Unraveling Turbulence: Modern Viewpoints On An Unsolved Problem

Arnold Sommerfeld Lectures (Munich 2023)

Outline

Leonardo da Vinci (1452-1519)

Turbulence

Turba is a Latin word for crowd. Turbulence originally refers to the disorderly motion of a crowd. Scientifically it refers to a complex and unpredictable motions of a fluid.

Turbulence

- Fluid turbulence is a major unsolved problem of physics.
- Emergent complex structure from simple rules (Newton's Second Law).

Turbulence

- Most fluid motions in nature at all scales are turbulent. Aircraft motions, river flows, atmospheric phenomena, astrophysical flows and even blood flows are some examples of set-ups where turbulent flows occur. **Why ?**
- Despite centuries of research, we still lack an analytical description and understanding of fluid flows in the non-linear regime. **Reductionism is not effective.**
- Insights to turbulence hold a key to understanding the principles and dynamics of non-linear systems with a large number of strongly interacting degrees of freedom far from equilibrium. **Probabilty measure ?**

Navier-Stokes Equations (1822)

- The incompressible Navier-Stokes (NS) equations provide a mathematical formulation of the fluid flow evolution:

$$\partial_t v^i + v^j \partial_j v^i = -\partial^i p + \nu \partial_{jj} v^i + F^i, \qquad \partial_i v^i = 0, \quad i = 1, ..., d \tag{1}$$

- v^i is the fluid velocity and p is the fluid pressure, ν is the kinematic viscosity and F^i is an external random force.
- The pressure is non-locally related to the velocity:

$$\nabla^2 p = -\partial_i v^j \partial_j v^i \tag{2}$$

Reynolds Number

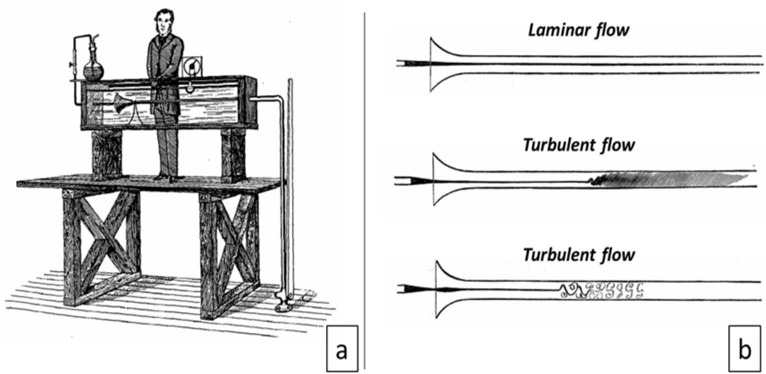
- An important dimensionless parameter in the study of fluid flows is the Reynolds number (1883)

$$\mathcal{R}_e = \frac{lv}{\nu} \qquad (3)$$

where l is a characteristic length scale, v is the velocity difference at that scale, and ν is the kinematic viscosity.

- The Reynolds number quantifies the relative strength of the non-linear interaction $v^j \partial_j v^i$ compared to the viscous term $\nu \partial_{jj} v^i$.
- When the Reynolds number is of order 10^3 or more, one observes numerically and experimentally a turbulent structure of the flow.
- This phenomenological observation is general, and fluid details are of no importance.

Transition to Turbulence



(Source: Wikipedia)

Turbulence in Nature

- Most flows in nature are turbulent. This is simple to see by noting that the kinematic viscosity of water is $\nu \simeq 10^{-6} \frac{m^2}{sec}$ and that of air is $\nu \simeq 1.5 \times 10^{-5} \frac{m^2}{sec}$. Thus, a medium size river has a Reynolds number $\mathcal{R}_e \sim 10^7$.

Turbulent Flows

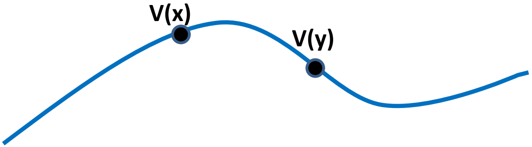
- The turbulent velocity field exhibits highly complex spatial and temporal structures and appears to be a random process. Thus, even though the NS equations are deterministic (in the absence of a random force), a single realization of a solution to the NS equations is unpredictable.
- Instead of studying individual solutions to the NS equations, one is led to consider the statistics of the solutions.

Statistical Properties

- Numerical and experimental data show that the statistical average properties exhibit a universal structure shared by all turbulent flows, independently of the details of the flow excitations.
- One defines the **inertial range** to be the range of distance scales $l \ll r \ll L$, where the scales l and L are determined by the viscosity and forcing, respectively.
- Turbulence at the inertial range of scales reaches a **steady state** that exhibits statistical homogeneity and isotropy.

The Statistical Approach

- Consider the statistics of velocity difference between points separated by a fixed distance.



Structure Functions

- Define the longitudinal velocity difference between points separated by a fixed distance $r = |\vec{r}|$

$$\delta v(r) = \big(\vec{v}(\vec{r}, t) - \vec{v}(0, t)\big) \cdot \frac{\vec{r}}{r} \tag{4}$$

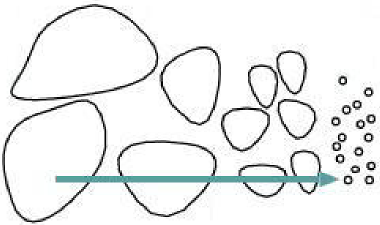
- The structure functions exhibit in the inertial range a scaling

$$S_n(r) = \langle(\delta v(r))^n\rangle \sim r^{\xi_n} \tag{5}$$

- The exponents ξ_n are **universal**, and depend only on the number of space dimensions.

K41 Theory

- In 1941 Kolmogorov argued that in three space dimensions the incompressible non-relativistic fluid dynamics in the inertial range follows a **cascade** breaking of large eddies to smaller eddies, called a direct cascade, where energy flux is being transferred from large eddies to small eddies without dissipation.



Scale Invariance

- Kolmogorov further assumed scale invariant statistics, that is

$$P(\delta v(r))\delta v(r) = F\left(\frac{\delta v(r)}{r^h}\right) \tag{6}$$

where $P(\delta v(r))$ is the probability density function, and h is a real parameter.

- Treating the mean viscous energy dissipation rate ϵ as a constant in the limit of infinite Reynolds number, he deduced a **linear scaling** of the exponents

$$\xi_n = \frac{n}{3} \tag{7}$$

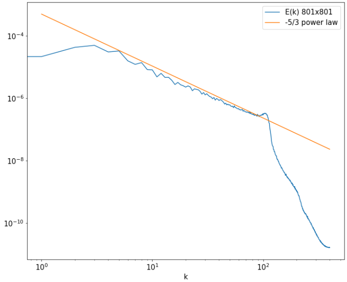
K41 Theory

- Longitudinal n-point functions:

$$S_n(r) \equiv \left\langle \left((\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{r}}{r} \right)^n \right\rangle \sim r^{\frac{n}{3}} \tag{8}$$

- Energy spectrum:

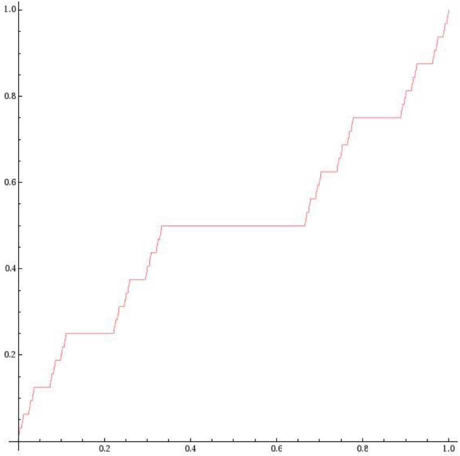
$$E(k) \sim k^{-5/3} \tag{9}$$



Intermittency

- All direct cascades are known numerically and experimentally to break scale invariance and do not simply follow Kolomogorov scaling.
- In two space dimensions the energy cascade is inverse, that is the energy flux is instead transferred to large scales.
- Kolmogorov's assumption that the random velocity field is **self-similar** is incorrect in direct cascades, but it seems to hold in the inverse cascade.
- The self-similarity assumption misses the **intermittency** of the turbulent flows.

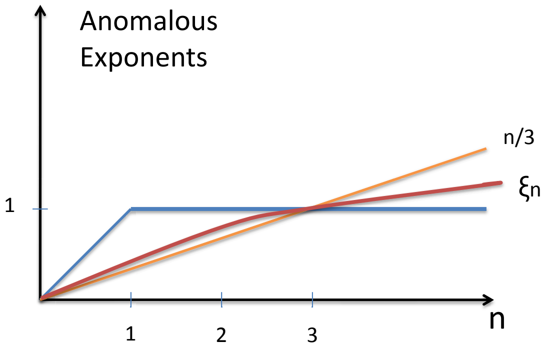
Intermittency (Schematic)



Cantor function

Anomalous Scaling

- The calculation of the anomalous exponents and their deviation from the Kolmogorov scaling is a major open problem.



Anomalous Scaling

- We propose and derive under certain assumptions an exact formula for the inertial range anomalous scalings ξ_n

$$\xi_n - \frac{n}{3} = \mathcal{G}^2(d)\xi_n(1-\xi_n) \tag{10}$$

- $\mathcal{G}(d)$ is a numerical real parameter that depends on the number of space dimensions $d \geq 2$.
- It quantifies intermittency and the deviation from Kolomogorov linear scaling $\xi_n = \frac{n}{3}$.

C. Eling, Y.O. JHEP **1509** (2015) 150
Y.O. JHEP **1711** (2017) 040
Y.O. Eur.Phys.J. **C78** (2018) no.8, 655
Y.O. arXiv:1809.10003 (Jacob Bekenstein: The Conservative Revolutionary)

Two Space Dimensions

- In two space dimensions the energy cascade is an **inverse cascade**, where the energy flux flows to scales larger than the injection scale.
- In this case, one has the energy spectrum agreeing with the Kolmogorov scaling $\xi_2 = \frac{2}{3}$.
- This implies that $\mathcal{G}^2 = 0$, and that all the other scaling exponents follow the Kolmogorov scaling $\xi_n = \frac{n}{3}$.

Three Space Dimensions

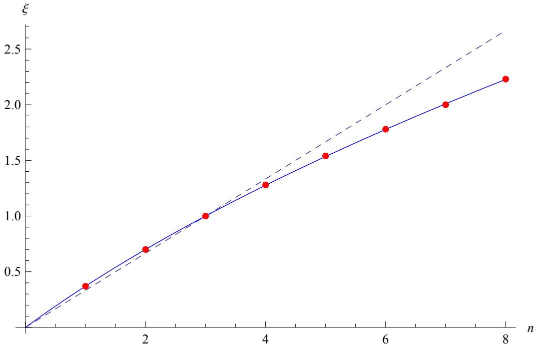


Figure: The dashed line represents Kolmogorov scaling. The best fit value of the free parameter $\mathcal{G}^2$ is about 0.161. The error on the data is about ± 1 percent (Benzi et.al. 1995).

Three Space Dimensions

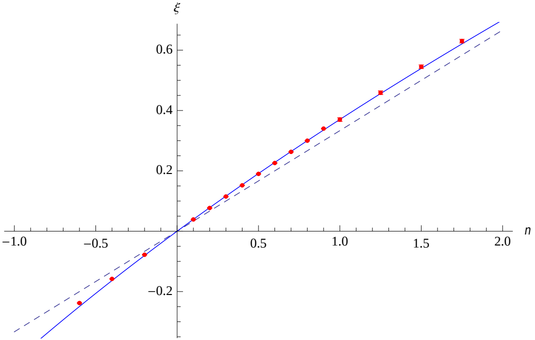


Figure: Fit to numerical data of numerical low moments (Chen et.al 2005). The dashed line represents Kolmogorov scaling. The best fit value of the free parameter $\mathcal{G}^2$ is about 0.159.

Four Space Dimensions

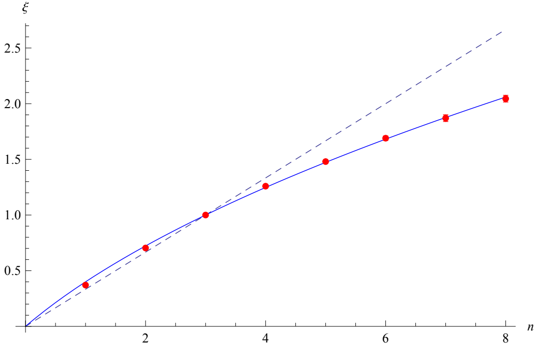


Figure: Fit to the 4d exponents given in (Gotoh et.al. 2007). The solid line is the 4d fit with $\mathcal{G}^2$ about 0.278.

Random Geometry

- Formula (10) is (Knizhnik-Polyakov-Zamolodchikov)-type relation (KPZ) that arises when coupling a dynamical system to a random geometry (1988):

$$d\mu_\gamma(x) \sim e^{\gamma\phi(x)} d\mu \tag{11}$$

- The Gaussian random field $\phi(x)$ has covariance $\phi(x)\phi(y) \sim -\log|x-y|$.

Scale Symmetry Breaking

- In the absence of a viscosity term, the (inviscid) NS equations (1) exhibit two scale symmetries of space and time:

$$x^i \to e^{\sigma} x^i \ , \qquad t \to e^{z\sigma} t \tag{12}$$

- The local energy dissipation $\epsilon(x) = \frac{\nu}{2} \left(\partial_i v^j + \partial_j v^i\right)^2$ (alternatively the flux) breaks spontaneously the symmetries of the inviscid NS equations to $z = \frac{2}{3}$:

$$\Delta_{K41}[v^i] = \frac{1}{3} \tag{13}$$

Inertial Range Dilaton

- The dilaton $\tau(x)$ is the fluctuation:

$$\epsilon(x) = \bar{\epsilon} e^{\delta \tau(x)} \tag{14}$$

- The dilaton action reads:

$$S_D(\tau, \hat{g}) = \frac{d}{\Omega_d (d-1)!} \int_M d^d x \sqrt{\hat{g}} \left(\tau \mathcal{P}_{\hat{g}} \tau + 2Q \mathcal{Q}_{\hat{g}} \tau \right) \tag{15}$$

- T. Levy, Y.O. JHEP **1806** (2018) 119, I. Hason, arXiv:1708.08294, T. Levy, Y.O., A. Raviv-Moshe JHEP **1812** (2018) 122, JHEP **1910** (2019) 006, A. Kislev, T. Levy, Y.O. JHEP **7** (2022) 1.

Dilaton Field Theory

- $\mathcal{P}_{\hat{g}}$ are the conformally covariant operators (GJMS 92):

$$\mathcal{P}_{\hat{g}} = (-\Delta)^{\frac{d}{2}} + \textit{lower order} \tag{16}$$

- $\mathcal{Q}_{\hat{g}}$ is the $\mathcal{Q}$-curvature scalar (Branson 91):

$$\mathcal{Q}_{\hat{g}} = \frac{1}{2(d-1)}(-\Box)^{\frac{d}{2}-1}R + \ldots \tag{17}$$

Dilaton Dressing

- The operators in the theory are K41 operators O_{K41} dressed by a dilaton factor:

$$O(x) = e^{d\alpha\tau} O_{K41}(x), \qquad \alpha = \gamma(1 - \Delta) \tag{18}$$

  where $d\Delta_{K41}$ is the undressed dimension of O_{K41}.
- We get the KPZ equation:

$$\Delta - \Delta_{K41} = \frac{\gamma^2}{2}\Delta(1 - \Delta) \tag{19}$$

Anomalous Scaling

- Consider the longitudinal structure functions S_n:

$$S_n(r) = \langle (\delta_r v)^n \rangle \sim r^{\xi_n} \tag{20}$$

where $\delta_r v$ is the longitudinal velocity difference between points separated by a fixed distance $r = |\vec{r}|$:

$$\delta_r v = \big(\vec{v}(\vec{r}, t) - \vec{v}(0, t)\big) \cdot \frac{\vec{r}}{r} \tag{21}$$

- The K41 scaling dimension of $(\delta_r v)^n$ is $\Delta_{K41} = \frac{n}{3}$, thus

$$\xi_n - \frac{n}{3} = \mathcal{G}^2(d)\xi_n(1 - \xi_n) \tag{22}$$

Trace Anomaly

Requiring that the inertial range universal structure and in particular the anomalous scalings should not depend on the forcing scale L:

$$a_{total} = a_{dilaton} + a_{K41} = 0 \tag{23}$$

and

$$\mathcal{G}^2(d) \simeq \frac{2}{\Omega_d (d-1)! |a_{K41}(d)|} \tag{24}$$

Summary

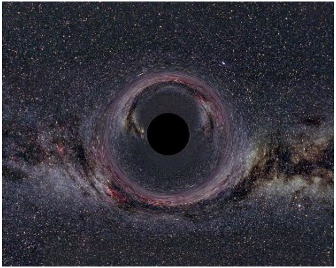
- Intermittency may be explained as a consequence of a random measure introduced by the local energy dissipation.
- This implies an exact formula for the anomalous scaling of turbulence:

$$S_n(r) = \langle (\delta_r v)^n \rangle \sim r^{\xi_n}$$

$$\xi_n - \frac{n}{3} = \mathcal{G}^2(d)\xi_n(1 - \xi_n)$$

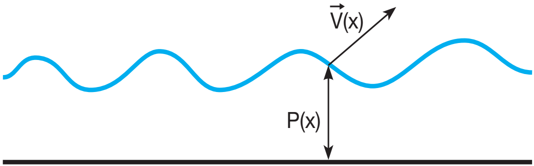
Black Hole Dynamics

- The existence of horizon is crucial: fields can fall into the black hole but cannot emerge, this breaks time reversal symmetry and allows Einstein equations to describe dissipative effects.



Geometrization of the Fluid Variables

- The dynamics of the event horizon is described by the Navier-Stokes equations (Damour (82), Bhattacharyya, Hubeny, Minwalla and Rangamani (08), Eling, Fouxon, Y.O. (09)).
- The fluid pressure and velocity in the geometrical picture :



Energy Power Spectrum

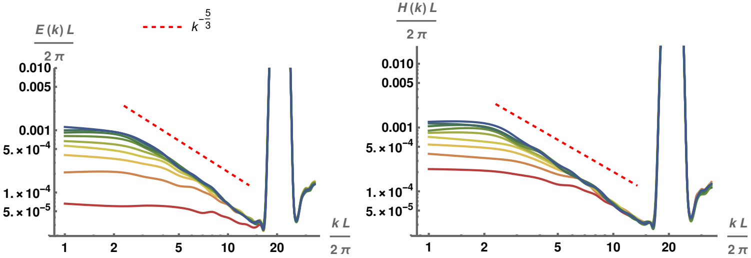


Figure: On the left: the double logarithmic plot of the ensemble averaged energy power spectrum $E(k)$ as a function of the wave vector k at various time steps , compared with the expected $k^{-5/3}$ scaling (red dotted line). On the right: the analogous plot for the energy power spectrum computed from the fluid velocity on the horizon.

With S. Waeber and A. Yarom (in peparation)

Local Energy Dissipation

- The local energy dissipation:

$$\epsilon(x) = \frac{\nu}{2}\left(\partial_i v^j + \partial_j v^i\right)^2 . \tag{25}$$

- The ensemble average of

$$\epsilon_r(x) = \frac{1}{Vol(B_d(r))}\int_{|x'-x|\leq r} d^d x' \epsilon(x') , \tag{26}$$

is independent of x by isotropy and of r by K41 scaling.

Holographic Local Energy Dissipation

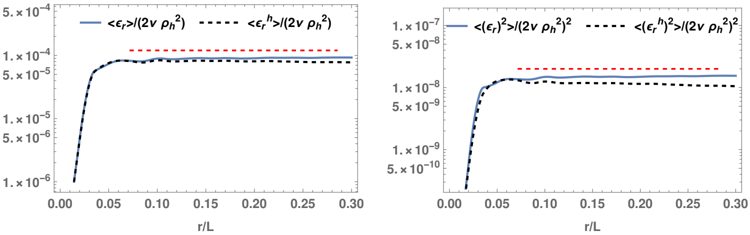


Figure: The first (left) and second (right) moment of the local energy dissipation ϵ_r (blue curve). As expected for two dimensional turbulence, ϵ_r and its higher moments show no scaling in the inertial range, indicated by the red dotted line. The black dotted curve shows the energy dissipation computed from the horizon.

Machine Learning of Fluid Flows

- Consider a non-linear PDE:

$$\partial_t \vec{v}(\vec{x}, t) = \mathcal{L} \vec{v}(\vec{x}, t) \tag{27}$$

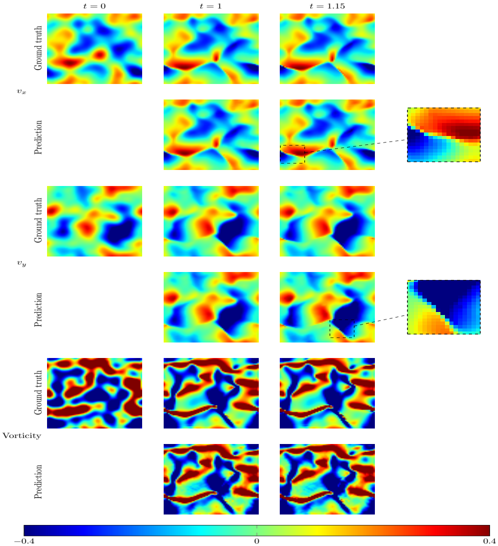
- A neural network evolves velocity fields, $\vec{v}(\vec{x}, t = 0)$ to a fixed time T

$$\Phi_T \vec{v}\left(\vec{x}, t = 0\right) = \vec{v}\left(\vec{x}, T\right) \tag{28}$$

  by learning from a set of $i = 1 \ldots N$ initial conditions sampled at $t = 0$, $\vec{v}_i\left(\vec{x}, t = 0\right)$, and their corresponding time-evolved solutions of $\vec{v}_i\left(\vec{x}, t = T\right)$.
- We generalized Φ_T, to propagate solutions at intermediate times, $0 \leq t \leq T$.

With R. Rotman, A. Dekel, R. Ber, L. Wolf (arXiv:2207.14366)

Machine Learning of Fluid Flows

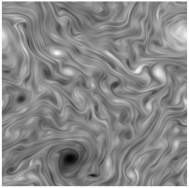
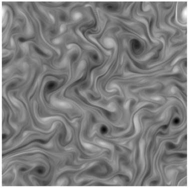
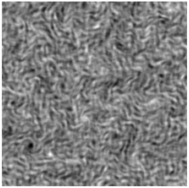
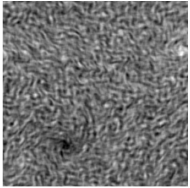


Machine Learning Complexity

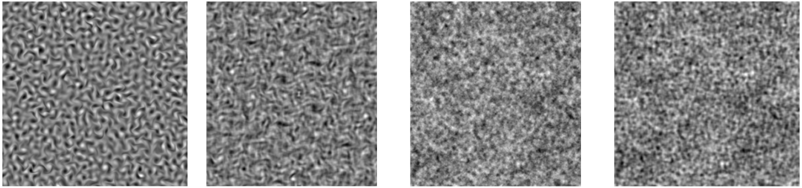
- We train neural networks to distinguish turbulence fluid configurations from chaotic ones, noise and real world images.
- What is the relative complexity of the various classification tasks involving turbulence?
- How does the pattern of complexity change with depth as we go inside the neural network? How does it compare with classifying real world images?
- Can we understand what features the neural network uses to distinguish chaos from turbulence?

With R. Janik and T. Whittaker arXiv:2211.15382
R. Janik and P. Witaszczyk (effective dimension)

Turbulence

Chaos and Noise



CNN

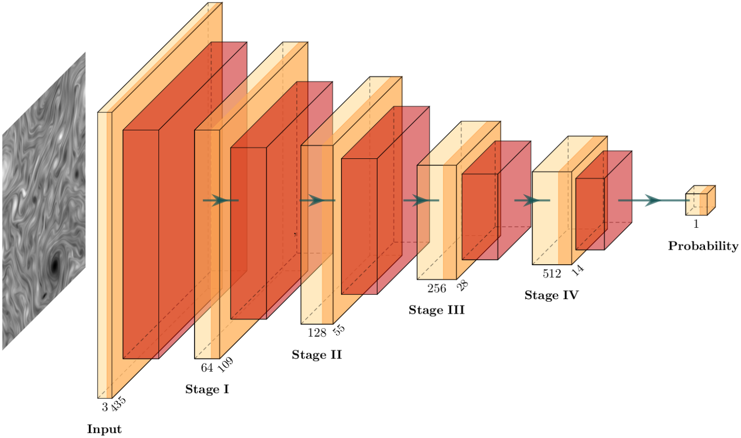


Figure: Schematic of the CNN. Each stage represents a set of convolutional layers.

Turbulence vs. Real World Images

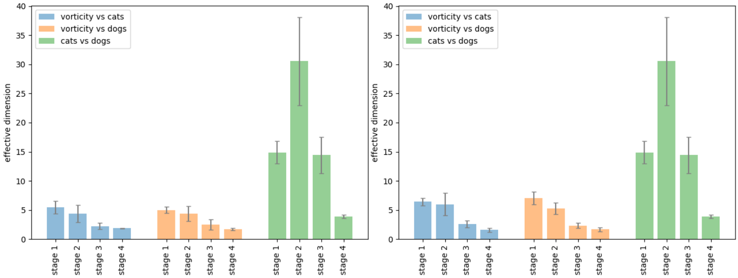


Figure: Left panel shows effective dimensions for images of weakly compressible turbulence vorticity vs. cats and dogs as well as for classifying between cats and dogs. Right panel shows the incompressible case.

Turbulence vs. Chaos

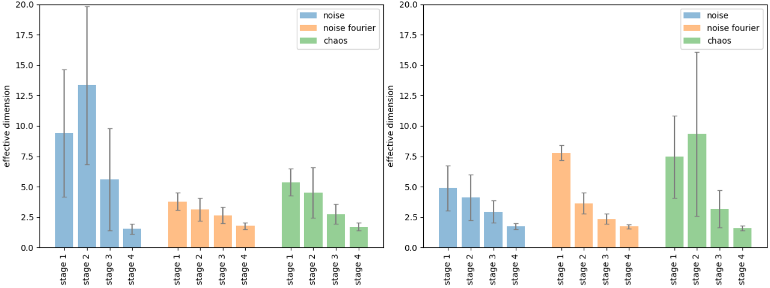


Figure: Effective dimensions for classifying weakly compressible turbulence vorticity (left) and incompressible turbulence vorticity (right).

Turbulence vs. Real World Images

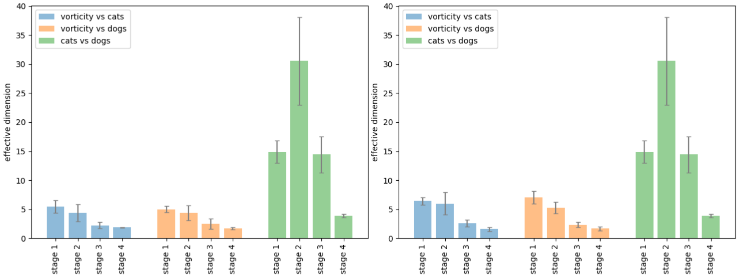


Figure: Left panel shows effective dimensions for images of weakly compressible turbulence vorticity vs. cats and dogs as well as for classifying between cats and dogs. Right panel shows the incompressible case.

Learning Statistical Turbulence

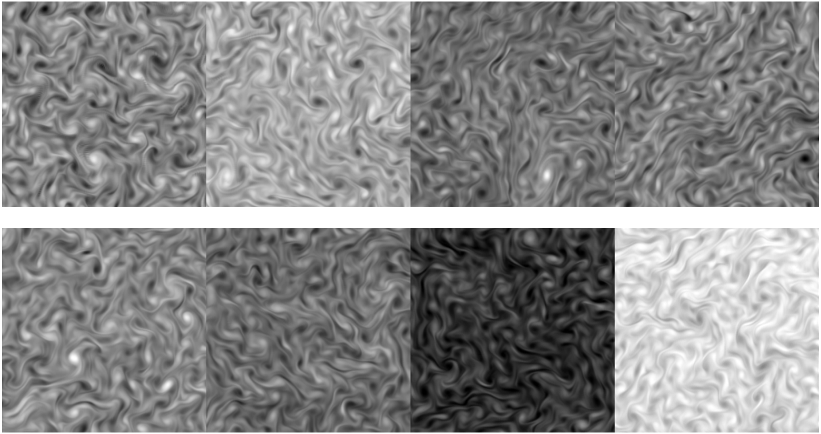


Figure: Four sample image patches from the training set (top) and four samples generated by the diffusion model (bottom).

Learning Statistical Turbulence

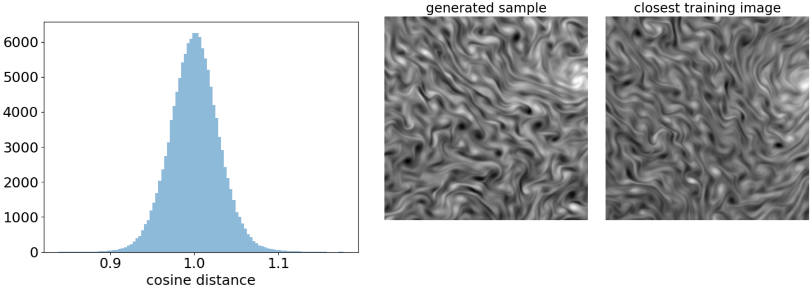


Figure: The histogram of cosine distances between 16 generated samples and the 8000 training images (left) and a pair of the most similar sample and training image (right).

Challenges

- Precision turbulence.
- Develop the field theory of turbulence.
- Holography - the fractal structure of the black hole horizon.
- Machine learning of turbulence statistical distribution.
- Superfluid turbulence.

Thank You