Topological Gravity and Matrix Models

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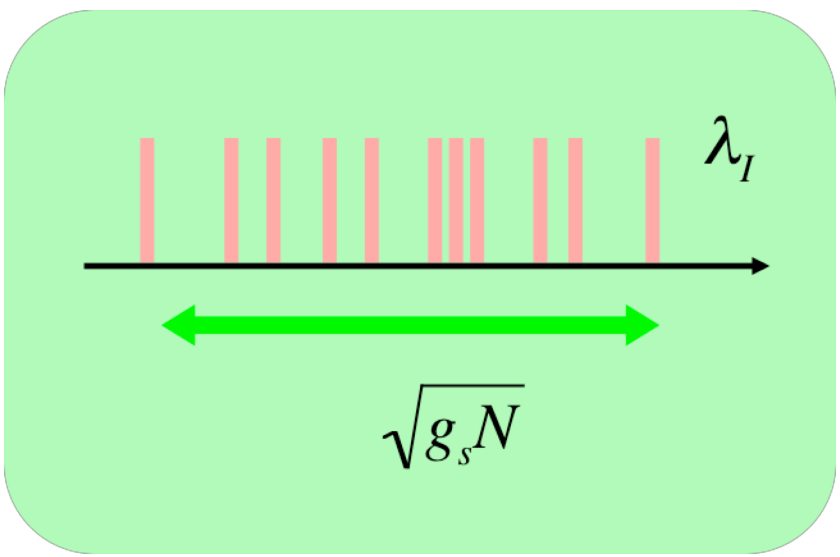
Based on work with E. Witten, arXiv:1804.03275 and in progress

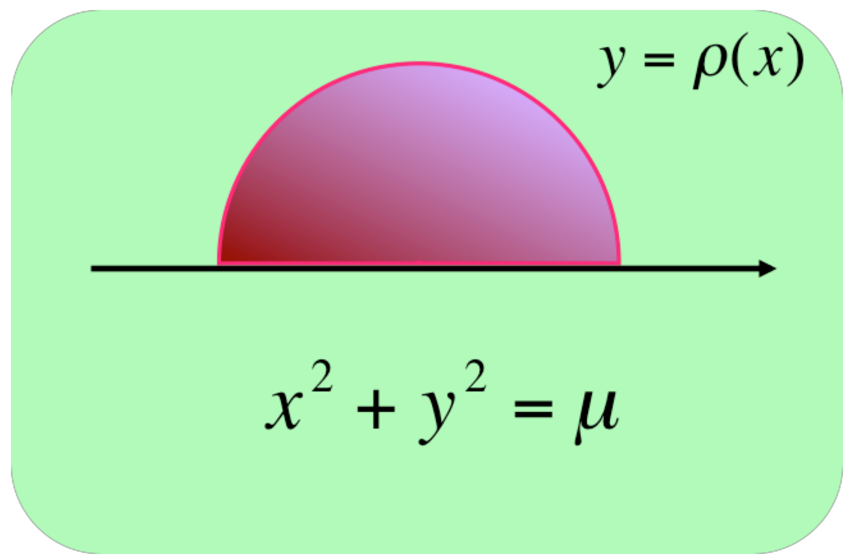
Wigner's Random Matrix Model

$$\lim_{N\to\infty} Z_N, \qquad Z_N = \frac{1}{\mathrm{Vol}\,U(N)} \int_{N\times N} d\Phi \cdot e^{-\mathrm{Tr}\,\Phi^2/g_s}$$

Eigenvalue distribution in 't Hooft limit

$$N \to \infty, \qquad g_s \to 0, \qquad N g_s = \mu = \mathit{fixed}$$

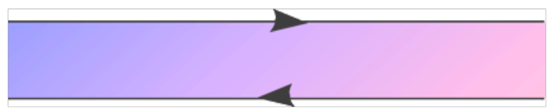




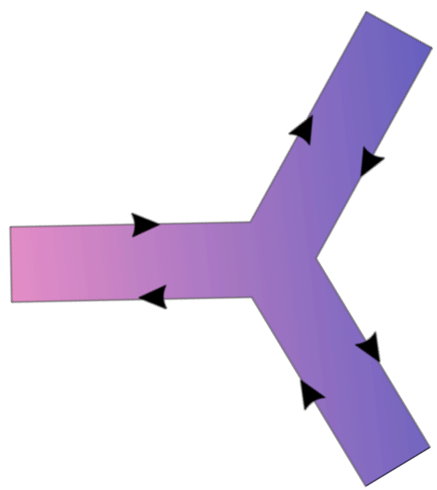
Matrix Models and 2d Gravity

$$\int d\Phi \; e^{-\frac{1}{g_s}\mathrm{Tr}(\Phi^2+\Phi^3)}$$

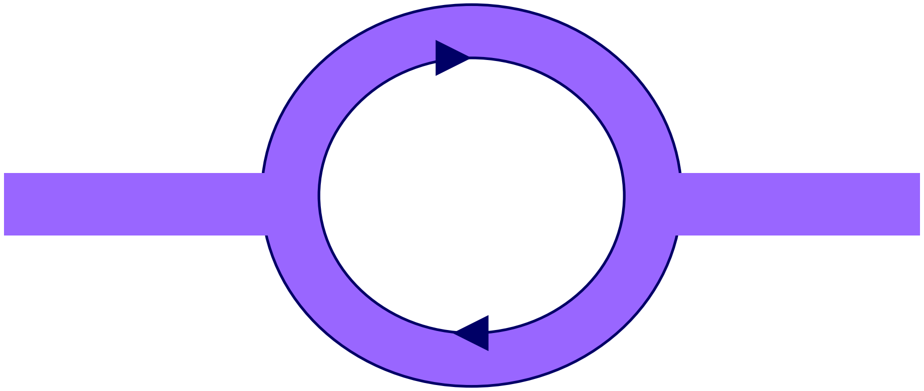
Φ_{ij}



$\mathrm{Tr}\ \Phi^3$

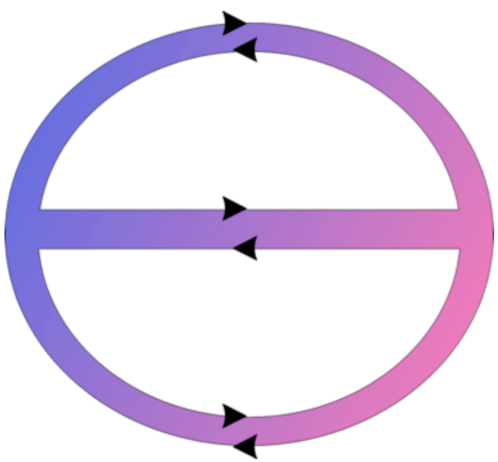


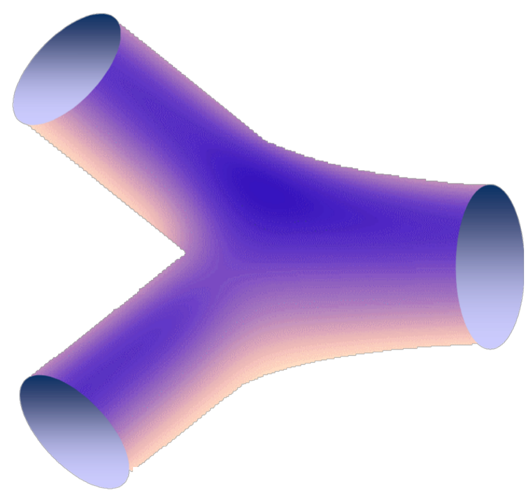
$\sqrt{g_s}$



$$g_s N = \mu = fixed$$

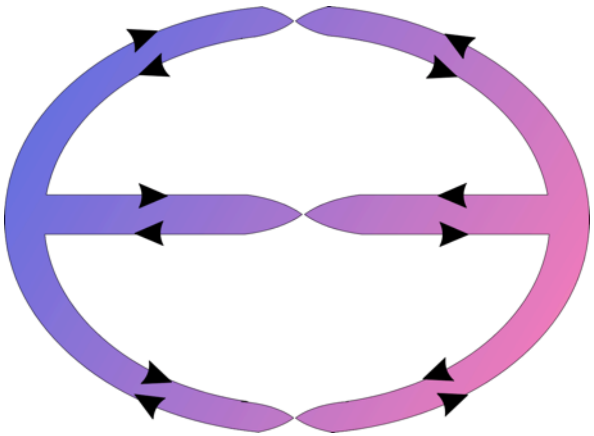
Planar diagrams

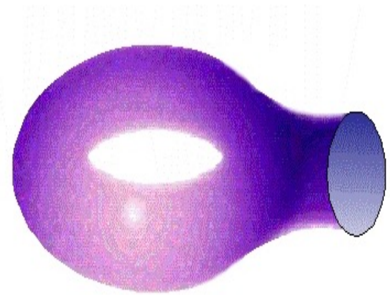




$$N^2$$

Non-planar diagrams

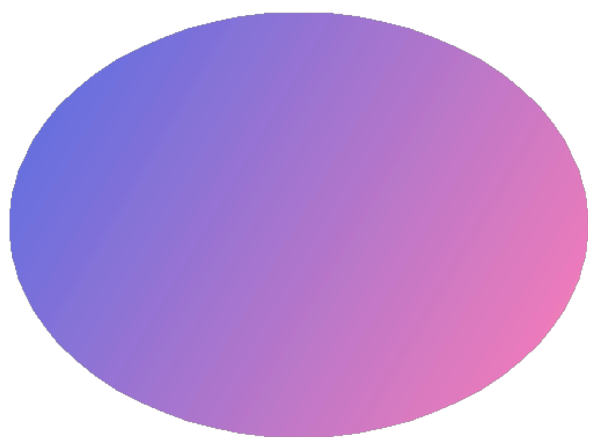




$g=1$

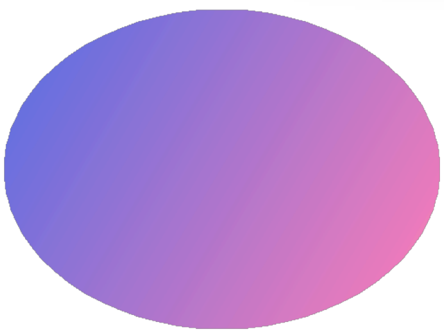
N^{2-2g}

Sum of diagrams

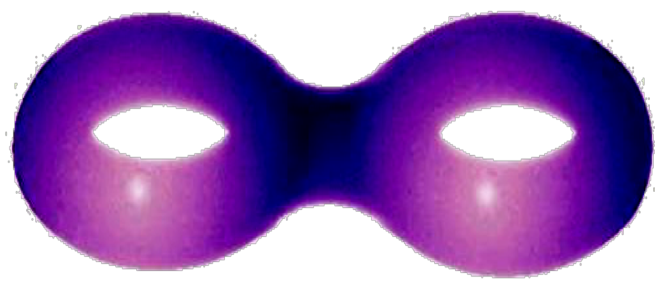


String Partition Function

$$Z = \exp \sum_{g \geq 0} g_s^{2g-2} F_g(\mu)$$

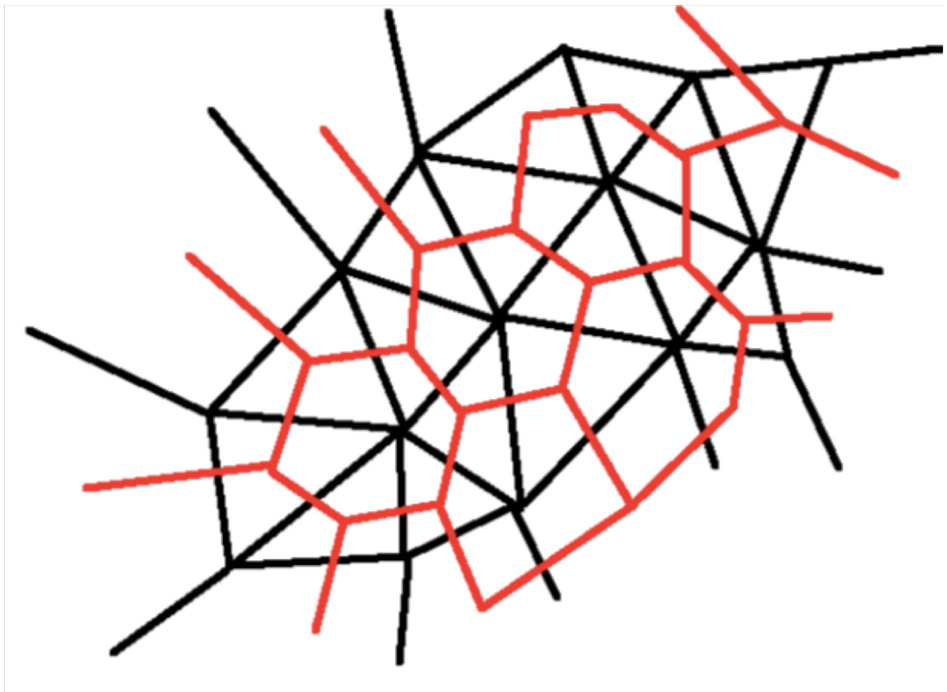


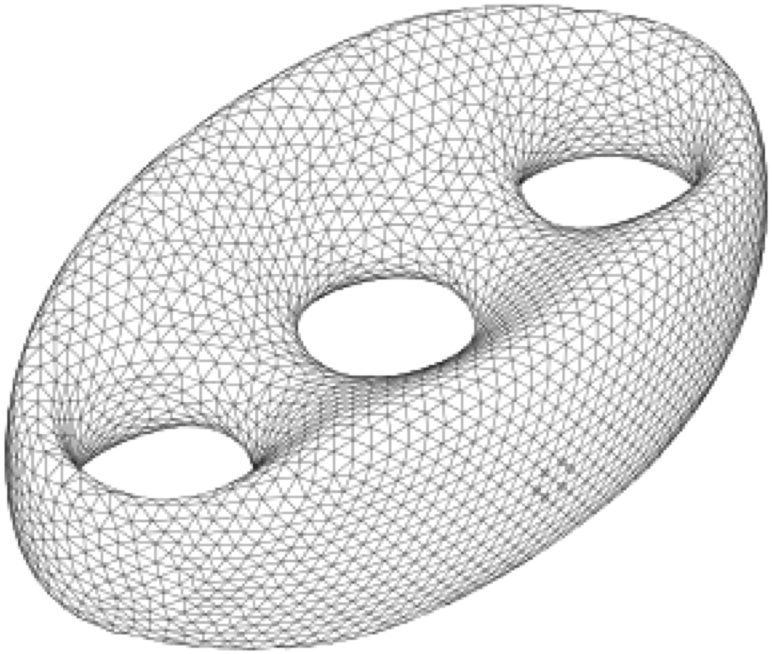
ribbon diagrams
1/N expansion



genus g
string surface

Triangulations of surface





Dual graph

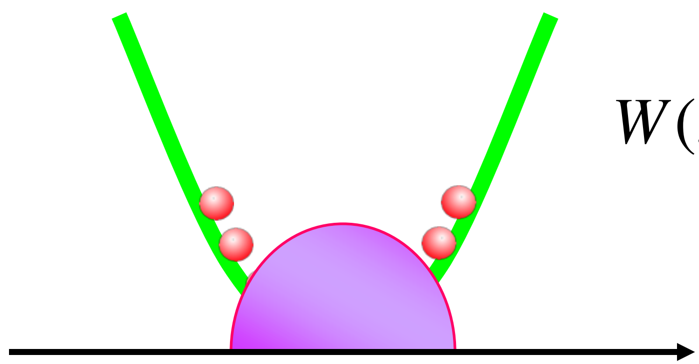
Limit of large number of vertices

Eigenvalue Dynamics

$$Z_{matrix} = \int d^N\lambda \cdot \prod (\lambda_I - \lambda_J)^2 \cdot e^{-\sum_I \lambda_I^2 / g_s}$$

Effective action (repulsive Coulomb force)

$$S_{eff} = \sum_I \lambda_I^2 - 2g_s \sum_{I<J} \log(\lambda_I - \lambda_J)$$



$$W(x) = x^2$$

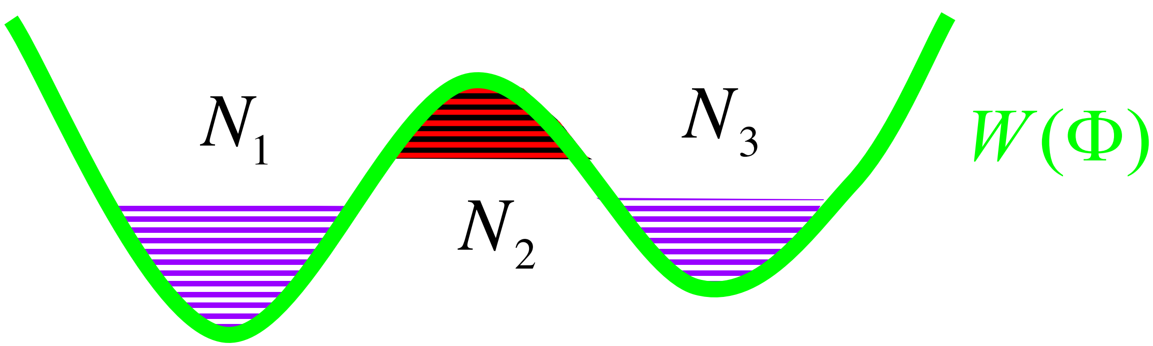
General Matrix Model

$$Z_{matrix} = \frac{1}{\mathrm{Vol}\, U(N)} \int_{N\times N} d\Phi \cdot e^{\mathrm{Tr}\, W(\Phi)/g_s}$$

’t Hooft limit

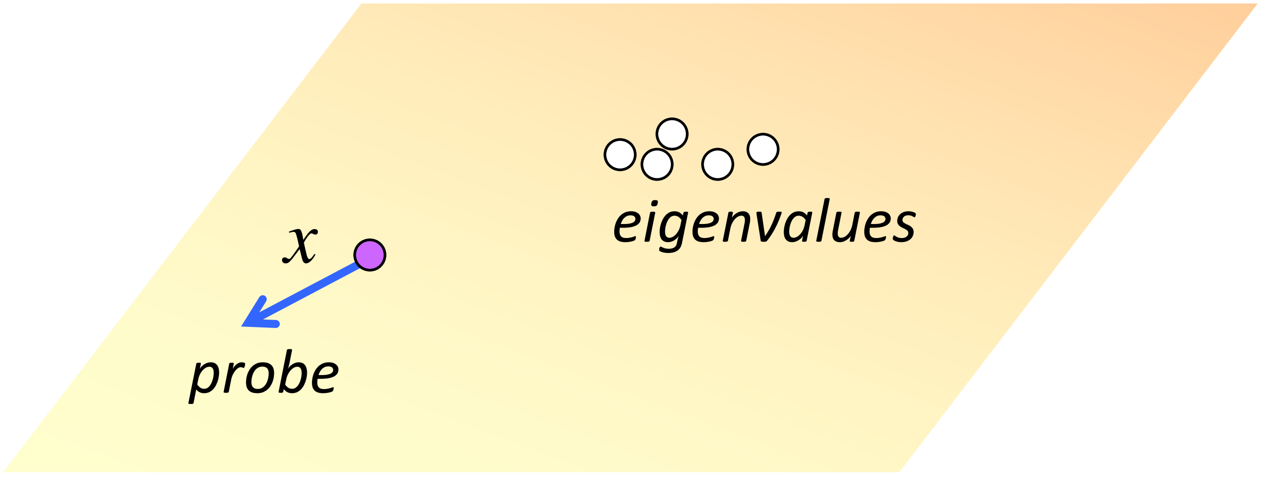
$$N_I \to \infty, \;\; g_s \to 0, \;\; N_I g_s = \mu_I = fixed$$

Filling fractions (perturbative expansion)



Resolvent

$$y dx = dS_{eff} = dW(x) + 2g_s \mathrm{Tr} \frac{dx}{\Phi - x}$$



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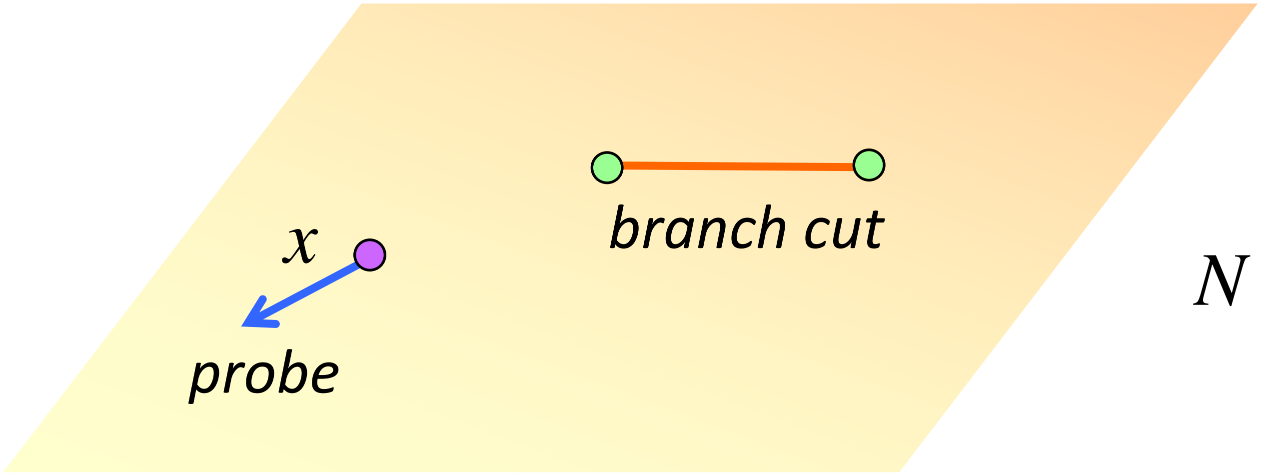
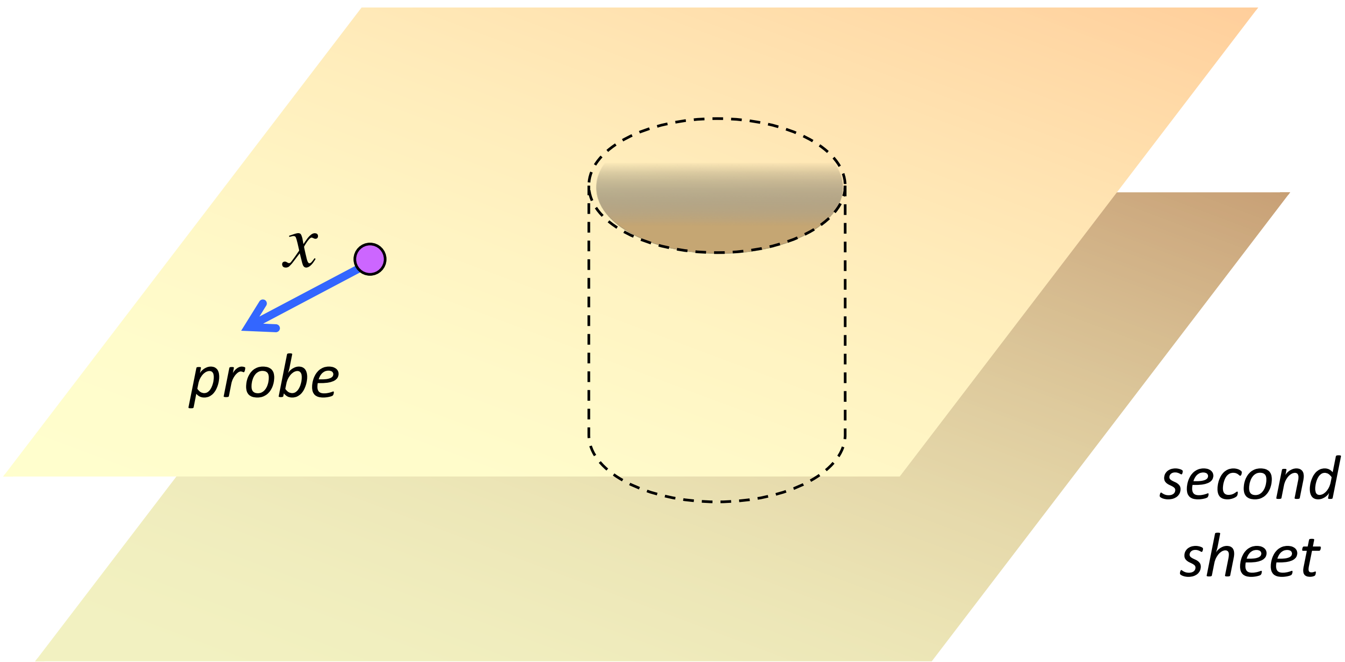


$N \to \infty$

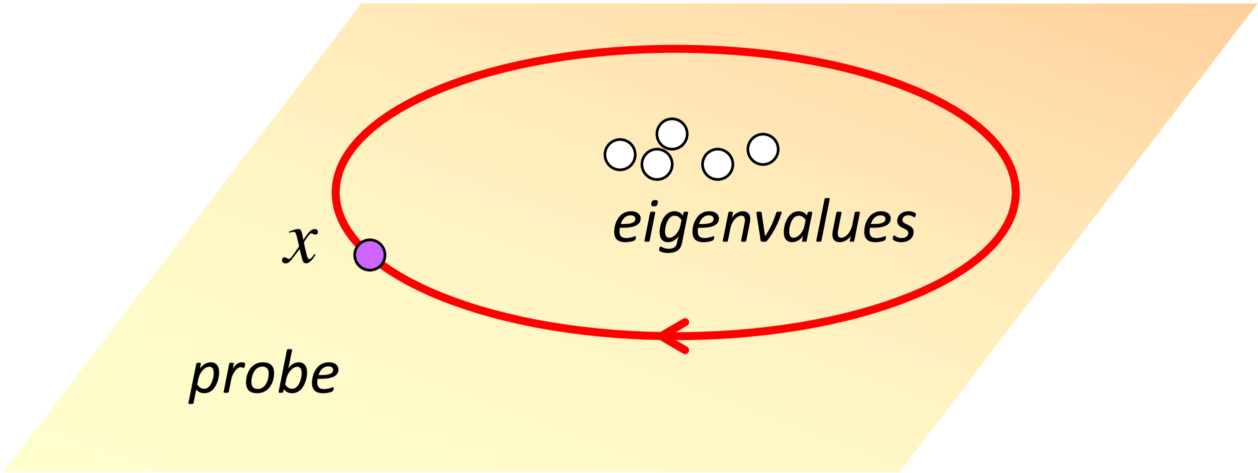
Effective Geometry

$$y^2 = W_n'(x)^2 + f_{n-1}(x) = P_{2n}(x)$$



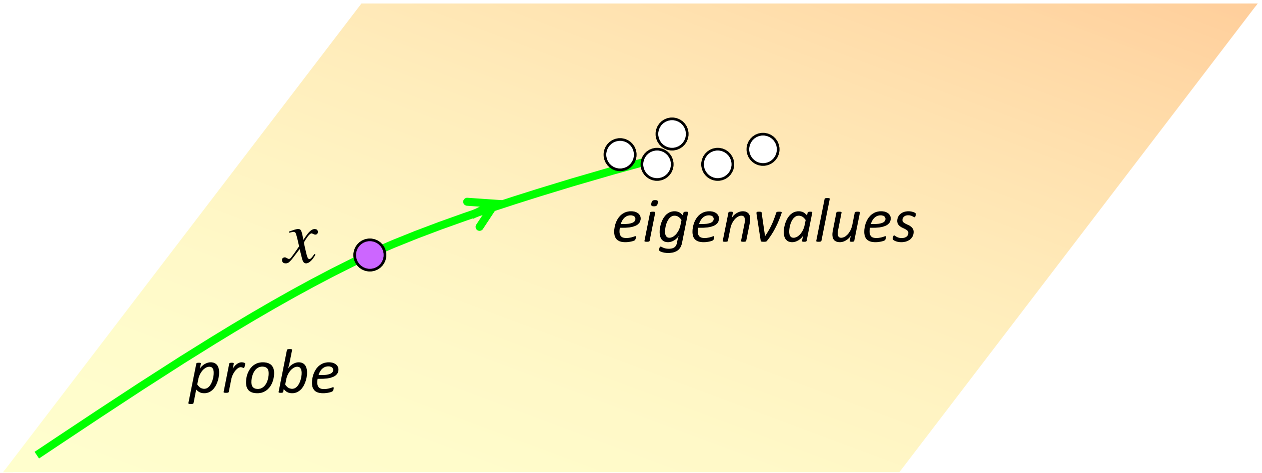
Periods (A)

$$\oint_{A_I} y\,dx = g_s N_I = \mu_I$$



Periods (B)

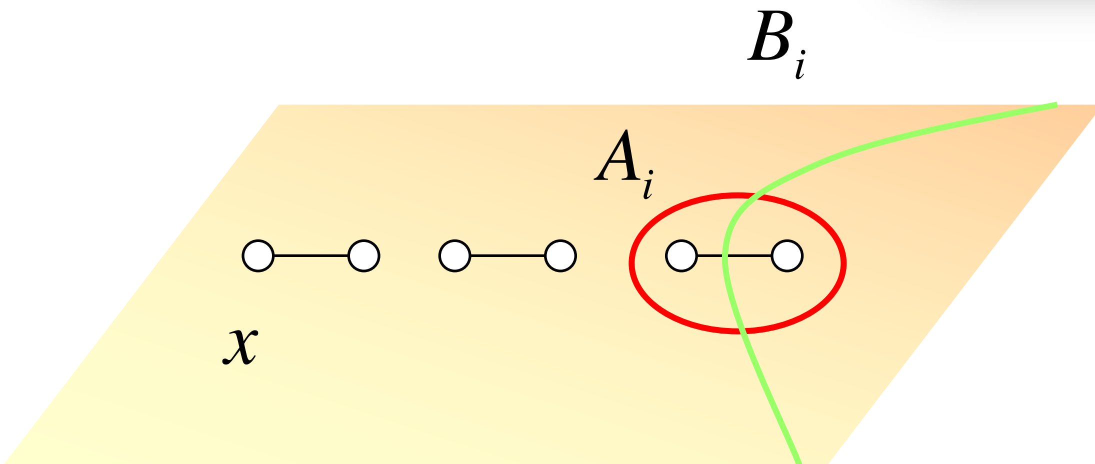
$$\int_{B_I} y\,dx = \frac{\partial F}{\partial \mu_I}$$

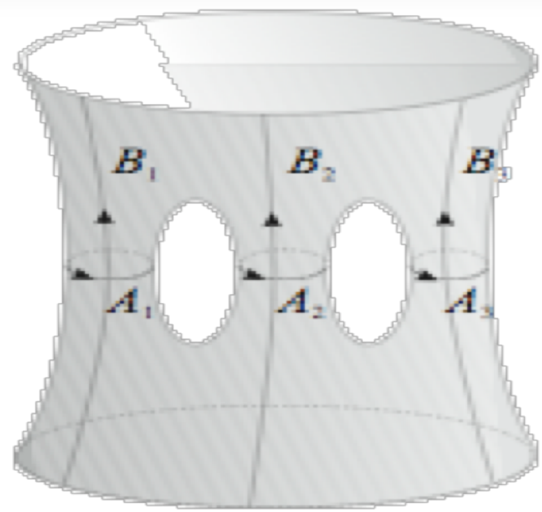


Spectral Curve

Hyperelliptic curve

$$\Sigma : y^2 = W'(x)^2 + f(x)$$





Quantum invariants of $\Sigma, \quad \omega = ydx$

$$\mu_i = \oint_{A_i} ydx \qquad \frac{\partial F_0}{\partial \mu_i} = \oint_{B_i} ydx$$

Spectral Curve

General phenomenon in matrix models (chains, external fields, beta-ensembles, SO/Sp, etc). Spectral curve

$$P(x,y) = 0$$

With meromorphic one-form

$$\omega = y dx$$

Emergent geometry of large *N* systems

Complex Curves in Phase Space

algebraic curve = level set

$$\Sigma : H(x,y)=0$$

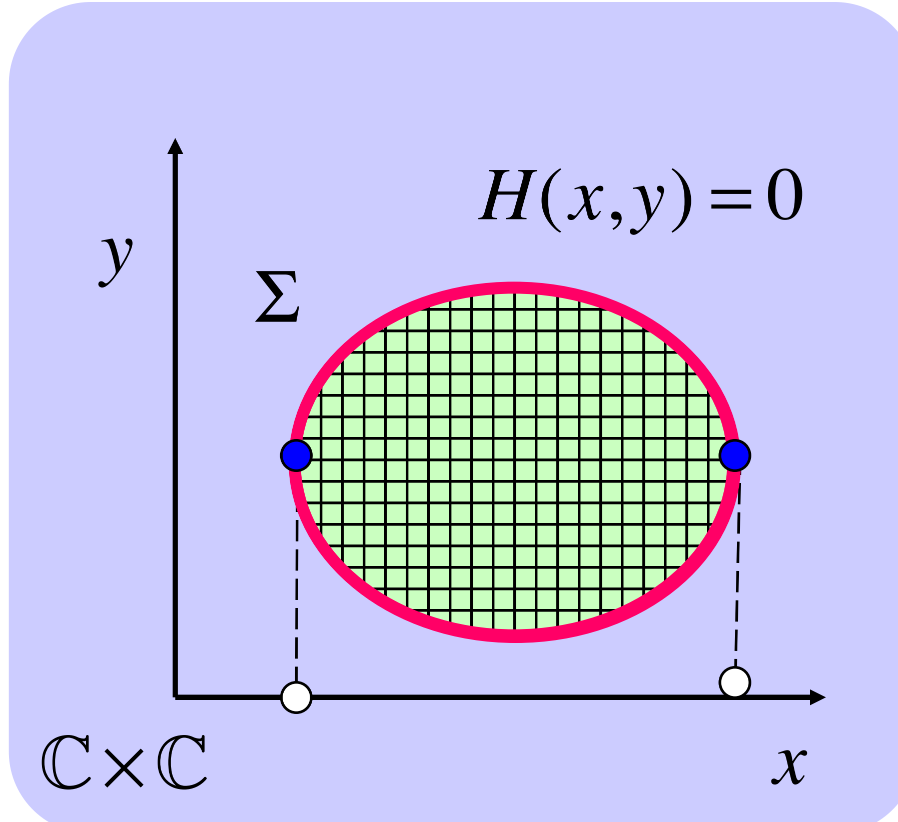
Hamilton-Jacobi theory

$$y = p(x)$$

branch pts = turning pts

Liouville form $\omega = y\,dx$

action $$S(x)=\int_{x_*}^{x}\omega$$



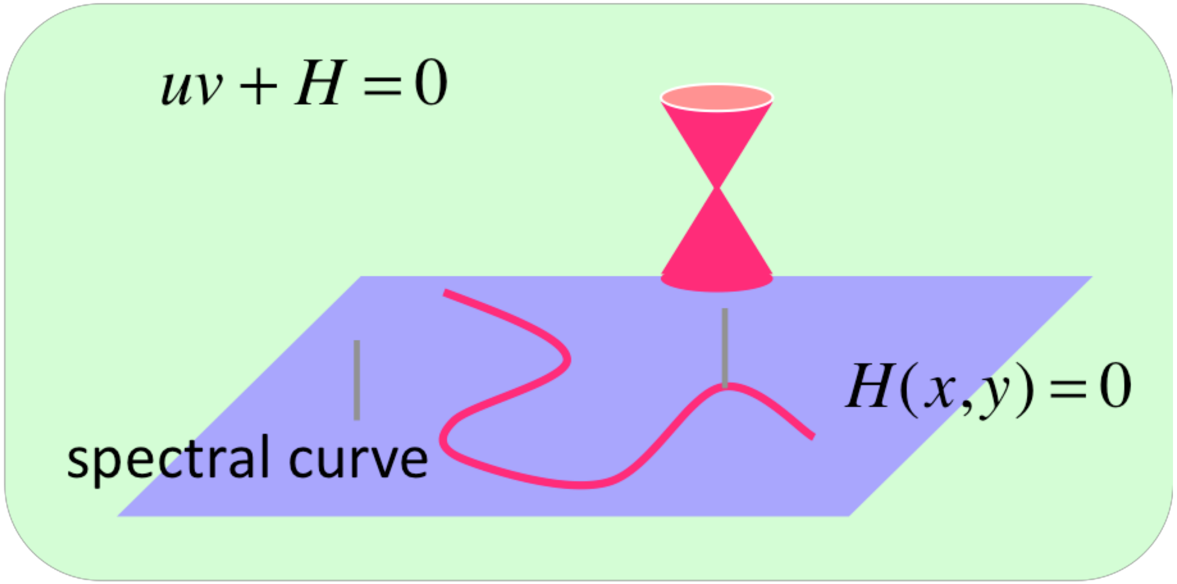
BS Quantization

$$dx \wedge dy \sim \hbar = g_s$$

Topological string on hypersurface CY_3

Calabi-Yau hypersurface in $\mathbb{C}^4$

$$X : uv + H(x,y) = 0$$

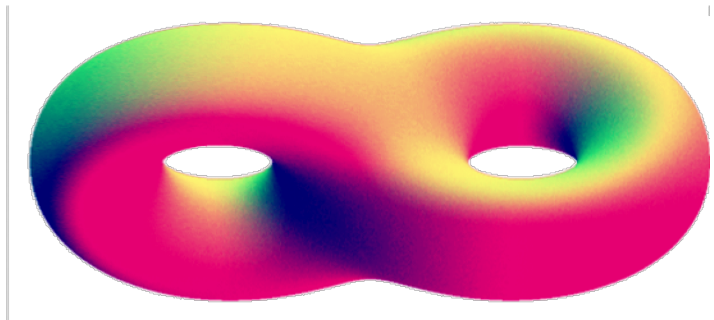


Period maps

$$\Omega = \frac{du}{u} \wedge dx \wedge dy \rightarrow \omega = y dx$$

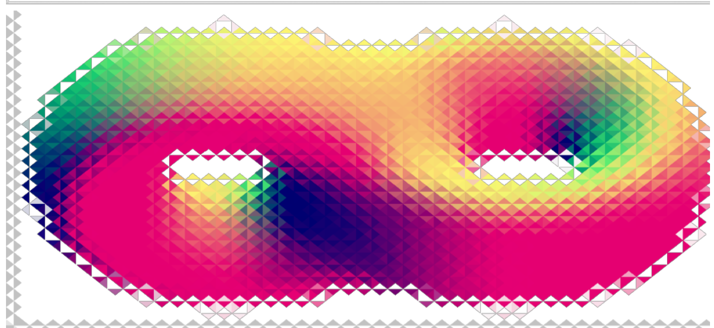
Quantum Curves

$N \to \infty$



$F_0(\mu)$

$O(1/N)$



$$\sum_{g\geq 0} N^{2-2g} F_g(\mu)$$

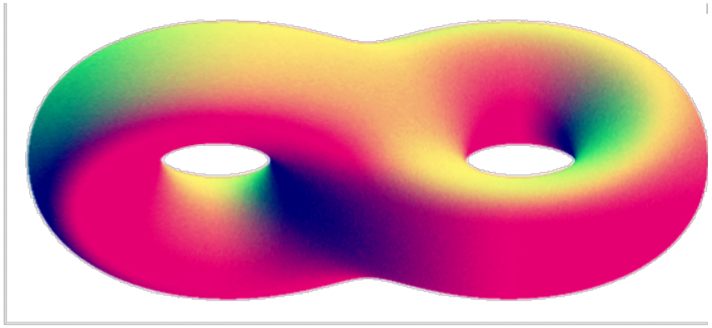
N finite

non-perturbative

Matrix models and chiral CFT

Eigenvalue density/matrix resolvent = collective boson field

$$\partial\varphi(x) = \mathrm{Tr}\frac{1}{x-\Phi}$$



$$\varphi(x)$$

Free scalar field

$$Z_{matrix} = \int D\varphi \cdot e^{-S[\varphi]}, \quad S = \int (\partial\varphi)^2 + O(\quad g_s)$$

Observables

$$\partial\varphi = \sum \mathrm{Tr}\Phi^n x^{-n-1} \qquad S + \sum t_n \mathrm{Tr}\Phi^n$$

Loop Equations & Virasoro Constraints

Invariance under diffeomorphism $x \longrightarrow \tilde{x}(x)$

Generated by stress-tensor

$$T(x) = \tfrac{1}{2}\left(\partial\varphi\right)^2(x)$$

Loop equations

$$\left\langle T(x)\right\rangle = W'(x)^2 + f(x)$$

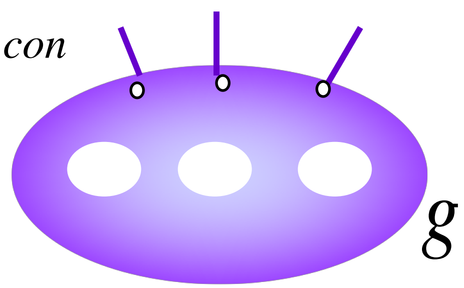
Eynard-Orantin: All-Genus Solution

Recursion relations for correlation functions, completely geometric

$$W(x_1,\cdots,x_n) = \left\langle \mathrm{Tr}\frac{1}{x_1-\Phi}\cdots\mathrm{Tr}\frac{1}{x_n-\Phi}\right\rangle_{con}$$

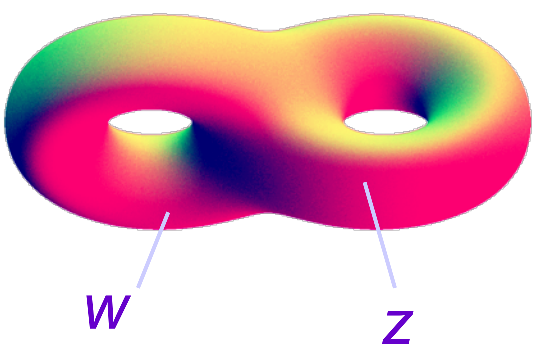
$$= \langle \partial\varphi(x_1)\cdots\partial\varphi(x_n)\rangle_{con}$$

$$= \sum_g g_s^{\,2g-2}\, W_g$$

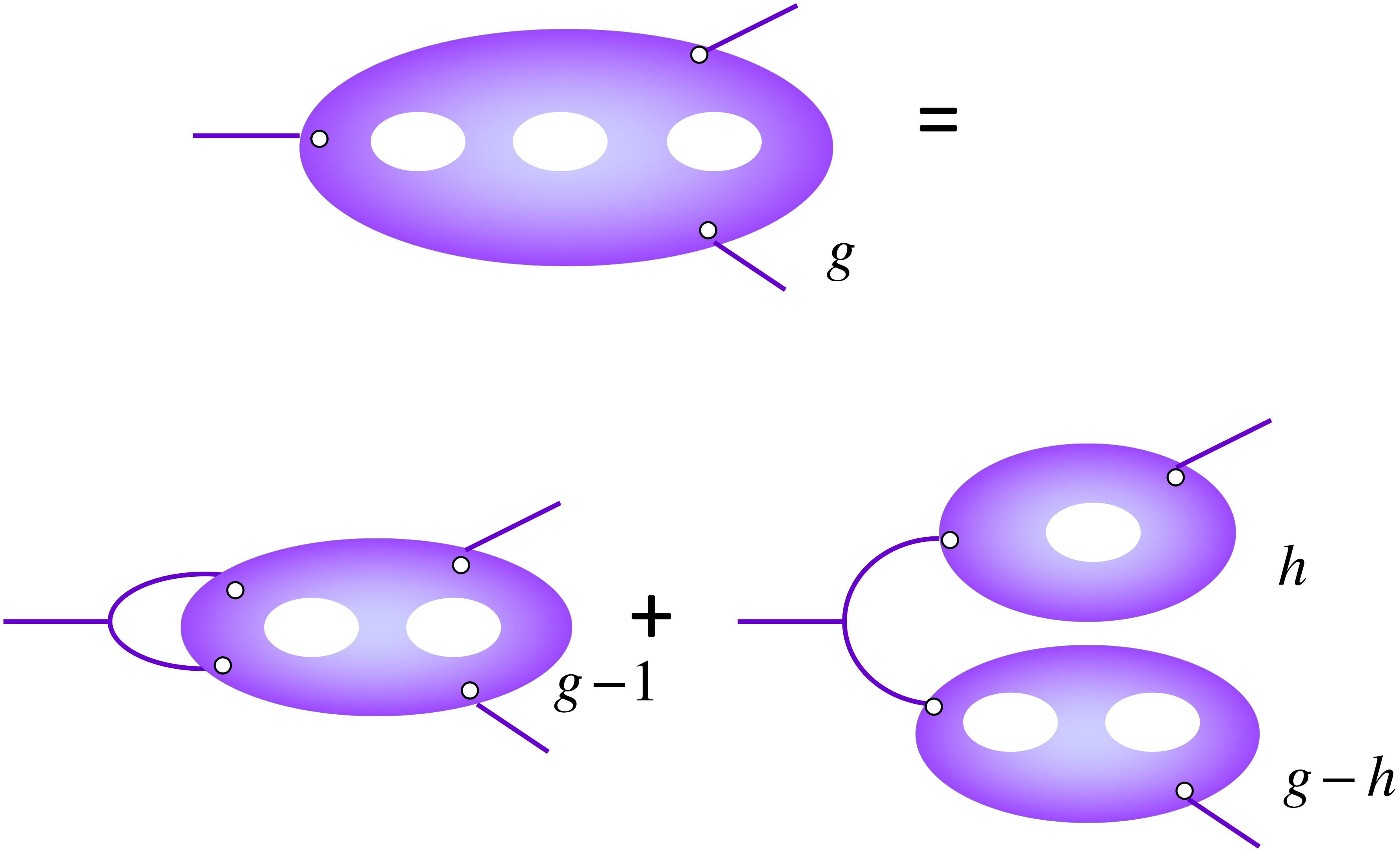


Lowers genus, adds more insertions. Reduces to Bergmann kernel

$$W_0(w,z) = B(w,z)dwdz$$

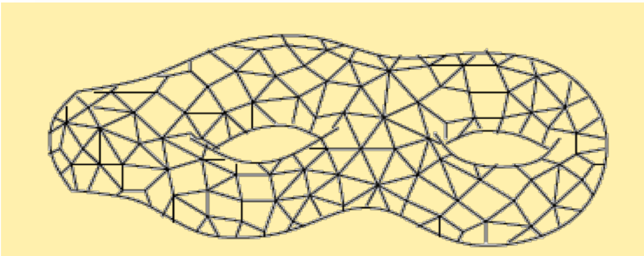


Recursion Relations



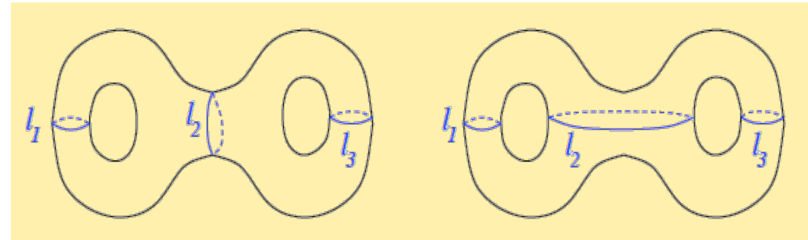
Quantum Spectral Curves

Random surfaces



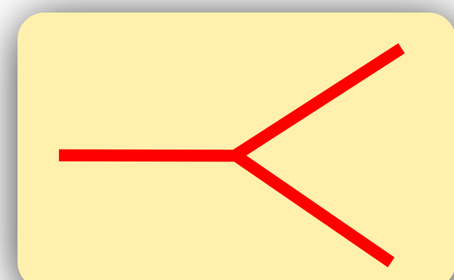
$$y^p = x^q$$

Geometry of $\mathcal{M}_g$



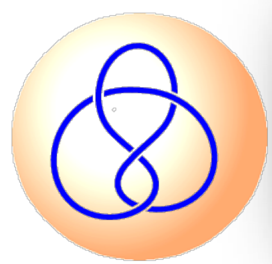
$$y = \sin\sqrt{x}$$

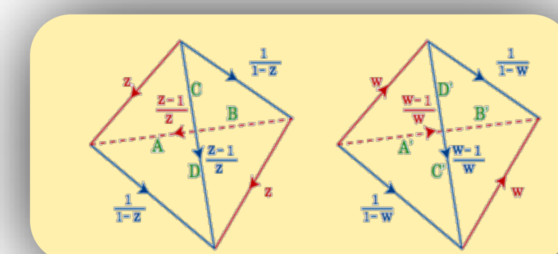
Toric Calabi-Yau



$$e^x + e^{-y} = 1$$

Hyperbolic knots





$$A(e^x, e^y) = 0$$

Double Scaling Limts

Random triangulations: 2d gravity coupled to (p,q) minimal CFT

$$P(x,y) = 0 \longrightarrow y^p = x^q + \dots$$

Pure 2d gravity (p,q) = (2,3)

$$S = \int \sqrt{g}R + \mu\sqrt{g} \qquad\qquad y^2 = x^3$$

Topological gravity (p,q) = (2,1)

$$y^2 = x$$

Topological Gravity

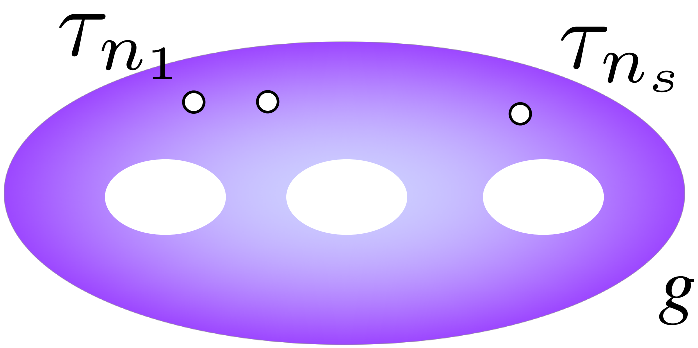
Moduli space of Riemann surfaces

$$\mathcal{M}_{g,s}, \qquad \dim \mathcal{M}_{g,s} = 3g - 3 + s$$

Observables

$$\tau_n = c_1(L)^n$$

Correlation functions



$$\langle \tau_{n_1} \cdots \tau_{n_s} \rangle_g = \int_{\mathcal{M}_{g,s}} c_1(L_1)^{n_1} \wedge \cdots \wedge c_1(L_s)^{n_s}$$

Maryam Mirzakhani's Work

Compute the volume of moduli space of surfaces

$$\mathcal{F}_g = \mathrm{vol}(\mathcal{M}_g)$$

Weil-Peterson metric: pick a constant negative curvature metric on the surface, equivalent to flat SL(2,R) connection.

$$\omega = \int_\Sigma \mathrm{Tr}(\delta A_1 \delta A_2)$$

Partition function

$$\mathcal{F}_g = \int_{\overline{\mathcal{M}_g}} e^{\omega}$$

Related to tautological classes

$$\omega = \pi_*(\tau_2)$$

Contact terms

$$\left\langle \exp\left(y\omega\right) \right\rangle = \left\langle \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k y^{k-1}}{(k-1)!} \tau_k \right) \right\rangle$$

Spectral curve (Eynard)

$$y = \frac{\sinh(\sqrt{\xi x})}{\sqrt{\xi}}$$

Jackiw-Teitelboim Gravity

Hyperbolic metrics

$$S = \int_{\Sigma} \phi \sqrt{g}(R+1) + \int_{\partial\Sigma} \phi_b K$$

Constant curvature metrics

$$R = -1$$

Dual to Schwarzian theory on the boundary, eqv SYK model

$$\int d\tau \ \{F, \tau\}$$

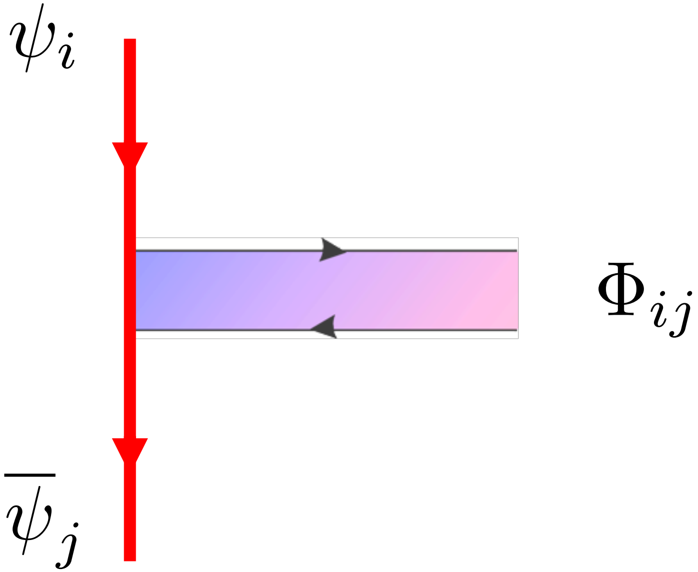
Witten-Stanford: $\rho(E) = \sinh \sqrt{E}$

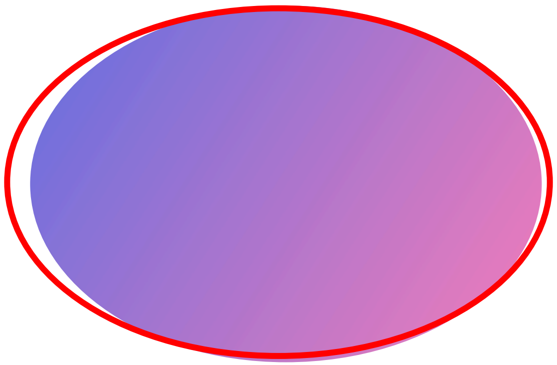
Open Strings

Introduce vector degrees of freedom

$$\int d\Phi d\psi d\overline{\psi}\; e^{\mathrm{Tr}W(\Phi)+\overline{\psi}(\Phi-z)\psi}$$

External quark line





Open Strings

Integrate out vector-valued variables

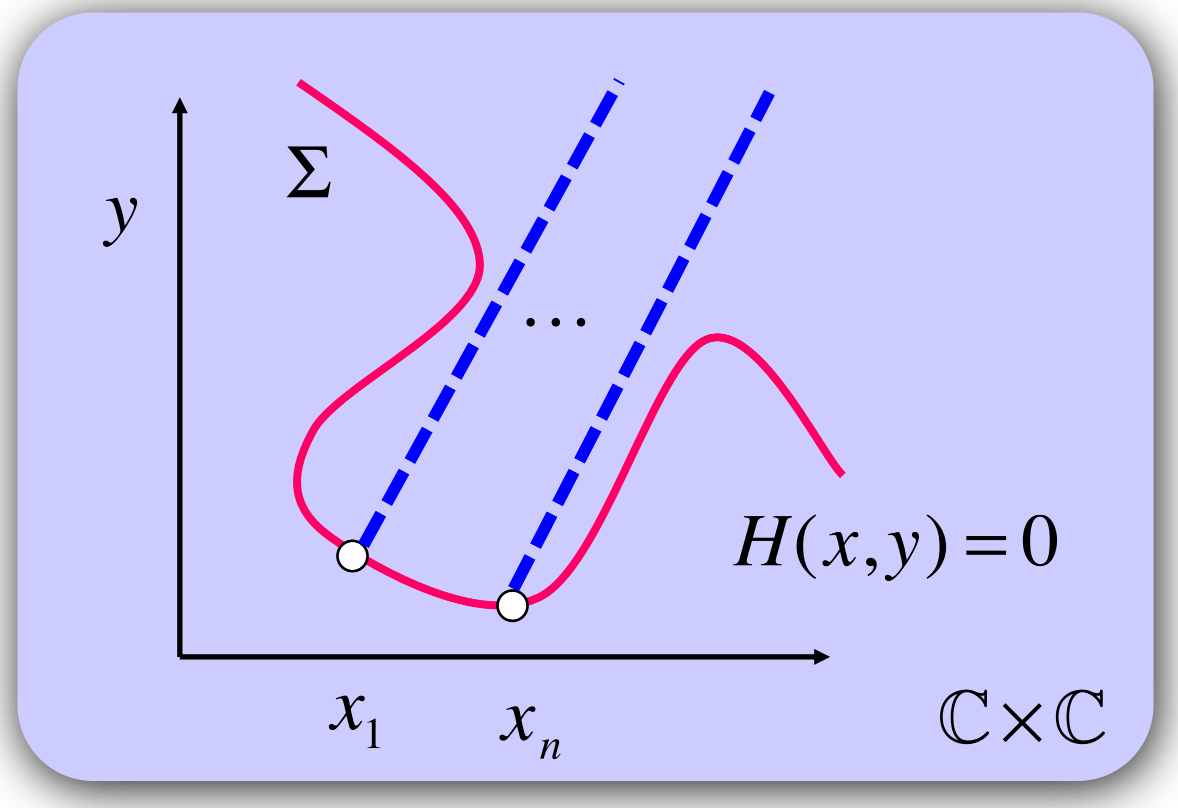
$$\int d\Phi \; e^{\mathrm{Tr} W(\Phi)} \det(\Phi - z)$$

Vertex operators = D-branes = fermions in chiral CFT

$$\psi(x) = e^{i\varphi(x)} = \det\left(x - \Phi\right)$$

$$\psi^*(x) = e^{-i\varphi(x)} = \det\left(x - \Phi\right)^{-1}$$

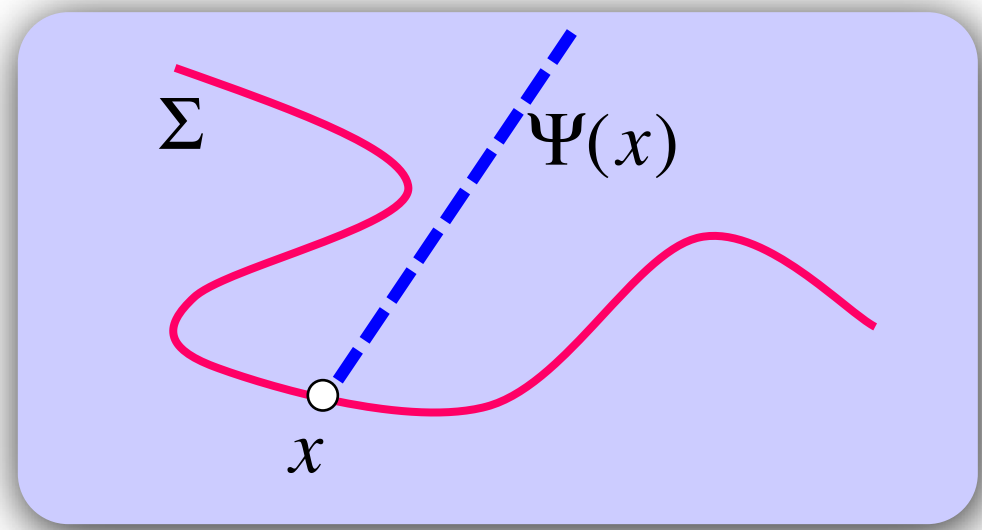
Topological D-Branes



Open string partition function

$$\Psi(x_1,\cdots,x_n)=\langle\psi(x_1)\cdots\psi(x_n)\rangle=\left\langle\prod_i \det(\Phi-x_i)\right\rangle$$

Quantum Wave Function



Single brane wave function $\Psi(x)$

$$\hat{H}\Psi = 0, \quad \hat{H} = \hat{H}(\hat{x}, \hat{y})$$

Quantization of phase space

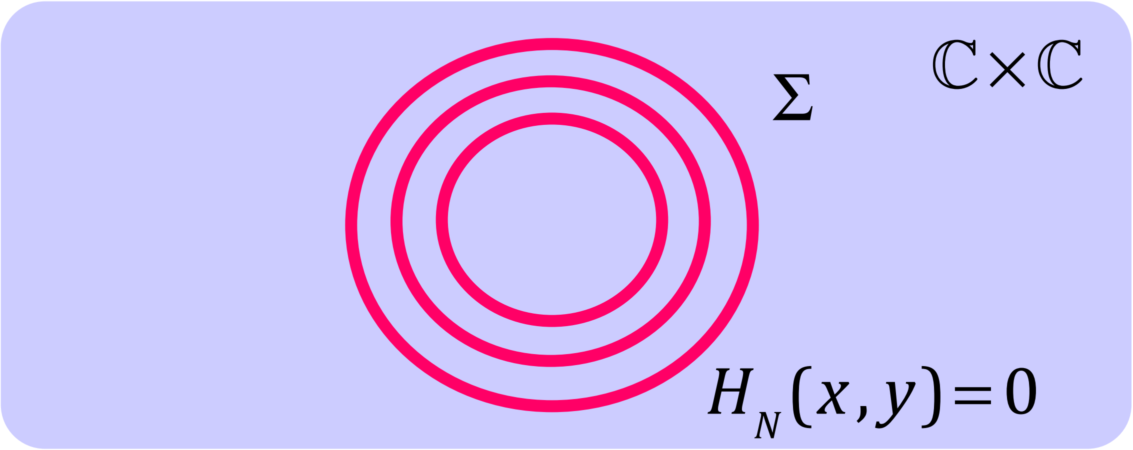
$$\hat{y} = -g_s \frac{\partial}{\partial x}, \quad [\hat{x}, \hat{y}] = g_s$$

Gaussian Matrix Model

$$\Psi_N(x) = \left\langle \det\left(\Phi - x\right)\right\rangle_N = H_{N-1}(x)\cdot e^{-x^2/2}$$

Eigenfunctions of harmonic oscillator

$$\left(-g_s^2 \frac{\partial^2}{\partial x^2} + x^2 - g_s(2N-1)\right)\Psi_N(x) = 0$$



Topological Gravity

Airy equation

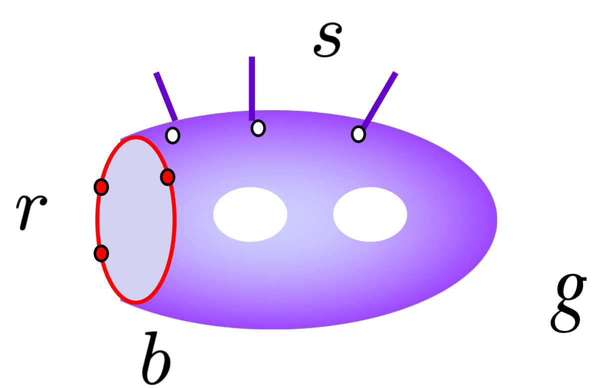
$$(\partial^2 - x)\Psi = 0$$

D-brane (open string) partition function

$$\Psi(z) = \int dp \; e^{izp + p^3/2}$$

Open Topological Strings

Surfaces with *b* holes, *s* bulk and *r* boundary punctures



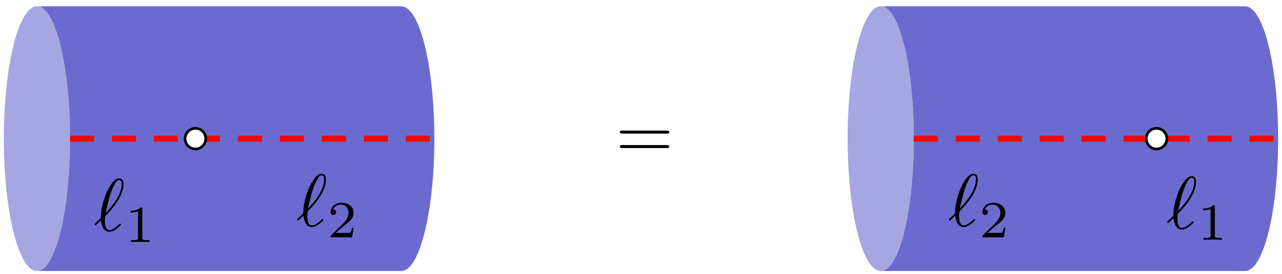
Attempted definition

$$\langle \tau_{n_1} \cdots \tau_{n_s} \sigma^r \rangle = \int_{\mathcal{M}_{g,b,s,r}} c_1(L_1)^{n_1} \cdots c_1(L_s)^{n_s}$$

Boundary puncture operator, dual to bdry cosm constant.

Open Topological String Anomaly

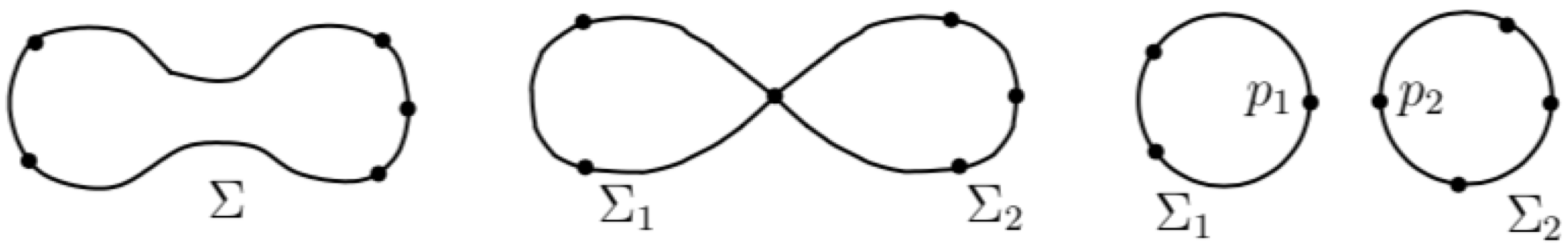
Moduli space in not orientable



$$d\ell_1 \wedge d\ell_2 = -d\ell_2 \wedge d\ell_1$$

Open Topological String Anomaly

Compactified moduli space has a boundary in (real) codim 1.
Degeneration of boundary:



Real dim *n* -3 -> dim *n* -4

LG-model

Matter for (p,1) model is topological gravity coupled to twisted N=2 SUSY minimal model (Landau-Ginzburg)

$$W(\phi) = \phi^p$$

In particular for "pure" topological gravity

$$W(\phi) = im\phi^2$$

After twisting *W* is spin one, so ϕ is spin ½ (in general spin 1/*p*).
Choice of spin structure

$$\phi \in S, \qquad S^{\otimes 2} = K$$

Closed Surfaces

Spin structures are odd or even, measured by the number of fermion zero modes

$$w = \dim H^0(\Sigma, S) = 0, 1 \ (mod\ 2)$$

Sum over spin structures for fixed genus *g*

$$\frac{1}{2}\sum_S (-1)^w = \frac{1}{2}\left(\frac{2^{2g} + 2^g}{2} - \frac{2^{2g} - 2^g}{2}\right) = 2^{g-1} = (\sqrt{2})^{2g-2}$$

Redefine the coupling constant

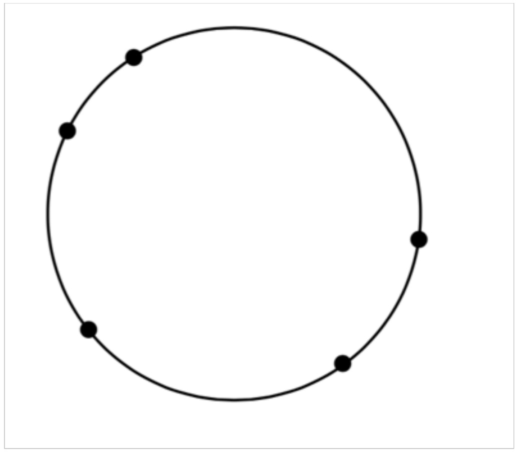
$$g_s \ \rightarrow \ \sqrt{2} g_s$$

Open Surfaces

Can we extend $(-1)^w$ to surfaces with boundaries?

Solved by mathematicians (Pandharipande, Solomon, Tessler)

Two choices of spin bundle on boundary. Include puntures that alternate between the two choices.



LG-model

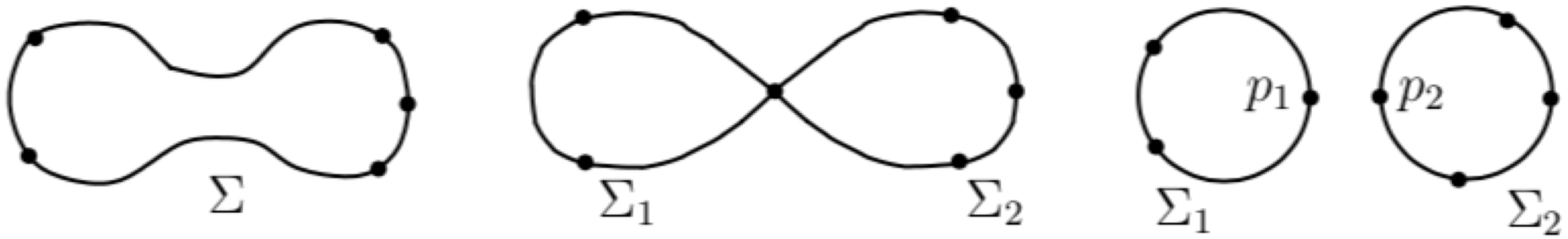
This prescription has a physical interpretation. In LG model there can be two choices of branes B, B', depending on the orientation. Solutions of instanton equation

$$\bar{\partial}\phi = \bar{W}'$$

Quantize: only physical states in B-B' channel.
Super gauge group interpretation

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, D \text{ even}, B, C \text{ odd}.$$

Anomalies



Matrix Models

Can derive the modified Virasoro constraints.

Open string partition function

$$\Psi(z) = \int dv \ \langle e^{v\sigma} \rangle$$

Satisfies

$$L_n \Psi(z) = \left(z^{n+1} \partial_z + \frac{1}{4}(n+1) z^n \right) \Psi(z)$$

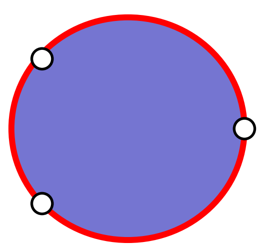
$$\left(\partial_z^2 - z + t_0 \right) \Psi(z) = 0$$

Matrix Models

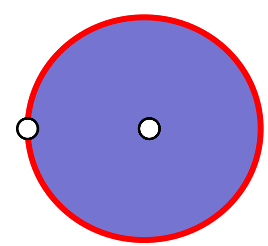
Consider the case of only puntures (bulk & boundary)

$$\langle e^{t_0\tau_0 + v\sigma} \rangle = e^{(t_0 v + v^3/3)/g_s}$$

Airy function!



$\langle \sigma^3 \rangle$



$\langle \sigma \tau_0 \rangle$

Conclusions

The relation between matrix models and topological
Gravity can be extended to open strings. Many subtle
Effects to take care of.

Open problems:

- Extend Mirzakhani’s work to unoriented strings. Problem: moduli space seems to be non-compact.

- Extend to topological supergravity. Moduli space of super-Riemann surfaces. Some preliminary results suggests that it’s related to the (-2,1) model, or LG with singular potential

$$W(\phi) = \frac{1}{\phi^2}, \qquad y^2 = \frac{1}{x}$$