Anyons

Beyond Bosons and Fermions

In quantum theory the notion of identity reaches a new level of precision and has profound dynamical significance.

Traditionally, the world has been divided between bosons (Bose-Einstein statistics) and fermions (Fermi-Dirac statistics).

Recently we've come to understand that there are other possibilities, generically called "anyons".

Anyons are realized in simple models and in some known material systems. They may open new possibilities for quantum engineering (topological quantum computing, "anyonics").

Bosons and Fermions

If two identical particles start at A, B and end at A′, B′, we must consider both (A → A′, B → B′) and (A → B′, B → A′) as possible accounts of what happened.

For bosons we add the amplitudes, for fermions we subtract the amplitudes.

(*By focusing on squares of amplitudes, we can assume $A' = A$, $B' = B$.*)

The "direct" and "exchange" processes are topologically distinct, so their relative weight is not determined classically.

Since $(\text{exchange})^2 = \text{direct}$, consistency *seems* to restrict the possible relative weights to ± 1.

Braid Group

One can have quantum-mechanical systems with reduced dimensionality.

In one space dimension, the notion of quantum statistics collapses.

In two space dimensions, there is a richer topology of trajectories.

While in 3 dimensions a double winding can be continuously deformed to triviality, in 2 dimensions that is not so. (Belt trick.)

The governing group is the braid group, instead of the permutation group.

Braid Group

crossing over operations σ_i

$\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$

$\sigma_1 \sigma_2 \sigma_1$

$\sigma_2 \sigma_1 \sigma_2$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ (Yang-Baxter)

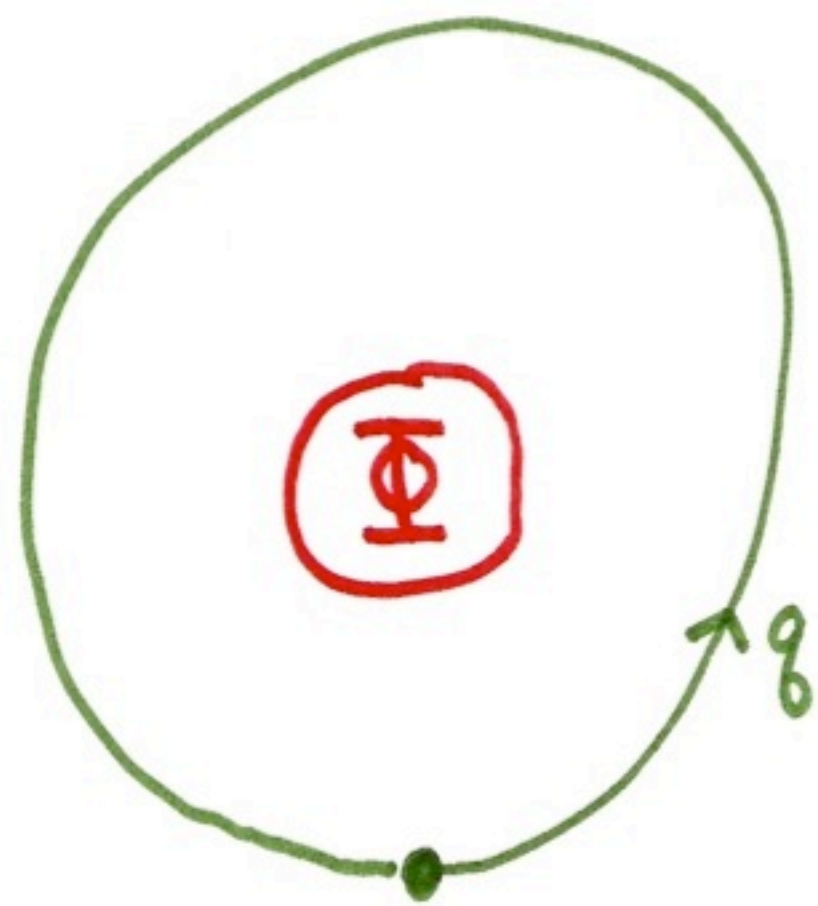
Symmetric Group

$\sigma_i^2 = 1$

Flux-Particle Anyons

The braid group admits 1-dimensional unitary representations with *any* phase $e^{i\theta}$. These define the classic anyons.

There is a nice dynamical realization of such anyons using flux and charge. It is related to the Aharonov-Bohm effect.



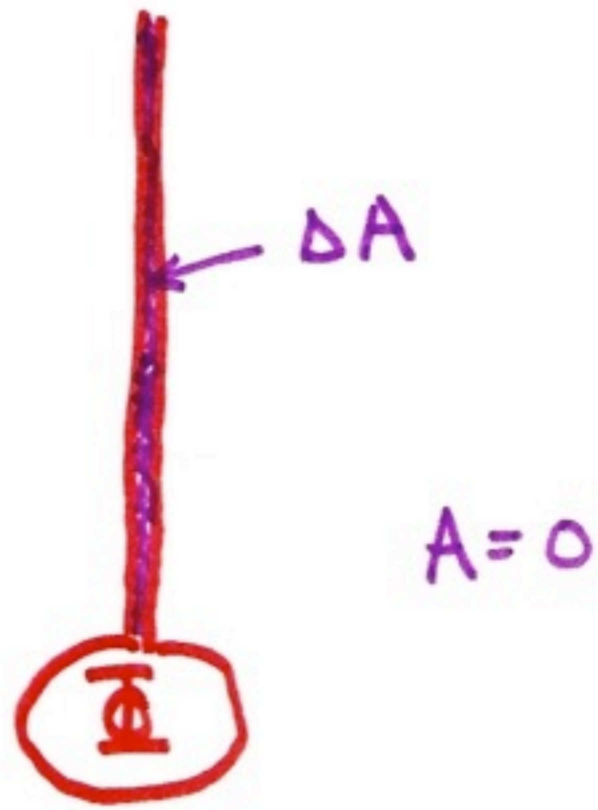
$$e^{iq\oint A\cdot v\,dt} = e^{iq\oint A\cdot dl} = e^{iq\Phi}$$

Flux-charge composites will, in general, be anyons.

Note:

In two dimensions, one can have flux "points".

The statistical interaction is essentially topological. One can capture it by attaching cuts to the flux points. When charged particles pass through the cuts, the amplitude for the trajectory gets multiplied by a phase.


ΔA
A = 0
Φ

A natural context for abelian anyons is gauge theories broken down to an abelian group, which may be finite.

Ordinary superconductors provide one (limited) example, with $U(1) \rightarrow Z_2$.

The fractional quantum Hall effect provides a rich set of examples. In the 1/m states, the quasiparticles are anyons with $\theta = 2\pi/m$.

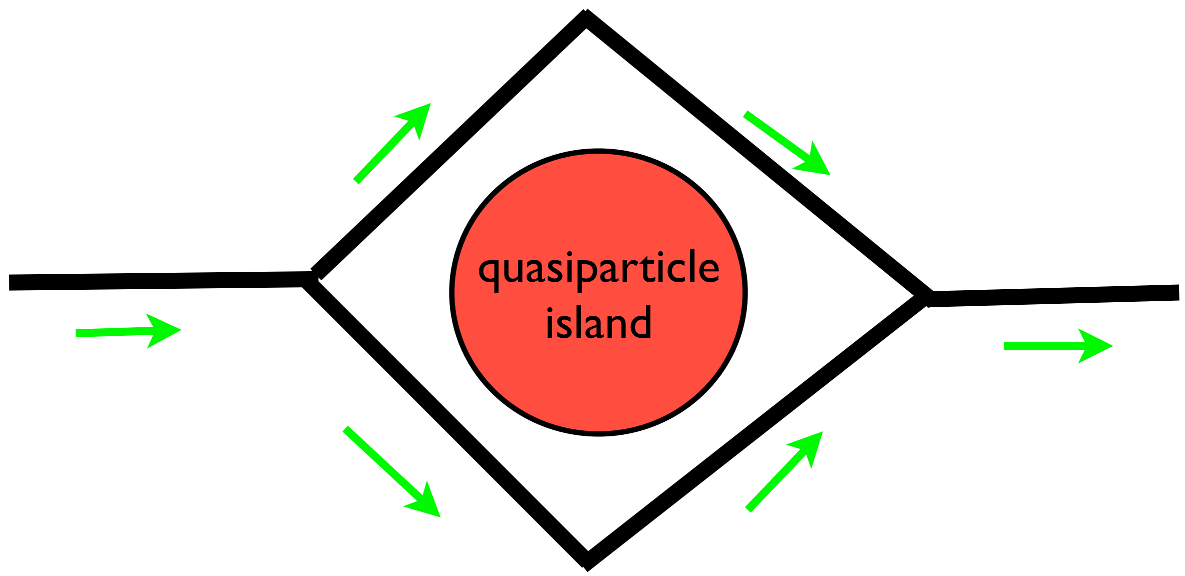
In principle, one can observe the anyon phase directly using interferometry, similar to what is done with SQUIDs.

One is detecting the “dynamical flux” associated with mutant statistics, in place of ordinary magnetic flux.

If the residual discrete gauge group is non-abelian, one will have nonabelian representations of the braid group, i.e. nonabelian anyons.

Nonabelian anyons are generally associated with ground-state degeneracy that grows exponentially with the number of anyons.

By moving them around in physical space, one navigates their quantum states in a controlled way exploring a large Hilbert space.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Majorana Anyons

Another promising approach to producing usable anyons is through circuits based on Majorana modes. This brings in some new and pretty ideas.

I will begin by reviewing the basic "Kitaev wire" construction.

$$a_j^\dagger, a_k \qquad 1 \leq j, k \leq N$$

$$\begin{aligned}
\{a_j, a_k\} &= \{a_j^\dagger, a_k^\dagger\} = 0 \\
\{a_j^\dagger, a_k\} &= \delta_{jk}
\end{aligned}$$

$$\begin{aligned}
\gamma_{2j-1} &= a_j + a_j^\dagger \\
\gamma_{2j} &= \frac{a_j - a_j^\dagger}{i} \\
\{\gamma_k, \gamma_l\} &= 2\,\delta_{kl}
\end{aligned}$$

$$H_0 \;=\; -\,i\sum_{j=1}^{N} \gamma_{2j-1}\gamma_{2j}$$

$$H_1 \;=\; -\,i\sum_{j=1}^{N-1} \gamma_{2j}\gamma_{2j+1}$$

In terms of the a-operators:

H_0 is simply occupation number.

H_1 is a combination of normal and superconducting hopping terms.

Like H_0 , H_1 can be written as a sum of occupation numbers, but of peculiar quasiparticles, not the original electrons.

Most importantly, H_1 does not contain γ_1 or γ_{2N} at all. Those operators create "Majorana modes" localized at the two ends.

Note that γ_1 and γ_{2N} are hermitean and square to 1. This is quite different from conventional fermions (or bosons).

Kitaev showed that H_1 is representative of a universality class. In general the Majorana modes extend over several lattice sites, but are exponentially localized, and have exponentially small energy.

The effective Hamiltonian is 0, but there is an algebra of hermitean operators

$$b_L^2 = 1\,;b_R^2 = 1\,;\ \{b_L, b_R\} = 0$$

If we join two wire ends, we get an effective interaction $H_{int.} \propto i\, b_L\, b_R$. That produces another occupation number; the Majorana modes are gone.

Now consider a junction of three wire ends. Is there a Majorana mode at the junction?

$$H \;=\; -\, i(\alpha\, b_1 b_2 + \beta\, b_2 b_3 + \gamma\, b_3 b_1)$$

$$\{b_j\,, b_k\} \;=\; 2\,\delta_{jk}$$

Note that we only include polynomials that are even in the b_j, on physical grounds.

We can realize the Clifford algebra using Pauli matrices, $b_j \rightarrow \sigma_j$.

But this does *not* yield a Majorana mode. The spectrum of this sum of σ_j is just $\pm(|\alpha|^2 + |\beta|^2 + |\gamma|^2)^{1/2}$.

Actually, the same problem arises for "junctions" with *one* end!

Something is missing ...

These representations lose the distinction between even and odd powers of b_j s!

To get the physics right, we must capture the implications of conservation of electron number parity P.

$$P \text{ ``}=\text{''} (-1)^{N_e}$$

$$\{P, b_j\} = 0$$

$$[P, H_{\text{eff.}}] = 0$$

$$P^2 = 1$$

Now consider the special operator

$$\Gamma \;\equiv\; i\, b_1 b_2 b_3$$

It satisfies

$$\begin{aligned} \Gamma^2 &= 1 \\ [H_{\mathrm{eff.}}, \Gamma\,] &= 0 \\ \{P, \Gamma\,\} &= 0 \end{aligned}$$

Γ has the right properties to create a Majorana mode:

It is hermitean and squares to 1

It commutes with the Hamiltonian

Importantly: it is *not* a function of the Hamiltonian!

"Majorana doubling" occurs not only for the ground state, but for all states, through the action of Γ.

If we diagonalize both H and P, then Γ connects degenerate states with $P = \pm 1$.

The junction spectrum is reminiscent of Kramers doubling.

A similar construction works for a junction of any odd number of wire ends. It does *not* depend on any single-particle approximation.

Indeed, Γ is associated with the *product* wave function.

Γ-operations, when extended to several locations, implement a mutant (Grassmann→Clifford: "quantum") form of supersymmetry, with anyons.

Majorana modes occur in other contexts, besides wire ends.

When they are attached to flux-points, they make nonabelian anyons.