D_1D_2} + \frac{c_2 + f_2(q)}{D_0D_1} + \frac{c_1 + f_1(q)}{D_0}$$

$$\int d^4q\, \frac{f_4(q)}{D_0D_1D_2D_3} = \int d^4q\, \frac{f_3(q)}{D_0D_1D_2} = \int d^4q\, \frac{f_2(q)}{D_0D_1} = \int d^4q\, \frac{f_1(q)}{D_0} = 0$$

$$A(q) \;\equiv\; \frac{\Delta_{0123}(q)}{D_0D_1D_2D_3} + \frac{\Delta_{012}(q)}{D_0D_1D_2} + \frac{\Delta_{01}(q)}{D_0D_1} + \frac{\Delta_0(q)}{D_0}$$

OPP-Integrand Reduction

Ossola, Papadopoulos, Pittau

Ellis, Giele, Kunszt

Giele, Kunszt, Melnikov

- OPP-decomposition

$$A_m = \int d^4q \frac{N(q)}{D_0 \dots D_{m-1}}$$

$$\begin{aligned} N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \Delta_{i_0 i_1 i_2 i_3}(q) \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\ &+ \sum_{i_0<i_1<i_2}^{m-1} \Delta i_0 i_1 i_2(q) \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\ &+ \sum_{i_0<i_1}^{m-1} \Delta i_0 i_1(q) \prod_{i \neq i_0, i_1}^{m-1} D_i \\ &+ \sum_{i_0}^{m-1} \Delta i_0(q) \prod_{i \neq i_0}^{m-1} D_i \end{aligned}$$

- $\Delta(q)$ are known polynomials
- c_i are the constant terms of Δ's

▷ Fitting c_i by numerical evaluating $N(q)$ at different values of $q \oplus$ system inversion

- q @ Quadruple-cut: $D_{i_0} = D_{i_1} = D_{i_2} = D_{i_3} = 0$

$$N(q) \quad = \quad \Delta_{i_0 i_1 i_2 i_3}(q) \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i$$

- q @ Triple-cut: $D_{i_0} = D_{i_1} = D_{i_2} = 0$

$$
\begin{aligned}
N(q) & - \sum_{i_0<i_1<i_2<i_3}^{m-1} \Delta_{i_0 i_1 i_2 i_3}(q) \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& = \Delta_{i_0 i_1 i_2}(q) \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i
\end{aligned}
$$

- q @ Double-cut: $D_{i_0} = D_{i_1} = 0$

$$
\begin{aligned}
N(q) &- \sum_{i_0<i_1<i_2<i_3}^{m-1} \Delta_{i_0 i_1 i_2 i_3}(q) \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&- \sum_{i_0<i_1<i_2}^{m-1} \Delta_{i_0 i_1 i_2}(q) \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&= \Delta_{i_0 i_1}(q) \prod_{i \neq i_0, i_1}^{m-1} D_i
\end{aligned}
$$

- q @ Single-cut: $D_{i_0} = 0$

$$
\begin{aligned}
N(q) &- \sum_{i_0<i_1<i_2<i_3}^{m-1} \Delta_{i_0 i_1 i_2 i_3}(q) \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&- \sum_{i_0<i_1<i_2}^{m-1} \Delta_{i_0 i_1 i_2}(q) \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&- \sum_{i_0<i_1}^{m-1} \Delta_{i_0 i_1}(q) \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&= \Delta_{i_0}(q)
\end{aligned}
$$

- OPP-reduction Ossola, Papadopoulos, Pittau (2006)

From the knowledge of the multi-variate polynomial-structure of the Integrand, all n-point coefficients can be determined by fitting a system of polynomial equations.

Advantage ▷ No integration required

Pitfall ▷ Numerical System Inversion $(\Delta \to 0)$

- Improved Reduction with DFT Ossola, Papadopoulos, Pittau, & P.M. (2008)

$$P_m(x) = c_0 + c_1 x + c_2 x^2 + \dots \; c_m x^m$$

▷ step 1: sample $P_m(x)$ at $(m+1)$ equidistant-points on the unit-circle, $P_{m,k} \equiv P_m(x_k)$,

$$x_k = \; e^{-2\pi i \frac{k}{(m+1)}} \qquad (k = 0, \dots, m) \; .$$

▷ step 2: find c_i from orthogonality (plane-waves):

$$c_\ell = \frac{1}{m+1} \sum_{k=0}^{m} P_{m,k} \; e^{2\pi i \frac{k}{(m+1)} \ell}$$

Cuts and Residues

For each cut $(ijk\cdots)$, $D_i = D_j = D_k = \cdots = 0$, a basis of four massless vectors

$$\left\{ e_1^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_4^{(ijk\cdots)} \right\}$$

$$\left(e_i^{(ijk\cdots)}\right)^2 = 0 \,, \qquad e_1^{(ijk\cdots)} \cdot e_3^{(ijk\cdots)} = e_1^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 0 \,,$$
$$e_2^{(ijk\cdots)} \cdot e_3^{(ijk\cdots)} = e_2^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 0 \,, \qquad e_1^{(ijk\cdots)} \cdot e_2^{(ijk\cdots)} = -e_3^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 1$$

The massless vectors $e_1^{(ijk\cdots)}$ and $e_2^{(ijk\cdots)}$ can be written as a linear combination of the two external legs at the edges of the propagator carrying momentum $q + p_i$, say K_1 and K_2, along the lines of [18]. In the case of double-cut, K_1 is the momentum flowing through the corresponding 2-point diagram, and K_2 is an arbitrary massless vector. In the case of single-cut both K_1 and K_2 are chosen as arbitrary vectors. In the case of quadruple-cut $(ijk\ell)$ we define

$$\begin{aligned} v_\perp^{(ijk\ell)} &= \left(K_3 \cdot e_4^{(ijk\ell)}\right)\, e_3^{(ijk\ell)} - \left(K_3 \cdot e_3^{(ijk\ell)}\right)\, e_4^{(ijk\ell)}, \\ v^{(ijk\ell)} &= \left(K_3 \cdot e_4^{(ijk\ell)}\right)\, e_3^{(ijk\ell)} + \left(K_3 \cdot e_3^{(ijk\ell)}\right)\, e_4^{(ijk\ell)} \,. \end{aligned} \tag{2.7}$$

The momentum K_3 is the third leg of the 4-point function associated to the considered quadruple-cut. To simplify our notation we will omit the indices of the cut $(ijk\cdots)$ whenever possible.

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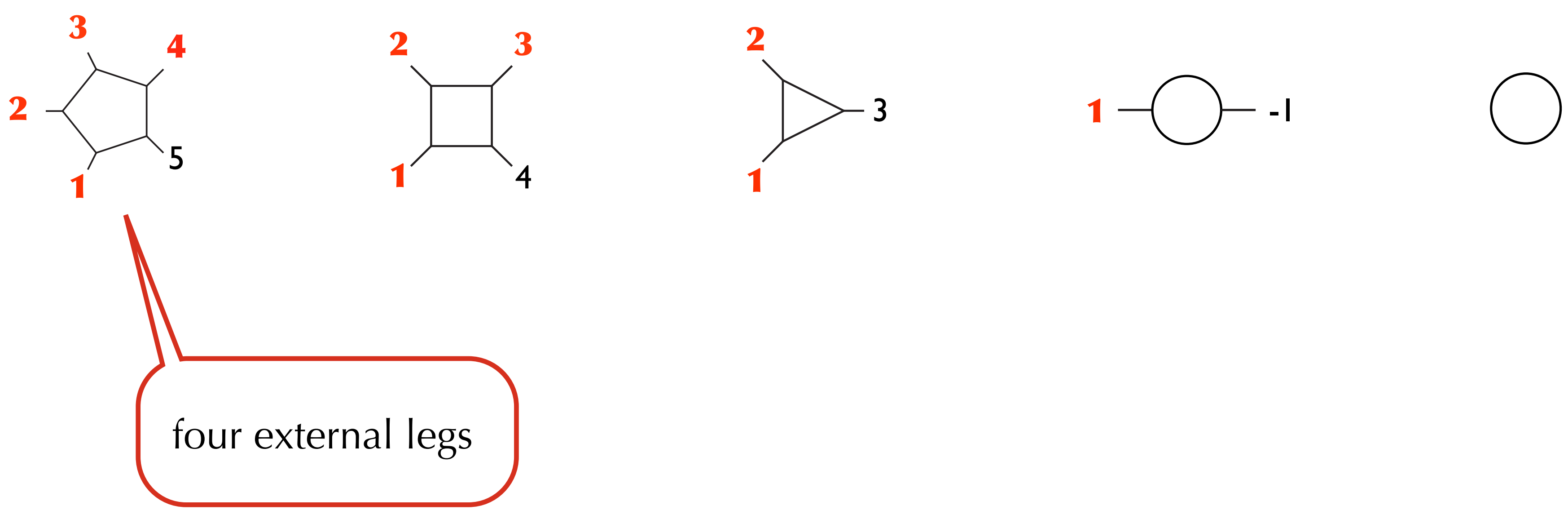

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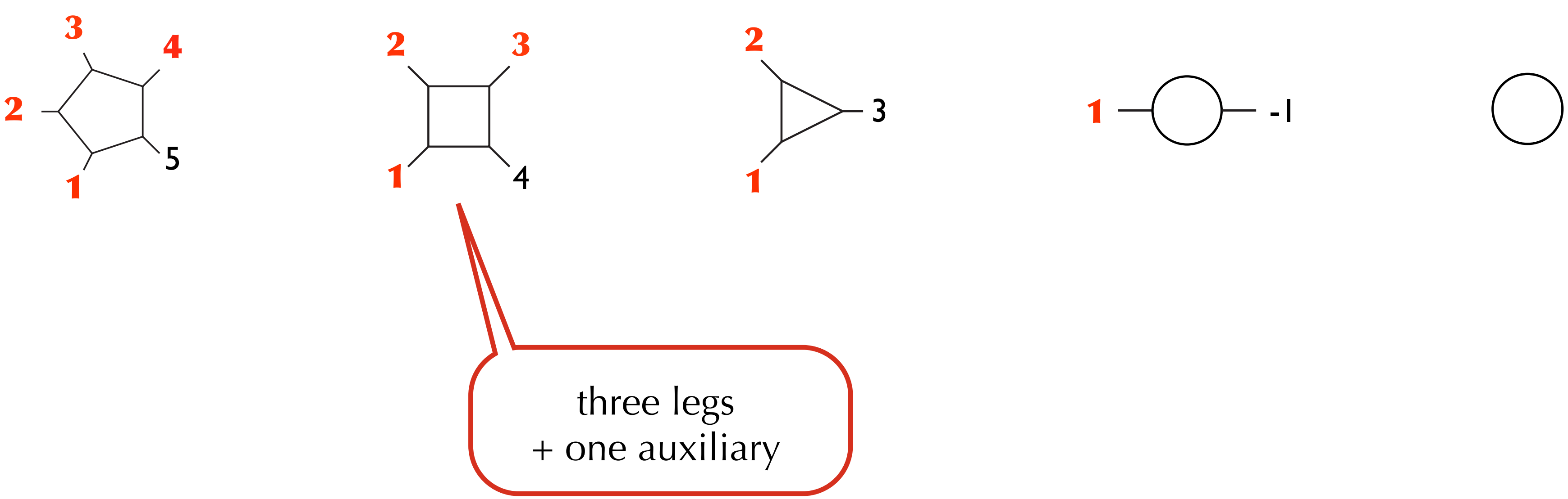

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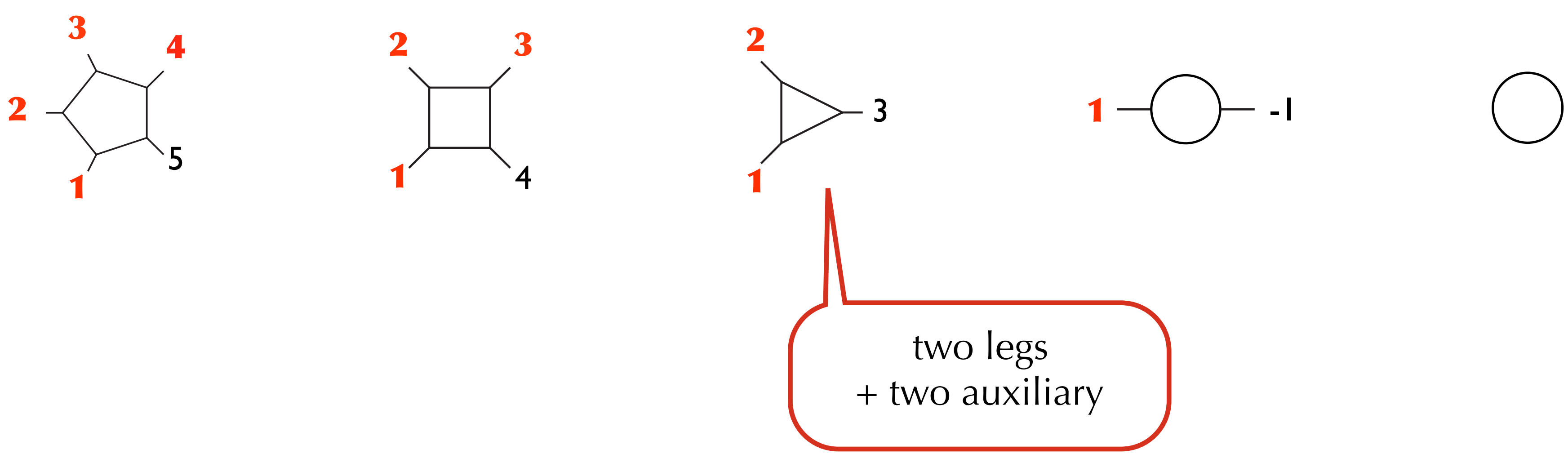

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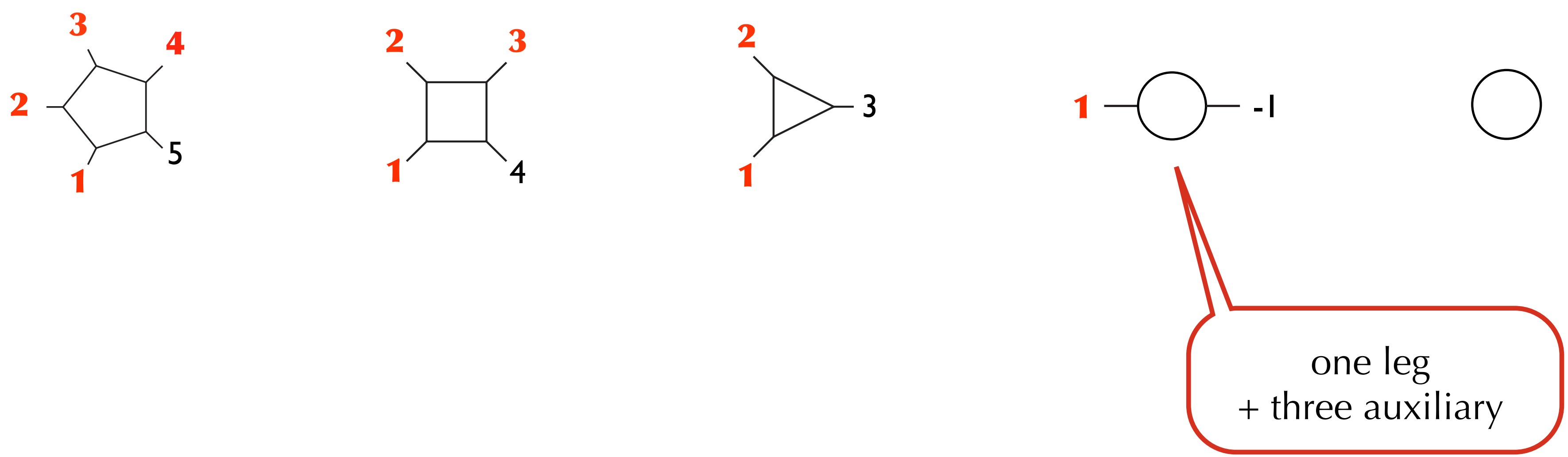
$$e_2^{(ijk\cdots)} \cdot e_3^{(ijk\cdots)} = e_2^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 0 \ , \qquad e_1^{(ijk\cdots)} \cdot e_2^{(ijk\cdots)} = -e_3^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 1$$



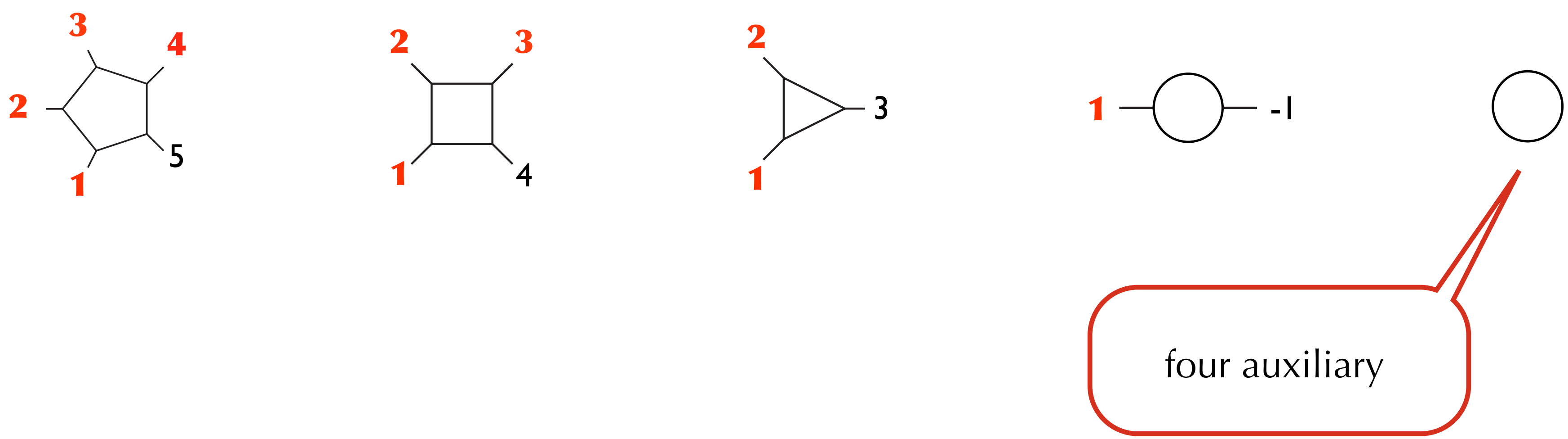
Cuts and Residues

For each cut $(ijk\cdots)$, $D_i = D_j = D_k = \cdots = 0$, a basis of four massless vectors

$$\left\{ e_1^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_4^{(ijk\cdots)} \right\}$$

$$\left(e_i^{(ijk\cdots)}\right)^2 = 0 \,, \qquad e_1^{(ijk\cdots)} \cdot e_3^{(ijk\cdots)} = e_1^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 0 \,,$$

$$e_2^{(ijk\cdots)} \cdot e_3^{(ijk\cdots)} = e_2^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 0 \,, \qquad e_1^{(ijk\cdots)} \cdot e_2^{(ijk\cdots)} = -e_3^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 1$$



Cuts and Residues

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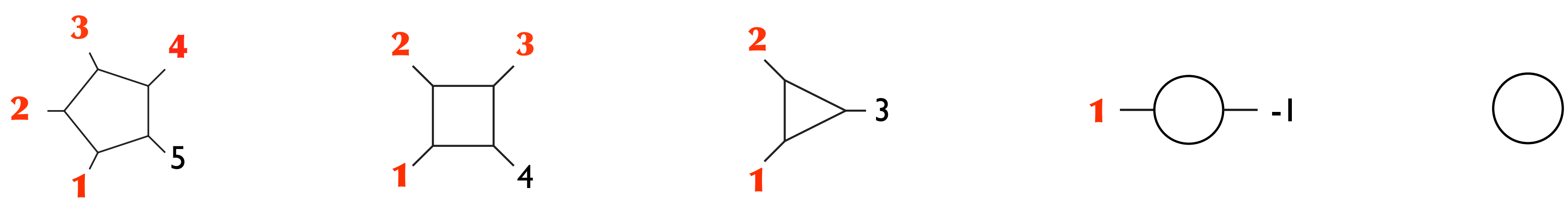

- Loop momentum decomposition

$$q + p_i = \sum_{\alpha=1}^{4} x_\alpha \ e_\alpha^{(ijk\cdots)}$$

The Shape of Residues

cut/legs	basis		Δ-variables (ISP's)
	external (p_i)	auxiliary (v_i)	
5	4	0	μ^2
4	3	1	$\mu^2, \quad q \cdot v_1$
3	2	2	$\mu^2, \quad q \cdot v_i \ (i = 1, 2)$
2	1	3	$\mu^2, \quad q \cdot v_i \ (i = 1, \ldots, 3)$
1	0	4	$\mu^2, \quad q \cdot v_i \ (i = 1, \ldots, 4)$

- ISP's = Irreducible Scalar Products:
  - q-components which can variate under cut-conditions
  - spurious: vanishing upon integration
  - non-spurious: non-vanishing upon integration $\Rightarrow$ MI's
- @ 1-Loop
  - $(q \cdot p_i)$ are ALL reducible
  - ISP's could be chosen to be ALL spurious   Pittau, de l'Aguila
  - n-ple cut identifies an n-point diagram

In general the residue of an m-point function is a a multivariate polynomial in μ^2 and the ISP's characterizing the residue. Each monomial has to be irreducible and its maximum rank has to be at most $(m+r-n)$. In the following we list the irreducible monomial entering each cut. For later convenience we give the decomposition of $g^{\mu\nu}$ in terms of the basis (2.6) and of the vectors (2.7),

$$g^{\mu\nu} = (e_1^\mu e_2^\nu + e_2^\mu e_1^\nu) - (e_3^\mu e_4^\nu + e_4^\mu e_3^\nu) \ , \tag{6.1}$$

$$g^{\mu\nu} = (e_1^\mu e_2^\nu + e_2^\mu e_1^\nu) + \frac{v^\mu \ v^\nu}{v^2} + \frac{v_\perp^\mu \ v_\perp^\nu}{v_\perp^2} \ . \tag{6.2}$$

- *Quintuple cut*, $(ijk\ell m)$ – The only irreducible monomial is μ^2. Indeed the residue of the quintuple cut does not have ISP's, thus the allowed monomials are $(\mu^2)^\alpha$. Moreover from eq. (6.1)

$$\begin{aligned}
(\mu^2)^\alpha &= \left[D_i + m_i^2 - p_i^2 - 2\,(q\cdot p_i) - q^2\right]^\alpha \\
&= \left[D_i + m_i^2 - p_i^2 - 2\,(q\cdot p_i) - 2\,(q\cdot e_1)(q\cdot e_2) + 2\,(q\cdot e_3)(q\cdot e_4)\right]^\alpha \\
&= \text{constant terms} + \text{RSP's} \ ,
\end{aligned} \tag{6.3}$$

  where the abbreviation "RSP's" means "reducible scalar products". This relation allows to express all the powers of μ^2 in terms of a particular one, $(\mu^2)^{\alpha_0}$. As in the renormalizable case we choose $\alpha_0 = 1$ in order to decouple the contribution of the pentagons from the computation of the coefficients of the boxes.

- *Quadruple cut*, $(ijk\ell)$ – The irreducible monomials are

$$(\mu^2)^\alpha \; ((q+p_i)\cdot v_\perp)^\beta \qquad \text{with } \beta = 0,1 \text{ and } \alpha = 0,1,2,\dots \;. \tag{6.4}$$

  Eq. (6.2) implies

$$\begin{aligned} ((q+p_i)\cdot v_\perp)^2 &= v_\perp^2 \left(q^2 - 2\,((q+p_i)\cdot e_1)((q+p_i)\cdot e_2) - \frac{((q+p_i)\cdot v)^2}{v^2} \right) \\ &= \text{constant terms} + \text{terms in } \mu^2 + \text{RSP's} \;, \end{aligned}$$

  therefore the terms with $\beta \geq 2$ are reducible.

- *Triple cut*, (ijk) – In this case the irreducible monomials are

$$(\mu^2)^\alpha \; ((q+p_i)\cdot e_{3,4})^\beta \qquad \text{with } \alpha,\, \beta = 0,1,2,\dots \;. \tag{6.5}$$

  The monomials containing both e_3 and e_4 are reducible. Indeed from eq. (6.1)

$$((q+p_i)\cdot e_3)\,((q+p_i)\cdot e_4) = \text{constant terms} + \text{terms in } \mu^2 + \text{RSP's} \;.$$

- *Double cut*, (ij) – The irreducible monomials are of the type

$$(\mu^2)^\alpha \ \ ((q+p_i)\cdot e_{3,4})^\beta \ \ ((q+p_i)\cdot e_2)^\gamma \qquad\qquad \text{with } \alpha,\beta,\gamma = 0,1,2,\ldots \ . \tag{6.6}$$

As in the previous case, the monomials depending on both e_3 and e_4 are reducible.

- *Single cut*, (i) – The irreducible monomials read as follows

$$(\mu^2)^\alpha \ \ ((q+p_i)\cdot e_{1,2})^\beta \ \ ((q+p_i)\cdot e_3)^\gamma \, ((q+p_i)\cdot e_4)^\delta \ \text{with } \alpha,\beta,\gamma,\delta = 0,1,\ldots \tag{6.7}$$

Eq. (6.1) allows to write

$$\begin{aligned}((q+p_i)\cdot e_1)\,((q+p_i)\cdot e_2) \ &= \ ((q+p_i)\cdot e_3)\,((q+p_i)\cdot e_4) \\ &\quad + \text{constant terms} + \text{terms in } \mu^2 + \text{RSP's} \ .\end{aligned}$$

Therefore the terms containing both e_1 and e_2 do not enter the parametrization of the residue.

The functions $\Delta(q,\mu^2)$ are parametrized in terms of the basis (2.6) and of the vectors (2.7):

$$\Delta_{ijk\ell m}(q,\mu^2) = c_{5,0}^{(ijk\ell m)}\;\mu^2\;, \tag{2.8}$$

$$\Delta_{ijk\ell}(q,\mu^2) = \Delta^R_{ijk\ell}(q,\mu^2) + c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)}\mu^2 + c_{4,4}^{(ijk\ell)}\mu^4\;, \tag{2.9}$$

$$\Delta_{ijk}(q,\mu^2) = \Delta^R_{ijk}(q,\mu^2) + c_{3,0}^{(ijk)} + c_{3,7}^{(ijk)}\mu^2\;, \tag{2.10}$$

$$\Delta_{ij}(q,\mu^2) = \Delta^R_{ij}(q,\mu^2) + c_{2,0}^{(ij)} + c_{2,9}^{(ij)}\mu^2\;, \tag{2.11}$$

$$\begin{aligned}\Delta_{i}(q,\mu^2) &= c_{1,0}^{(i)} + c_{1,1}^{(i)}((q+p_i)\cdot e_1) + c_{1,2}^{(i)}((q+p_i)\cdot e_2) \\ &\quad + c_{1,3}^{(i)}((q+p_i)\cdot e_3) + c_{1,4}^{(i)}((q+p_i)\cdot e_4)\;.\end{aligned} \tag{2.12}$$

For later convenience, we define the *reduced* polynomials Δ^R as,

$$\Delta^R_{ijk\ell}(q,\mu^2) = \left(c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)}\;\mu^2\right)(q+p_i)\cdot v_\perp\;, \tag{2.13}$$

$$\begin{aligned}\Delta^R_{ijk}(q,\mu^2) &= \left(c_{3,1}^{(ijk)} + c_{3,8}^{(ijk)}\mu^2\right)(q+p_i)\cdot e_3 + \left(c_{3,4}^{(ijk)} + c_{3,9}^{(ijk)}\mu^2\right)(q+p_i)\cdot e_4 \\ &\quad + c_{3,2}^{(ijk)}((q+p_i)\cdot e_3)^2 + c_{3,5}^{(ijk)}((q+p_i)\cdot e_4)^2 \\ &\quad + c_{3,3}^{(ijk)}((q+p_i)\cdot e_3)^3 + c_{3,6}^{(ijk)}((q+p_i)\cdot e_4)^3\;,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\Delta^R_{ij}(q,\mu^2) &= c_{2,1}^{(ij)}(q+p_i)\cdot e_2 + c_{2,2}^{(ij)}((q+p_i)\cdot e_2)^2 \\ &\quad + c_{2,3}^{(ij)}(q+p_i)\cdot e_3 + c_{2,4}^{(ij)}((q+p_i)\cdot e_3)^2 \\ &\quad + c_{2,5}^{(ij)}(q+p_i)\cdot e_4 + c_{2,6}^{(ij)}((q+p_i)\cdot e_4)^2 \\ &\quad + c_{2,7}^{(ij)}((q+p_i)\cdot e_2)((q+p_i)\cdot e_3) + c_{2,8}^{(ij)}((q+p_i)\cdot e_2)((q+p_i)\cdot e_4)\,.\end{aligned} \tag{2.15}$$

Spurious terms of the boxes

Again, we focus on the cut (0123) assuming $p_0 = 0$. The spurious terms are

$$\int d^d\bar{q}\frac{(q\cdot v_\perp)}{D_0D_1D_2D_3}, \qquad \int d^d\bar{q}\frac{\mu^2(q\cdot v_\perp)}{D_0D_1D_2D_3}. \tag{6}$$

We consider the Lorentz decompositions

$$\begin{aligned} I^\mu_{0123} &\equiv \int d^d\bar{q}\frac{\bar{q}^\mu}{D_0D_1D_2D_3} = \sum_{j=1}^{3} p_j^\mu D_j \\ I^{\mu\nu\rho}_{0123} &\equiv \int d^d\bar{q}\frac{\bar{q}^\mu\bar{q}^\nu\bar{q}^\rho}{D_0D_1D_2D_3} = \sum_{i,j,k=1}^{3} p_i^\mu p_j^\nu p_k^\rho D_{ijk} + \sum_{i=1}^{3}(\hat{g}^{\mu\nu}p_i^\rho + \hat{g}^{\mu\rho}p_i^\nu + \hat{g}^{\rho\nu}p_i^\mu)D_{00i} \end{aligned}$$

The first spurious integral vanish since $p_i \cdot v_\perp = 0$ and

$$\int d^d\bar{q}\frac{(q\cdot v_\perp)}{D_0D_1D_2D_3} = \sum_{j=1}^{3}(v_\perp\cdot p_j)D_j = 0. \tag{7}$$

The integral with μ^2 in the numerator vanishes as well. We can show it using

$$\int d^d\bar{q}\frac{\mu^2(q\cdot v_\perp)}{D_0D_1D_2D_3} = -\tilde{g}_{\mu\nu}(v_\perp)_\rho\, I^{\mu\nu\rho}_{0123}, \tag{8}$$

and the properties (1).

Spurious terms of the triangles

For simplicity we consider the cut (012) and we assume $p_0 = 0$. The generalization is straightforward. The spurious terms are

$$\int d^d\bar{q}\frac{(q\cdot e_{3,4})}{D_0D_1D_2}, \quad \int d^d\bar{q}\frac{(q\cdot e_{3,4})^2}{D_0D_1D_2}, \quad \int d^d\bar{q}\frac{(q\cdot e_{3,4})^3}{D_0D_1D_2}, \int d^d\bar{q}\frac{\mu^2(q\cdot e_{3,4})}{D_0D_1D_2}, \tag{4}$$

We consider the Lorentz decompositions

$$\begin{aligned}
I^{\mu}_{012} &\equiv \int d^d\bar{q}\frac{\bar{q}^\mu}{D_0D_1D_2} = \sum_{j=1}^{2} p_j^\mu C_j \\
I^{\mu\nu}_{012} &\equiv \int d^d\bar{q}\frac{\bar{q}^\mu\bar{q}^\nu}{D_0D_1D_2} = \sum_{i,j=1}^{2} p_i^\mu p_j^\nu C_{ij} + \hat{g}^{\mu\nu}C_{00} \\
I^{\mu\nu\rho}_{012} &\equiv \int d^d\bar{q}\frac{\bar{q}^\mu\bar{q}^\nu\bar{q}^\rho}{D_0D_1D_2} = \sum_{i,j,k=1}^{2} p_i^\mu p_j^\nu p_k^\rho C_{ijk} + \sum_{i=1}^{2}(\hat{g}^{\mu\nu}p_i^\rho + \hat{g}^{\mu\rho}p_i^\nu + \hat{g}^{\rho\nu}p_i^\mu)C_{00i}
\end{aligned}$$

Using the relations $e_{3,4}^2 = (e_{3,4},\cdot p_1) = (e_{3,4},\cdot p_2) = 0$, it is easy to realize that

$$\begin{aligned}
\int d^d\bar{q}\frac{(q\cdot e_{3,4})}{D_0D_1D_2} &= \sum_{j=1}^{2}(e_{3,4}\cdot p_j)C_j = 0 \\
\int d^d\bar{q}\frac{(q\cdot e_{3,4})^2}{D_0D_1D_2} &= \sum_{i,j=1}^{2}(e_{3,4}\cdot p_i)(e_{3,4}\cdot p_j)C_{ij} + e_{3,4}^2C_{00} = 0, \\
\int d^d\bar{q}\frac{(q\cdot e_{3,4})^3}{D_0D_1D_2} &= \sum_{i,j,k=1}^{2}(e_{3,4}\cdot p_i)(e_{3,4}\cdot p_j)(e_3\cdot p_k)C_{ijk} \\
&+ \sum_{i=1}^{2} 3e_{3,4}^2(e_{3,4}\cdot p_i)C_{00i} = 0.
\end{aligned}$$

The integrals with μ^2 in the numerator vanish. It can be easily shown using

$$\int d^d\bar{q}\frac{\mu^2(q\cdot e_{3,4})}{D_0D_1D_2} = -\tilde{g}_{\mu\nu}(e_{3,4})_\rho\, I^{\mu\nu\rho}_{012} \tag{5}$$

and the properties (1).

Spurious terms of the bubbles

For simplicity we consider the cut (01) and we assume $p_0 = 0$. The generalization is straightforward. The spurious terms are

$$\int d^d\bar{q}\frac{(q\cdot e_{3,4})}{D_0D_1}, \quad \int d^d\bar{q}\frac{(q\cdot e_{3,4})^2}{D_0D_1}, \quad \int d^d\bar{q}\frac{(q\cdot e_{3,4})(q\cdot e_2)}{D_0D_1}. \tag{3}$$

Using the Lorentz decompositions

$$\begin{aligned} I_{01}^{\mu} &\equiv \int d^d\bar{q}\frac{\bar{q}^\mu}{D_0D_1} = p_1^\mu B_1 \\ I_{01}^{\mu\nu} &\equiv \int d^d\bar{q}\frac{\bar{q}^\mu\bar{q}^\nu}{D_0D_1} = \hat{g}^{\mu\nu}B_{00} + p_1^\mu p_1^\nu B_{11} \end{aligned}$$

we get

$$\begin{aligned} \int d^d\bar{q}\frac{(q\cdot e_{3,4})}{D_0D_1} &= (p_1\cdot e_{3,4})B_1 = 0, \\ \int d^d\bar{q}\frac{(q\cdot e_{3,4})^2}{D_0D_1} &= e_{3,4}^2 B_{00} + (p_1\cdot e_{3,4})(p_1\cdot e_{3,4})B_{11} = 0, \\ \int d^d\bar{q}\frac{(q\cdot e_2)(q\cdot e_{3,4})}{D_0D_1} &= (e_2\cdot e_{3,4})B_{00} + (p_1\cdot e_2)(p_1\cdot e_{3,4})B_{11} = 0. \end{aligned}$$

The integrals vanish since $e_{3,4}^2 = (e_2\cdot e_{3,4}) = (p_1\cdot e_{3,4}) = 0$

Spurious terms of the tadpoles

The spurious terms are

$$\int d^d\bar{q}\frac{((q+p_i)\cdot e_a)}{D_i}, \qquad (a = 1, \cdots, 4). \tag{2}$$

The integrand is an odd function integrated over an even domain. Therefore the integrals vanish.

Neglecting terms of $\mathcal{O}(\epsilon)$, the one loop amplitude can be written in terms of master integrals and of the coefficients of $\Delta_{ijk\ell m}$, $\Delta_{ijk\ell}$, Δ_{ijk}, Δ_{ij}, and Δ_i,

$$\begin{aligned}
\mathcal{A}_n = & \sum_{i<j<k<\ell}^{n-1} \left\{ c_{4,0}^{(ijk\ell)} I_{ijk\ell} + c_{4,4}^{(ijk\ell)} I_{ijk\ell}[\mu^4] \right\} \\
& + \sum_{i<j<k}^{n-1} \left\{ c_{3,0}^{(ijk)} I_{ijk} + c_{3,7}^{(ijk)} I_{ijk}[\mu^2] \right\} \\
& + \sum_{i<j}^{n-1} \left\{ c_{2,0}^{(ij)} I_{ij} + c_{2,1}^{(ij)} I_{ij}[(q+p_i)\cdot e_2] + c_{2,2}^{(ij)} I_{ij}[((q+p_i)\cdot e_2)^2] + c_{2,9}^{(ij)} I_{ij}[\mu^2] \right\} \\
& + \sum_{i}^{n-1} c_{1,0}^{(i)} I_i \ ,
\end{aligned} \tag{2.16}$$

where

$$I_{i_1\cdots i_k}[\alpha] \equiv \int d^d\bar{q} \frac{\alpha}{D_{i_1}\cdots D_{i_k}}, \qquad I_{i_1\cdots i_k} \equiv I_{i_1\cdots i_k}[1]. \tag{2.17}$$

As already noted in [5, 8, 12], some of the terms appearing in the integrand decomposition (2.5) vanish upon integration. They are called *spurious* and do not contribute to the amplitude (2.16). Beside the scalar boxes, triangles, bubbles and tadpoles, the other master integrals are the linear and quadratic two-points functions [40, 41] and the integrals containing powers of μ^2 in the numerator. The latter can be traded with higher dimensional integrals [40, 41]

$$I_{i_1 \cdots i_k}[(\mu^2)^r f(q,\mu^2)] = \frac{1}{\pi^r} \prod_{\kappa=1}^{r} \left(\kappa - 3 + \frac{d}{2} \right) \int d^{d+2r}\bar{q} \frac{f(q,\mu^2)}{D_{i_1} \cdots D_{i_k}} \ . \tag{2.18}$$

As already noticed in [42], eq. (2.16) is free of scalar pentagons.

...in our example...

$$\mathcal{A}_n = \int A(q) \quad \equiv \int \frac{\Delta_{0123}(q)}{D_0 D_1 D_2 D_3} + \int \frac{\Delta_{012}(q)}{D_0 D_1 D_2} + \int \frac{\Delta_{01}(q)}{D_0 D_1} + \int \frac{\Delta_0(q)}{D_0}$$

Master Integrals in the 4D-decomposition