Technische Universität München
Ludwig-Maximilians-Universität München

Master Thesis
Theoretical and Mathematical Physics

The problem of equivalence of different gauges in External Current QED

Author:
Benedikt Wegener

Supervisor:
Dr. Wojciech Dybalski

2nd Supervisor:
Dr. Dirk André Deckert

December 18, 2018

Contents

1 Introduction

In Classical Electrodynamics it is well known that a change of gauge of the electromagnetic potential $A'_\mu(x) = A_\mu + \partial_\mu \chi(x)$ does not have any influence on the experimental outcome, since the electromagnetic field strength tensor is left invariant. In Quantum Electrodynamics this issue is more controversial. On the one hand, there is a widespread belief that the gauge should not matter, on the other hand, the awareness that there is no satisfactory formulation of this proof or fact si growing.
In fact, standard text book discussions of quantization of the electromagnetic field start with fixing the gauge. In striking contrast to the claimed arbitrariness of the function χ above, essentially with the Coulomb and the Lorentz gauge only two gauge fixing conditions are discussed.
Even though there are few elaborations on the unitary equivalence of gauges such as [HLL94] and [NTO94], the authors usually ignore mathematical pathologies that occur in the course of the discussion.
The goal of this thesis is to shed some light on the problem of gauge invariance from a mathematical point of view in the simple case of the quantized electromagnetic field coupled to an external current j.
To address this problem, we will review Dirac's elaboration on Singular Systems [Dir64] in the first part of Chapter 2. Dirac explains that a system has intrinsic constraints, that force the physical quantities to a submanifold, whenever the Legendre transformation is not a local isomorphism. Moreover, he introduces a classification of the constraints, from which the origin of gauge freedom and the necessity to fix the gauge in order to define a consistent quantization procedure become clear.
In order to speak of a Quantum Theory, we need to fix a quantization map that describes the transition from the Classical to the Quantum Theory. The standard procedure in textbooks is the so-called canonical quantization. However, in the usual ansatz to define such a quantization map, Groenewold showed that one encounters contradictions due to the non-commutativity of observables in Quantum Theories [Gro46].
In the second part of Chapter 2, we obviate these contradictions by identifying classical observables with Weyl elements, which are bounded functions of the fields. Field theories are then uniquely defined by specifying a regular representation of the Weyl-C^*-algebra, the C^*-algebra generated by the observables.
In this formalism, the quantization is defined by replacing the classical commutative Weyl-C^*-algebra with some non-commutative analogue and specifying a regular representation. Unitary equivalence of gauges can hence be addressed on the level of regular representations of Weyl-C^*-algebras.

Applying Dirac's procedure of Singular Systems to the classical Lagrangian of the electromagnetic field reproduces the gauge freedom of the electromagnetic potential $A_\mu(x) \to A_\mu + \partial_\mu \chi(x)$. In this thesis, we will focus on the quantization of two different gauges. On the one hand, the extensively studied Coulomb gauge and on the other hand the seldom discussed Axial gauge.
While the representation of the Coulomb gauge observables turns out to be well-defined, there are severe singularities for the Axial gauge representation which lead to ill-definiteness of the observables. This discussion can be found in Chapter 3.
Mund, Schroer and Yngvarssn [MSY05] suggested a method of smoothing these singularities such that the representation is well defined. We adopt this idea to justify a regularized Axial gauge representation, in the course of which the problem of unitary equivalence of the different gauges under consideration can be stated on a rigorous level. The problem of obtaining the Yngvarson-Mund-Schroer smearing by canonical quantization of a certain classical theory will be discussed in Chapters 4 and 5.
Finally, in Chapter 6 and 7, we are in the position to investigate the unitary equivalence of different gauges rigorously. We will show in Chapter 6 that such equivalence holds if the total electric charge of the system is zero. Interestingly, for non-zero electric charge the equivalence appears to fail due to a different low-energy behaviour of the two gauges as will be proven in Chapter 7.

2 Preliminaries

This chapter serves as introduction to the basic concepts and formalisms that we will make use of to address the problem of the unitary equivalence of different gauges.

2.1 Conventions

First, let us point out which conventions are used in this work.
The spacetime will be assumed to be the pair $(Mink_4, \eta)$ consisting of the 4-dimensional Minkowski space and the Minkowski metric η with signature $(+,-,-,-)$.
Different types of indices will appear in this work. Greek indices are 4-vector indices, i.e. indices labeling the components of a 4-vector and run from 0 to 3, whereas Latin indices denote spatial components running from 1 to 3.
Whenever there appear indices that neither indicate 4-vector, nor spatial components, we point out their range.
Furthermore, Heaviside units, i.e. $c = \hbar = \mu_0 = \epsilon_0 = 1$, are used in this work.

2.2 Constrained Systems and Gauge freedom

Singular Systems

In this section, we provide an introduction to the concept of Singular Systems where the standard procedure of computing the time evolution fails. We will elaborate on this issue and we will expose that gauge field theories are a special class of such Singular Systems. The first one to investigate the properties of such system was Paul Dirac in his famous Lectures on Quantum Mechanics [Dir64]. The discussion of Singular Systems in point mechanics and Field Theory follows mostly the book of Sundermeyer [Sun82], the Bachelor Thesis [Frä11] and [Thi07], chapter 24.
A physical system is usually described on some n-dimensional configuration space M and a Lagrange function $TM \to \mathbb{C}$ on its tangent bundle. For the following discussion we need to assume that the Lagrange function is not explicitly time dependent. Assuming TM has local coordinates (q^i, v^i) we can write:

$$L = L(q^i(t), v^i(t)) \tag{2.1}$$

where we identified with v^i the velocity of q^i.
As it is well known the principle of least action leads to Euler-Lagrange equations that

govern the time evolution:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^i \partial v^j} v^j - \frac{\partial^2 L}{\partial v^i \partial v^j} \dot{v}^j \overset{!}{=} 0 \tag{2.2}$$

In this form it is obvious that the accelerations $\dot{v}^i$ can only be uniquely expressed in terms of q^j and v^j if the Hessian matrix

$$G_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j} \tag{2.3}$$

is invertible. In other words if it has non-vanishing determinant.

Definition 2.1. A system is called *singular* if the Hessian matrix of the Lagrange function L with respect to the velocities

$$G_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j} \tag{2.4}$$

has vanishing determinant.

The issues arising for a singular system become more obvious if one tries to deal with the system in the frame of Hamilton mechanics. It is well known that Hamilton mechanics offer an equivalent way of describing physical systems. The Hamilton function is usually a function $H : T^*M \to \mathbb{C}$ on the cotangent bundle. The transition from the Lagrange to the Hamilton function is obtained via the Legendre transformation

Definition 2.2. The map

$$\begin{aligned} \rho_L : TM &\to T^*M \\ (q^i, v^i) &\mapsto (q^i, p_i = \frac{\partial L}{\partial v^i}) \end{aligned} \tag{2.5}$$

is called *Legendre transformation*.

Remark 2.3. *We note that the Jacobi matrix of the Legendre-transformation with respect to the velocities $J(\rho_L)$ coincides with the Hessian matrix G. Hence, we can give an equivalent definition of a singular system in terms of the Legendre transformation.*

Definition 2.4. (i) A physical system is called *singular* if the Legendre transformation ρ_L is not a local isomorphism.

(ii) The Hamilton function H for a non-singular system is a function:

$$\begin{aligned} H : T^*M &\to \mathbb{C} \\ (q^i, p_i) &\mapsto v^i p_i - L(q^i, v^i) \end{aligned} \tag{2.6}$$

where the velocities $v^i(q^j, p_j)$ are expressed in terms of positions and its canonical momenta.

For singular Lagrangians it is *not* possible to solve all velocities in terms of the canonical momenta and positions. In this case it is not clear how the Hamilton function is defined. We will now pretend that the Hamilton function is a function on a larger space than T^*M by formally following the definition.
As it turns out the Hamilton function will be a function on $T^*\overline{M}$ for some smaller configuration space $\overline{M}$. $T^*\overline{M}$ will be interpreted as the physical phase space. The derivation of these results will be discussed in the following.

Let us assume that the rank of G is constant and $rk(G) = n - r$ where n is the dimension of the configuration space M and $r \in \mathbb{N}$ is some number. Then we can (at least locally due to the inverse function theorem) express $n - r$ velocities in terms of momenta and position:

$$v^a = u^a(q^i, p_a, v^j) \tag{2.7}$$

with $i \in \{1, \dots, n\}, a \in \{1, \dots, n-r\}, j \in \{n-r, \dots, n\}$.
Furthermore, inserting this in the remaining equations $p_j = \frac{\partial L}{\partial v^j}$ cannot depend on v^j since then the rank of G would exceed $n - r$. Hence, we get

$$p_j = \left(\frac{\partial L}{\partial \dot{q}^j}\right)_{\dot{q}^a = u^a(q^i, p_a, v^j)} := \pi_j(q^i, p_i) \tag{2.8}$$

which shows that the p_a are not independent.

Definition 2.5. The Hamilton function

$$H' : T^*M \oplus TM \to \mathbb{C} \tag{2.9}$$

$$H'(q^i, p_i, v^j) = \left(p_i v^i - L(q^i, \dot{q}^i)\right)_{\dot{q}^a = u^a(q^i, p_i, v^j)} \tag{2.10}$$

is called *primary Hamiltonian*.

Proposition 2.6. *H' is linear in v^j with coefficients $\phi_j = p_j - \pi_j(q^i, p_i)$*

Proof. Differentiating H' with respect to v^j gives the result, see [Thi07]. □

Remark 2.7. *We decompose H' as*

$$H'(q^i, p_i, v^j) = \tilde{H}(q^i, p_i) + v^j \phi_j(q^i, p_i), \tag{2.11}$$

where $\tilde{H}$ is independent of the velocities v^j.

Definition 2.8. The functions:

$$\phi_j = p_j - \pi_j(q^i, p_i) \tag{2.12}$$

are called *primary constraints*.

The reason that ϕ_j are called constraints is a consequence of the equations of motion

Theorem 2.9. *The Hamilton equations are*

$$\dot{q}^i = \frac{\partial \tilde{H}}{\partial p_i} + v^j \frac{\partial \phi_j}{\partial p_i}, \dot{p}_i = -\frac{\partial \tilde{H}}{\partial q^i} - v^j \frac{\partial \phi_j}{\partial q^i}, \phi_j = 0. \tag{2.13}$$

They are are equivalent to the Euler-Lagrange equations.

This theorem can be proved by carefully following the definitions and hence we will omit the technical details.

Remark 2.10. *The equations of motion show that the v^j's do not follow any dynamical trajectory and thus are arbitrary c-number functions which may be interpreted as* Lagrange multipliers *that constrain the system to the submanifold defined via $\{\phi_j = 0\}$.*
Hence, the constrained phase space is coordinatized by the positions q^a and the canonical momenta p^a which we equip with the standard symplectic structure:

$$\{q^a, p_b\} = \delta^a_b \tag{2.14}$$
$$\{q^a, q^b\} = \{p_a, p_b\} = \{q^a, v^j\} = \{p_a, v^j\} = 0 \tag{2.15}$$

such that we can express the e.o.m in the well known compact form:

$$\dot{p}_i = \{p_i, H'\} \tag{2.16}$$
$$\dot{q}^i = \{q^i, H'\} \tag{2.17}$$

For a general observable $F \in C^\infty(\mathcal{M})$ the equation of motion reads:

$$\dot{F} = \{F, H'\} = \{F, \tilde{H}\} + v^j \{F, \phi_j\} \tag{2.18}$$

Remark 2.11. *The configuration space for the physical system is not M since there are additional constraints that restrict the system to some submanifold.*
Recalling (2.7) and the fact that $\phi_j \approx 0$ show that the Hamilton function restricted to the physical phase space is independent of the velocities v^i. Hence, the physical Hamilton function H is a function on $T^*\bar{M}$ for some configuration space $\bar{M}$. We will see that the ϕ_j are not necessarily *the only constraints defining $\bar{M}$.*

Notation. *The symbol $\approx$ indicates weak equalities. These are equalities that hold in the virtue of the constraints, i.e. on the constraint submanifold. Hence, we should also write $\phi_j \approx 0$.*

Dirac-Bergmann Algorithm and Secondary Constraints

The result that the primary constraints vanish on the physical submanifold is consistent with the time evolution if and only if:

$$\dot{\phi}_k = \{\phi_k, H'\} = \{\phi_k, \tilde{H}\} + v^j \{\phi_k, \phi_j\} \approx 0 \tag{2.19}$$

Otherwise the system would leave the submanifold $\{\phi_j = 0\}$ after some time and violate the constraints.
In order to fulfil this equation we can distinguish between three cases. With the definition $M_{rs} = \{\phi_r, \phi_s\}$ we have:

(*i*) $\{\phi_j, \tilde{H}\} \neq 0, \det(M) \neq 0$
All Lagrange multipliers are (weakly) fixed $v^r = -M^{rs}\{\phi_s, \tilde{H}\}$ and the equations of motion become $\dot{F} = \{F, \tilde{H}\} - \{F, \phi_r\}M^{rs}\{\phi_s, H\}$, where M^{rs} denotes the inverse matrix of M_{rs}.

(*ii*) $\{\phi_j, \tilde{H}\} \neq 0, \det(M) = 0$
To be discussed further.

(*iii*) $\{\phi_j, \tilde{H}\} = 0$
Some Lagrange multipliers are fixed, depending on the rank of M.

In the following we will discuss the second case further. Let $rk(M) = r - t$, then there are t nulleigenvectors e_s of M such that:

$$\dot{\phi}_k e^k_s = \{\phi_k, \tilde{H}\}e^k_s + v^j\{\phi_k, \phi_j\}e^k_s \tag{2.20}$$

$$= \{\phi_k e^k_s, \tilde{H}\} \overset{!}{\approx} 0 \tag{2.21}$$

For each e_s this *could* yield conditions ϕ_s that are linearly independent of the primary constraints. Such conditions are called *secondary constraints*.
After finding the secondary constraints we need to repeat the whole procedure and eventually we find further linear independent consistency constraints that assure $\dot{\phi}_s \approx 0$. Those constraints are called *tertiary constraints*.
This method breaks down after having at most $2n$ linear independent constraints in total because then the phase space is constraint to a discrete set of points. This procedure is called *Dirac-Bergmann algorithm*.

First and Second class constraints

Definition 2.12. A function F on the phase space is called *first class* if

$$\{F, \phi_j\} \approx 0 \tag{2.22}$$

for all constraints ϕ_j. Otherwise it is called *second class*.
Analogously, we call a constraint *first class* or *second class* respectively if its constraint function ϕ_j is first respectively second class.

Remark 2.13. *We will see that the distinction between primary and secondary constraints is not substantial. The distinction between first and second class constraints, however, will play an essential role in the discussion of gauge freedom.*

We note that linear combinations of constraints again are constraints. Hence, one may think of constraints as a vector space.
Assume that we have k constraints in total and thus M is a $k \times k$-matrix. If $\det(M) \neq 0$ then every constraint is of second class.
However, if $\det(M) = 0$ we assume that the matrix M has rank $k - l$ for some $l \in \mathbb{N}$ and we can find a basis such that M has l zero rows and columns. That means that we can construct l first class constraints. It is not possible to find a basis such that we have more than l first class constraints since then the rank of M would be smaller that $k - l$. We say that we have constructed a maximal set of first class constraints.
In the following we assume that we are always working with constraints such that we have a maximal set of first class constraints. We rename all such first class constraints as γ_m and second class constraints as ξ_μ. Here and in the rest of this section, the indices m and μ do not indicate spatial or 4-vector components. They run over the number of the respective constraints.

Remark 2.14. *Working with a maximal set of first class constraints we always have an even number of second class constraints. Assume the matrix M_s of Poisson brackets of the second class constraints. This has to have non vanishing determinant. Otherwise we could construct a first class constraint from linear combinations of second the second class constraints which is a contradiction to the maximality of the first class constraints.*
Since the Poisson bracket is anti-symmetric, so is M_s. As anti-symmetric matrix with non zero determinant M_s has to have an even number of rows which means there has to be an even number second class constraints.

Second class constraints

Let us have a closer look at the second class constraints and assume for this purpose that only such constraints are present.
Since $\det(M) \neq 0$ the Lagrange multipliers are fixed. Inserting the fixed Lagrange multipliers to the time evolution gives:

$$\{F, H'\} = \{F, \tilde{H}\} - \{F, \xi_\mu\} M^{\mu\nu} \{\xi_\nu, \tilde{H}\} \tag{2.23}$$

It is obvious that the Poisson bracket structure is not compatible with the constraints since in general $\{F, \phi_k\} \not\approx 0$. However, we have $\phi_k \approx 0$ which is not captured by the Poisson bracket. Hence, we need to introduce a new modified symplectic structure that respects the constraints. (2.23) motivates the definition of a new bracket:

Definition 2.15. The *Dirac bracket* is a modified Poisson bracket and defined between two phase space functions $f, g \in C^\infty(M)$ via:

$$\{f, g\}_D = \{f, g\} - \{f, \xi_\mu\} M^{\mu\nu} \{\xi_\nu, g\} \tag{2.24}$$

Proposition 2.16 (Properties of the Dirac bracket)**.** *The Dirac bracket has the same algebraic properties as the Poisson bracket (antisymmetry, linearity, Jacobi and product rule). Moreover, $\{f, \xi_\mu\}_D = 0$ holds strongly for every $f \in C^\infty(M)$.*
The difference to the Poisson bracket is that the canonical relation $\{q^i, p_j\} = \delta^i_j$ is no longer satisfied for the Dirac bracket.

Remark 2.17. *Up to now we are dealing with the whole phase space M having the dynamics restricted to a submanifold. Actually, we want to work with the reduced phase space $\bar{M}$ containing only the physical degrees of freedom. Unfortunately, in the most cases $\bar{M}$ is very difficult to construct. However, Maskawa and Nakajima [MN76] were able to show that the Dirac bracket acts on the constrained phase space as the Poisson bracket on the reduced phase space.*

First class constraints

Assume that we do not only have second class constraints but also some first class constraints γ_m. The time evolution then reads:

$$\{F, H'\} = \{F, \tilde{H}\} + v^k\{F, \gamma'_k\} - \{F, \xi_\mu\}\tilde{M}^{\mu\nu}\{\xi_\nu, \tilde{H}\} \tag{2.25}$$

where $\tilde{M}$ is the matrix consisting of Poisson brackets of the second class constraints. Since we can deal with second class constraints in this case in the same way as we already discussed and the presence of second class constraints does not influence the way of dealing with first class constraints, we will assume for the further discussion that only first class constraints are present.
We note that in (2.25) only primary first class constraints γ'_k appear. It is first conjectured by Dirac [Dir64] and shown in numerous examples that in many situations it is physically correct to consider the *secondary* first class constraints as well by adding them to the primary Hamiltoninan H' using additional Lagrange multipliers λ^a although it does not follow strictly from the formalism:

Definition 2.18. The extended Hamiltonian H_e is

$$H_e = \tilde{H} + \lambda^m\gamma_m \tag{2.26}$$

and is assumed to govern the time evolution:

$$\{F, H_e\} = \{F, \tilde{H}\} + \lambda^m\{F, \gamma_m\} \tag{2.27}$$

Since λ^m are Lagrange multipliers, the time evolution is not unique. For an infinitesimal time evolution, we have $F(t) = F + (\{F, \tilde{H}\} + \lambda^m\{F, \gamma_m\})t + \mathcal{O}(t^2)$. Since, the λ^m are arbitrary, for infinitesimal time evolution the difference between two functions arising from F is:

$$\delta_\epsilon F = \epsilon^m\{F, \gamma_m\} \tag{2.28}$$

Hence, two functions differing in terms of $\{F, \gamma_m\}$ describe the *same* observable. This motivates

Definition 2.19. (i) First class constraints are generators of *gauge transformations*.

(ii) A function $F \in C^\infty(M)$ is called observable if $\{F, \gamma_m\} \approx 0\ \forall m$.

Remark 2.20. *Notice that every second class constraint classically eliminates one degree of freedom while every first class constraint removes two. The reason for this is that they do not only delete degrees of freedom but also compute gauge orbits. However, since the number of second class constraints is always even, the reduced phase space has always again an even number of physical degrees of freedom.*

2.3 The Idea of Canonical Quantization

After having discussed the main properties of classical Singular Systems, our goal is to perform the transition from Classical to Quantum Theory. So far a working hypothesis is to define a Quantum Theory by fixing a quantization map that describes the transition from the Classical to the Quantum Theory.
In this thesis, we will work with the so called *Canonical Quantization*. In a posterior chapter, we will describe the procedure of Canonical Quantization in a different formalism on a more rigorous level. The purpose of this chapter is to give a first impression of the chosen quantization and the problems one faces in the presence of constraints.
Historically, this procedure was first introduced by Dirac in his famous book on QM [Dir30]. The basic idea may be summarized as

$$\{\cdot, \cdot\} \to -\frac{i}{\hbar}[\cdot, \cdot] \tag{2.29}$$

which is to be read as: map each classical observable to an operator on a suitable Hilbert space in such a manner that the Poisson bracket is mapped to $-\frac{i}{\hbar}$ times the commutator of the corresponding operators. The idea (2.29) is formulated for an arbitrary choice of units. In the units that are used in this thesis the pre-factor in front of the commutator is just $-i$.
There are many attempts to embed the formal idea (2.29) in a rigorous mathematical context. In a very popular approach, one assumes that the operators corresponding to classical observables are at least symmetric. That means, we need to fix a suitable Hilbert space $\mathcal{H}$ and find a linear map $Q : C^\infty(M) \to SYM(\mathcal{H})$ that intertwines the Poisson bracket on $C^\infty(M)$ with the commutator on $SYM(\mathcal{H})$. Due to the Stone-von Neumann theorem one can additionally demand that the position and momentum are mapped to the standard form of the position and momentum operators.
For this approach, Groenewold proved the famous No Go theorem [Gro46] which states that there are only certain Lie-subalgebras of $C^\infty(M)$, for that (2.29) can be satisfied.
We refer to [Giu03] for a modern perspective on the canonical quantization procedure in the this formalism. The author works mathematically rigorous and amongst others proves the No Go theorem.

2.3.1 The Idea of Canonical Quantization of Singular Systems

As we have discussed in the previous section a singular system is constrained. The procedure of Canonical Quantization is not consistent with those constraints. This comes from the fact that the Poisson bracket does not respect the constraints in the sense that the Poisson bracket of a constraint function with some other function does in general not vanish (not even weakly). Hence, the quantization of such a bracket does not vanish weakly as well.
The commutator of the corresponding operators on the other hand does vanish weakly by construction. Hence, the quantization procedure for singular systems needs to be modified.

Systems with only Second class constraints

If only second class constraints are present we have seen in the previous section that one can construct the Dirac bracket on the constrained phase space such that the constraints are strongly implemented. It turned out that the Poisson brackets calculated on the reduced phase space equal the Dirac brackets on the original phase space. But the reduced phase space is nothing but the physical space of independent variables.
Hence, it is reasonable to modify the formal requirement (2.29) to:

$$\{\cdot,\cdot\}_D \to -\frac{i}{\hbar}[\cdot,\cdot] \tag{2.30}$$

This modification does however not remove the problems of ordering ambiguities after quantization. The results of the Groenewold theorem remain valid.

Systems with First class constraints

The quantization of systems with first class constraints is more subtle. We have seen that first class constraints generate gauge transformation that do not change the physical state. Hence, there is more that just one set of canonical variables that describes a physical state. This ambiguity is not fixed by the transition from Poisson to Dirac bracket since this only respects second class constraints. For this reason it is not possible to transfer the previously discussed procedure for only second class constraints to the case where first class constraint are present as well.
In order to define Canonical Quantization we need to eliminate this ambiguity. This is done by imposing external restrictions on the canonical coordinates, so called *gauge conditions*. These gauge conditions have to be chosen such that there is a one-to-one correspondence between physical states and canonical coordinates that are left independent after imposing the additional restrictions. It is admissible to bring in such external restrictions since they merely remove the ambiguity of the observables by fixing the the Lagrange multipliers and do not affect the gauge-independent quantities.
In the following, we will again denote by ζ_μ the second class and by γ_m the first class

constraints of the system under consideration. In order to fix the gauge completely, the conditions have to fulfil two requirements:

(i) *Invertibility*:
Let Ω_m denote the gauge conditions. The requirement of invertibility then reads:

$$\det(\{\Omega_n, \gamma_m\}) \neq 0 \tag{2.31}$$

(ii) *Attainability*:
There must exist a transformation from the arbitrary values of the gauge variables to those satisfying Ω_n.

The gauge conditions also have to satisfy the consistency relations, which are now formulated with the Dirac brackets:

$$\{\Omega_n, H_e\}_D = \{\Omega_n, \tilde{H}\}_D + \lambda^n\{\Omega_n, \gamma_m\} \approx 0 \tag{2.32}$$

This fixes the Lagrange arbitrary multipliers λ^n

$$\lambda^m = -G'^{nm}\{\Omega_n, \tilde{H}\}_D \tag{2.33}$$

with $G'_{nm} = \{\Omega_n, \gamma_m\}$ such that the time evolution becomes

$$\begin{aligned}\{F, H_e\}_D &= \{F, \tilde{H}\}_D - \{F, \gamma_m\}G'^{mn}\{\Omega_n, \tilde{H}\}_D \\ &= \{F, \tilde{H}^*\} - \{F, \gamma_m\}G'^{mn}\{\Omega_n, \tilde{H}^*\}\end{aligned} \tag{2.34}$$

with $\tilde{H}^* = \tilde{H} - \xi_\mu M^{\mu\nu}\{\xi_\nu, \tilde{H}\}$ the so called first class conjugate.

Corollary 2.21. *The time evolution of observables does not change by fixing the gauge.*

Proof. Recall that an observable F is defined via $\{F, \gamma_m\} \approx 0$. Hence, the latter part that fixes the Lagrange multipliers vanish in the time evolution of an observable. □

Proposition 2.22. *Let the first class constraints and the gauge conditions be collected in a vector $\phi_v := (\gamma_m, \Omega_n)$ (the index v again does not indicate 4-vector components) and call the corresponding constraint matrix $G_{vw} = \{\phi_v, \phi_w\}$. Then G has the form*

$$G = \begin{pmatrix} 0 & -G'^T \\ G' & * \end{pmatrix}, \tag{2.35}$$

where $*$ is some block matrix that will not play a role. Hence, G is invertible due to the invertibility of G'. Let further $\{\cdot,\cdot\}_{D^*}$ denote the Dirac bracket corresponding to G. Then the time evolution (2.34) can be rewritten:

$$\{F, H_e\}_D \approx \{F, \tilde{H}^*\}_{D^*} \tag{2.36}$$

Proof. The inverse of G has the form:

$$G^{-1} = \begin{pmatrix} C & G'^{-1} \\ -(G'^T)^{-1} & 0 \end{pmatrix}, \tag{2.37}$$

where C is some block matrix.

$$\{F, H_e\}_D \approx \{F, \tilde{H}^*\} - \{F, \gamma_m\} G'^{mn} \{\Omega_n, \tilde{H}^*\} - \{F, \gamma_m\} C^{nm} \{\gamma_m, H^*\} \tag{2.38}$$
$$+ \{F, \Omega_m\} G'^{nm} \{\gamma_n, H^*\} \tag{2.39}$$
$$= \{F, \tilde{H}^*\} - \{F, \phi_\nu\} G^{\mu\nu} \{\phi_\mu, H^*\} \tag{2.40}$$
$$= \{F, \tilde{H}^*\}_{D^*} \tag{2.41}$$

Note that we have inserted a zero in the first line since $\{F, \gamma_m\} \approx 0$ and γ_m is first class which means $\{\gamma_m, \tilde{H}^*\} \approx \{\gamma_m, \tilde{H}\}$ and the Dirac-Bergmann procedure ensures that this vanishes. □

By introducing the gauge conditions Ω_n we have turned the first class constraints to second class ones. Hence, there are no more generators of gauge transformations and the time evolution becomes unique. The time evolution of physical quantities however is not affected.

Proposition 2.23. *Collecting all constraints and gauge condition in a vector $\sigma_k = (\gamma_m, \Omega_n, \xi_\mu)$ and defining the corresponding constraint matrix $M_{ij} = \{\sigma_i, \sigma_j\}$ whose Dirac bracket will be denoted by $\{\cdot, \cdot\}_{D_{gf}}$ allows us to write the time evolution from Proposition 2.22 in the compact form:*

$$\frac{d}{dt} F = \{F, \tilde{H}\}_{D_{gf}} \tag{2.42}$$

Proof. The proof is quite long and does not give any new insights. It can for instance be found in [RR10]. □

That means that fixing the gauge removes all first class constraints and leaves us with a system with only second class constraints without changing physical information. The Canonical Quantization for Systems with only second class constraints can now be applied.

Remark 2.24. *One needs to be careful by fixing the gauge in the above described manner. Gribov [Gri78] showed that it is in general not possible to find global gauge fixing such that every gauge orbit is intersected exactly once. This is called Gribov ambiguity.*
However, this ambiguity is only present in non-abelian gauge theories. Since we will only be dealing with QED which turns out to feature an abelian gauge symmetry, we will not have to worry about this.

2.4 Formal Transition to Field Theory

In the last sections, we have discussed the treatment of singular mechanical systems. In this chapter, we will discuss the relation to Field Theories on a non-rigorous level and how the results from the previous sections about the treatment of Singular Systems can be generalized to Field Theories.

2.4.1 From Point Mechanics to Field Theories

To define a Field Theory first we need to define what a field is. Unfortunately, there is no generic definition that works in every situation. One may think of a field as some kind of "function" on the spacetime. In practice, the space of fields $\mathcal{F}$ is defined for every theory individually. We will discuss one rigorous approach that is suitable for the purposes of this thesis in the next section.
The main object of classical Field Theory is the Lagrangian which is a functional of the fields and their time derivatives

$$L(t) = L(\phi_i(x,t), \dot{\phi}_i(x,t)) \tag{2.43}$$

The indices indicate the number of the fields. Since we restrict ourselves to local Field Theories it is possible to rewrite the Lagrangian as volume integral over the Lagrange density

$$L = \int d^3x \; \mathcal{L}(\phi_i(x), \nabla\phi_i(x), \dot{\phi}_i(x)) \tag{2.44}$$

where the Lagrange density is now a function of the fields and their spatial and time derivatives.
To pass to the Hamiltonian formulation we need to define the canonical momenta. To do so we introduce the notion of the functional derivative:

$$\frac{\delta}{\delta\phi^i} = \frac{\partial}{\partial\phi^i} - \partial_j \frac{\partial}{\partial(\partial_j\phi^i)} \tag{2.45}$$

The canonical momentum π_i conjugate to ϕ^i then is the functional derivative of the Lagrangian with respect to the time derivative $\dot{\phi}_i$:

$$\pi_i := \frac{\delta L}{\delta\dot{\phi}^i} \tag{2.46}$$

The Hamiltonian then is a functional of the fields and their canonical momenta

$$H = H(\phi_i(x), \pi_i(x)) \tag{2.47}$$

and is defined via the Legendre transformation

$$H = \int d^3x \; \pi_i\dot{\phi}^i - L = \int d^3x \; \mathcal{H}(\phi_i, \pi_i, \nabla\phi_i, \nabla\pi_i) \tag{2.48}$$

where the function $\mathcal{H}$ is called the Hamilton density. We can read off that it is defined via $\mathcal{H} = \pi_i \dot{\phi}^i - \mathcal{L}$.
The equations of motion are governed by the principle of least action. Where the action is:

$$\mathcal{A} = \int d^4x \mathcal{L} = \int d^4x \left(\pi_i \dot{\phi}^i - \mathcal{H} \right) \tag{2.49}$$

The standard procedure of variational calculus yields:

$$\dot{\phi}^i = \frac{\partial H}{\partial \pi_i} - \partial_j \frac{\partial H}{\partial(\partial_j \pi_i)} = \frac{\delta H}{\delta \pi_i} \tag{2.50}$$

$$\dot{\pi}_i = -\frac{\partial H}{\partial \phi^i} + \partial_j \frac{\partial H}{\partial(\partial_j \phi^i)} = -\frac{\delta H}{\delta \phi^i} \tag{2.51}$$

To simplify the notion of the equations of motion we introduce the Poisson bracket of two functionals $F(\phi^i, \pi_i)$ and $G(\phi^i, \pi_i)$:

$$\{F, G\} := \int d^3z \left(\frac{\delta F}{\delta \phi^i(z)} \frac{\delta G}{\delta \pi_i(z)} - \frac{\delta F}{\delta \pi_i(z)} \frac{\delta G}{\delta \phi^i(z)} \right) \tag{2.52}$$

It is understood that the Poisson bracket and later the Dirac bracket are always evaluated at same time.
The fundamental functional derivatives are:

$$\frac{\delta \phi^i(x)}{\delta \pi_j(y)} = \frac{\delta \pi_i(x)}{\delta \phi^j(y)} = 0 \tag{2.53}$$

$$\frac{\delta \phi^i(x)}{\delta \phi^j(y)} = \frac{\delta \pi_j(x)}{\delta \pi_i(y)} = \delta^i_j \delta^{(3)}(x-y) \tag{2.54}$$

Hence, the equations of motion can be written as:

$$\dot{F} = \int d^3x \left(\frac{\delta F}{\delta \phi^i} \dot{\phi}^i + \frac{\delta F}{\delta \pi_i} \dot{\pi}_i \right) = \{F, H\} \tag{2.55}$$

2.4.2 Constrained Field Theories

From now on constraints are taken into account. Again, we have to face the fact that the constraint functions become densities depending on the fields, their canonical momenta and spatial derivatives. They are coupled to the Hamiltonian via Lagrange multiplier fields:

$$H = \tilde{H} + \int d^3x \, \lambda^r(x) \chi_r(x) \tag{2.56}$$

where λ^r are the Lagrange multiplier fields and χ_r the constraint densities.
Respecting the constraints in the action one immediately sees that we get the equations of motion:

$$\dot{F} = \{F, \tilde{H}\} + \int d^3x \; \lambda^r(x)\{F, \chi_r\} \tag{2.57}$$

The Dirac-Bergmann algorithm to identify secondary constraints and the classification into first and second class constraints is the same in Field Theory as before. Thus, we will not write down every result in detail.
The above discussion can be generalized to the following rules for the transition to field theory:

1. The phase space variables and Lagrange multipliers are replaced by coordinate depending fields
2. A phase space function becomes a functional of the canonical fields
3. Whenever there is a summation of the canonical fields or Lagrange multipliers there has to be a volume integral over the sum

From these rules we can immediately read off the expression for the Dirac bracket for a field theory:

$$\{F, G\}_D = \{F, G\} - \int d^3z d^3z' \; \{F, \gamma_m(z)\}(M^{-1})^{mn}(z, z')\{\gamma_n(z'), G\} \tag{2.58}$$

where the γ_k denote the second class constraints and $M_{mn}(x, y) = \{\gamma_m(x), \gamma_n(y)\}$ is the constraint matrix. The inverse M^{-1} is meant via the integral relation:

$$\int d^3z M_{mn}(x, z)(M^{-1})^{nj}(z, y) = \delta^j_m \delta^{(3)}(x - y) \tag{2.59}$$

In the same way one sees that the gauge transformations generated by the first class constraints take the infinitesimal form:

$$\delta_\epsilon F = \int d^3y \; \epsilon(y)\{F, \chi(y)\} \tag{2.60}$$

2.5 Canonical Quantization of Weyl Systems

The purpose of this section is define a rigorous approach to Field theory and to fix a quantization map.
In the first subsection, we discuss a rigorous treatment of Classical Systems. In this course, one comes across the Weyl element as observables. In the second subsection, we will discuss the algebra that is generated by the Weyl elements. We will explain, that specifying a regular representation of the Weyl algebra amounts to describing a physical system.
In the last subsection, we will discuss a way of defining Canonical Quantization rigorously.

2.5.1 Weyl Elements in Classical Field Theory

The results of this subsection are taken from [HR15]. For the most part, they can be found in chapter 8.
In the last sections, we have discussed the usual approach to Field Theory by replacing discrete quantities in the point-particle mechanics by densities.
In this chapter, we will discuss a different approach. The advantage is that naturally a classical version of the Weyl algebra arises which allows us to define the Canonical Quantization procedure on a more rigorous level.
Unfortunately, this extended formalism does not allow for the usual Hamilton and Lagrangian formulation.

We will see later that for finding finite energy solutions of the Maxwell equations, it is necessary to work with the real direct sum Hilbert space:

$$\mathcal{R} := L^2(\mathbb{R}^3, \mathbb{R}^3) \oplus L^2(\mathbb{R}^3, \mathbb{R}^3) \tag{2.61}$$

in which we combine the electric and magnetic fields as 6-tuple:

$$\Psi := (\vec{E}, \vec{B}) \tag{2.62}$$

Moreover, we will discuss that the electric and magnetic field are related via the Maxwell equations. Since those equations are differential equations, we need to assume certain differentiability and regularity properties.
In order to make sense of differentiation of fields, one extends the phase space from $\mathcal{R}$ using so-called Gelfand triples. This formalism has the advantage that it also covers the discussion of discrete distributions.

Definition 2.25. A *Gelfand triple*

$$E \subset \mathcal{R} \subset E' \tag{2.63}$$

consists of $\|\cdot\|$-dense locally convex subspace E of a Hilbert space $\mathcal{R}$, such that the topology on E is finer the $\|\cdot\|$-topology, and its topological dual E'. Note that the second implication is weak*-dense.

We will subsequently refer to E as the space of test functions. This space needs to be chosen individually for each problem and represents the degrees of freedom of the physical problem. In our case of the electromagnetic field, it will be chosen such that it satisfies the above mentioned differentiation and regularity conditions.
The extended phase space is now taken to be the topological dual space E' of the test function space E. The field Hilbert space $\mathcal{R}$ is injectively embedded in the extended field space via

$$\mathrm{i} : \mathcal{R} \to E' \tag{2.64}$$

$$\xi \mapsto (\xi, \cdot) \tag{2.65}$$

where $(\cdot,\cdot)$ denotes the scalar product on $\mathcal{R}$. Since $E \subset \mathcal{R}$ is dense, the mapping is injective.
As we have discussed earlier the time evolution is governed by the Poisson bracket in the usual framework. To allow for such a formulation on the extended Phase space as well, we need to assume that the test function space is equipped with an additional structure.

Definition 2.26. Let E be a real vector space. A map

$$\sigma : E \times E \to \mathbb{R} \tag{2.66}$$

is called *pre-symplectic structure* if it is anti-symmetric:

$$\sigma(f,g) = -\sigma(g,f) \qquad \forall f,g \in E \tag{2.67}$$

The two-tuple (E,σ) then is called *pre-symplectic space*.
A pre-symplectic structure σ' is called *symplectic strucutre* if it is non-degenerate:

$$\sigma'(f,g) = 0 \quad \forall g \in E \Rightarrow f = 0 \tag{2.68}$$

Then (E,σ') is denoted a *symplectic space*.

The phase space E' is equipped with the weak-$*$-topology, by which we have $E'' = E$.

Definition 2.27. By assigning to each field configuration its expectation value, the space of observables is the topological dual of the phase space.
The most relevant observables are the *smeared fields*. For each $f \in E$, we define:

$$\phi^0(f) : E' \to \mathbb{R} \tag{2.69}$$
$$F \mapsto F(f) \tag{2.70}$$

In general the smeared fields are unbounded on the phase space E' and hence, it is useful to work with the classical Weyl elements:

Definition 2.28. Let $f \in E$ and ϕ^0 as above, then the corresponding classical Weyl element is defined as map:

$$W^0(f) : E' \to \mathbb{C}_1 \tag{2.71}$$
$$F \mapsto e^{i\phi^0(f)[F]} = e^{iF(f)} \tag{2.72}$$

We are now able to define the fundamental (Poisson) bracket relations:

$$\{\phi^0(f), \phi^0(g)\} = \sigma(f,g)\mathbb{1} \tag{2.73}$$
$$\{W^0(f), W^0(g)\} = \sigma(f,g)W^0(f+g) \tag{2.74}$$

where $\mathbb{1}$ is the unit function with $F[\mathbb{1}] = 1$ for all $F \in E'$.
Call

$$\widetilde{\mathcal{W}}(E,0) := LH\{W^0(f), f \in E\}, \tag{2.75}$$

where LH indicates the linear hull. In the next chapter, we will discuss an additional structure on $\widetilde{W}(E,\sigma)$, which we will make use of for the definition of the quantization procedure.

Remark 2.29. *In this formalism, the replacement of the Poisson bracket by the Dirac bracket results in a modification of the (pre-)symplectic form.*

For many applications, including (Q)ED, a complex test function space is essential.

Definition 2.30. Let (E,σ) be a pre-symplectic space. A real linear operator $J : E \to E$ is called *complex structure* if it satisfies:

$$J^2 = -\mathbb{1} \tag{2.76}$$

$$\sigma(J\cdot, J\cdot) = \sigma(\cdot,\cdot) \tag{2.77}$$

$$\sigma(f, Jf) \geq 0 \qquad \forall f \in E \tag{2.78}$$

Assume that (E,σ) is a pre-symplectic space with complex structure j, then we can define a multiplication with $z \in \mathbb{C}$ on E via:

$$z \cdot f = Re(z)f + Im(z)Jf \qquad \forall f \in E \tag{2.79}$$

In particular, we have

$$Jf = if \qquad \forall f \in E \tag{2.80}$$

which turns E to a complex vector space, which we denote by E_J. The pre-symplectic structure allows for defining a complex semi-inner product

$$\sigma_J(f,g) = \sigma(f, Jg) + i\sigma(f,g) \tag{2.81}$$

which becomes an inner product if and only if σ is symplectic. In this case, the complexified test function space (E_J, σ_J) becomes a pre-Hilbert space. The previous discussion can be generalized to a complexified test function space straightforwardly.

Remark 2.31. *Let (E,σ) be a pre-symplectic space with complex structure J, then there is a homeomorphism*

$$E' \to E'_J \tag{2.82}$$

$$F(\cdot) \mapsto L_F(\cdot) := \frac{1}{\sqrt{2}}\left(F(\cdot) - iF(J\cdot)\right) \tag{2.83}$$

with the inverse

$$F(\cdot) = \frac{1}{\sqrt{2}}\left(L_F(\cdot) + \overline{L_F(\cdot)}\right) = \sqrt{2}\left(Re(L_F(\cdot))\right) \tag{2.84}$$

In the following, we will drop the index J when working with a complexified test function space.

Definition 2.32. Let ϕ^0 denote the classical fields as before. Then we introduce the *classical creation* and *annihilation fields* via:

$$a^0(f) := \frac{1}{\sqrt{2}}\left(\phi^0(f) + i\phi^0(Jf)\right) \tag{2.85}$$

$$a^{0\dagger}(f) := \frac{1}{\sqrt{2}}\left(\phi^0(f) - i\phi^0(Jf)\right) \tag{2.86}$$

for each $f \in E$.
By construction, they satisfy:

$$a^0(f)[F] = \overline{L_F(f)}, \qquad a^{0\dagger}(f)[F] = L_F(f) \tag{2.87}$$

Remark 2.33. *The latter property shows that the smeared classical creation and operation fields are the complex conjugate of each other for each $f \in E$.*
Furthermore, using the fundamental Poisson bracket, one can show:

$$\{a^0(f), a^{0\dagger}(g)\} = -i\sigma_J(f,g)\mathbb{1} \tag{2.88}$$

$$\{a^0(f), a^0(g)\} = 0 = \{a^{0\dagger}(f), a^{0\dagger}(g)\} \tag{2.89}$$

for all test functions $f, g \in E$.

2.5.2 Weyl C^*-algebra, Weyl systems and Representations

The results of this subsection can be found in [HR15], mostly in chapter 16 - 18. Hence, we will omit the proofs. The interested reader can find them in the abovementioned literature.
In this section, we will treat $\hbar$ as a real variable. We will need this later to discuss the relation between classical and quantum physics in the limit $\hbar \to 0$. In the subsequent chapters, we will return to the units, that we defined earlier with $\hbar = 1$.
In the last section, we have introduced the commutative algebra $\tilde{\mathcal{W}}(E,0)$ for a pre-symplectic space (E,σ). In Quantum Theories, the observables do not commute in general. For applications in Quantum Theories, we define a non-commutative version of $\tilde{W}(E,0)$.
Equip the linear hull of $W(f), f \in E$ with a $*$-operation and a twisted product

$$W(f)^* = W(-f) \tag{2.90}$$

$$W(f)W(g) = e^{-\frac{i}{2}\hbar\sigma(f,g)}W(f+g) \tag{2.91}$$

for some $\hbar \in \mathbb{R}$. These relations are called *Weyl relations*. Then call the linear hull $\{W(f), f \in E\} = \tilde{\mathcal{W}}(E,\hbar\sigma)$.
Due to the Weyl relations, every polynomial of $W(f)$'s reduces to a linear combination making $\widetilde{\mathcal{W}}(E,\hbar\sigma)$ a $*$-algebra. To pursue one the construction of a C^*-algebra related to $\tilde{\mathcal{W}}(E,0)$, we need the notion of a representation

Definition 2.34. A *representation* $(\Pi, \mathcal{H})$ of a $*$-algebra A is a $*-$homomorphism $\Pi : A \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$,i. e. for $W, W_1 \in A$

1. linearity: $\Pi(aW + bW_1) = a\Pi(W) + b\Pi(W_1)$
2. multiplicativity: $\Pi(WW_1) = \Pi(W)\Pi(W_1)$
3. symmetry: $\Pi(W^*) = \Pi(W)^*$

Moreover, if $\Pi(1) = \mathbb{1}$, the representation is called *unital*.

Proposition 2.35. *There is a unique C^*-norm $\|\cdot\|$ on $\widetilde{\mathcal{W}}(E, \hbar\sigma)$ such that every representation and state is $\|\cdot\|$-continuous. It is given by:*

$$\|A\| = \sup\{\|\Pi(A)\|, \Pi \text{ is a representation}\} \qquad \forall A \in \widetilde{\mathcal{W}}(E, \hbar\sigma) \tag{2.92}$$

Definition 2.36. Let (E, σ) be a pre-symplectic space. Then the completion of $\widetilde{\mathcal{W}}(E, \hbar\sigma)$ in the unique C^*-norm

$$\mathcal{W}(E, \hbar\sigma) := \overline{\widetilde{\mathcal{W}}(E, \hbar\sigma)}^{\|\cdot\|} \tag{2.93}$$

is called the *Weyl C^*-algebra* over the pre-symplectic space $(E, \hbar\sigma)$.

The Weyl C^*-algebra is assumed to be the abstract version of the algebra of observables. We note that the C^*-algebra of classical observables arises from the case $\hbar = 0$. In practice, we are interested in concrete realizations of the Weyl C^*-algebra:

Definition 2.37. Let (E, σ) be a pre-symplectic space and $\hbar \in \mathbb{R}$. Suppose $E \ni f \mapsto W_\kappa^\hbar(f)$ to be mapping from E to the unitary operators over some non-trivial Hilbert space $\mathcal{H}_\kappa$ satisfying the Weyl relations.
Then the 2-tuple $(W_\kappa^\hbar, \mathcal{H}_\kappa)$ is called a *Weyl system* over $(E, \hbar\sigma)$.
A Weyl system $(W_\kappa, \mathcal{H}\kappa)$ over $(E, \hbar\sigma)$ is called regular, if for each $f \in E$ the mapping $\mathbb{R} \ni t \to W_\kappa(tf)$ is strongly continuous.

Definition 2.38. A representation of a C^*-algebra A is called *irreducible* if the only invariant closed subspaces of $\mathcal{H}$ under the action of $\Pi(A)$ are the trivial ones $\{0\}$ and $\mathcal{H}$.

Lemma 2.39. *Irreducibility of a representation $(\Pi, \mathcal{H})$ is equivalent to one of the following conditions:*

1. *Every non-zero vector $\Psi \in \mathcal{H}$ is cyclic, i.e.*

$$\overline{\Pi(A)\Psi} = \mathcal{H} \tag{2.94}$$

2. *The commutant of $\Pi(A)$: $\Pi(A)' = \mathbb{C}\mathbb{1}$ is trivial.*

Proof. See e.g. [Dyb17]. □

Definition 2.40. Two representations $(\Pi_1, \mathcal{H}_1)$ and $(\Pi_2, \mathcal{H}_2)$ are called *unitary equivalent* if there is a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that:

$$\Pi_2(\cdot) = U\Pi_1(\cdot)U^{-1} \tag{2.95}$$

We will state a general theorem of irreducible representations that will prove to be useful at a later point of this thesis.

Theorem 2.41. *Let $\Pi : A \to \mathcal{B}(\mathcal{H})$ be a irreducible representation of some C^*-algebra A and $x_1, \ldots, x_j$ be an linearly independent set in $\mathcal{H}$ and $y_1, \ldots, y_j \in \mathcal{H}$, then there is an element $W \in A$ such that $\Pi(A)x_i = y_i$ for all $i \in \{1, \ldots, n\}$.*
If $y_i = Vx_i$ for some unitary operators V, then A can be chosen to be unitary as well.

Proof. See [KR86], Thm. 10.2.1. □

Definition 2.42. If for each $f \in E$ the one-parameter group $\{\Pi(W(tf)); t \in \mathbb{R}\}$ is strongly continuous, then the representation Π is called a regular representation of $\mathcal{W}(E, \hbar\sigma)$.

In the language of the representations, we have a nice connection between regular representations of $\mathcal{W}(E, \sigma)$ over a pre-symplectic space (E, σ) and Weyl systems.

Proposition 2.43. *The Weyl systems $(W_\Pi, \mathcal{H}_\pi)$ over a pre-symplectic space (E, σ) are in one-to-one correspondence to the regular representations of the Weyl C^*-algebra $\mathcal{W}(E, \sigma)$.*

Moreover, we can rephrase this proposition to find another property that characterizes the Weyl C^*-algebra over a pre-symplectic space (E, σ) uniquely.

Proposition 2.44. *Let (E, σ) be a pre-symplectic space and $\hbar \in \mathbb{R}$. Then $\mathcal{W}(E, \hbar\sigma)$ is the unique C^*-algebra (up to *-isomorphisms) generated by non-zero elements $W(f), f \in E$, satisfying the following two uniqueness assumptions:*

1. *The elements $W(f), f \in E$, fulfil the Weyl relations*
2. *Every Weyl system $(W_\kappa, \mathcal{H}_\kappa)$ over $(E, \hbar\sigma)$ arises from a representation $(\Pi_\kappa, \mathcal{H}_\kappa)$ of $\mathcal{W}(E, \hbar\sigma)$ via*

$$W_\kappa(f) := \Pi_\kappa(W(f)) \qquad \forall f \in E \tag{2.96}$$

Theorem 2.45 (Stone's Theorem)**.** *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group. Then there exists a unique operator $A : \mathcal{D}_A \to \mathcal{H}$, that is self-adjoint on $\mathcal{D}_A$ and such that*

$$\forall t \in \mathbf{R} : \qquad U_t = e^{itA} \tag{2.97}$$

Conversely, every self-adjoint operator generates a strongly continuous one-parameter unitary group in the above mentioned way.

Remark 2.46. *The infinitesimal generator A of a strongly continuous unitary group$(U_t)_{t\in\mathbf{R}}$ may be computed as:*

$$A\psi = -i \lim_{t\to 0} \frac{U(t)\psi - \psi}{h} \tag{2.98}$$

We will write $A = -i \left.\frac{dU(t)}{dt}\right|_{t=0}$. The domain of the so-called generator A is

$$dom(A) := \{\psi \in \mathcal{H} | \lim_{t\to 0} \frac{1}{t} \left(U(t) - \mathbb{1}\right) \Psi \text{ exists in norm}\} \tag{2.99}$$

Definition 2.47. Let $(\mathcal{H}_\Pi, W_\Pi)$ be a regular representation of $\mathcal{W}(E, \hbar\sigma)$. Then we associate it with the quantum field system $(\Phi^\hbar_\Pi, \mathcal{H}_\Pi)$ via Stone's Theorem:

$$\Phi^\hbar_\Pi(f) = -i \left.\frac{dW_\Pi(tf)}{dt}\right|_{t=0}, \qquad f \in E \tag{2.100}$$

Proposition 2.48. *Let $f, g \in E$, then*

$$[\Phi^\hbar_\Pi(f), \Phi^\hbar_\Pi(g)] = i\hbar\sigma(g, f)\mathbb{1} \tag{2.101}$$

on $dom(\Phi^\hbar_\Pi(f)) \cap dom(\Phi^\hbar_\Pi(g))$.

Definition 2.49. If E carries a complex structure, then we can repeat the procedure of the complexified phase space as described in the classical case. If Π is a regular representation of $\mathcal{W}(E, \hbar\sigma)$ for a pre-symplectic space, then we define the creation and annihilation operators

$$a^\hbar_\Pi(f) = \frac{1}{\sqrt{2}} \left(\Phi_\Pi(f) + i\Phi_\Pi(Jf)\right) \tag{2.102}$$

$$a^{\hbar\dagger}_\Pi(f) = \frac{1}{\sqrt{2}} \left(\Phi_\Pi(f) - i\Phi_\Pi(Jf)\right) \tag{2.103}$$

for all $f \in E$.

Proposition 2.50. *$a^\hbar_\Pi(f)$ and $a^{\hbar\dagger}_\Pi(f)$ are closed and mutually adjoint to each other for each $f \in E$. Moreover, on the common domain they satisfy:*

$$[a^\hbar_\Pi(f), a^{\hbar\dagger}_\Pi(f)] = \hbar\sigma_J(f, g)\mathbb{1}_\Pi \tag{2.104}$$

$$[a^\hbar_\Pi(f), a^\hbar_\Pi(f)] = 0 = [a^{\hbar\dagger}_\Pi(f), a^{\hbar\dagger}_\Pi(f)] \tag{2.105}$$

2.5.3 Canonical Weyl Quantization

In the formalism that we have chosen, the procedure of canonical quantization can be described rigorously. The results are taken from [HR15], chapter 16 and 19.
Recall from section 2.3 that the idea of canonical quantization is to assign to each

classical observable a quantum observable. Hence, we will throughout this thesis understand QED as quantized electrodynamics.
In general, a quantum observable is a self-adjoint operator on a fixed Hilbert space. The correspondence between classical and quantum observables is such that most of the algebraic properties are preserved. The main difference from an algebraic point of view is that the quantum algebra of observable is no longer commutative.

Definition 2.51. The $\mathbb{R}$-linear mapping $\phi^0 : f \to \phi^0(f)$ from E into the $\mathbb{R}$-valued functions on E' , as specified in the preceding subsection, constitutes the *classical field system* over the pre-symplectic test function space (E, σ).

The fact that the field system specified by ϕ^0 is called the classical one comes from the fact that the fields $\phi^0(f), \phi^0(g)$ and the Weyl operators are commutative.
The quantum field system then arises in the same manner with the slight modification that the fields are no longer commuting

Definition 2.52. Let (E, σ) be a pre-symplectic space, with $\sigma \neq$ and $0 \neq \hbar \in \mathbb{R}$. Suppose $E \ni f \to \phi^\hbar_\kappa(f)$ to be an $\mathbb{R}$-linear mapping from E into the self-adjoint operators on some nontrivial complex Hilbert space $\mathcal{H}_\kappa$ such that the canonical commutation relations

$$[\phi^\hbar_\kappa(f), \phi^\hbar_\kappa(g)] = i\hbar\sigma(f,g)\mathbb{1} \tag{2.106}$$

on $dom(\phi^\hbar_\kappa(f)) \cap dom(\phi^\hbar_\kappa(g))$ are fulfilled. Then the tuple $(\phi^\hbar_\kappa, \mathcal{H}_\kappa)$ is called a *canonical quantum field system* or a representation of the CCR over $(E, \hbar\sigma)$.

Remark 2.53. *We discuss the properties of an explicit example of a field operator ϕ in greater detail in the next subsection.*

Definition 2.54. Suppose a classical field system ϕ^0 over a pre-symplectic space (E, σ), with $\sigma \neq 0$.
Then a canonical field quantization of ϕ^0 is its replacement by a canonical quantum field system of the form $(\phi^\hbar_\kappa, \mathcal{H}_\kappa)$ over $(E, \hbar\sigma)$ with $\hbar \neq 0$.

Unfortunately, the quantization of fields is only the first step of a complete quantization procedure because it does not cover, how to quantize arbitrary non-linear functions of the fields. As we seen, this issue does not occur for the Weyl algebra since polynomials of Weyl elements break down to linear combinations due the Weyl relations. Hence, it is customary, to work with a quantization of the Weyl algebra and define the fields in the respective representation as generators of strongly continuous one-parameter subgroups.

Definition 2.55. Let (E, σ) be a pre-symplectic space and associated with it the commutative classical Weyl C*-algebra $\mathcal{W}(E, 0)$. Under the *canonical Weyl quantization* we understand a mapping

$$\mathcal{W}(E, 0) \to \mathcal{W}(E, \hbar\sigma) \tag{2.107}$$

with $\hbar \neq 0$ such that:

$$\sum_i c_i W^0(f_i) \mapsto \sum_i c_i W^\hbar(f_i) \tag{2.108}$$

The transition from the classical to the quantum Weyl algebra is then followed by specifying a regular representation Π which depends on the physical situation.
The quantization of the fields is then given by:

$$\phi^0(f) \mapsto \phi^\hbar_\Pi(f), \qquad f \in E \tag{2.109}$$

Remark 2.56. *Provided E carries a complex structure, then the classical annihilation and creation operators are mapped to the quantum creation and annihilation operators:*

$$a^{0(\dagger)}(f) \mapsto a^{\hbar(\dagger)}_\Pi(f), \qquad f \in E \tag{2.110}$$

Remark 2.57. *We have not yet clarified in which sense the canonical Weyl quantization is related the symbolic replacement*

$$\{\cdot,\cdot\} \to -\frac{i}{\hbar}[\cdot,\cdot] \tag{2.111}$$

that we introduced earlier.

In the literature the problem of different quantization procedures is an extensively discussed issue. A popular approach, that addresses the issue of canonical quantization on a rigorous level, is Deformation Quantization. The following discussion is taken from [BR10].
In classical theories, the algebra observables is usually equipped with two operations. On one side, one has the usual commutative product and on the other side the Poisson bracket, under which the algebra becomes a Lie algebra. Such an algebra is called *Poisson algebra*.
In the formalism, that we chose, the Poisson algebra of the classical system is $\widetilde{\mathcal{W}}(E,0)$ whose norm closure is the widely discussed Weyl C^*-algebra.

Definition 2.58 (Strict Deformation Quantization)**.** A *Strict Deformation Quantization* $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar\in I}$ of a Poisson algebra $(\mathcal{P},\{\cdot,\cdot\})$ consists for each $\hbar \in I \subset \mathbb{R}$ of a C^*-algebra $\mathcal{A}^\hbar$ with norm $\|\cdot\|_\hbar$ and of a linear, $*$-preserving map (the quantization map)

$$Q_\hbar : \mathcal{P} \to \mathcal{A}^\hbar \tag{2.112}$$

such that $Q_0 : \mathcal{P} \hookrightarrow \mathcal{A}^0$ is the identical embedding and such that the following conditions are satisfied:

1. Dirac's condition:

$$\lim_{\hbar\to 0}\left\| \frac{[Q_\hbar(A), Q_\hbar(B)]}{i\hbar} - Q_\hbar(\{A,B\})\right\|_\hbar = 0, \qquad \forall A, B \in \mathcal{P} \tag{2.113}$$

2. von Neumann's condition:

$$\lim_{\hbar \to 0} \|Q_\hbar(A)Q_\hbar(B) - Q_\hbar(AB)\|_\hbar = 0, \qquad \forall A, B \in \mathcal{P} \tag{2.114}$$

3. $\hbar \to \|Q_\hbar(A)\|_\hbar$ is continuous for each $A \in \mathcal{P}$

4. Deformation condition: The map $Q_\hbar : \mathcal{P} \to \mathcal{A}^\hbar$ is injective and its image is a sub-$*$-algebra of $\mathcal{A}^\hbar$.

Dirac's condition for a strict deformation quantization is a weakened form of the idea of replacing the Poisson bracket by the commutator.
The canonical Weyl quantization satisfies all of the foregoing conditions and hence is an admissible quantization procedure in the sense of deformation quantization.
We note that the canonical Weyl quantization is far from being a unique quantization procedure. It is, for instance, possible to modify the quantization map by an appropriate quantization factor: $Q_\hbar^\omega : f \to \omega(f,\hbar)W^\hbar(f)$. The particular choice of the quantization factor determines the operator ordering of the fields. For example the factors $\omega(f,\hbar) = e^{-\frac{1\pm 2}{4}\hbar\|f\|^2}$ lead to Wick normal and anti-normal ordering respectively.

2.6 Fock Space and Fock Representation

In contrast to the last section, where we chose a systematic approach to a quantized Field Theory, we take in this section the more popular point of view and start with the definition of (quantum) annihilation and creation operators and a certain Hilbert space which is called *Fock Space*. In this section, we will return to Heaviside units, in particular $\hbar = 1$.

In the following section, we will understand under $\mathfrak{h}$ a complex Hilbert space which we will refer to as one-particle space. The idea of the Fock Space is to construct n-particle Hilbert space from $\mathfrak{h}$ as the tensor product of the one-particle spaces. Anyway, one needs to respect the behaviour of the states under the parity operation, i.e. we need to make sure if we are dealing with Bosons or Fermions.
Depending on whether our particle is a Boson or Fermion the state space of a pair of these particles is either $E_{s,2}(\mathfrak{h} \otimes \mathfrak{h})$ or $E_{a,2}(\mathfrak{h} \otimes \mathfrak{h})$, where $E_{s,2}$ is the projection onto the vectors invariant under permutation on $\mathfrak{h} \otimes \mathfrak{h}$ and $E_{2,a}$ is the projection onto the vectors that change sign under the permutation.
For Bosons the n-particle state space is the symmetric subspace of the n-fold tensor product of the one-particle space $\mathfrak{h}^n = E_{s,n}(\mathfrak{h} \otimes \cdots \otimes \mathfrak{h})$, where the operator $E_{s,n}$ sometimes is called symmetrization operator on $\mathfrak{h}^n$ and is defined by [Dyb17]

$$E_{s,n} = \frac{1}{n!} \sum_{\sigma \in P_n} \sigma \tag{2.115}$$

where P_n is the symmetric group of degree n. We call $\mathfrak{h}_s^n = E_{s,n}(\mathfrak{h} \otimes \cdots \otimes \mathfrak{h})$ the n-particle state.

Definition 2.59. The symmetric Fock Space $\Gamma_s(\mathfrak{h})$ over the $\mathfrak{h}$ is defined as the infinite direct sum of Hilbert spaces

$$\Gamma_s(\mathfrak{h}) = \bigoplus_{n \geq 0} \mathfrak{h}_s^n = \bigoplus_{n \geq 0} E_{s,n}(\bigotimes^n \mathfrak{h}) \tag{2.116}$$

with $\mathfrak{h}^0 = \mathbb{C}\Omega$ where Ω is called the vacuum vector.

The basic operators on the Fock Space are the creation and annihilation operators. They are defined via

Definition 2.60. For any $f \in \mathfrak{h}$, we define the bosonic creation

$$a^\dagger(f) : \mathfrak{h}_s^n \to \mathfrak{h}_s^{n+1} \tag{2.117}$$

$$E_s^n(\bigotimes_{i=1}^n f_i) \mapsto E_s^{n+1}(f \otimes \bigotimes_{i=1}^n f_i) \tag{2.118}$$

and annihilation operator:

$$a(f) : \mathfrak{h}_s^n \to \mathfrak{h}_s^{n-1} \tag{2.119}$$

$$E_s^n(\bigotimes_{i=1}^n f_i) \mapsto \sum_{i=1}^n \langle f, f_i \rangle \, E_s^{n-1}(f_1 \otimes \ldots \hat{f_i} \otimes \cdots \otimes f_n) \tag{2.120}$$

Note that $a^\dagger(f)$ depends linearly on f while $a(f)$ depends antilinearly on f.

Definition 2.61. We call $\Gamma_s(\mathfrak{h}) \supset \Gamma_s^{fin}(\mathfrak{h}) := \{\Psi \in \Gamma_s(\mathfrak{h}); \Psi^{(n)} = 0 \text{ except for finitely many } n\}$ the finite symmetric Fock Space.
Note that $\Gamma_s^{fin}(\mathfrak{h}) \subset \Gamma_s(\mathfrak{h})$ is dense [Att].

Definition 2.62. Using Definition 2.60 we can extend the definition of $a^\dagger(f)$ and $a(f)$ to $\Gamma_s^{fin}(\mathfrak{h})$ where they are unbounded operators.
Let $u, v \in \mathfrak{h}$ and $f, g \in \Gamma(\mathfrak{h})$. Then the following relations hold [Att]:

$$\langle a^\dagger(u) f, g \rangle = \langle f, a(u) g \rangle \tag{2.121}$$

$$[a(u), a^\dagger(v)] = 0 \tag{2.122}$$

$$[a(u), a^\dagger(v)] = \langle u, v \rangle \, 1 \tag{2.123}$$

which implies that $a(u)$ and $a^\dagger(u)$ are mutually adjoint on $\Gamma_s^{fin}(\mathfrak{h})$.

Lemma 2.63. *For $u \in \mathfrak{h}$ note that the operators $a(u)$ and $a^\dagger(u)$ are closable since they have a densely defined adjoint.*
The operator $a^\dagger(u)$ is the adjoint operator of $a(u)$ where by $a^\dagger(u), a(u)$ we mean the closures.

Proof. See [Att]. □

Definition 2.64. Let $\mathfrak{u} : a \to b$ be a unitary operator between two Hilbert spaces, then $\mathfrak{u}$ extends to an untiary operator $\Gamma(\mathfrak{u}) : \Gamma_{(s)}(a) \to \Gamma(b)_{(s)}$ defined via:

$$\Gamma(\mathfrak{u})|_{\Gamma^{(n)}} = \mathfrak{u} \otimes \cdots \otimes \mathfrak{u} \tag{2.124}$$

$$\Gamma(\mathfrak{u})\Omega_1 = \Omega_2 \tag{2.125}$$

The unitary operator $\Gamma(\mathfrak{u})$ is called the *second quantization* of $\mathfrak{u}$.
We have the useful relations:

$$\Gamma(\mathfrak{u})a(f)\Gamma(\mathfrak{u})^\dagger = a(\mathfrak{u}f) \tag{2.126}$$

$$\Gamma(\mathfrak{u})a^\dagger(f)\Gamma(\mathfrak{u})^\dagger = a^\dagger(\mathfrak{u}f) \tag{2.127}$$

Theorem 2.65.

1. *For each $f \in \mathfrak{h}$*

$$\frac{1}{\sqrt{2}}(a(f) + a^\dagger(f)) := \Phi_S(f) \tag{2.128}$$

is essentially self adjoint on Γ_{fin} and called the Segal quantization.

2. *Let $\Psi \in \Gamma_{fin}$ and $f, g \in \mathfrak{h}$. Then:*

$$\Phi_S(f)\Phi_S(g)\Psi - \Phi_S(f)\Phi_S(g)\Psi = iIm\,\langle f, g\rangle\,\Psi \tag{2.129}$$

Further if $W_F(f)$ denotes the unitary Fock-Weyl *operator $e^{i\Phi_S(f)}$, then:*

$$W_F(f+g) = W_F(f)W_F(g)e^{-\frac{i}{2}Im\langle f,g\rangle} \tag{2.130}$$

3. *If $f_n \to f$ in $\mathfrak{h}$, then:*

$$W_F(f_n)\Psi \to W_F(f)\Psi \;\forall\Psi \in \Gamma_s(\mathfrak{h}) \tag{2.131}$$

$$\Phi_S(f_n)\Psi \to \Phi_S(f)\Psi \;\forall\Psi \in \Gamma_{fin} \tag{2.132}$$

Proof. See [RS75], Thm X.41. □

Definition 2.66. A *conjugation* C on a complex Hilbert space $\mathcal{H}$ is an antilinear isometry $C : \mathcal{H} \to \mathcal{H}$ such that $C^2 = 1$.

Definition 2.67. Assume that there is a conjugation C on $\mathfrak{h}$ and define $\mathfrak{h}_C := \{f \in \mathfrak{h}; Cf = f\}$. For each $f \in \mathfrak{h}_C$, we define $\phi(f) := \Phi_S(f)$ and $\pi(f) := \Phi_S(if)$. The map $f \mapsto \Phi(f)$ is called *canonical field* over $(\mathfrak{h}, C)$ and the map $f \mapsto \pi(f)$ is called *canonical momentum* over $(\mathfrak{h}, C)$.

Theorem 2.68. *Let $\mathfrak{h}$ be a complex Hilbert space with a conjugation C and $\phi(\cdot)$ and $\pi(\cdot)$ be the canonical fields. Then:*

1. *a) For each $f \in \mathfrak{h}_C$, $\phi(f)$ is essentially self-adjoint on Γ_{fin}*

   b) $\{\phi(f), f \in \mathfrak{h}_C\}$ is a commuting family of self-adjoint operators

   c) Ω is a cyclic vector for $\{\phi(f), f \in \mathfrak{h}_C\}$

   d) If $f_n \to f$, then:

$$\phi(f_n)\Psi \to \phi(f)\Psi \ \forall \Psi \in \Gamma_{fin} \tag{2.133}$$

   and

$$W_F(f_n)\Psi \to W_F(f)\Psi \ \forall \Psi \in \Gamma_s \tag{2.134}$$

2. *The above properties hold with $\phi(\cdot)$ replaced by $\pi(\cdot)$.*

3. *If $f, g \in \mathfrak{h}_C$, then:*

$$[\phi(f), \pi(f)]\Psi = i\langle f, g\rangle \Psi \ \forall \Psi \in \Gamma_{fin} \tag{2.135}$$

$$W_F(f)W_F(Cg) = e^{-i\langle f,g\rangle} W_F(Cg)W_F(f) \tag{2.136}$$

Proof. See [RS75], Thm. X.43. □

Proposition 2.69. 1. *Let $f, g \in \mathcal{H}$, then:*

$$W(f)dom(\Phi_S(g)) = dom(\Phi_S(g)) \tag{2.137}$$

$$W(f)\Phi_S(f)W^{\dagger}(f) = \Phi_S(g) - Im(\langle f, g\rangle)\mathbb{1} \tag{2.138}$$

2. *The expectation value in the vacuum state is:*

$$\langle \Omega, W(f)\Omega\rangle = e^{-\frac{1}{4}\|f\|^2} \tag{2.139}$$

Proof. See [HR15], Proposition 18.5-6. □

2.6.1 Fock Representation

In this section, we will specify a certain representation of $\mathcal{W}(E, \sigma)$ for a pre-symplectic space (E, σ) which is called *Fock representation*. The results are taken from and the respective proofs can be found in [HR15], chapter 18.5.
We will subsequently assume that there is a complex structure on E given by the complex unit i. We will use this complex structure to deduce a complex Hilbert space from (E, σ) which we will build the previously discussed Fock Space on.

Lemma 2.70. *The null-space $ker_\sigma := \{f \in E; \sigma(f,g) = 0,\ \forall g \in E\}$ is a complex subspace of the complexified phase space.*

That allows us to take the quotient $E_\sigma := E_i \mod ker_\sigma$. Together with the scalar product $\langle f,g\rangle = \sigma(f,ig) + i\sigma(f,g)$ the quotient E_σ is a complex pre-Hilbert space. Note that $\sigma(f,g) = Im(\langle f,g\rangle)$. The norm-completion $\mathcal{H} := \overline{E_\sigma}^{\|\cdot\|}$ then is a complex Hilbert space.

Remark 2.71. *If (E,σ) is a symplectic space, then $ker_\sigma = 0$ and hence $E_\sigma = E_i$.*

Definition 2.72. For each $\hbar > 0$ there exists a unique representation $(\Pi_F, \Gamma_s(\mathcal{H}))$ of the Weyl C^*-algebra $\mathcal{W}(E,\hbar\sigma)$ such that:

$$\Pi_F(W^\hbar(f)) = W_F(\sqrt{\hbar}[f]), \qquad \forall f \in E \tag{2.140}$$

Proposition 2.73. *The following assertion for the Fock representation $(\Pi_F, \Gamma_s(\mathcal{H}))$ of $\mathcal{W}(E,\hbar\sigma)$ is valid*

$$\Phi^\hbar_{\Pi_F} = \hbar\Phi_S \tag{2.141}$$

and the same is true for the annihilation and creation operators.

Lemma 2.74. *Let $\mathfrak{h}$ be a separable Hilbert space with conjugation J, then the set*

$$\{e^{i\Phi_S(f)}; f \in \mathfrak{h}\} \tag{2.142}$$

is irreducible in $\Gamma_s(\mathfrak{h})$.

Proof. [RS75], Appendix 7, Lemma 1. □

Theorem 2.75. *Let (E,σ) be symplectic space satisfying the assumptions of the above construction. Then the Fock representation as defined in Definition 2.72 is irreducible.*

Proof. First of all, we notice that $\{e^{i\Phi_s(f)}; f \in \mathcal{H}\}$ is irreducible due to Lemma 2.74.
Moreover $E \subset \mathcal{H}$ is dense and we have $\{e^{i\Phi_s(f)}; f \in \mathcal{H}\}' \subset \{e^{i\Phi_s(f)}; f \in E\}'$.
Assume that there is $A \in \mathcal{B}(\Gamma_s(\mathcal{H}))$ such that $A \in \{e^{i\Phi_s(f)}; f \in E\}'$. For every $g \in \mathcal{H}$, there is a sequence $(g_n)_n \subset E$ such that $g_n \to g$. The continuity of the fields Theorem 2.65, 1.d) tells us that:

$$[A, \Pi_F(W^\hbar(g))]\Psi = [A, \lim_{n\to\infty} \Pi_F({}^\hbar W(g_n))]\Psi \tag{2.143}$$
$$= \lim_{n\to\infty} [A, \Pi_F({}^\hbar W(g_n))]\Psi \tag{2.144}$$
$$= 0 \tag{2.145}$$

That means $\{e^{i\Phi_s(f)}; f \in \mathcal{H}\}' \supset \{e^{i\Phi_s(f)}; f \in E\}'$ and hence $\{e^{i\Phi_s(f)}; f \in \mathcal{H}\}' = \{e^{i\Phi_s(f)}; f \in E\}' = \mathbb{C}\mathbb{1}$. □

2.6.2 Fock Space over $L^2(\mathbb{R}^d)$

In physics one often works with $\mathfrak{h} = L^2(\mathbb{R}^d)$ for some $d \in \mathbb{N}$. We will briefly discuss the exact definition of the creation and annihilation operator on that specific Fock Space. The reference for this subsection is [Dyb17].
We will only discuss the bosonic Fock Space.

Lemma 2.76.

$$\bigotimes^n L^2(\mathbb{R}^d) \cong L^2(\mathbb{R}^{nd}) \tag{2.146}$$

Proof. See e.g. [Ara18]. □

Hence, we have $\mathfrak{h}_s^n \cong L_s^2(\mathbb{R}^{nd})$. Next, define a domain $D \subset \Gamma_s^{fin}(L^2(\mathbb{R}^d))$:

$$D := \{\Psi \in \Gamma_s^{fin}(L^2(\mathbb{R}^d)); \Psi^{(n)} \in \mathcal{S}(\mathbb{R}^{nd}) \; \forall n \in \mathbb{N}\} \tag{2.147}$$

Now, for each $p \in \mathbb{R}^d$ we define an operator $a(p) : D \to \Gamma(L^2(\mathbb{R}^d))$ by

$$(a(p)\Psi)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1}\Psi^{(n+1)}(p, k_1, \dots, k_n) \tag{2.148}$$

Note that the adjoint of $a(p)$ is not densely defined since formally:

$$(a^\dagger(p)\Psi)^{(n)}(k_1, \dots, k_n) = \frac{1}{\sqrt{n}} \sum_{l=1}^n \delta(p - k_l)\Psi^{(n-1)}(k_1, \dots, k_{l-1}, k_{l+1}, \dots, k_n) \tag{2.149}$$

Anyway, the creation and annihilation operators are well defined as quadratic forms on $D \times D$. Let $g \in \mathcal{S}(\mathbb{R}^d)$:

$$a(g) = \int d^d p \; a(p)\overline{g(p)} \tag{2.150}$$

$$a^\dagger(p) = \int d^d p \; a^\dagger(p)g(p) \tag{2.151}$$

These expressions give well-defined operators on D which can be extended to $\Gamma^{fin}(L^2(\mathbb{R}^d))$. Let $g \in \mathcal{S}(\mathbb{R}^d)$ and $\Psi \in \Gamma^{fin}(L^2(\mathbb{R}^d))$. Then they act as follows:

$$(a(g)\Psi)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1} \int d^d p \; a(p)\overline{g(p)}\Psi^{(n+1)}(p, k_1, \dots, k_n) \tag{2.152}$$

$$(a^\dagger(g)\Psi)^n(k_1, \dots, k_n) = \frac{1}{\sqrt{n}} \sum_{l=1}^n g(k_l)\Psi^{(n-1)}(k_1, \dots, k_{l-1}, k_{l+1}, \dots, k_n) \tag{2.153}$$

These expressions can be used to define $a(g)$ and $a^\dagger(g)$ for $g \in L^2(\mathbb{R}^2)$. Since these operators leave $\Gamma^{fin}(L^2(\mathbb{R}^d))$ invariant, we can compute the commutator on this domain for $f, g \in L^2(\mathbb{R}^d)$:

$$[a(f), a^\dagger(g)] = \langle f, g \rangle \, 1 \tag{2.154}$$

2.6.3 Fock Space of the photon

In this thesis we are particularly interested in the description of photons. In the literature, the Fock Space of a photon is usually assumed to be the product of L^2-spaces. Hence, we need to extend last subsection's results to define the Fock Space of the photon.

Definition 2.77. Let $H_0 := \{x \in Mink_4, x \cdot x = 0, x_0 > 0\} \subset Mink_4$ be called the *mass-0-hyperboloid*. We note that H_0 is Lorentz-invariant. Define further the homeomorphism $\zeta : H_0 \to \mathbb{R}^3 - \{0\}, (x_0, \vec{x}) \mapsto \vec{x}$. Then the (upto multiples) unique Lorentz-invariant measure on H_0 is given by

$$\Omega_0(E) = \int_{\zeta(E)} \frac{d^3x}{\omega(x)} \tag{2.155}$$

for any measurable $E \subset Mink_4$ and with $\omega(x) = |\vec{x}|$, see [RS75], IX.8 and Thm. IX.37.

For the Photon, we take $\tilde{\mathfrak{h}} := L^2(H_0, d\Omega_0) \otimes \mathbb{C}^3$ as one-particle space. It is, however, helpful to make use of the specific form of the Lorentz invariant measure to carry the fields to $L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$.
We can identify any $f \in L^2(H_0, d\Omega_0)$ with $f(\omega(x), x) \in L^2(\mathbb{R}^3)$. Moreover, the map

$$\tilde{U} : L^2(H_0, d\Omega_0) \to L^2(\mathbb{R}^3) \tag{2.156}$$

$$f \mapsto \frac{f(\omega(x), x)}{\sqrt{\omega(x)}} \tag{2.157}$$

is unitary onto $L^2(\mathbb{R}^3)$. Hence, $U = \bigoplus_{i \leq 3} \tilde{U}$ is a unitary map from $\tilde{\mathfrak{h}}$ onto $\mathfrak{h} := L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$, see [RS75], X.7.
From Definition 2.64, we know that $\Gamma(U) : \Gamma_s(\tilde{\mathcal{H}}) \to \Gamma_s(\mathcal{H})$ is a unitary mapping:

$$\Gamma(U)\tilde{a}^{(\dagger)}(f)\Gamma(U)^\dagger = a^{(\dagger)}\left(\frac{f(\omega(x), x)}{\sqrt{\omega(x)}}\right) \tag{2.158}$$

where $\tilde{a}^{(\dagger)}$ are the creation and annihilation operators on $\Gamma(\tilde{\mathcal{H}})$ respectively. For further discussion, by $f(x)$ we mean $f(\omega(x), x)$.

Definition 2.78. For convenience, we will be ignoring the Maxwell equations, that constrain the one particle and Fock Space, at this point and work with the extended one particle space $\mathfrak{h}$ [BJ87].
For subsequent discussions the *transversal one-particle space* $\mathfrak{h}_T \subset L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$ will play an essential role

$$\mathfrak{h}_T := \{f \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^3; k \cdot \hat{f}(k) = 0 \text{ a.e.}\} \cong L^2(\mathbb{R}^3) \otimes \mathbb{C}^2, \tag{2.159}$$

where a.e. is short notation for almost everywhere.

Lemma 2.79.

$$L^2(\mathbb{R}^3) \otimes_{\mathbb{C}} \mathbb{C}^2 \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \tag{2.160}$$

Proof. As $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}$, we can make us of the distributivity of tensor product over the direct sum [Rot09] to find an isomorphism $L^2(\mathbb{R}^3) \otimes_{\mathbb{C}} \mathbb{C}^2 \cong (L^2(\mathbb{R}^3) \otimes \mathbb{C}) \oplus (L^2(\mathbb{R}^3) \otimes \mathbb{C})$. Using $L^2(\mathbb{R}^3) \otimes C \cong L^2(\mathbb{R}^3)$ we find the isomorphism. □

Lemma 2.80.

$$\Gamma(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)) \cong \Gamma(L^2(\mathbb{R}^3)) \otimes \Gamma(L^2(\mathbb{R}^3)) \tag{2.161}$$

Proof. See [Att], Thm. 8.7. □

2.6.4 The Transversal Test function space

In the later discussion, we will need to fix a test function space for describing QED. It will turn out that L, that we will define in this subsection, will serve as test function space for the transversal observables.

Definition 2.81. Let $\mathfrak{h} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$ with C defined as $Cf(\cdot) = \overline{f}(-\cdot)$. Then define the transversal one-particle space $\mathfrak{h}_T := \{f \in \mathfrak{h}; k \cdot \hat{f} = 0 \text{ a.e.}\}$ and $\mathfrak{h}_{T,C}$ in the obvious way.
Then, we call $\Phi(P_T(\cdot))$ and $\pi(P_T(\cdot))$ *transversal fields* or equivalently $f \mapsto \Phi(f)$ and $f \mapsto \pi(f)$ for $f \in \mathfrak{h}_T$ the *transversal canonical field* and *transversal canonical momentum*.

To define the algebra of transversal observables, we need to fix a test function space that reflects all degrees of freedom of the theory. We proceed analogue to [BJ87]:

$$L_\Phi = \omega^{-\frac{1}{2}} curl(\widehat{\mathcal{S}_{\mathbb{R}}(\mathbb{R}^3)} \otimes \mathbb{C}^3) \subset \mathfrak{h}_T \tag{2.162}$$

$$L_\pi = \omega^{\frac{1}{2}} P_T(\widehat{\mathcal{S}(\mathbb{R}^3)} \otimes C^3) \subset \mathfrak{h}_T \tag{2.163}$$

Then consider the subspace of $\mathfrak{h}_T$:

$$L = (1+C)L_\Phi + (1-C)L_\pi \tag{2.164}$$

Lemma 2.82. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. The following are equivalent:*

1. $f(x) = f(-x)$ *for almost all* $x \in \mathbb{R}^n$
2. $\hat{f}(\xi) \in \mathbb{R}^n$ *for almost all* $\xi \in \mathbb{R}^n$

Proof. The proof is immediate from:

$$\overline{\hat{f}}(\xi) = \int_{\mathbb{R}^n} \frac{d^n x}{(2\pi)^{\frac{n}{2}}} \overline{f}(x) e^{ix\cdot\xi} \tag{2.165}$$

$$= \int_{\mathbb{R}^n} \frac{d^n x}{(2\pi)^{\frac{n}{2}}} \overline{f}(-x) e^{-ix\cdot\xi} \tag{2.166}$$

□

This means that the conjugation is defined such that the inverse Fourier transformation of $g \in (1+C)L_\phi \in \mathcal{S}_\mathbb{R}(\mathbb{R}^3, \mathbb{R}^3)$ and $f \in (1-C)L_\pi \in i\mathcal{S}_\mathbb{R}(\mathbb{R}^3, \mathbb{R}^3)$.
It is evident that $\sigma(\cdot,\cdot) = Im(\langle\cdot,\cdot\rangle_{\mathfrak{h}_T})$ defines a symplectic structure turning L into a symplectic space. That means, we can decompose each $\hat{h} \in L$ via: $\hat{h} = \omega^{\frac{1}{2}}\widehat{Re(h)} + i\omega^{-\frac{1}{2}}\widehat{Im(f)}$ with $\omega^{\frac{1}{2}}Re(h) \in (1+C)L_\phi$ and $i\omega^{-\frac{1}{2}}\widehat{Im(f)} \in (1-C)L_\pi$.

Lemma 2.83. *L is dense in $\mathfrak{h}_T$ and hence defines a test function space in the aforementioned sense.*

Proof. We notice that $\omega^{\pm\frac{1}{2}}\mathcal{S}(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$ ([Ara18], Thm. 10.5) and hence L is dense in $\mathfrak{h}_T$. □

Since L is dense in $\mathfrak{h}_T$ and admits the complex structure i, we can apply the results of the last section to find

Corollary 2.84. *Let $W(L,\sigma)$ be the Weyl algebra of the symplectic space L as defined in (2.164). Then the Fock representation $(\Pi, \Gamma_s(\mathfrak{h}_T))$ (with $\hbar = 1$) defined via*

$$\Pi(W(f)) = e^{i\Phi_S(f)}, \qquad f \in L \tag{2.167}$$

is an irreducible representation of $W(L, Im(\langle\cdot,\cdot\rangle_{\mathfrak{h}_T}))$.

3 Different gauges in External Current QED

In this chapter, we will apply the results about the quantization of Singular Systems to External Current Electrodynamics, that we introduce in the first section. We will deduce in the second section that our model has a gauge freedom which allows for different choices of gauge fixing.
In the third section, we will elaborate on the famous Coulomb gauge condition. We will also discuss the well-definiteness of the Dirac bracket in this gauge. Thus, we can specify a representation and quantization of the observables in this gauge.
In the fourth section, we repeat the procedure for the so-called Axial gauge. However, the Dirac bracket turns out to be very singular hence not well-defined. Hence, the discussion in this section is on a formal level. These singularities will also appear in the representation of the observable algebra.

3.1 Introduction to External Current ED

For explicit computations, we will apply the approach from Chapter 1. However, we have also stated that we are able to gain mathematical rigour in the description of the fields at the expense of lacking a Langranigan and Hamiltonian formulation.
Nevertheless, the standard approach to discuss (Q)ED is by stating a Lagrange density. Hence, we will derive the results from the usual textbook formulation of (Q)ED, for which we will mostly follow [DEF$^+$99], and afterwards embed those results in the aforementioned rigorous formalism.
Note that $\mathbb{R}^3$ as manifold inherits a star operator mapping:

$$* : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3) \tag{3.1}$$

Additionally, Ω^1 and Ω^2 can be identified with vector fields on $\mathbb{R}^3$ via:

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \leftrightarrow adx + bdy + cdz \leftrightarrow ady \wedge dz - bdx \wedge dz + cdx \wedge dy \tag{3.2}$$

For a fixed time, the electric and the magnetic fields are described as vector field on $\mathbb{R}^3$. Using the above result, we can identify the electric field as time-varying 1-form and the magnetic field as time-varying 2-form on $\mathbb{R}^3$.
To take the time dependence of the fields into account, we will think of the fields as

being defined on the Minkowski space $Mink_4$.
To give the Lagrangian formulation of Maxwell theory we define a new 2-form in $Mink_4$ that respects the transformation behaviour:

$$F = B + E \wedge dx^0 \tag{3.3}$$

The Maxwell equations in terms of F then read

$$dF = 0 \tag{3.4}$$
$$d * F = 0 \tag{3.5}$$

for the free electromagnetic field. In reality the Maxwell equations allow to couple to a current $j \in \Omega^3(Mink_4)$ which is constrained to be conserved $dj = 0$ and have compact spatial support. To construct a Lagrangian formulation we introduce a 1-form $A \in \Omega^1(Mink_4)$ such that:

$$F = dA \tag{3.6}$$

Note that the first Maxwell equation is automatically satisfied due to $d^2 = 0$. If we introduce the Lagrangian

$$L = -\frac{1}{2} F \wedge *F - A \wedge *j \tag{3.7}$$

the second Maxwell equation follows from an action principle. At this point we notice that neither the Lagrangian nor the Maxwell field change under $A \to A + df$ for $f \in \Omega^0(Mink_4)$. We will discuss this feature more extensively in the next section. Furthermore, we notice that this ambiguity in the choice of A leads to the fact that such a theory is described using principal bundles. We will not elaborate on this any further.

3.2 External Current ED as Singular System

It is helpful to clarify, how the physical and mathematical notations are related in order to make sense of the computations.

Remark 3.1. *Suppose, we have a Gelfand triple $E \subset \mathcal{R} \subset E'$, then the smeared fields are elements of E'. The idea of physicists' notation is to represent the fields by a "quadratic" form. Thus, if we use the physics notation, it is related to the formalism from the previous chapter by the scalar product (2.65) for some test function meaning $\phi(f)" = " \langle \phi(x), f(x) \rangle_{\mathcal{R}}$.*

We start with the Lagrange density of Classical Electrodynamics in components

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j \cdot A \tag{3.8}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor expressed in components and we assume that the current j is Schwartz class.

Observation 3.2. *The field strength tensor* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ *is the component notion of (3.6).*

Computation. In local coordinates, we can write the differential form $A = A_\mu dx^\mu$ where A_μ are the components of A and 0-forms and dx^μ denote the coordinate differentials of the local coordinate system of $(x^0, \ldots, x^3)$. This gives

$$dA = d(A_\mu dx^\mu) \tag{3.9}$$
$$= \partial_\nu A_\mu dx^\nu \wedge dx^\mu. \tag{3.10}$$

The components then give $(dA)_{\sigma\lambda} = F_{\sigma\lambda} = \partial_\sigma A_\lambda - \partial\lambda A_\sigma$.
For the spatial components, we have: $F_{ij} = \partial_i A_j - \partial_j A_i$. Recall from physics textbooks that $B = curl(A) = \nabla \times A$ and we see that the spatial components of F are the components of B.
Also recall from physics textbooks $E_i = \partial_0 A_i - \partial_i A_0$. Thus:

$$E \wedge dt = (\partial_0 A_i - \partial_i A_0) dx^i \wedge dt \tag{3.11}$$
$$= F_{0i} dx^i \wedge dt. \tag{3.12}$$

On $Mink_4$ the field strength tensor can also1 be displayed as matrix:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{3.13}$$

□

The Euler-Lagrange equations for the dynamical variables A^μ are:

$$\partial_\nu F^{\nu\mu} = j^\mu \tag{3.14}$$

Due to the anti-symmetry of F, it is evident that (3.14) contracted with ∂_μ gives the conservation of j:

$$\partial_\mu j^\mu = 0 \tag{3.15}$$

Statement 3.3. *The total charge carried by the external current* $Q := \int_{\mathbb{R}^3} d^3x\ j_0(x)$ *is conserved.*

Computation. Since we assumed that j vanishes at infinity, we can use the Gauss theorem to find:

$$\partial_0 Q = \partial_0 \int_{\mathbb{R}^3} d^3x\ j_0(x) = \int_{\mathbb{R}^3} d^3x\ \partial_i j^i(x) = 0 \tag{3.16}$$

□

In quantum theories it is more convenient to work with the Hamilton formalism. As discussed the transition from the Lagrangian to the Hamiltonian is performed via the Legendre transformation

$$\pi_{A^i} =: \pi_i = \frac{\delta \mathcal{L}}{\delta\left(\partial^0 A^i\right)} = F_{i0} = E_i \tag{3.17}$$

$$\pi_0 = 0 \tag{3.18}$$

Computation.

$$\frac{\delta L}{\delta\left(\partial^0 A^\sigma\right)} = -\frac{1}{2} F_{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial\left(\partial^0 A^\sigma\right)} \tag{3.19}$$

$$= -\frac{1}{2} F_{\mu\nu} \frac{\partial\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)}{\partial\left(\partial^0 A^\sigma\right)} \tag{3.20}$$

$$= -\frac{1}{2} F_{\mu\nu}\left(\delta_0^\mu \delta_\sigma^\nu - \delta_0^\nu \delta_\sigma^\mu\right) \tag{3.21}$$

$$= -\frac{1}{2}\left(F_{0\sigma} - F_{\sigma 0}\right) = F_{\sigma 0} \tag{3.22}$$

This is true for $\sigma \in \{0, \ldots, 3\}$. For $\sigma = 0$, this in particular gives $\pi_0 = F_{00} = 0$ due to the anti-symmetry of F. □

which gives the (primary) Hamilton density:

$$\mathcal{H} = -\frac{1}{2}\pi^i \pi_i + \frac{1}{4} F^{ij} F_{ij} + j_i A^i + A^0(-\partial_i \pi^i + j^0) + v^0 \pi_0 \tag{3.23}$$

Note that the magnetic field in terms of the vector potential is given by:

$$B_i = \frac{1}{2}\epsilon_i^{jk} F_{kj} \tag{3.24}$$

Statement 3.4. *(3.23) yields a local $U(1)$ gauge symmetry.*

For the proof, we need to apply the procedure described in section 2.2 to the Lagrangian (3.8).

Observation 3.5. *The Lagrangian (3.8) is singular.*

Computation. We need to prove $\det\left(\frac{\delta^2 L}{\partial(\partial_0 A_\mu)\partial(\partial_0 A_\nu)}\right) = 0$. Since $\pi^0 = \frac{\delta L}{\delta(\partial_0 A_0)} = 0$, the first row and column in $\frac{\delta^2 L}{\delta(\partial_0 A_\mu)\delta(\partial_0 A_\nu)}$ are zero which implies that its determinant is vanishing. □

That means that the Hamiltonian (3.23) is constrained. We will compute the set of constraints using the Dirac-Bergmann procedure.

Observation 3.6. *The set of constraints on (3.23) is:*

$$\pi_0 \approx 0 \tag{3.25}$$

$$\partial_i \pi^i - j_0 \approx 0 \tag{3.26}$$

Computation. We already have seen that $\pi_0 \approx 0$ is the only primary constraint. We need to find the secondary constraints that are implied by that constraint:

$$\begin{aligned}
\{\pi^0(y), H\} &= \int d^3x \{\pi^0(y), \mathcal{H}(x)\} \\
&= \int d^3x \; (\partial_i \pi^i - j_0(x))\delta^{(3)}(x-y) \\
&= (\partial_i \pi^i - j_0)(y)
\end{aligned} \tag{3.27}$$

This secondary constraint implies:

$$\begin{aligned}
\{(\partial_i \pi^i - j_0)(y), H\} &= \int d^3x \; \partial_i \{\pi^i(y), \mathcal{H}(x)\} - \partial_0 j_0(y) \\
&= \int d^3x \partial_i \{\pi^i(y), (\frac{1}{4} F^{nm} F_{nm} + j_n A^n)(x)\} - \partial_0 j_0(y) \\
&= \int d^3x \; \partial_i \left(\frac{1}{2} \left(\partial^m \delta^{ni} - \partial^n \delta^{mi} \right) F_{nm}(x) - j^i(x) \right) \delta^{(3)}(x-y) - \partial_0 j_0(y) \\
&= \int d^3x \; \partial_i \left(\frac{1}{2} \left(\partial^m F_m{}^i - \partial^n F^i{}_n)(x) \right) - j^i(x) \right) \delta^{(3)}(x-y) - \partial_0 j_0(y) \\
&= \int d^3x \partial_i \left(\partial^m F_m{}^i(x) - j^i(x) \right) \delta^{(3)}(x-y) - \partial_0 j_0(y) \\
&= -\int d^3x \; \partial_i j^i(x) \delta^{(3)}(x-y) - \partial_0 j_0(y) \\
&= 0
\end{aligned} \tag{3.28}$$

We used that fact that F_{ij} is anti-symmetric. □

Observation 3.7. *The set of constraints from Observation 3.6 is of first class.*

Computation. The constraints obviously Poisson commute since both only depend on the canonical momenta:

$$\{\pi_0(x), (\partial_i \pi^i - j_0)(y)\} = 0 \tag{3.29}$$

□

As we have already discussed first class constraints generate gauge transformations. For explicit computations, we need to fix the gauge. In order to choose a consistent gauge fixing, we need to compute the gauge transformations.

Statement 3.8. *The first class constraints (3.25) and (3.26) give rise to the following gauge transformation:*

$$A_i \to A_i + \partial_i \phi \tag{3.30}$$

$$A_0 \to A_0 + \phi' \tag{3.31}$$

for some $\phi, \phi' \in C^\infty(Mink_4)$.

Computation of Statement 3.8 and Statement 3.4. First, we note that Observation 3.7 implies that the gauge group is abelian.
The infinitesimal gauge transformations generated by (3.25) read:

$$\delta_{\epsilon_1} A_\mu(x) = \int d^3y\, \epsilon_1(y)\{A_\mu(x), \pi_0(y)\} = \epsilon_1(x)\delta_{\mu 0} \tag{3.32}$$

$$\delta_{\epsilon_1} \pi_\mu(x) = \int d^3y\, \epsilon_1(y)\{\pi_\mu(x), \pi_0(y)\} = 0 \tag{3.33}$$

and correspondingly for (3.26) we obtain:

$$\delta_{\epsilon_2} A_\mu(x) = \int d^3y\, \epsilon_2(y)\{A_\mu(x), (\partial_i \pi^i - j_0)(y)\} = \delta_{\mu i}\, \partial_i \epsilon_2(x) \tag{3.34}$$

$$\delta_{\epsilon_2} \pi_\mu(x) = \int d^3y\, \epsilon_1(y)\{\pi_\mu(x), (\partial_i \pi^i - j_0)(y)\} = 0 \tag{3.35}$$

One often additionally assumes that the field strength tensor to be invariant under the gauge transformations and hence identifies $\phi' = \partial_0 \phi$.
Assume that the gauge group is called G and its corresponding Lie algebra $L(G) \cong T_e(G)$.
Since $\phi : Mink_4 \to \mathbb{R}$, then $i\phi$ must map into the complex line $i\mathbb{R}$. Topologically, $U(1)$ may be identified with the circle $U(1)$. The tangent space at the identity is obtained by differentiating curves $t \mapsto e^{it\theta}$ at $t = 0$ for all real numbers $\theta(x)$. This gives $L(U(1)) = i\mathbb{R}$ the complex line as Lie algebra.
Exponentiating of the Lie algebra element $i\phi$ leads to:

$$g(x) = e^{i\phi(x)} \in U(1) \tag{3.36}$$

This allows us to write the gauge transformation as

$$A_\mu \to A_\mu + ig\partial_\mu g^{-1} \tag{3.37}$$

with $g\partial_\mu g^{-1} \in L(U(1))$. □

Remark 3.9. *From this formalism there is no reason that we identified the gauge as $U(1)$ and not for example $\mathbb{R}$. This, however, becomes clear if one works in a different setting including for example Dirac fields. We will identify the gauge group in this thesis as $U(1)$ to be consistent with the literature.*

Statement 3.10. *The Hamilton density in terms of the electric and magnetic field weakly is:*

$$\mathcal{H} = -\frac{1}{2}\left(\pi_i(x)\pi^i(x) + B_i(x)B^i(x)\right) + j_i A^i \tag{3.38}$$

Computation. Note:

$$\begin{aligned} 4 \cdot B_i B^i &= (\epsilon_{ijk}F^{kj})(\epsilon^i_{\ nm}F^{mn}) && (3.39)\\ &= -(\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn})F^{kj}F^{mn} && (3.40)\\ &= -F^{mn}(F_{mn} - F_{nm}) = -2F^{mn}F_{mn} && (3.41) \end{aligned}$$

Inserting this to (3.23) and respecting the constraint (3.25) and (3.26) gives the result □

Definition 3.11. In physical situations it is natural to assume the energy to be bounded, i.e. a bounded Hamiltonian, meaning $\int_{\mathbb{R}^3} \mathcal{H} < \infty$ [Spo04]. In particular, this implies:

$$-\int_{\mathbb{R}^3} \pi_i \pi^i < \infty \tag{3.42}$$

$$-\int_{\mathbb{R}^3} B_i B^i < \infty \tag{3.43}$$

In other words, assuming the energy of the system being finite implies: $\pi, B \in L^2(\mathbb{R}^3, \mathbb{R}^3)$.

3.3 Coulomb gauge

For explicit computations, we need to remove the gauge freedom. A very prominent choice of gauge fixing in textbooks is the so-called *Coulomb gauge* $\partial_i A^i = 0$ which we will discuss in this section.
In the last section, we discussed that it is natural to assume $\pi, B \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. In the formalism that we are working with, this means that the smeared fields $\pi(f) \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. The fundamental object in the discussion of gauge freedom in Electrodynamics, however, is the vector potential A. Hence, we will start the discussion with the extended test function space to be $\tilde{E}_{ext} = \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, Mink_4) + i\mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, Mink_4)$. The naive pre-symplectic structure is $Im(\langle \cdot, \cdot \rangle)$. With the help of this test function space, we will compute the Dirac bracket which will serve as a modified pre-symplectic structure on the modified test function space $E_{ext} = \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3) + i\mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$.
We will later see, why it is more convenient to work with L from (2.164) as test function space.

As we saw in Observation 3.6, there are two first class constraints. Consequently, we need a second gauge condition. For reasons of simplicity of the formulas, we choose: $\Delta A_0 + j_0 \approx 0$. Anyway, we will show that the Dirac bracket of the spatial components does not depend on the second gauge condition. This reflects the fact that

A_0 is completely arbitrary as it has arbitrary time evolution. Summarizing, we have the constraints and gauge conditions:

$$\begin{array}{ll} \pi_0 \approx 0 & \partial_i A^i \approx 0 \\ \partial^i \pi_i - j_0 \approx 0 & \Delta A_0 + j_0 \approx 0 \end{array}$$

Statement 3.12. *The corresponding Dirac brackets are:*

$$\{A_i(x), \pi_j(y)\}_D = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right) \delta^{(3)}(x-y) \tag{3.44}$$

with all the other Dirac brackets of the canonical fields vanishing.

Remark 3.13. *As explained in Appendix B, the operator* $\frac{\partial_i \partial_j}{\Delta} = R_i R_j : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$*,where* R_i *is a so-called Riesz-operator, is well defined.*
Hence, the smeared Dirac bracket for $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3)$

$$\{A_i(f), \pi_j(g)\} = \delta_{ij} \langle f, g \rangle_{L^2} - \langle f, R_i R_j g \rangle_{L^2} = \delta_{ij} \langle f, g \rangle_{L^2} - \langle R_i R_j f, g \rangle_{L^2} \tag{3.45}$$

makes sense.

$$\left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right) \delta^{(3)}(x-y) := \delta^{(3)\perp}(x-y) \tag{3.46}$$

is called transverse Delta *in the literature.*

Computation. We need to compute the constraint matrix. The fundamental Poisson brackets are

$$\{A_\mu(f), \pi_\nu(g)\} = \delta_{\mu\nu} \langle f, g \rangle \tag{3.47}$$

for test functions $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3)$. We represent this fact by formally writing

$$\{A_\mu(x), \pi_\nu(y)\} = \delta_{\mu\nu} \delta^{(3)}(x-y) \tag{3.48}$$

because:

$$\langle f, \delta \star g \rangle = \langle f, g \rangle \tag{3.49}$$

Using this formal expressions, we find:

$$M(x,y) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Delta \delta^{(3)}(x-y) \tag{3.50}$$

The inverse obviously reads:

$$M^{-1}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \frac{1}{\Delta}\delta^{(3)}(x-y) \tag{3.51}$$

which gives the Dirac brackets:

$$\begin{aligned} \{A_i(x), \pi_j(y)\} &= \delta_{ij}\delta^{(3)}(x-y) - \iint dzdz' \, \{A_i(x), \partial_n A^n(z)\}\frac{1}{\Delta}\delta^{(3)}(z-z')\{\partial_m A^m(z'), \pi_j(y)\} \\ &= \delta_{ij}\delta^{(3)}(x-y) - \iint dzdz' \, \partial_i \delta^{(3)}(x-z)\frac{1}{\Delta}\delta^{(3)}(z-z')\partial_j\delta^{(3)}(y-z') \\ &= \left(\delta_{ij} - \frac{\partial_i\partial_j}{\Delta}\right)\delta^{(3)}(x-y) \end{aligned} \tag{3.52}$$

The vanishing of the other Dirac brackets is straightforward to verify with the same methods. □

Observation 3.14. *The Dirac bracket $\{A_i, \pi_j\}_D$ is independent from the choice of the second gauge condition.*

Computation. Let $f(A,\pi)$ be some function of A and π and assume that it is an admissible second gauge condition. The constraint matrix M with $f(A,\pi)$ as second gauge condition then is

$$M(x,y) = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \Delta & \beta \\ 0 & -\Delta & 0 & \gamma \\ -\alpha & -\beta & -\gamma & 0 \end{pmatrix} \delta^{(3)}(x-y) \tag{3.53}$$

with some entries α, β, γ. This constraint matrix has the inverse:

$$M^{-1}(x,y) = \begin{pmatrix} 0 & -\frac{\gamma}{\Delta\alpha} & \frac{\beta}{\Delta\alpha} & -\frac{1}{\alpha} \\ \frac{\gamma}{\Delta\alpha} & 0 & -\frac{1}{\Delta} & 0 \\ -\frac{\beta}{\Delta\alpha} & \frac{1}{\Delta} & 0 & 0 \\ \frac{1}{\alpha} & 0 & 0 & 0 \end{pmatrix} \delta^{(3)}(x-y) \tag{3.54}$$

It is evident that the Dirac bracket of A_i and π_j only depends on the $(3,2), (3,4)$ and $(4,2)$ components of M^{-1}. Since (3.51) and (3.54) coincide in these components, the Dirac bracket does not change under a change of the second gauge condition.
One should note that due to the assumption of $f(A,\pi)$ being an admissible gauge condition implies that $\alpha \neq 0$. Hence, the inverse is well defined. □

Statement 3.15. *Observe that the complex structure on the naive test function space $E = \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3) + i\mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$ is given by the complex unit i. Hence, according to the proceedings in subsection 2.5.1, there are classical creation and annihilation operators on the phase space. A representation of the fields in terms of the classical creation and annihilation operator is given by*

$$A_0(f) = -\langle \frac{j_0(x)}{\Delta}, f\rangle \tag{3.55}$$

$$A(f) = \frac{1}{\sqrt{2}}\left(a(P_T(\omega^{-\frac{1}{2}}C\widehat{f})) + a^\dagger(P_T(\omega^{-\frac{1}{2}}\widehat{f}))\right) = \phi^0(\omega^{-\frac{1}{2}}P_T(\hat{f})) \tag{3.56}$$

$$\pi(g) = \frac{1}{\sqrt{2}}\left(a(P_T(i\omega^{\frac{1}{2}}C\widehat{g})) + a^\dagger(P_T(i\omega^{\frac{1}{2}}\widehat{g}))\right) - \langle\frac{\nabla}{\Delta}j_0, g\rangle_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \tag{3.57}$$

$$= \phi^0(i\omega^{\frac{1}{2}}P_T(\hat{g})) - i\,\langle\frac{1}{|k|^2}\widehat{j_0}, k\cdot\hat{g}\rangle_{L^2(\mathbb{R}^3,\mathbb{R})} \tag{3.58}$$

for $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$. Here and in the following computations, a c-number is obviously meant to be a multiple of the identity.
In the physics literature this result is often symbolically displayed as

$$A_i(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{1}{2\omega(k)}}\epsilon_i^n(k)\left[a_n(k)e^{ik\cdot x} + a_n^\dagger(k)e^{-ik\cdot x}\right] \tag{3.59}$$

$$\pi_i(x) = -i\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{\omega(k)}{2}}\epsilon_i^n(k)\left[a_n(k)e^{ik\cdot x} - a_n^\dagger(k)e^{-ik\cdot x}\right] - \frac{\partial_i j_0(x)}{\Delta} \tag{3.60}$$

with the polarization vectors $\vec{\epsilon}^n$ that together with $\frac{\vec{k}}{|k|}$ build an orthogonal basis for $\mathbb{R}^3$.

Observation 3.16. *The polarization vectors $\vec{\epsilon^n}(k)$ satisfy the completeness relation:*

$$\sum_{n=1}^{2}\epsilon_i^n(k)\epsilon_j^n(k) = \delta_{ij} - \frac{k_i k_j}{k^2} \tag{3.61}$$

Computation. One can check this statement by acting with the basis vectors $(\epsilon^1(k), \epsilon^2(k), \frac{\vec{k}}{|k|})$ on both sides of the equation. Since one gets the result that both sides of the equation coincide for this particular basis one can verify the result for an arbitrary vector acting on either sides of the equation by taking superpositions of the basis elements. □

Computation of Statement 3.15. First of all, we note that the symplectic structure in E is given by $\sigma = Im(\langle\cdot,\cdot\rangle)$ as discussed in Definition 2.72. Then, we use the results from subsection 2.5.1 and compute for f, g as above:

$$\{A(f), \pi(g)\} = \{\phi^0(\omega^{-\frac{1}{2}} P_T(\hat{f})), \phi^0(i\omega^{\frac{1}{2}} P_T(\hat{g})) - i\,\langle \frac{1}{|k|^2}\widehat{j_0}, k \cdot \hat{g}\rangle\} \tag{3.62}$$

$$= \langle \omega^{-\frac{1}{2}} P_T(\hat{f}), \omega^{\frac{1}{2}} P_T(\hat{g})\rangle \tag{3.63}$$

$$= \langle P_T(\hat{f}), P_T(\hat{g})\rangle \tag{3.64}$$

$$= \langle \hat{f} - \frac{k \cdot \hat{f}}{|k|^2} k, \hat{g} - \frac{k \cdot \hat{g}}{|k|^2} k\rangle \tag{3.65}$$

$$= \langle \hat{f}, \hat{g}\rangle - \langle \frac{k \cdot \hat{f}}{|k|}, \frac{k \cdot \hat{g}}{|k|}\rangle \tag{3.66}$$

which is the smeared version of the transversal Delta in momentum space.

We can also reproduce the result in the physics formalism. In this formalism the classical creation and annihilation operator satisfy the formal Dirac bracket $\{a_n(k), a_m^\dagger(k')\} = i\delta_{nm}\delta^{(3)}(k-k')$. Then

$$\begin{aligned}\{A_i(x), \pi_j(y)\} &= -i \iint \frac{d^3k d^3k'}{(2\pi)^3} \frac{1}{2} \sqrt{\frac{\omega(k')}{\omega(k)}} \epsilon_i^n(k) \epsilon_j^m(k') \\ &\quad \left(-\{a_n(k), a_m^\dagger(k')\} e^{i(k\cdot x - k'\cdot y)} + \{a_n^\dagger(k), a_m(k')\} e^{-i(k\cdot x - k'\cdot y)}\right) \\ &= \int \frac{d^3k}{(2\pi)^3} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) e^{ik\cdot(x-y)} \\ &= \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right) \delta^{(3)}(x-y)\end{aligned} \tag{3.67}$$

where the derivatives are meant in the distributional sense and the inverse Laplacian as in Appendix B. □

Remark 3.17. *The additional term in (3.58) is well defined. As explained in Appendix B the Riesz potential operator $\Delta^{-\frac{1}{2}} : \mathcal{S}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ and Riesz operators $R_j : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ are well defined. Hence their composition is an operator: $R_j \Delta^{-\frac{1}{2}} : \mathcal{S}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$.*

Remark 3.18. *In this and the following two chapters, we will be working with the classical fields. As discussed in section 2.5, we are interested in the Weyl elements for the quantization procedure. However, we know that the fields and the Weyl elements are related via the exponential $e^{i\cdot}$ which means the Weyl element corresponding to $\pi(f)$ is $e^{i\pi(f)}$.*

Notation. *We call:*

$$\pi^f(g) = \phi^0(i\omega^{\frac{1}{2}} P_T(\hat{g})) \tag{3.68}$$

for $g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$ the (representation of the) free canonical momenta.

Observation 3.19. *The magnetic field in the above representation is:*

$$B(f) = curl(A(f)) \tag{3.69}$$
$$= \phi^0(\omega^{-\frac{1}{2}}(\widehat{curl(f)})) \tag{3.70}$$
$$= \phi^0(i\omega^{-\frac{1}{2}}(k \times \hat{f})) \tag{3.71}$$

In the physics literature, the magnetic field is often displayed as:

$$B_i(x) = i \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{1}{2\omega(k)}} \epsilon_{ijk} k_j \epsilon_k^n(k) \left[a_n(k) e^{ik\cdot x} - a_n^\dagger(k) e^{-ik\cdot x} \right] \tag{3.72}$$

Observation 3.20. *We call:*

$$H^f = \int d^3x \mathcal{H}^f(x) = \frac{1}{2} \int d^3x \left((\pi^{f,i} \pi_i^f)(x) + (B_i B^i)(x) \right) \tag{3.73}$$
$$= \int d^3k \; \omega(k) \sum_{n=1}^{2} a_n^\dagger(k) a_n(k) \tag{3.74}$$

the (representation of the) free Hamiltonian.

Computation.

$$\begin{aligned}
\int d^3x \; \left(\pi^{f,i}\pi_i^f\right)(x) = &- \iiint \frac{d^3x d^3k d^3k'}{(2\pi)^3} \frac{1}{2} \sqrt{\omega(k)\omega(k')} \epsilon_i^n(k) \epsilon^{m,i}(k') \\
&\left[a_n(k) e^{ik\cdot x} - a_n^\dagger(k) e^{-ik\cdot x} \right] \left[a_m(k') e^{ik'\cdot x} - a_m^\dagger(k') e^{-ik'\cdot x} \right] \\
= &- \int d^3k \frac{\omega(k)}{2} \Big(\epsilon_i^n(k) \epsilon^{i,m}(k) \Big(- a_n(k) a_m^\dagger(k) - a_n^\dagger(k) a_m(k) \Big) \\
&+ \epsilon_i^n(k) \epsilon^{m,i}(-k) \Big(a_n(k) a_m(-k) + a_n^\dagger(k) a_m^\dagger(-k) \Big) \Big) \\
= &- \int d^3 \; \omega(k) \Big(\sum_{n=1}^{2} a_n^\dagger(k) a_n(k) + \frac{\epsilon_i^n(k) \epsilon^{m,i}(-k)}{2} \\
&\Big(a_n(k) a_m(-k) + a_n^\dagger(k) a_m^\dagger(-k) \Big) \Big)
\end{aligned} \tag{3.75}$$

$$\begin{aligned}
\int d^3x \left(B_i(x)B^i(x)\right) = & -\iint \frac{d^3xd^3kd^3k'}{(2\pi)^3}\frac{1}{2}\sqrt{\frac{1}{\omega(k)\omega(k')}} \left(k^ik'_i\epsilon^{n,j}(k)\epsilon^m_j(k') - k'_i\epsilon^{n,i}(k)k_j\epsilon^{m,j}(k'))\right) \\
& \left[a_n(k)e^{ik\cdot x} - a_n^\dagger(k)e^{-ik\cdot x}\right]\left[a_m(k')e^{ik'\cdot x} - a_m^\dagger(k')e^{-ik'\cdot x}\right] \\
= & \int d^3k\frac{\omega(k)}{2}\Big(\epsilon^{n,i}(k)\epsilon^m_i(k)\Big(- a_n(k)a_m^\dagger(k) - a_n^\dagger(k)a_m(k)\Big) \\
& + \epsilon^{n,i}(k)\epsilon^m_i(-k)\Big(a_n(k)a_m(-k) + a_n^\dagger(k)a_m^\dagger(-k)\Big)\Big) \\
= & \int d^3k\, \omega(k)\Big((-\sum_{n=1}^{2}a_n^\dagger(k)a_n(k) + \frac{\epsilon^{n,i}(k)\epsilon^m_i(-k)}{2} \\
& \Big(a_n(k)a_m(-k) + a_n^\dagger(k)a_m^\dagger(-k)\Big)\Big) \qquad (3.76)
\end{aligned}$$

One sees that both contributions coincide up to a sign in front of the second addend in the integral which leads to the cancelling of these terms. □

Observation 3.21. *The (representation of the) total Hamiltonian is given by:*

$$H^C = H^f + H^I \tag{3.77}$$

where H^f is the free Hamiltonian and H^I may be interpreted as an interaction Hamiltonian taking the form:

$$H^I = \int d^3x \left(\frac{1}{2}j_0(x)\frac{j_0(x)}{\Delta} + j_i(x)A^i(x)\right) \tag{3.78}$$

Computation. We easily see:

$$\pi_i(x) = \pi_i^f(x) - \frac{\partial_i j_0(x)}{\Delta} \tag{3.79}$$

and hence:

$$\left(\pi^i\pi_i\right)(x) = \left(\pi^{f,i}\pi_i^f\right)(x) - 2\pi_i^f(x)\frac{\partial_i j_0(x)}{\Delta} + \left(\frac{\partial_i j_0(x)}{\Delta}\right)\left(\frac{\partial^i j_0(x)}{\Delta}\right) \tag{3.80}$$

Integrating this gives us:

$$\int d^3x \left(\pi^i\pi_i\right)(x) = \int d^3x \left(\pi^{f,i}\pi_i^f\right)(x) + j_0(x)\frac{j_0(x)}{\Delta} \tag{3.81}$$

where we integrated by parts once in the second and third addend respectively. Additionally, we used the fact: $\partial^i \cdot \pi_i^f = 0$ which comes from the fact that $\vec{k} \perp \vec{\epsilon}^n(k)$.

Using Observation 3.20, we find the Hamiltonian to be:

$$H = \frac{1}{2}\int d^3x \left(\left(\pi^{f,i}\pi_i^f\right)(x) + \vec{B}(x)^2 + j_0\frac{j_0(x)}{\Delta} + 2\, j_i(x)A^i(x)\right) \tag{3.82}$$

$$= \int d^3k\, \omega(k) \sum_{n=1}^{2} a_n^\dagger(k)a_n(k) + \int d^3x \left(\frac{1}{2}j_0(x)\frac{j_0(x)}{\Delta} + j_i(x)\cdot A^i(x)\right) \tag{3.83}$$

$$= H^f + A(j) + \frac{1}{2}\langle j_0, \frac{j_0}{\Delta}\rangle \tag{3.84}$$

$$= H^f + \phi^0(\omega^{-\frac{1}{2}}P_T(j)) + \frac{1}{2}\langle j_0, \frac{j_0}{\Delta}\rangle \tag{3.85}$$

□

Statement 3.22. *In the representation of Statement 3.15 the Maxwell equations are satisfied.*

Comp. Since the Bianchi identity follows from the antisymmetry of F, it is satisfied by definition. What is left to show is:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{3.86}$$

First we check this identity for $\nu \neq 0$. Then we find:

$$\begin{aligned}
\partial_\mu F^{\mu i}(x) &= \partial_k F^{ki}(x) - \{\pi^i(x), H\} \\
&= \partial_k F^{ki}(x) - \int d^3y\, \frac{1}{2}\{\pi^i(x), F_{nm}(y)\}F^{nm}(y) + j^i(y)\delta^{(3)}(x-y) \\
&= \partial_k F^{ki}(x) - \int d^3y\, \frac{1}{2}\left((-\partial_n\delta^i_m + \partial_m\delta^i_n)\delta^{(3)}(x-y)\right)F^{nm}(y) + j^i(x) \\
&= \partial_k F^{ki}(x) - \partial_n F^{ni}(x) + j^i(x) \\
&= j^i(x)
\end{aligned} \tag{3.87}$$

We used that the Hamiltonian is first class (see Definition 2.12) and hence the Poisson bracket and the Dirac bracket coincide.
For $\nu = 0$, we only need to verify the Gauss law

$$\begin{aligned}
\partial_\mu F^{\mu 0} &= \partial_i \pi^i \\
&= j_0
\end{aligned} \tag{3.88}$$

what one easily reads from Statement 3.15 since the polarization vectors are orthogonal to $\vec{k}$. □

Notation. *With $\cdot$ between two vector valued quantities, we will indicate the standard scalar product on $\mathbb{R}^n$.*

Statement 3.23 (Quantized Fields). *Let the notation be as in section 2.6 with* $\mathcal{H} = \Gamma_s(\mathfrak{h})$, $\hbar = 1$ *and* $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$. *Then the quantized fields in the Coulomb gauge are:*

$$A^C(f) = \Phi_S(\omega^{-\frac{1}{2}} P_T(\hat{f})) \tag{3.89}$$

$$\pi^C(g) = \Phi_S(i\omega^{\frac{1}{2}} P_T(\hat{g})) - i \left\langle \frac{1}{|k|^2}\widehat{j_0}, k \cdot \hat{g} \right\rangle_{L^2(\mathbb{R}^3,\mathbb{R})} \tag{3.90}$$

$$B(f) = \Phi_S(\omega^{-\frac{1}{2}} (\widehat{curl(f)})) \tag{3.91}$$

Computation. We will only prove $[A(f), \pi(g)] = \int d^3x f_T(x) g_T(x) := \langle f_T, g_T \rangle_{L^2}$ for $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$. The verification for the other commutators is analogously.

$$\begin{aligned}
[A(f), \pi(g)] =& [a\left((2\omega)^{-\frac{1}{2}} P_T(\hat{\bar{f}})\right), a^\dagger\left(i(\frac{\omega}{2})^{\frac{1}{2}} P_T(\hat{g})\right)] && (3.92)\\
&+ [a^\dagger\left((2\omega)^{-\frac{1}{2}} P_T(\hat{f})\right), a\left(i(\frac{\omega}{2})^{\frac{1}{2}} P_T(\hat{\bar{g}})\right)] && (3.93)\\
=& \frac{1}{2}\langle P_T(\hat{\bar{f}}), iP_T(\hat{g})\rangle - \frac{1}{2}\langle iP_T(\hat{\bar{g}}), P_T(\hat{f})\rangle && (3.94)\\
=& \frac{i}{2}\left(\langle P_T(\hat{\bar{f}}), P_T(\hat{g})\rangle + \langle P_T(\hat{\bar{g}}), P_T(\hat{f})\rangle\right) && (3.95)\\
=& i\,\langle P_T(\hat{\bar{f}}), P_T(\hat{g})\rangle && (3.96)\\
=& i\,\langle f_T, g_T\rangle_{L^2(\mathbb{R}^3,\mathbb{R})} && (3.97)
\end{aligned}$$

□

3.4 Axial gauge

In the literature, the condition $A_3 \approx 0$ ([Wei95], [HLL94]) is often called *Axial gauge*. We understand a more general class of gauge fixing as Axial gauge. In particular, for fixed $e \in \mathbb{R}^3$, the two external conditions on the right are what we will call Axial gauge. In this setting, we have a total of four constraints:

$$\begin{array}{ll} \pi_0 \approx 0 & e_i A^i \approx 0 \\ \partial^i \pi_i - j_0 \approx 0 & -e_i \pi^i + e_i \partial^i A_0 \approx 0 \end{array}$$

Observation 3.24. *The constraints are of second class and the Dirac brackets are*

$$\{A_i(x), \pi_j(y)\}_D = \left(\delta_{ij} - \frac{e_j \partial_i}{e_k \partial^k}\right) \delta^{(3)}(x-y) \tag{3.98}$$

$$\{A_0(x), A_i(y)\}_D = \left(\frac{e_k e^k \partial_i}{(e_j \partial^j)^2} - \frac{e_i}{(e_j \partial^j)}\right) \delta^{(3)}(x-y) \tag{3.99}$$

with the residual Dirac brackets vanishing.

Remark 3.25. *Already at this point, we notice that (3.98) is not even well-defined for the smeared fields due to the singularities in* $\frac{1}{e \cdot k}$ *for* $k \neq 0$.
Hence, the following computations need to be interpreted on a formal level. After changing to the momentum space with the Fourier transformation, we will apply the formal transformation rule $\frac{1}{e_i \partial^i} \to \frac{-i}{e \cdot k}$ *which is justified by Proposition B.12 on some open region G that does not contain the orthogonal space of e.*
Hence, the Dirac bracket and, as it will turn out, the observables are well defined for test functions that are supported in such a region G. Unfortunately, such functions are not dense in $L^2(\mathbb{R}^3)$ *and hence, do not qualify for a test function space in the sense of Definition 2.25.*

Computation. We will simply compute the constraint matrix and its inverse to show that every constraint is second class.
The constraint matrix then becomes:

$$M(x,y) = \begin{pmatrix} 0 & 0 & 0 & -(e_i \partial^i) \\ 0 & 0 & -(e_i \partial^i) & 0 \\ 0 & (e_i \partial^i) & 0 & -e_i e^i \\ (e_i \partial^i) & 0 & e_i e^i & 0 \end{pmatrix} \delta^{(3)}(x-y) \tag{3.100}$$

It is easy to check that the inverse is:

$$M^{-1}(x,y) = \begin{pmatrix} 0 & \frac{e_j e^j}{(e_k \partial^k)} & 0 & 1 \\ -\frac{e_j e^j}{(e_k \partial^k)} & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \frac{1}{(e_i \partial^i)} \delta^{(3)}(x-y) \tag{3.101}$$

Now, we are able to compute the Dirac brackets:

$$\begin{aligned} \{A_i(x), \pi_j(y)\}_D &= \delta_{ij}\delta^{(3)}(x-y) - \iint d^3z d^3z' \{A_i(x), (\partial_n \pi^n - j_0)(z)\} M^{-1}_{23}(z,z') \{e_m A^m(z'), \pi_j(y)\} \\ &= \delta_{ij}\delta^{(3)}(x-y) - \iint d^3z d^3z' \partial_i \delta^{(3)}(x-z) \left(\frac{-\delta^{(3)}(z-z')}{e_k \partial^k} \right) e_j \delta^{(3)}(y-z') \\ &= \delta_{ij}\delta^{(3)}(x-y) - \int d^3z \left(\frac{e_j \partial_i}{e_k \partial^k} \delta^{(3)}(x-z) \right) \delta^{(3)}(z-y) \\ &= \left(\delta_{ij} - \frac{e_j \partial_i}{e_k \partial^k} \right) \delta^{(3)}(x-y) \end{aligned} \tag{3.102}$$

and:

$$\begin{aligned} \{A_0(x), A_i(y)\}_D = -\Bigg(\iint d^3z d^3z' \; & \{A_0(x), \pi_0(z)\} \big(M^{-1}_{12}(z,z') \{\partial_j \pi^j(z'), A_i(y)\} \\ & + M^{-1}_{14}(z,z') e_j \{(-\pi^j + \partial^j A_0)(z'), A_i(y)\} \Bigg) \end{aligned}$$

$$= -\Bigg(\iint d^3z d^3z' \, \delta^{(3)}(x-z) \bigg(\frac{e_k e^k}{(e_j \partial^j)^2} \delta^{(3)}(z-z')(-\partial_i)\delta^{(3)}(z'-y) + \frac{1}{(e_j\partial^j)} \delta^{(3)}(z-z') e_i \delta^{(3)}(z'-y) \bigg) \Bigg)$$

$$= -\Bigg(\int d^3z' \left(\frac{e_k e^k}{(e_j \partial^j)^2} \delta^{(3)}(x-z')(-\partial_i)\delta^{(3)}(z'-y) \right) + \frac{e_i}{(e_j\partial^j)} \delta^{(3)}(x-y) \Bigg)$$

$$= \left(\frac{e_k e^k \partial_i}{(e_j\partial^j)^2} - \frac{e_i}{(e_j \partial^j)} \right) \delta^{(3)}(x-y) \tag{3.103}$$

□

Observation 3.26. *The Dirac bracket of the canonical fields A_i, π_j in the spatial axial gauge is independent of the second gauge condition.*

Computation. Assume that we removed of degree of gauge freedom with the axial gauge condition. Then, we need one more external constraint to remove the residual gauge freedom. Let this external constraint be denoted by $f(A, \pi)$.
The constraint matrix then reads:

$$M(x,y) = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & -(e_i\partial^i) & \beta \\ 0 & (e_i\partial^i) & 0 & \gamma \\ -\alpha & -\beta & -\gamma & 0 \end{pmatrix} \delta^{(3)}(x-y) \tag{3.104}$$

having the inverse:

$$M^{-1}(x,y) = \begin{pmatrix} 0 & -\frac{\gamma}{\alpha}(e_i\partial^i)^{-1} & \frac{\beta}{x}(e_i\partial^i)^{-1} & -\frac{1}{\alpha} \\ \frac{\gamma}{\alpha}(e_i\partial^i)^{-1} & 0 & -(e_i\partial^i)^{-1} & 0 \\ -\frac{\beta}{\alpha}(e_i\partial^i)^{-1} & (e_i\partial^i)^{-1} & 0 & 0 \\ \frac{1}{\alpha} & 0 & 0 & 0 \end{pmatrix} \delta^{(3)}(x-y) \tag{3.105}$$

As before, the Dirac bracket of A_i and π_j only depends on the $(2,3)$, $(3,4)$ and $(4,2)$ components of M^{-1}. Since (3.101) and (3.105) are identical in these components, so is the Dirac bracket $\{A_i, \pi_j\}$ in both cases. □

Statement 3.27. *A representation of the Dirac bracket relations in terms of the classical fields* *or annihilation and creation operators respectively is:*

$$A_0(h) = -\phi^0(i\omega^{\frac{1}{2}} P_T(e)\widehat{\frac{h}{e \cdot \partial}}) + \langle \frac{j_0}{(e \cdot \partial)^2}, h \rangle \tag{3.106}$$

$$A(f) = \phi^0(\omega^{-\frac{1}{2}} P_T(\hat{f})) - \phi^0(\omega^{-\frac{1}{2}} P_T(e\frac{k \cdot \hat{f}}{e \cdot k})) \tag{3.107}$$

$$\pi(g) = \phi^0(i\omega^{\frac{1}{2}} P_T(\hat{g})) - i \langle \frac{1}{e \cdot k}\widehat{j_0}, e \cdot \hat{g} \rangle_{L^2(\mathbb{R}^3, \mathbb{R})} \tag{3.108}$$

for $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$ *and* $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R})$. *In the physics notation, this is:*

$$A_0(x) = -\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega(k)}{2}} \frac{e^i \epsilon_i^n(k)}{e_j k^j} \left[a_n(k) e^{ik \cdot x} + a_n^\dagger(k) e^{-ik \cdot x}\right] + \frac{j_0(x)}{(e^i \partial_i)^2} \tag{3.109}$$

$$A_i(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{1}{2\omega(k)}} \left(\epsilon_i^n(k) - \frac{k_i}{e_j k^j} e^i \epsilon_i^n(k)\right) \left[a_n(k) e^{ik \cdot x} + a_n^\dagger(k) e^{-ik \cdot x}\right] \tag{3.110}$$

$$\pi_i(x) = -i \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega(k)}{2}} \epsilon_i^n(k) \left[a_n(k) e^{ik \cdot x} - a_n^\dagger(k) e^{-k \cdot x}\right] + \frac{e_i}{e^j \partial_j} j_0(x) \tag{3.111}$$

Computation. The proof is analogous to the proof of Statement 3.15.

$$\begin{aligned}
\{A(f), \pi(g)\} &= \{\phi^0(\omega^{-\frac{1}{2}} P_T(\hat{f})) - \phi^0(\omega^{-\frac{1}{2}} P_T(e\frac{k \cdot \hat{f}}{e \cdot k})), \phi^0(i\omega^{\frac{1}{2}} P_T(\hat{g}))\} && (3.112)\\
&= \langle \omega^{-\frac{1}{2}} \left(P_T(\hat{f}) - P_T(e\frac{k \cdot \hat{f}}{e \cdot k})\right), \omega^{\frac{1}{2}} P_T(\hat{g}) \rangle && (3.113)\\
&= \langle P_T(\hat{f}) - P_T(e\frac{k \cdot \hat{f}}{e \cdot k}), P_T(\hat{g}) \rangle && (3.114)\\
&= \langle \hat{f} - \frac{k \cdot \hat{f}}{|k|^2} k - e\frac{k \cdot \hat{f}}{e \cdot k} + \frac{k \cdot \hat{f}}{|k|^2} k, \hat{g} - \frac{k \cdot \hat{g}}{|k|^2} k \rangle && (3.115)\\
&= \langle \hat{f}, \hat{g} \rangle - \langle \frac{k \cdot \hat{f}}{|k|}, \frac{k \cdot \hat{g}}{|k|} \rangle && (3.116)\\
&= \langle \hat{f}, \hat{g} \rangle - \langle \frac{k \cdot \hat{f}}{e \cdot k}, e \cdot \hat{g} \rangle && (3.117)
\end{aligned}$$

which is the smeared version of the Dirac bracket in Observation 3.24 in momentum space.

The verification of (3.103) is analogous:

$$\{A_0(h), A(f)\} = \{-\phi^0(i\omega^{\frac{1}{2}} P_T(e)\widehat{\frac{h}{e\cdot\partial}}), \phi^0(\omega^{-\frac{1}{2}} P_T(\hat{f})) - \phi^0(\omega^{-\frac{1}{2}} P_T(e\frac{k\cdot\hat{f}}{e\cdot k}))\} \tag{3.118}$$

$$= \langle P_T(e)\widehat{\frac{h}{e\cdot\partial}}, P_T(\hat{f} - e\frac{k\cdot\hat{f}}{e\cdot k})\rangle \tag{3.119}$$

$$= \langle\widehat{\frac{h}{e\cdot\partial}}, e\cdot\hat{f} - \frac{k\cdot\hat{f}}{e\cdot k}\rangle - \langle\frac{k}{|k|}\cdot\widehat{\frac{h}{e\cdot\partial}}, \frac{k\cdot\hat{f}}{|k|} - \frac{k\cdot\hat{f}}{|k|}\rangle \tag{3.120}$$

$$= \langle\widehat{\frac{h}{e\cdot\partial}}, e\cdot\hat{f} - \frac{k\cdot\hat{f}}{e\cdot k}\rangle \tag{3.121}$$

$$= i\,\langle\hat{h}, \frac{e\cdot\hat{f}}{e\cdot k} - \frac{k\cdot\hat{f}}{(e\cdot k)^2}\rangle \tag{3.122}$$

□

Observation 3.28. *The magnetic field in the above representation is:*

$$B(f) = curl(A(f)) \tag{3.123}$$

$$= \phi^0(\omega^{-\frac{1}{2}}(\widehat{curl(f)})) \tag{3.124}$$

$$= \phi^0(i\omega^{-\frac{1}{2}}(k\times\hat{f})) \tag{3.125}$$

for $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3)$.
In the physics literature, the magnetic field is often displayed as:

$$B_i(x) = i\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{1}{2\omega(k)}}\epsilon_{ijk}k_j\epsilon_k^n(k)\left[a_n(k)e^{ik\cdot x} - a_n^\dagger(k)e^{-ik\cdot x}\right] \tag{3.126}$$

Observation 3.29. *The Hamiltonian Statement 3.10 then is:*

$$H = H^f + H^I \tag{3.127}$$

where again H^f *is the free Hamiltonian and* H^I *can be interpreted as interaction Hamiltonian and has the form:*

$$H^I = \int d^3x\, \frac{e^i\pi_i^f(x)}{e^k\partial_k}j_0(x) + \frac{1}{2}j_0(x)\frac{1}{(e^j\partial_j)^2}j_0(x) + j_i(x)A^i(x) \tag{3.128}$$

Computation. Using the notation from the beginning of the chapter, we can rewrite π_i in the following way:

$$\pi_i(x) = \pi_i^f(x) + \frac{e_i}{e_j\partial^j}j_0(x) \tag{3.129}$$

Hence, we find:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\pi_i\pi^i + \frac{1}{4}F^{ij}F_{ij} + j_iA^i \\
&= \frac{1}{2}(\pi_i^f + \frac{e_i}{e^j\partial_j}j_0)(\pi^{f,i} + \frac{e^i}{e^k\partial_k}j_0) + F^{f,ij}F_{ij}^f + j_iA^i \\
&= \mathcal{H}^f + \frac{e^i\pi_i^f}{e^j\partial_j}j_0 + \frac{1}{2}\frac{e_i}{e^j\partial_j}j_0\frac{e^i}{e^k\partial_k}j_0 + j^iA_i \\
&= \mathcal{H}^f + \frac{e^i\pi_i^f}{e^j\partial_j}j_0 + \frac{1}{2}j_0\frac{1}{(e^j\partial_j)^2}j_0 + j_iA^i
\end{aligned} \tag{3.130}$$

□

Statement 3.30. *In the representation in Statement 3.27 the Maxwell equations are satisfied.*

Computation. Since the Bianchi identity follows from the antisymmetry of F, it is satisfied by definition. What is left to show is:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{3.131}$$

First we check this identity for $\nu \neq 0$. Then we find:

$$\begin{aligned}
\partial_\mu F^{\mu i}(x) &= \partial_k F^{ki}(x) - \{\pi^i(x), H\} \\
&= \partial_k F^{ki}(x) - \int d^3y\, \frac{1}{2}\{\pi^i(x), F_{nm}(y)\}F^{nm}(y) + j^i(y)\delta^{(3)}(x-y) \\
&= \partial_k F^{ki} - \int d^3y\, \frac{1}{2}\Big((-\partial_n\delta^i_m + \partial_m\delta^i_n)\delta^{(3)}(x-y)\Big)\, F^{nm}(y) + j^i(x) \\
&= \partial_k F^{ki}(x) - \partial_n F^{ni}(x) + j^i(x) \\
&= j^i(x)
\end{aligned} \tag{3.132}$$

We used that the Hamiltonian is first class (see Definition 2.12) and hence the Poisson bracket and the Dirac bracket coincide.
For $\nu = 0$ we only need to prove that the Gauss law is satisfied

$$\begin{aligned}
\partial_\mu F^{\mu 0} &= \partial_i\pi^i \\
&= j_0
\end{aligned} \tag{3.133}$$

what one easily reads from Statement 3.27 since the polarization vectors are orthogonal to $\vec{k}$. □

Statement 3.31 (Quantized Fields). *Let the notation be as in section 2.6 with* $\mathcal{H} = \Gamma_s(\mathfrak{h})$, $\hbar = 1$ *and* $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$. *Then the quantized fields in the Axial gauge are:*

$$A^e(f) = \Phi_S(\omega^{-\frac{1}{2}} P_T(\hat{f})) - \Phi_S(\omega^{-\frac{1}{2}} P_T(e\frac{k \cdot \hat{f}}{e \cdot k})) \tag{3.134}$$

$$\pi^e(g) = \Phi_S(i\omega^{\frac{1}{2}} P_T(\hat{g})) - i \left\langle \frac{1}{e \cdot k}\widehat{j_0}, e \cdot \hat{g} \right\rangle_{L^2(\mathbb{R}^3, \mathbb{R})} \tag{3.135}$$

$$B(f) = \Phi_S(\omega^{-\frac{1}{2}}(\widehat{curl(f)})) \tag{3.136}$$

Computation. This directly follows from the definition of the canonical Weyl quantization subsection 2.5.3 and Statement 3.27 and Observation 3.28.
To check that they indeed satisfy the correct commutation relation, on proceeds analogously to the Computation of Equation 3.3. □

4 Discretely smeared Axial gauge

In the last chapter, we have discussed the representation of the magnetic and the electric field in two different gauges. While the representation and the involved quantities for the Coulomb gauge were well defined, the discussion of the Axial gauge was on a formal level since the inverse operator $\frac{1}{e \cdot k}$ is not well defined on $\mathcal{S}(\mathbb{R}^3)$.
To get rid of these singularities, we adapt a strategy of smoothing the Axial gauge that was originally introduced in [MSY05]. The authors in this paper however used a different formalism.
In this chapter, we will discuss the discretely smeared Axial gauge. We will need the results of this chapter to define the continuously smeared Axial gauge in the next chapter. In the first section, we will double the degrees of freedom and in the course of that the gauge group. This allows for imposing the Axial gauge condition twice.
In the second section, we will generalize this procedure the case that we add arbitrary many countable degrees of freedom.

4.1 Twofold discretely smeared Axial gauge

The idea of the smearing out the Axial gauge is to enlarge the naive degrees of freedom by allowing for two different vector potentials with the same field strength tensor. Since the physical information of the vector potential A is fully encoded in its differential $dA = F$, the field strength tensor, we assume that we can decompose the vector potential into a sum $A = A^1 + A^2$ of two vector potentials having the same differential or field strength tensor respectively. This gives us the possibility to impose independent gauge conditions on A^1 and A^2 respectively.
This idea of extending the degrees of freedom will be generalized to arbitrary many vector potentials in the next step. Finally, we will justify that we can replace the quantities with the singularities by some Riemann-similar sum which converges to an integral expression which removes the singularities.
The goal of this section is to show that if we extend the naive phase space in the aforementioned way and impose the Axial gauge condition on each vector potential with different gauge vectors $e_1, e_2 \in \mathbb{R}^3$ respectively, then we have:

Statement 4.1. *The representation of the canonical fields in the above decomposition with gauge* *conditions $e^i_1 A^1_i = 0$ and $e^i_2 A^2_i = 0$ is*

$$A = \frac{1}{2}(A^{e_1} + A^{e_2}) \tag{4.1}$$

$$\pi = \frac{1}{2}(\pi^{e_1} + \pi^{e_2}) \tag{4.2}$$

where $\cdot^{e_i}$ is the respective representation with the condition $e^k_i A^{e_i}_k = 0$ we developed in section 3.4.

In order to prove this theorem we will need several computational steps which we will elaborate on separately:

Observation 4.2. *Expressing $A = A^1 + A^2$ as the sum of two independent vector fields we have a six dimensional gauge freedom.*

Computation. We choose new coordinates:

$$A^+ := A^1 + A^2 \tag{4.3}$$

$$A^- := A^1 - A^2 \tag{4.4}$$

In the new coordinates the Lagrange density is:

$$\mathcal{L} = -\frac{1}{4}F^+_{\mu\nu}F^{+,\mu\nu} - j^\mu A^+_\mu \tag{4.5}$$

It is obvious that the Legendre transformation for A^-_μ is not bijective because (4.5) is independent of A^-. In particular, we have:

$$\pi^-_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^{-,\nu})} = 0 \tag{4.6}$$

The computations for the canonical momentum of π^+ are identical to those in section 3.2. Hence, we have

$$\pi^+_\mu \approx F^+_{0\mu} \tag{4.7}$$

which implies the primary constraint $\pi^+_0 \approx 0$. The extended Hamilton density is:

$$\mathcal{H} = -\frac{1}{2}\pi^+_i \pi^{+,i} + \frac{1}{4}F^+_{ij}F^{+,ij} - A^+_0(\partial^i \pi^+_i - j_0) + j^i A^+_i + v^{-,\mu}\pi^-_\mu + v^{+,0}\pi^+_0 \tag{4.8}$$

The fundamental Poisson brackets are:

$$\{A^\pm_\mu(x), \pi^\pm_\nu(y)\} = \delta_{\mu\nu}\delta^{(3)}(x-y) \tag{4.9}$$

$$\{A^\pm_\mu(x), \pi^\mp_\nu(y)\} = 0 \tag{4.10}$$

Analogously to the computations in section 3.2, $\pi_0^+ \approx$ implies the Gauss law $\partial^i \pi_i^+ - j_0 \approx 0$.
In conclusion, we have six constraints

$$\pi_0^+ \approx 0 \to \partial^i \pi_i^+ - j_0 \approx 0 \tag{4.11}$$
$$\pi_\mu^- \approx 0 \tag{4.12}$$

which obviously are first class. □

In the course of extending the phase space, we assumed that the field strength tensors F^1 and F^2 corresponding to A^1 and A^2 respectively are identical. In the new coordinates this reads $F^1 - F^2 = F^- \approx 0$. We start by implementing the vanishing of the spatial components $F_{ij}^- \approx 0$. We will call them consistency conditions and treat them like gauge fixing functions.

Observation 4.3. *Implementing the consistency constraints $F_{ij}^- \approx 0$ gives a $U(1) \times U(1)$ gauge symmetry of the form:*

$$A_\mu^+ \to A_\mu^+ + \partial_\mu \phi \tag{4.13}$$
$$A_\mu^- \to A_\mu^- + \partial_\mu \phi' \tag{4.14}$$

Computation. First of all we note that the constraints (4.11) generate a $U(1)$ gauge symmetry for A^+ of the form (4.13) as we have discussed in section 3.2.
Furthermore, we note:

$$\partial_1 F_{23}^- = \partial_2 \partial_1 A_3^- - \partial_3 \partial_1 A_2^- \tag{4.15}$$
$$= \partial_2 F_{13}^- - \partial_3 F_{12}^- \tag{4.16}$$

This means that for every $g \in \mathcal{S}_\mathbb{R}(\mathbb{R}^3)$, we have $F_{23}^-(\partial_1 g) = 0$ in the sense that $F_{23}^-(\partial_1 g)[G] = 0$ every field configuration G. We know that $E'' = E$ and hence $F_{23}^-(\partial_1 g)[G] = G(h)$ for all $G \in E'$ for some $h \in \mathcal{S}_\mathbb{R}(\mathbb{R}^3)$. This implies $h = 0$.
Now, for $F_{23}^-(g)[G] = G(f)$ for all $G \in E'$ and some $f \in \mathcal{S}_\mathbb{R}(\mathbb{R}^3)$ such that $\partial_1 f = 0$. The only solution to this is $f = 0$ and hence we write $F_{23}^- = 0$.
It is obvious that π_0^+, $\partial^i \pi_0^+ - j_0$ and π_0^- are Poisson commuting with F_{ij}^- and hence they remain first class after imposing the consistency constraints.
π_k^- on the other does not Poisson commute with F_{ij}^-. The constraint matrix for the constraint functions $\{\pi_1^-, \pi_2^-, \pi_3^-, F_{12}^-, F_{13}^-\}$ has the form:

$$m(x,y) = \begin{pmatrix} 0 & 0 & 0 & \partial_2 & \partial_3 \\ 0 & 0 & 0 & -\partial_1 & 0 \\ 0 & 0 & 0 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 \\ -\partial_3 & 0 & \partial_1 & 0 & 0 \end{pmatrix} \delta^{(3)}(x-y) \tag{4.17}$$

It is straightforward to check that $rk(m) = 4$ and that the kernel is spanned by $\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ 0 \\ 0 \end{pmatrix}$ which means that the constraint $\partial^i \pi_i^-$ is first class while the others are second class. That means that the residual gauge freedom for A^- is generated by π_0^- and $\partial^i \pi_i^-$ which results in a gauge freedom of the form (see section 3.2):

$$A_\mu^- \to A_\mu^- + \partial_\mu \phi \tag{4.18}$$

□

Up to now, we have imposed the consistency conditions $F_{ij}^- \approx 0$ which removed two degrees of gauge freedom. Anyway, there are still first class constraints and hence there is still gauge freedom. We call the gauge freedom that is left after already removing some first class constraints *residual gauge (freedom)*.

Observation 4.4. *The Axial gauge fixing $e_1 \cdot A^1 \approx 0$ and $e_2 \cdot A^2 \approx 0$ together with the consistency conditions $F_{0i}^- \approx 0$ fixes the residual gauge completely having the constraints:*

$$1)\ \pi_0^+ \approx 0 \to 2)\ \partial^i \pi_i^+ - j_0 \approx 0 \tag{4.19}$$

$$3)\ e_1^i(A_i^+ + A_i^-) \approx 0 \to 4)\ e_1^i(-\pi_i^+ + \partial_i(A_0^+ + A_0^-)) \approx 0 \tag{4.20}$$

$$5) - 8)\ \pi_\mu^- \approx 0 \tag{4.21}$$

$$9) - 10)\ F_{1j}^- \approx 0 \tag{4.22}$$

$$11)\ e_2^i(A_i^+ - A_i^-) \approx 0 \to 12)\ e_2^i(-\pi_i^+ + \partial_i(A_0^+ - A_0^-)) \approx 0 \tag{4.23}$$

Computation. First, we rewrite the gauge fixing functions in the new coordinates $e_1^i A_i^1 = \frac{1}{2} e_1^i(\vec{A}_i^+ + \vec{A}_i^-)$ and $e_2 \cdot A^2 = \frac{1}{2} e_2^i(A_i^+ - A_i^-)$.
Recalling the definition of F^-, we have $F_{0i}^- = \partial_0 A_i^- - \partial_i A_0^-$. To compute the Dirac bracket, we need to express F_{0i}^- in the canonical coordinates. Note from (4.6) that we can not express $\partial_0 A_i^-$ can not be expressed in terms of the canonical coordinates. Recall from section 2.2 that $\partial_0 A_i^- := v_i^-$ then acts as Lagrange multiplier in the Hamiltonian and hence $F_{0i}^- = v_i^- - \partial_i A_0^-$.
Now, we see that the consistency conditions $F_{0i}^- \approx 0$ is not a condition on the canonical coordinates but rather on the coordinates and the Lagrange multipliers v_i^-. Hence, can not imply this constraint in the usual form using Dirac brackets.
First of all, we note that all constraints $1) - 12)$ are second class (see Observation 4.6 for the computation). Thus, the gauge freedom is removed and the Lagrange multipliers v_i^- are fixed.
Recall from section 2.2 that the v_i^- are fixed such that the time derivative of the

constraints vanish. One computes:

$$\{e_1^i(A_i^+ + A_i^-), H\} = e_1^i\left(-\pi_i^+ + \partial_i A_0^+ + v_i^-\right) \tag{4.24}$$

$$\{e_2^i(A_i^+ - A_i^-), H\} = e_2^i\left(-\pi_i^+ + \partial_i A_0^+ - v_i^-\right) \tag{4.25}$$

Inserting the constraint $F_{0i}^- = v_i^- - \partial_i A_0^- \approx 0$ in (4.24) and (4.25) respectively gives the constraints 4) and 12). Hence, with the gauge conditions 4) + 12), we make sure that the consistency conditions $F_{0i}^- \approx$ are satisfied.
To check that these conditions remove the whole gauge freedom is subject to the next computation. It is implied by the fact that the constraint matrix is invertible. □

Remark 4.5. *In the following, we will apply many concepts of the study of matrices over $\mathbb{R}$ to the constraint matrix. We justify this by recalling the property of the Fourier transformation that derivatives of the delta distribution are mapped to the corresponding polynomials in the Fourier-variables $k^\alpha \in \mathbb{R}$.*
Hence, the entries of the constraint matrix can equivalently be viewed as real polynomials, for which the used results apply.

Observation 4.6. *The constraint matrix has the form:*
$M(x,y) =$

$$\begin{pmatrix}
0 & 0 & 0 & -e_1^i\partial_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_2^i\partial_i \\
0 & 0 & -e_1^i\partial_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_2^i\partial_i & 0 \\
0 & e_1^i\partial_i & 0 & -e_1^i e_{1,i} & 0 & e_{11} & e_{12} & e_{13} & 0 & 0 & 0 & -e_1^i e_{2,i} \\
e_1^i\partial_i & 0 & -1 & 0 & e_1^i\partial_i & 0 & 0 & 0 & 0 & 0 & e_1^i e_{2,i} & 0 \\
0 & 0 & 0 & -e_1^i\partial_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_2^i\partial_i \\
0 & 0 & -e_{11} & 0 & 0 & 0 & 0 & 0 & \partial_2 & \partial_3 & e_{21} & 0 \\
0 & 0 & -e_{12} & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & e_{22} & 0 \\
0 & 0 & -e_{13} & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & e_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\partial_3 & 0 & \partial_1 & 0 & 0 & 0 & 0 \\
0 & e_2^i\partial_i & 0 & -e_1^i e_{2,i} & 0 & -e_{21} & -e_{22} & -e_{23} & 0 & 0 & 0 & -e_2^i e_{2,i} \\
e_2^i\partial_i & 0 & e_1^i e_{2,i} & 0 & -e_2^i\partial_i & 0 & 0 & 0 & 0 & 0 & e_2^i e_{2,i} & 0
\end{pmatrix} \delta^{(3)}(x-y) \tag{4.26}$$

For the further computations, we need to note that we are only interested in the Dirac brackets of the fields A_μ^+ and the corresponding momenta since by construction they are fields carring physical information while A_μ^- are auxiliary fields to impose the gauge conditions.
Thus, we do not need to know the whole inverse constraint matrix to compute the physical relevant brackets but only some particular components. We will make use of a basic theorem from linear algebra.

Theorem 4.7. *Let $A \in \mathbb{C}^{n\times n}$ an invertible matrix. Then the inverse of A is*

$$A^{-1} = \frac{1}{\det A} adj(A) \tag{4.27}$$

where $adj(A)$ is the adjugate matrix of A given by

$$adj(A)_{ij} = (-1)^{i+j}\tilde{A}_{ij} \tag{4.28}$$

where $\tilde{A}_{ij}$ is the determinant of the (i,j)-minor of A.

Proof. The statement can be found in most textbooks in linear algebra, e.g.[HJKM03], chapter 5. □

To apply this theorem to our particular problem, we first need to compute the determinant of the constraint matrix and the components we are interested in. For simplicity, we will ignore the $\delta^{(3)}(x-y)$ in the computations.

Statement 4.8.

$$\det(M) = 16\,\partial_1^2(e_1^i\partial_i)^4(e_2^i\partial_i)^4 \tag{4.29}$$

Computation. The proof is consecutive application of Laplace's formula. One successively removes the first two rows and the first two columns by deleting $e_1^i\partial_i$ and $e_2^i\partial_i$ respectively. We end up with a sum of 16 terms having the form:

$$-\frac{(e_1^k\partial_k)^4(e_2^k\partial_k)^4}{(e_i^k\partial_k)(e_j^k\partial_k)}\det\begin{pmatrix} 0 & 0 & 0 & \partial_2 & \partial_3 & e_{i1} \\ 0 & 0 & 0 & -\partial_1 & 0 & e_{i2} \\ 0 & 0 & 0 & 0 & -\partial_1 & e_{i3} \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & 0 \\ -\partial_3 & 0 & \partial_1 & 0 & 0 & 0 \\ -e_{j1} & -e_{j2} & -e_{j3} & 0 & 0 & 0 \end{pmatrix} \quad i,j\in\{1,2\} \tag{4.30}$$

To arrive at this form for every addend one eventually needs to swap some rows and columns.
Now, the matrix is a block matrix with two vanishing blocks. Hence, the addends simplify to:

$$\begin{aligned} &-\frac{(e_1^k\partial_k)^4(e_2^k\partial_k)^4}{(e_i^k\partial_k)(e_j^k\partial_k)}\det\begin{pmatrix} \partial_2 & \partial_3 & e_{i1} \\ -\partial_1 & 0 & e_{i2} \\ 0 & -\partial_1 & e_{i3} \end{pmatrix}\det\begin{pmatrix} -\partial_2 & \partial_1 & 0 \\ -\partial_3 & 0 & \partial_1 \\ -e_{j1} & -e_{j2} & -e_{j3} \end{pmatrix} && (4.31)\\ &= -\frac{(e_1^k\partial_k)^4(e_2^k\partial_k)^4}{(e_i^k\partial_k)(e_j^k\partial_k)}(\partial_1(\vec{e}_i\cdot\nabla))(-\partial_1(\vec{e}_j\cdot\nabla)) && (4.32)\\ &= \partial_1^2(e_1^k\partial_k)^4(e_2^k\partial_k)^4 && (4.33) \end{aligned}$$

Thus, every addend contributes the same amount and we find:

$$\det(M) = 16\,\partial_1^2(e_1^k\partial_k)^4(e_2^k\partial_k)^4 \tag{4.34}$$

□

We need to specify which components of M^{-1} contribute to the Dirac brackets of A^+_μ and π^+_ν. In other words, we need to determine the constraints that do not Poisson commute with the respective fields.

Observation 4.9. *The components of M^{-1} contributing to the Dirac brackets of A^+_μ and π^+_ν are combinations of* $1, 2, 3, 4, 11$ *and* 12.

Computation. The fundamental Poisson bracket relations tell us that the only constraint functions that do not Poisson commute with the relevant fields are:

$$A^+_0 : \pi^+_0 \tag{4.35}$$
$$A^+_i : \partial^i \pi_i - j_0,\ e^i_1(-\pi^+_i + \partial_i \cdot (A^+_0 + A^-_0)),\ e^i_2(-\pi^+_i + \partial_i \cdot (A^+_0 - A^-_0)) \tag{4.36}$$
$$\pi^+_i : e^i_1(A^+_i + A^-_i),\ e^i_2(A^+_i - A^-_i) \tag{4.37}$$

□

Observation 4.10. *The only non-zero components contributing to the Dirac brackets are $M^{-1}_{1,2}$, $M^{-1}_{1,4}$, $M^{-1}_{1,12}$, $M^{-1}_{2,3}$ and $M^{-1}_{2,11}$.*

Proof. The idea of this proof is to show that the other components vanish. With help of Theorem 4.7, it is sufficient that the respective minors of M vanish. That is equivalent to showing that the respective minors have linear dependent rows and hence vanishing determinant.

- $M^{-1}_{4,2} = M^{-1}_{4,3} = M^{-1}_{4,11} = M^{-1}_{4,12} = 0$:
  If one deletes the fourth row, then the first and the fifth column in Equation 4.26 are linearly dependent. Hence, $M^{-1}_{4,k} = 0$ for all $k \neq 1, 5$.
  Hence, from Observation 4.9 we see that $M^{-1}_{4,1}$ (or equivalently $M^{-1}_{1,4}$ due to the anti-symmetry of M^{-1}) is the only 4-component that can contribute to the Dirac bracket.

- $M^{-1}_{12,2} = M^{-1}_{12,3} = M^{-1}_{12,11} = 0$:
  If one deletes the twelfth row, then the first and the fifth column in Equation 4.26 are linearly dependent. Hence, $M^{-1}_{12,k} = 0$ for all $k \neq 1, 5$.
  Hence, from Observation 4.9 we see that $M^{-1}_{12,1}$ (or equivalently $M^{-1}_{1,12}$) is the only 12-component that can contribute to the Dirac bracket.

- $M^{-1}_{11,3} = M^{-1}_{11,1} = 0$:
  If one deletes the eleventh row, then the second, sixth, seventh and eighth columns have non-zero entries only in the third, ninth and tenth entry.
  Hence, these four column vectors can not be linearly independent. That means that $M^{-1}_{11,k} = 0$ for $k \neq 2, 6, 7, 8$.

Since the constraint matrix (4.26) is anti-symmetric the inverse is anti-symmetric as well. Hence, we know the residual relevant non-zero components of M^{-1} due to the anti-symmetry. □

Observation 4.11. *The Dirac bracket of the zero component A_0^+ and the canonical momenta π^+ is vanishing:*

$$\{A_0^+, \pi_i^+\}_D = 0. \tag{4.38}$$

Computation. We already know

$$\{A_\mu^+(x), \pi_\nu^+(y)\} = -\delta_{\mu\nu}\delta^{(3)}(x-y) \tag{4.39}$$

and hence A_0^+ and π_i^+ Poisson commute. Hence, the only components contributing to the Dirac bracket arise from M^{-1}. Anyway, in Observation 4.10 we proved that the components corresponding to the constraints that do not Poisson commute with A_0^+ or π_i^+ (compare Observation 4.9) vanish. □

In order to determine the Dirac brackets, we need to compute the value of the relevant components.

Proposition 4.12. *The non-zero components from Observation 4.10 have the following values:*

$$M_{1,2}^{-1} = \frac{1}{4}\left(\frac{e_2^i e_{2,i}}{(e_1^j\partial_j)^2} + \frac{e_1^i e_{1,i}}{(e_2^i\partial_i)^2} + \frac{2\, e_1^i e_{2,i}}{(e_1^i\partial_i)(e_2^i\partial_i)}\right) \tag{4.40}$$

$$M_{1,4}^{-1} = \frac{1}{2}\frac{1}{e_1^i\partial_i} \tag{4.41}$$

$$M_{1,12}^{-1} = \frac{1}{2}\frac{1}{e_2^i\partial_i} \tag{4.42}$$

$$M_{2,3}^{-1} = \frac{1}{2}\frac{1}{e_1^i\partial_i} \tag{4.43}$$

$$M_{2,11}^{-1} = \frac{1}{2}\frac{1}{e_2^i\partial_i} \tag{4.44}$$

Proof. Using Theorem 4.7 it is sufficient to compute the determinant of the respective minors since we computed the determinant $\det(M)$.
We will start by discussing the latter four components. To do this we need to recall the proof of Statement 4.8. Each minor multiplied with the value of the respective component of M appeared as addend of the determinant .
As discussed, every addend contributed the same amount to the determinant. Hence, we know:

$$\tilde{M}_{1,4} = \tilde{M}_{2,3} = \frac{1}{2}\frac{\det(M)}{e_1^i\partial_i} \tag{4.45}$$

$$\tilde{M}_{1,12} = \tilde{M}_{2,11} = \frac{1}{2}\frac{\det(M)}{e_2^i\partial_i} \tag{4.46}$$

The minor $\tilde{M}1, 2$ was computed with Mathematica. One can equivalently, compute the determinant with consecutive application of the Laplace's formula. The computation does not give any new insights. Hence, we will not discuss the technical details any further. □

Now, we have all ingredients to determine the residual Dirac brackets:

Statement 4.13.

$$\{A_i^+(x), \pi_j^+(y)\}_D = \left[\delta_{ij} - \frac{1}{2}\left(\frac{e_{1,j}}{e_1^k\partial_k} + \frac{e_{2,j}}{e_2^k\partial_k}\right)\partial_i\right]\delta^{(3)}(x-y) \tag{4.47}$$

$$\begin{aligned}\{A_0^+(x), A_i^+(y)\}_D =& \Bigg[\frac{1}{4}\left(\frac{e_2^i e_{2,i}}{(e_1^k\partial_k)^2} + \frac{e_1^i e_{1,i}}{(e_2^k\partial^k)^2} + \frac{2\, e_1^l e_{2,l}}{(e_1^k\partial_k)(e_2^k\partial_k)}\right)\partial_i \\ & - \frac{1}{2}\left(\frac{e_{1,i}}{e_1^k\partial_k} + \frac{e_{2,i}}{e_2^k\partial_k}\right)\Bigg]\delta^{(3)}(x-y)\end{aligned} \tag{4.48}$$

Computation. Recalling the definition of the Dirac bracket and Proposition 4.12 we only need to compute the Poisson brackets mentioned in the proof of Observation 4.9.
It is straight forward to verify that the following holds:

$$\{A_0^+(x), \pi_0^+(y)\} = \delta^{(3)}(x-y) \tag{4.49}$$

$$\{A_i^+(x), \partial^k\pi_k^+(y) - j_0(y)\} = \partial_i\delta^{(3)}(x-y) \tag{4.50}$$

$$\{A_i^+(x), e_j^k(-\pi_k^+ + \partial_k(A_0^+ \pm A_0^-))(y)\} = -\, e_{j,i}\delta^{(3)}(x-y) \tag{4.51}$$

$$\{\pi_i^+(x), e_j^k(A_k^+ \pm A_k^-)(y)\} = -\, e_{j,i}\delta^{(3)}(x-y) \tag{4.52}$$

Inserting the results from Proposition 4.12 and those we just stated to the definition of the Dirac bracket gives the result. □

Computation of Statement 4.1. We need to prove that the representation in Statement 4.1 satisfies Statement 4.13.
In particular, with help of the results of the last section, we have for $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$

$$\{A^+(f), \pi^+(g)\} = \frac{1}{4}\{A^{e_1}(f) + A^{e_2}(f), \pi^{e_1}(g) + \pi^{e_2}(g)\} \tag{4.53}$$

$$= \frac{1}{4}\left(4\,\langle\hat{f},\hat{g}\rangle - 2\,\langle\frac{k\cdot\hat{f}}{e_1\cdot k}, e_1\cdot\hat{g}\rangle - 2\,\langle\frac{k\cdot\hat{f}}{e_2\cdot k}, e_2\cdot\hat{g}\rangle\right) \tag{4.54}$$

$$= \langle\hat{f},\hat{g}\rangle - \frac{1}{2}\left(\langle\frac{k\cdot\hat{f}}{e_1\cdot k}, e_1\cdot\hat{g}\rangle - \langle\frac{k\cdot\hat{f}}{e_2\cdot k}, e_2\cdot\hat{g}\rangle\right) \tag{4.55}$$

which is the smeared version of (4.47) in momentum space.
In the same way, one verifies that the correct bracket relation (4.48) using the same techniques as in Equation 3.4. □

Remark 4.14. *In the physics notation, the representation of the conjugate momentum to the expansion of A^+_μ is:*

$$\pi^+_j(x) = -i \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega(k)}{2}} \epsilon^n_j(k) \left[a_n(k)e^{ik\cdot x} + a^\dagger_n(k)e^{-ik\cdot x}\right] + \left(\frac{e_{1,j}}{e^k_1\partial_k} + \frac{e_{2,j}}{e^k_2\partial_k}\right)\frac{j_0(x)}{2} \tag{4.56}$$

4.2 n-fold discretely smeared Axial gauge

The next step of smearing out the Axial gauge is to generalize the results of the last section for arbitrary many copies of the vector potential, on which the Axial gauge condition is implemented respectively.
That means, we decompose the vector potential $A^+ = \sum_{i=1}^{n} A^i$ as sum of n vector potentials with the same field strength tensor $F^i = dA^i$.
This system has gauge freedom which allows for imposing the Axial gauge fixing $e_i \cdot A^i \approx 0$ for each addend. The proof of the statements for arbitrary $n \in \mathbb{N}$ will be made using induction and using the results of the last section as induction beginning.

Statement 4.15. *A representation of the canonical fields in the above decomposition with the Axial gauge conditions is:*

$$A^+ = \frac{1}{n}\sum_{k=1}^{n} A^{e_k} \tag{4.57}$$

$$A^+_0 = \frac{1}{n}\sum_{k=1}^{n} A^{e_k,f}_0 - \frac{1}{n^2}\left(\sum_{j,k=1}^{n} \frac{e^i_j e_{k,i}}{(e_j\partial)(e_k\partial)}\right) j_0 \tag{4.58}$$

$$\pi^+ = \frac{1}{n}\sum_{k=1}^{n} \pi^{e_k}_i = \pi^f + \frac{1}{n}\left(\sum_{k=1}^{n} \frac{e_k}{e_k\partial}\right) j_0 \tag{4.59}$$

Observation 4.16. *Assume, the vector potential $A^+ = \sum_{i=1}^{n} A^i$ is decomposed as the sum of n vector potentials. Then we have a $\underbrace{U(1)\times\cdots\times U(1)}_{n-times}$ gauge symmetry after imposing the consistency conditions $\tilde{F}^k_{ij} \approx 0, k \in \{2,\dots,n\}$. We explain in the proof how $\tilde{F}^k$ is defined.*

Computation. The induction beginning for $n = 2$ is Observation 4.3.
To simplify the computations we again introduce new coordinates:

$$A^+ = \sum_{i=1}^{n} A^i \tag{4.60}$$

$$\tilde{A}^k = A^1 - A^k,\ k \in \{2,\dots,n\} \tag{4.61}$$

These new coordinates have the advantage that only A^+ is a dynamical field while the $\tilde{A}^k$ are auxiliary fields to impose the gauge condition. In order to express the Lagrangian in the new coordinates we need to express the old coordinates in terms of the new ones. One easily checks:

$$A^1 = \frac{1}{n}\left(A^+ + \sum_{i=2}^{n} \tilde{A}^i\right) \tag{4.62}$$

$$A^k = \frac{1}{n}\left(A^+ + \sum_{i\neq k}^{n} \tilde{A}^i - (n-1)\tilde{A}^k\right) \tag{4.63}$$

Thus, we have the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^+_{\mu\nu}F^{+,\mu\nu} - j^\mu A^+_\mu \tag{4.64}$$

We immediately see that (4.64) is independent of $\tilde{A}^k, 2 \leq k \leq n$ and hence $\tilde{\pi}^k_\mu = 0$ are four primary constraints which generate complete gauge freedom of $\tilde{A}^k$.
$\tilde{F}^k = d\tilde{A}^k$ is defined as the field strength tensor of corresponding to $\tilde{A}^k$. Analogously to the explanation in the last section, the decomposition is only admissible if $F^l \approx F^m$ for $l, m \in \{1, \ldots, n\}$. In the new coordinates, these assumptions are equivalent to saying $\tilde{F}^k \approx 0, k \in \{2, \ldots, n\}$.
Hence, we start by imposing $\tilde{F}^k_{ij} \approx 0$. We can repeat the computation from Observation 4.3. Since $\tilde{F}^k_{ij}$ only depends on $\tilde{A}^k$, it is obvious that it Poisson commutes with $\tilde{\pi}^i, i \neq k$.
Thus, the constraint matrix $\tilde{M}$ for the constraints $\tilde{\pi}^k_\mu \approx 0$ and $\tilde{F}^k_{ij} \approx 0$, consists of block matrices:

$$\tilde{M} = \begin{pmatrix} 0_{2\times2} & 0_{6\times2} & \ldots\ldots\ldots\ldots & & 0_{6\times2} \\ 0_{6\times2} & \kappa & 0_{6\times6} & \ldots\ldots & 0_{6\times6} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0_{6\times2} & 0_{6\times6} & \ldots\ldots & 0_{6\times6} & \kappa \end{pmatrix} \tag{4.65}$$

with:

$$\kappa = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & 0_{4\times4} & & & \partial_2 & \partial_3 \\ & & & & -\partial_1 & 0 \\ & & & & 0 & -\partial_1 \\ \hline 0 & -\partial_2 & \partial_1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$0_{2\times2}$}} \\ 0 & -\partial_3 & 0 & \partial_1 & & \end{array}\right) \tag{4.66}$$

To determine the residual first class constraints, we need to find the kernel of $\tilde{M}$. Since the first two rows are zero rows, vectors with entries in the first two columns are in the kernel and hence $\pi_0^+ \approx 0$ and $\partial^i \pi_i^+ \approx 0$ are still first class. As we know, these constraints generate the well-known $U(1)$ gauge freedom for A_μ^+.
We can make use of the block structure of $\tilde{M}$ and reproduce the residual basis element of the kernel $ker(\tilde{M})$ from the kernel of κ. Since the first row and column respectively are zero, vectors with entries in the first component vanish. Hence, $\tilde{\pi}^k \approx 0, 2 \leq k \leq n$ is still a first class constraint. Recalling the computation of the kernel of m as defined in (4.17), we assert that the only remaining first class constraint is $\partial^i \tilde{\pi}_i^k$ for each $2 \leq k \leq n$.

In conclusion, for each k we have two first class constraints $\tilde{\pi}_0^k \approx 0$ and $\partial^i \tilde{\pi}_i^k \approx 0$. It is evident that these constraints Poisson commute with $\tilde{A}^i, i \neq k$ and with A^+. Thus, they generate the known $U(1)$ gauge symmetry for $\tilde{A}^n$. □

Observation 4.17. *Imposing the Axial gauge for every vector potential A^j and the consistency conditions $\tilde{F}_{ij}^k \approx 0$ fixes the gauge completely.*
The set of constraints is given recursively. For $n = 2$, the set of constraints is stated in Observation 4.4. For each auxiliary vector potential $A^k, k \in \{2, \ldots, n\}$, there are eight additional constraints

$$k1) - k4)\ \tilde{\pi}_\mu^k \approx 0 \tag{4.67}$$

$$k5) - k6)\ \tilde{F}_{1j}^k \approx 0 \tag{4.68}$$

$$k7)\ n \cdot e_k^i A_i^k \approx 0 \tag{4.69}$$

$$k8) e_k^i \left(-\pi_i^+ + n \cdot \partial_i A_0^k \right) \approx 0 \tag{4.70}$$

with:

$$A^+ = \sum_{i=1}^{n} A^i \tag{4.71}$$

$$\tilde{A}^k = A^1 - A^k,\ k \in \{2, \ldots, n\} \tag{4.72}$$

One has to note that the old constraints need to be expressed in the new coordinates as well.

Computation. First, we note that replacing the condition $e_k^i A_i^k \approx 0$ by $n \cdot e_k^i A_i^k \approx 0$ does not change the physical situation. We choose the factor n to simplify the computations. The induction beginning for $n = 2$ is given by Observation 4.4.
Hence, we only need to make the induction step $n - 1 \to n$. As already stated the Lagrangian in the new coordinates is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^+ F^{+,\mu\nu} - j^\mu A_\mu^+ \tag{4.73}$$

In analogy to the computations of the last section, the Hamiltonian can be expressed recursively. Obviously A^+ are the only fields having a dynamical share in the Lagrangian and $\tilde{\pi}^n \approx 0$ are primary constraints.

$$\mathcal{H}_n = \mathcal{H}_{n-1} + \tilde{v}^{n,\mu}\tilde{\pi}^n_\mu \tag{4.74}$$

where $\mathcal{H}_n$ is the extended Hamiltonian for n smearing vectors. $\mathcal{H}_2$ is given by (4.8). The Axial gauge conditions in the new coordinates are of the form:

$$e^i_k A_{k,i} = \frac{1}{n} e^i_k \left(A^+_i + \sum_{j \neq k}^{n} \tilde{A}^j_i - (n-1)\tilde{A}^k_i \right) \tag{4.75}$$

$$e^i_1 A_{1,i} = \frac{1}{n} e^i_1 \left(A^+_i + \sum_{j}^{n} \tilde{A}^j_i \right) \tag{4.76}$$

These conditions fix the spatial gauge freedom of A^n. This also means that the Lagrange multipliers are fixed such that the time derivative vanishes. One computes:

$$\{n \cdot e^i_k A^k_i, H\} = e^i_k \left(-\pi^+_i + \partial_i A^+_0 + \sum_{j \neq k}^{n} \tilde{v}^j_i - (n-1)\tilde{v}^k_i \right) \approx 0 \tag{4.77}$$

$$\{n \cdot e^i_1 A^1_i, H\} = e^i_1 \left(-\pi^+_i + \partial_i A^+_0 + \sum_{j=2}^{n} \tilde{v}^j_i \right) \approx 0 \tag{4.78}$$

Imposing $\tilde{F}^k_{0i} = \tilde{v}^k_i - \partial_i \tilde{A}^k_0 \approx 0$ we get:

$$e^i_k \left(-\pi^+_i + \partial_i (A^+_0 + \sum_{j \neq k}^{n} \tilde{A}^j_0 - (n-1)\tilde{A}^k_0) \right) = e^i_k \left(-\pi^+_i + n\partial_i A^k_0 \right) \approx 0 \tag{4.79}$$

$$e^i_1 \left(-\pi^+_i + \partial_i (A^+_0 + \sum_{j}^{n} \tilde{A}^j_0) \right) = e^i_k \left(-\pi^+_i + n\partial_i A^1_0 \right) \approx 0 \tag{4.80}$$

Hence, the gauge conditions $k8)$ make sure that the consistency conditions $\tilde{F}^k_{0i} \approx 0$ are satisfied. □

Observation 4.18. *The constraint matrix has the form:*

$$M_n = \begin{pmatrix} A & B_2 & \cdots\cdots\cdots & & & B_n \\ -B_2^T & K_1^n & C_{2,3} & \cdots\cdots & & C_{2,n} \\ -B_3^T & C_{3,2} & K_2^n & C_{3,4} & \cdots & C_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -B_{n-1}^T & C_{n-1,2} & \cdots & C_{n-1,n-2} & K_{n-1}^n & C_{n-1,n} \\ -B_n^T & C_{n,2} & \cdots\cdots & & C_{n,n-1} & K_n^n \end{pmatrix} \tag{4.81}$$

$$= \left(\begin{array}{ccc|c} & & & B_n \\ & M_{n-1}^* & & C_{2,n} \\ & & & \vdots \\ & & & C_{n-1,n} \\ \hline -B_n^T & C_{n,2} & \ldots \quad C_{n,n-1} & K_n^n \end{array}\right) \tag{4.82}$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 & -e_1^i\partial_i \\ 0 & 0 & -e_1^i\partial_i & 0 \\ 0 & e_1^i\partial_i & 0 & -e_1^i e_{1,i} \\ e_1^i\partial_i & 0 & e_1^i e_{1,i} & 0 \end{pmatrix} \tag{4.83}$$

$$B_k = \begin{pmatrix} 0 & \cdots\cdots\cdots & & & & & 0 & -e_k^i\partial_i \\ 0 & \cdots\cdots\cdots & & & & 0 & -e_k^i\partial_i & 0 \\ 0 & e_{1,1} & e_{1,2} & e_{1,3} & 0 & 0 & 0 & -e_1^i e_{k,i} \\ e_1^i\partial_i & 0 & \cdots\cdots & & & 0 & e_1^i e_{k,i} & 0 \end{pmatrix} \tag{4.84}$$

$$C_{k,m} = \left(\begin{array}{cccccc|cc} & & & & & & 0 & -e_m^i\partial_i \\ & & & & & & -e_{m,1} & 0 \\ & & & 0_{6\times 6} & & & -e_{m,2} & \vdots \\ & & & & & & -e_{m,3} & \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ \hline 0 & e_{k,1} & e_{k,2} & e_{k,3} & 0 & 0 & 0 & -e_k^i e_{m,i} \\ e_k^i\partial_i & 0 & \cdots\cdots & & & 0 & e_k^i e_{m,i} & 0 \end{array}\right) \tag{4.85}$$

$$K_j^m = \left(\begin{array}{c|c} 0_{4\times 4} & U_j^m = \begin{pmatrix} 0 & 0 & 0 & (m-1)e_j^i\partial_i \\ \partial_2 & \partial_3 & (m-1)e_{j,1} & 0 \\ -\partial_1 & 0 & (m-1)e_{j,2} & 0 \\ 0 & -\partial_1 & (m-1)e_{j,3} & 0 \end{pmatrix} \\ \hline -(U_j^m)^T & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_j^i e_{j,i} \\ 0 & 0 & e_j^i e_{j,i} & 0 \end{matrix} \end{array}\right) \tag{4.86}$$

In M^*_{n-1} we changed $K_j^{n-1} \to K_j^n$ to respect that the coordinates are defined differently. We used the notation $e_j^i \partial_i := e_j \partial$. We will adapt this notation in the subsequent steps for sake of clarity.

Computation. We will use induction to prove this statement. We only need to do the induction step since we already proved the induction beginning in Observation 4.6. For the induction step we need to compute the Poisson brackets of the constraints involving those from Observation 4.17. These brackets are the components that occur if one adds a vector potential in the decomposition in Observation 4.16. Using the fundamental Poisson bracket, it is straightforward to verify the form of the constraint matrix. The computations of A, B_2 and K_1^2 can be found in Observation 4.6. The computations for B_k and K_k^n are analogously. For K_i^n, we will exemplary show the computation for $(K_k^n)_{1,8}$:

$$\{\tilde{\pi}_0^k(x), e_k^i(-\pi_i^+ + n \cdot \partial_i A_0^k)(y)\} = n(e_k^i \partial_i)\{\tilde{\pi}_0^k(x), A_0^k(y)\} \tag{4.87}$$

$$= e_k^i \partial_i \{\tilde{\pi}_0^k(x), \left(A_0^+ + \sum_{i \neq k}^{n} \tilde{A}_0^i - (n-1)\tilde{A}_0^k \right)(y)\} \tag{4.88}$$

$$= (n-1) e_k^i \partial_i \delta^{(3)}(x-y). \tag{4.89}$$

For $C_{k,m}$ we will exemplary compute $(C_{k,m})_{2,7}$:

$$\{\tilde{\pi}_1^k(x), n \cdot e_m^i A_i^m(y)\} = \{\tilde{\pi}_1^k(x), e_m^i \left(A_i^+ + \sum_{j \neq m}^{n} \tilde{A}_i^j - (n-1)\tilde{A}_i^m \right)(y)\} \tag{4.90}$$

$$= \{\tilde{\pi}_1^k(x), e_m^i \tilde{A}_i^k(y)\} \tag{4.91}$$

$$= -e_{m,1}\delta^{(3)}(x-y) \tag{4.92}$$

□

Lemma 4.19. *Let $H \in \mathbb{R}^{n\times n}$ be some quadratic matrix. Let H further be divided in block form*

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.93}$$

such that D^{-1} exists. Then:

$$\det(H) = \det(D)\det(A - BD^{-1}C) \tag{4.94}$$

Proof. See e.g. [Mey01], Chapter 6.2. □

Observation 4.20.

$$\det M_n = (n^2 \partial_1)^{2(n-1)} \prod_{i=1}^{n} \left(e_i^j \partial_j \right)^4 \tag{4.95}$$

Proof. We will again use induction to prove this statement. The induction beginning is Statement 4.8.
For the induction step we will use Lemma 4.19 with $D = K_n^n$. Thus, the first step is to show that K_n^n is invertible. Since the determinant is invariant under changes of columns we can change the first four columns with the last four ones without changing the determinant. Using Lemma 4.19 we find:

$$\det(K_n^n) = (\det(U_n^n))^2 \tag{4.96}$$

Now, it is straightforward to check:

$$\det(U_n^n) = (n-1)^2 \partial_1 (e_n^i \partial_i)^2 \tag{4.97}$$

Hence:

$$\det(K_n^n) = (n-1)^4 \partial_1^2 (e_n^i \partial_i)^4 \tag{4.98}$$

Lemma 4.19 tells us:

$$\det(M_n) = \det(K_n^n) \det(M_{n-1}^* - \begin{pmatrix} B_n \\ C_{2,n} \\ \vdots \\ C_{n-1,n} \end{pmatrix} (K_n^n)^{-1} \begin{pmatrix} -B_n^T \\ -C_{2,n} \\ \vdots \\ -C_{n-1,n} \end{pmatrix}^T) \tag{4.99}$$

One checks:

$$(K_n^n)^{-1} = \tag{4.100}$$

$$\begin{pmatrix}
0 & \frac{\partial_1}{(n-1)^2(e_n\partial)} & \frac{\partial_2}{(n-1)^2(e_n\partial)} & \frac{\partial_3}{(n-1)^2(e_n\partial)} & 0 & 0 & 0 & -\frac{1}{n-1} \\
-\frac{\partial_1}{(n-1)^2(e_n\partial)} & 0 & 0 & 0 & e_{n,2} & e_{n,3} & \frac{\partial_1}{n-1} & 0 \\
-\frac{\partial_2}{(n-1)^2(e_n\partial)} & 0 & 0 & 0 & \frac{e_{n,2}\partial_2+e_n\partial}{\partial_1} & \frac{e_{n,3}\partial_2}{\partial_1} & \frac{\partial_2}{\partial_1} & 0 \\
-\frac{\partial_3}{(n-1)^2(e_n\partial)} & 0 & 0 & 0 & \frac{\partial_3 e_{n,2}}{\partial_1} & \frac{\partial_3 e_{n,3}+e_n\partial}{\partial_1} & \frac{\partial_3}{n-1} & 0 \\
0 & -e_{n,2} & -\frac{e_{n,2}\partial_2+e_n\partial}{\partial_1} & -\frac{\partial_3 e_{n,2}}{\partial_1} & 0 & 0 & 0 & 0 \\
0 & -e_{n,3} & -\frac{e_{n,3}\partial_2}{\partial_1} & -\frac{\partial_3 e_{n,3}+e_n\partial}{\partial_1} & 0 & 0 & 0 & 0 \\
0 & -\frac{\partial_1}{n-1} & -\frac{\partial_2}{n-1} & -\frac{\partial_3}{n-1} & 0 & 0 & 0 & 0 \\
\frac{1}{n-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \frac{1}{e_n\partial} \tag{4.101}$$

We need to compute three different types of products of matrices, namely those appearing in (4.99).
One checks

$$B_n \cdot (K_n^n)^{-1} \cdot B_n^T = \begin{pmatrix} 0 & 0 & 0 & -\frac{e_1\partial}{n-1} \\ 0 & 0 & -\frac{e_1\partial}{n-1} & 0 \\ 0 & \frac{e_1\partial}{n-1} & 0 & *_1 \\ \frac{e_1\partial}{n-1} & 0 & -*_1 & 0 \end{pmatrix} \tag{4.102}$$

with $*_1 = \frac{(e_n^i e_{n,i})(e_1\partial)^2+2(n-1)(e_1^i e_{n,i})(e_1\partial)(e_n\partial)}{(n-1)^2(e_n\partial)^2}$ and

$$B_n \cdot (K_n^n)^{-1} \cdot (-C_{m,n})^T = \tag{4.103}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e_m\partial}{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{e_m\partial}{n-1} & 0 \\ 0 & -\frac{e_{n,1}(e_1\partial)}{(n-1)e_n\partial} & -\frac{e_{n,2}(e_1\partial)}{(n-1)e_n\partial} & -\frac{e_{n,3}(e_1\partial)}{(n-1)e_n\partial} & 0 & 0 & 0 & *_2 \\ -\frac{e_1\partial}{n-1} & 0 & 0 & 0 & 0 & 0 & -*_2 & 0 \end{pmatrix} \tag{4.104}$$

with $*_2 = \frac{(e_1\partial)(e_m\partial)(e_n^i e_{n,i})+(n-1)e_n\partial\left((e_1^i e_{n,i})e_m\partial+(e_m^i e_{n,i})e_1\partial\right)}{(n-1)^2(e_n\partial)^2}$ and

$$C_{j,n} \cdot (K_n^n)^{-1} \cdot (-C_{m,n})^T = \tag{4.105}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e_m\partial}{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(e_m\partial)e_{n,1}}{(n-1)e_n\partial} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(e_m\partial)e_{n,2}}{(n-1)e_n\partial} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(e_m\partial)e_{n,3}}{(n-1)e_n\partial} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{(e_j\partial)e_{n,1}}{(n-1)e_n\partial} & -\frac{(e_j\partial)e_{n,3}}{(n-1)e_n\partial} & -\frac{(e_j\partial)e_{n,1}}{(n-1)e_n\partial} & 0 & 0 & 0 & *_3 \\ \frac{e_j\partial}{n-1} & 0 & 0 & 0 & 0 & 0 & -*_3 & 0 \end{pmatrix} \tag{4.106}$$

with $*_3 = \frac{(e_m\partial)(e_j\partial)(e_n^i e_{n,i})+(n-1)(e_n\partial)\left((e_m^i e_{n,i})e_j\partial+(e_j^i e_{n,i})e_m\partial\right)}{(n-1)^2(e_n\partial)^2}$. Now, we need to compute the sum of matrices $M_n' := M_{n-1}^* - \begin{pmatrix} B_n \\ C_{1,n} \\ \vdots \\ C_{n-1,n} \end{pmatrix} (K_n^n)^{-1} \begin{pmatrix} -B_n^T \\ -C_{1,n}^T \\ \vdots \\ -C_{n-1,n}^T \end{pmatrix}^T$. We will do this computation again component wise and recognize that the sum is strongly related to M_{n-1} which allows us to compute the determinant of M_n using the induction hypothesis and hence complete the proof. To make the relation to M_{n-1} as clear as possible we introduce new vectors:

$$\tilde{e}_k^i = \frac{(n-1)(e_n\partial)e_k^i + (e_k\partial)e_n^i}{(n-1)(e_n\partial)} \tag{4.107}$$

Using this definition the component have the form:

$$A' = \begin{pmatrix} 0 & 0 & 0 & -\tilde{e}_1\partial \\ 0 & 0 & -\tilde{e}_1\partial & 0 \\ 0 & \tilde{e}_1\partial & 0 & -\tilde{e}_1^i\tilde{e}_{1,i} \\ \tilde{e}_1\partial & 0 & \tilde{e}_1^i\tilde{e}_{1,i} & 0 \end{pmatrix} \tag{4.108}$$

$$
B'_m = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\tilde{e}_m\partial \\ 0 & 0 & 0 & 0 & 0 & 0 & -\tilde{e}_m\partial & 0 \\ 0 & \tilde{e}_{1,1} & \tilde{e}_{1,2} & \tilde{e}_{1,3} & 0 & 0 & 0 & -\tilde{e}_1^i\tilde{e}_{m,i} \\ \tilde{e}_1\partial & 0 & 0 & 0 & 0 & 0 & \tilde{e}_1^i\tilde{e}_{m,i} & 0 \end{pmatrix} \tag{4.109}
$$

$$
C'_{j,m} = \left(\begin{array}{c|c} 0_{6\times 6} & \begin{matrix} 0 & -\tilde{e}_m^i\partial_i \\ -\tilde{e}_{m,1} & 0 \\ -\tilde{e}_{m,2} & \vdots \\ -\tilde{e}_{m,3} & \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & \tilde{e}_{j,1} & \tilde{e}_{j,2} & \tilde{e}_{j,3} & 0 & 0 \\ \tilde{e}_j^i\partial_i & 0 & \multicolumn{3}{c}{\dots\dots\dots\dots} & 0 \end{matrix} & \begin{matrix} 0 & -\tilde{e}_j^i\tilde{e}_{m,i} \\ \tilde{e}_k^i\tilde{e}_{m,i} & 0 \end{matrix} \end{array}\right) \tag{4.110}
$$

$$
K'^n_j =
$$

$$
\left(\begin{array}{c|c} 0_{4\times 4} & \tilde{U}_j^n = \begin{pmatrix} 0 & 0 & 0 & (n-2)(\tilde{e}_j\partial) \\ \partial_2 & \partial_3 & (n-1)\tilde{e}_{j,1} - \frac{(\tilde{e}_j\partial)e_{n,1}}{e_n\partial} & 0 \\ -\partial_1 & 0 & (n-1)\tilde{e}_{j,2} - \frac{(\tilde{e}_j\partial)e_{n,2}}{e_n\partial} & 0 \\ 0 & -\partial_1 & (n-1)\tilde{e}_{j,3} - \frac{(\tilde{e}_j\partial)e_{n,3}}{e_n\partial} & 0 \end{pmatrix} \\ \hline -(U_j^m)^T & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{e}_j^i\tilde{e}_{j,i} \\ 0 & 0 & \tilde{e}_j^i\tilde{e}_{j,i} & 0 \end{matrix} \end{array}\right) \tag{4.111}
$$

We recognize that the primed matrices A', B'_m and $C'_{j,m}$ have the same form as their unprimed counterparts and one can transform from the unprimed matrices to the primed ones via:

$$
e_j \to \tilde{e}_j \tag{4.112}
$$

For K'^n_j this is not true since there are extra terms in seventh row and column. Anyway, we will show that these components appear in the determinant only contracted with the partial derivations such that we can replace $(n-1)\tilde{e}_j - \frac{(\tilde{e}_j\partial)e_n}{e_n\partial} \to (n-2)\tilde{e}_j$ without changing the determinant. Thus, we can use the induction hypothesis to compute the determinant.

Observation 4.21. *Replacing $(n-1)\tilde{e}_j - \frac{(\tilde{e}_j\partial)e_n}{e_n\partial} \to (n-2)\tilde{e}_j$ in K'^n_j does not change the determinant of M'_n.*

Computation. We note that if we change indices in $\tilde{A}^k$, then the constraint matrix M'_n stays form invariant, only the indices in the block matrices B_k, C_{km} and K_k^m are changed.

This means that it is sufficient to prove this statement for one particular k since the proof generalizes to any $2 \leq k \leq n-1$ due to the symmetry of M'_n.
Since M'_n and K'^n_j are antisymmetric and K'^n_j is a diagonal element of M'_n the computations for the columns will up to possible sign differences also apply to the rows. Hence, we will restrict ourselves to the discussion of the columns.
Since the partial derivatives in the fifth and sixth column of K'^n_j are the only non-zero entries in these columns in M'_n, we can easily express $\det(M'_n)$ in terms of the minors of the fifth column of K'^n_j. We can repeat this procedure for the sixth column of K'^n_j and find:

$$\det(M'_n) = -\partial_1\Big(\partial_1 \det((M'_n)^{5,6}_{3,4}) + \partial_2 \det((M'_n)^{5,6}_{2,4}) + \partial_3 \det((M'_n)^{5,6}_{2,3})\Big) \tag{4.113}$$

where $(M'_n)^{i,j}_{a,b}$ indicates the matrix where in M'_n the columns corresponding to the i-th and j-th columns and the rows corresponding to the a-th and b-th row of K'^n_j are erased. Expanding the determinants in the row respective row of $\{2,3,4\}$ of K'^n_j that has not been erased gives:

$$\det(M'_n) = \partial_1 \left(((n-1)\tilde{e}^i_j - \frac{(\tilde{e}_j\partial)e^i_n}{e_n\partial})\partial_i \det((M'_n)^{5,6,7}_{2,3,4}) + \dots \right) \tag{4.114}$$

where the dots indicate that there are also different minors appearing in this expansion. Anyway, after doing this procedure for the columns and rows of K'^n_j respectively, those minors are independent of $(n-1)\tilde{e}_j - \frac{(\tilde{e}_j\partial)e_n}{e_n\partial}$.
Using

$$\left((n-1)\tilde{e}^i_j - \frac{(\tilde{e}_j\partial)e^i_n}{e_n\partial} \right) \partial_i = (n-2)(\tilde{e}_j\partial) \tag{4.115}$$

completes the proof. □

With Observation 4.21 we can use K'^n_j with $(n-2)\tilde{e}_j$ as entries to compute the determinant. We will call this matrix K^{*n}_j. One recognizes that K^{*n}_j becomes K^{n-1}_j under the transformation:

$$\tilde{e}_j \to e_j \tag{4.116}$$

Hence, we have the equality:

$$\det(M'_n(e_1, \dots, e_n)) = \det(K^n_n) \cdot \det(M_{n-1}(\tilde{e}_1, \dots, \tilde{e}_{n-1})) \tag{4.117}$$

The induction hypothesis allows us to compute:

$$\det(M_{n-1}(\tilde{e}_1, \dots, \tilde{e}_{n-1})) = \left((n-1)^2\partial_1^2\right)^{2(n-2)} \prod_{i=1}^{n-1} (\tilde{e}_i\partial)^4 \tag{4.118}$$

Using

$$\tilde{e}_j\partial = \frac{(n-1)(e_n\partial)(e_j\partial) + (e_n\partial)(e_j\partial)}{(n-1)e_n\partial} = \frac{n}{n-1}e_j\partial \tag{4.119}$$

simplifies (4.118) to:

$$\det(M'_n(e_1,\ldots,e_n)) = \frac{n^{4(n-1)}}{(n-1)^4}\partial_1^{4(n-2)}\prod_{i=1}^{n-1}(e_i\partial)^4 \tag{4.120}$$

Hence, we have

$$\det(M_n) = \det(K_n^n)\cdot\det(M'_n) = \left(n^2\partial_1^2\right)^{2(n-1)}\prod_{i=1}^{n}(e_i\partial)^4 \tag{4.121}$$

which completes the proof. □

Having determined the determinant of M_n we can use Theorem 4.7 to compute the inverse M_n^{-1}. Since our goal is to construct the Dirac brackets we can restrict ourselves to only compute the components of M_n^{-1} that contribute to the Dirac bracket of the dynamical quantities A_μ^+ and π_i^+.

Remark 4.22. *The only constraint functions that do not Poisson commute with A_μ^+ or π_i^+ are:*

$$A_0^+ : \pi_0^+ \tag{4.122}$$

$$A_i^+ : \partial^i\pi_i^+ - j_0, e_j^i(-\pi_i^+ + n\cdot\partial_i A_0^j) \tag{4.123}$$

$$\pi_i^+ : e_j^i A_i^j \tag{4.124}$$

Hence, it is sufficient to know the components of M_n^{-1} corresponding to any combination of the above mentioned constraint functions to construct the Dirac brackets.
In order to simplify the discussion further we note that the matrix M_n is form invariant under exchange of indices $i \leftrightarrow j$, $2 \leq i, j \leq n$ in $\tilde{A}^i$. Thus, the knowledge of all components of M_n^{-1} corresponding to the constraint functions for some particular $j \geq 2$ allows us to construct those components for every $i \geq 2$.
Following the same argument we see that it is sufficient to compute only one component of M_n^{-1} corresponding to $e_j^i(-\pi_i^+ + n\cdot\partial_i A_0^j)$ and $e_k^i A_i^k$ for $2 \leq j \neq k \leq n$. The others can be reconstructed due to the symmetry of M and the accompanying symmetry of M^{-1} under exchange of indices.
However, we need to respect the fact that M_n is not form invariant under exchange of $1 \leftrightarrow j$ in $\tilde{A}^j$ for $j \geq 2$.
Hence, we can reconstruct every required component from knowledge of $(M_n^{-1})_{1,2}$, $(M_n^{-1})_{1,3}$, $(M_n^{-1})_{1,4}$, $(M_n^{-1})_{1,11}$, $(M_n^{-1})_{1,12}$, $(M_n^{-1})_{2,3}$, $(M_n^{-1})_{2,11}$, $(M_n^{-1})_{3,4}$, $(M_n^{-1})_{3,12}$, $(M_n^{-1})_{4,11}$, $(M_n^{-1})_{11,12}$ and $(M_n^{-1})_{11,20}$.

To compute this we will use Theorem 4.7 to use induction. The induction beginning is given by Observation 4.10 and Proposition 4.12.

Observation 4.23. *Let $(m_n)_{a,b}$ be the (a,b)-minor matrix of M_n, $1 \leq a, b \leq 8n-4$. Then we have*

$$\det((m_n)_{a,b}) = \det(K_n^n) \det \left(\left(M_{n-1}^* - \begin{pmatrix} B_n \\ C_{2,n} \\ \vdots \\ C_{n-1,n} \end{pmatrix} (K_n^n)^{-1} \begin{pmatrix} -B_n^T \\ C_{n,2} \\ \vdots \\ C_{n,n-1} \end{pmatrix}^T \right)_{a,b} \right) \tag{4.125}$$

where the subscript $_{a,b}$ denotes that the a-th row and b-th column is deleted.

Computation. We chose n such that there is neither a row nor a column of K_n^n deleted. Hence, we can still use Lemma 4.19 to compute the determinant since K_n^n is invertible. We have

$$\det((m_n)_{a,b}) = \det(K_n^n) \det \left(\left(M_{n-1}^* \right)_{a,b} - \begin{pmatrix} B_n \\ C_{2,n} \\ \vdots \\ C_{n-1,n} \end{pmatrix}_a (K_n^n)^{-1} \begin{pmatrix} -B_n^T \\ C_{n,2} \\ \vdots \\ C_{n,n-1} \end{pmatrix}_b^T \right) \tag{4.126}$$

where the subscripts indicate the deletion of the respective row or column.
Without loss of generality, let $C_{j,n}$ and $C_{m,n}$ for some integers j, m be the matrices in which the row or column respectively is deleted. Then due to the definition of matrix multiplication, we have:

$$(C_{j,n})_a (K_n^n)^{-1} (C_{m,n})_b = \left(C_{j,n} (K_n^n)^{-1} C_{n,m} \right)_{a,b} \tag{4.127}$$

Hence:

$$\begin{pmatrix} B_n \\ C_{2,n} \\ \vdots \\ C_{n-1,n} \end{pmatrix}_a (K_n^n)^{-1} \begin{pmatrix} -B_n^T \\ C_{n,2} \\ \vdots \\ C_{n,n-1} \end{pmatrix}_b^T = \left(\begin{pmatrix} B_n \\ C_{2,n} \\ \vdots \\ C_{n-1,n} \end{pmatrix} (K_n^n)^{-1} \begin{pmatrix} -B_n^T \\ C_{n,2} \\ \vdots \\ C_{n,n-1} \end{pmatrix}^T \right)_{a,b} \tag{4.128}$$

Furthermore, it is obvious that the operation of deleting rows and columns is linear. □

Observation 4.24.

$$(\tilde{M}_3)_{11,20} = 0 \tag{4.129}$$

Computation. It is straightforward to check that the first and 13-th of row M_3 are equal if one deletes the last column. Hence, $(\tilde{M}_3)_{11,20} = 0$. □

Observation 4.25. *The only non-zero components from the above mentioned ones are* $(M_n^{-1})_{1,2}$, $(M_n^{-1})_{1,4}$, $(M_n^{-1})_{1,12}$, $(M_n^{-1})_{2,3}$ *and* $(M_n^{-1})_{2,11}$.

Proof. We need to show $(\tilde{M}_n)_{ab} = 0$ for every combination a, b that is required to construct the Dirac bracket but is not mentioned in the statement. Here $(\tilde{M}_n)_{ab}$ is the (a, b)-minor of M_n.
We use induction to prove this statement. The induction beginning is given by Observation 4.10 and Observation 4.24 respectively.
The induction hypothesis is: $(\tilde{M}_n)_{i,j} = 0 \; \forall n \in \mathbb{N}$ where (i, j) is one of the respective pairs.
For the induction step we will make use of Observation 4.23. Additionally, we note that Observation 4.21 can also be applied to $\det\left(\left(M^*_{n-1} - \begin{pmatrix} B_n \\ C_{2,n} \\ \vdots \\ C_{n-1,n} \end{pmatrix} (K_n^n)^{-1} \begin{pmatrix} -B_n^T \\ C_{n,2} \\ \vdots \\ C_{n,n-1} \end{pmatrix}^T \right)_{a,b}\right)$ for $1 \leq a, b \leq 8n - 4$. Hence, we have:

$$(\tilde{M}_n)_{a,b} = \det(K_n^n) \det((M_{n-1}(\tilde{e}_j))_{a,b}) \tag{4.130}$$

In this form we can use the induction hypothesis for $\det(M_{n-1}(\tilde{e}_j)_{a,b})$ which completes the proof.

□

Observation 4.26. *The Dirac bracket of* A_0^+ *and the canonical momenta* π_i^+ *vanishes:*

$$\{A_0^+(x), \pi_i^+(y)\}_D = 0 \tag{4.131}$$

Proof. Since A_0^+ and π_i^+ are Poisson commuting and the only components of M_n^{-1} appearing in their Dirac bracket (see Remark 4.22) are $(M_n^{-1})_{1,3}$ and $(M_n^{-1})_{8k+3}$, $2 \leq k \leq n$, which are vanishing (see Observation 4.25), the Dirac bracket of A_0^+ and π_i^+ is also vanishing. □

Observation 4.27. *The non-zero components in Observation 4.25 have the values:*

$$(M_n^{-1})_{1,2} = \frac{1}{n^2} \left(\sum_{j,k=1}^{n} \frac{e_j^i e_{k,i}}{(e_j\partial)(e_k\partial)} \right) \tag{4.132}$$

$$(M_n^{-1})_{1,4} = \frac{1}{n} \frac{1}{e_1\partial} \tag{4.133}$$

$$(M_n^{-1})_{1,12} = \frac{1}{n} \frac{1}{e_2\partial} \tag{4.134}$$

$$(M_n^{-1})_{2,3} = \frac{1}{n} \frac{1}{e_1\partial} \tag{4.135}$$

$$(M_n^{-1})_{2,11} = \frac{1}{n} \frac{1}{e_2\partial} \tag{4.136}$$

Computation. We prove this statement using induction. The induction beginning is given by Proposition 4.12.
For the induction step we note that the statement of Observation 4.21 is still valid for our computations. Hence, following the argumentation as before the induction step breaks down to replacing $e_j \to \tilde{e}_j$ in (4.132)-4.136. Using additionally $\tilde{e}_j\partial = \frac{n}{n-1}e_j\partial$, the statements (4.133)-(4.136) become clear.
For (4.132), we have:

$$
\begin{aligned}
(M_n^{-1})_{1,2} &= (M_{n-1}^{-1}(\tilde{e}_j))_{1,2} && (4.137)\\
&= \frac{1}{(n-1)^2}\left(\sum_{j,k=1}^{n-1}\frac{\tilde{e}_j^i\tilde{e}_{k,i}}{(\tilde{e}_j\partial)(\tilde{e}_k\partial)}\right) && (4.138)\\
&= \frac{1}{n^2}\left(\sum_{j,k=1}^{n-1}\frac{\tilde{e}_j^i\tilde{e}_{k,i}}{(e_j\partial)(e_k\partial)}\right) && (4.139)\\
&= \frac{1}{n^2}\Big(\sum_{j,k=1}^{n-1}\frac{1}{(e_j\partial)(e_k\partial)}\big(e_j^ie_{k,i}+\frac{e_j^ie_{n,i}(e_k\partial)}{(n-1)(e_n\partial)}+\frac{e_k^ie_{n,i}(e_j\partial)}{(n-1)(e_n\partial)} && (4.140)\\
&\qquad +\frac{e_n^ie_{n,i}}{((n-1)e_n\partial)^2}\big)\Big) && (4.141)\\
&= \frac{1}{n^2}\left(\sum_{j,k=1}^{n}\frac{e_j^ie_{k,i}}{(e_j\partial)(e_k\partial)}\right) && (4.142)
\end{aligned}
$$

□

Observation 4.28. *The Dirac brackets of the canonical fields are:*

$$\{A_i^+(x),\pi_j^+(y)\} = \left(\delta_{ij}-\frac{1}{n}\left(\sum_{k=1}^{n}\frac{e_{k,j}}{e_k^l\partial_l}\right)\partial_i\right)\delta^{(3)}(x-y) \tag{4.143}$$

$$\{A_0^+(x),A_i^+(y)\} = \left(\frac{1}{n^2}\left(\sum_{j,k=1}^{n}\frac{e_j^ie_{k,i}}{(e_j\partial)(e_k\partial)}\right)\partial_i-\frac{1}{n}\left(\sum_{k=1}^{n}\frac{e_{k,j}}{e_k^l\partial_l}\right)\right)\delta^{(3)}(x-y) \tag{4.144}$$

Computation. Recalling the definition of the Dirac bracket, Remark 4.22 and Observation 4.25 we see that the only non zero components of M_n^{-1} appearing in the Dirac bracket of A_i^+ and π_j^+ are the components $(M_n^{-1})_{2,8k+3}$, $1 \leq k \leq n$.
Using Observation 4.27 and inserting the values of the non-zero components to the Dirac bracket gives the result.

Repeating the same procedure we find that the non zero components appearing in the Dirac bracket of A_0^+ and A_i^+ are $(M_n^{-1})_{1,2}$ and $(M_n^{-1})_{1,8k+4}$, $1 \leq k \leq n$. The values of those components are given in Observation 4.27. Inserting them to the definition of the Dirac bracket gives the result. □

Computation of Statement 4.15. We need to show that this representations satisfy the Dirac bracket relations that we constructed. The computation is analogous to the one for Statement 4.1 and hence will be omitted. □

Observation 4.29. *The magnetic field in the above representation for $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$ is*

$$B(f) = curl(A(f)) \tag{4.145}$$

$$B(f) = \frac{1}{n}\sum_{i=1}^{n} \phi^0(\omega^{-\frac{1}{2}}(\widehat{curl(f)})) \tag{4.146}$$

$$= \phi^0(i\omega^{-\frac{1}{2}}(k \times \hat{f})) \tag{4.147}$$

and hence coincides with the magnetic field in the Coulomb as well as the Axial gauge.

Statement 4.30. *The Maxwell equations in this representation are satisfied.*

Computation. The Hamiltonian of Statement 3.10 in this representation is:

$$H = H^f + H^I \tag{4.148}$$

with

$$H^I = \frac{1}{n}\sum_{i=1}^{n} \pi(\frac{e_i}{e_i \cdot \partial} j_0) - \frac{1}{2}\langle \frac{1}{n}\sum_{i=1}^{n} \frac{e_i}{e_i \cdot k}\widehat{j_0}, \frac{1}{n}\sum_{i=1}^{n} \frac{e_i}{e_i \cdot k}\widehat{j_0}\rangle + A(j) \tag{4.149}$$

The verification of the Maxwell equations is analogous to the computations for Statement 3.30. □

5 YSM-type smeared Axial gauge

In Chapter 3, we have seen that the representation of the Weyl algebra in the Axial gauge is not well defined due to certain singularities. In the last chapter 4, we justified that one can work in a more general setting of the discretely smeared Axial gauge. However, the representation, that we introduced in this chapter, is still only defined on a formal level.

In the first section, we will discuss the limit of the scalar product term of the discretely smeared Axial gauge representation and show that it converges for $n \to \infty$ to a well defined expression under certain technical assumptions that we will specify. The the observables in the smeared Axial gauge turn out to have a similar form as in [MSY05]. We will use these results to define a representation of the observable algebra. We will refer to this kind of gauge as the smeared Axial gauge.

In the third section, we will discuss different possibilities of smearing out the Axial gauge. In particular, we will show that there are ways to smear out the gauge such that the representation of the observables are not manifestly equivalent to the representation in the Coulomb gauge.

In the last section, we will explain that the differences in the representation only come from the transversal shares of the fields. Hence, we will elaborate on the representation of the transversal observables. We will see that the representation of the Weyl algebra of the transversal fields reduces to the Fock representation.

5.1 Smeared Inverse Differential Operator

In the last chapter, we have discussed the representation of the canonical fields in the discretely smeared Axial gauge. Since, we are interested in the relations between the observables in the different gauges, we will concentrate on the electric field or the canonical momentum respectively. We will not consider the magnetic field in this chapter, because we have seen that the representations of the magnetic field in the gauges under consideration coincide.

For a test function f, we found the representation of the canonical momentum in the discretely smeared Axial gauge:

$$\pi(f) = \pi^f(f) - \frac{1}{n} \sum_{j=1}^{n} \langle \frac{e_j}{e_j^k \partial_k} j_0, f \rangle \tag{5.1}$$

$$= \pi^f(f) - i\frac{1}{n}\sum_{j=1}^{n}\langle \frac{e_j}{e_j\cdot k}\widehat{j_0}, \hat{f}\rangle \tag{5.2}$$

From (5.2), we see that only the scalar product share of the representation is changed in the smearing process. The goal of this chapter is to justify the smearing out of the Inverse Differential Operator such that it maps the Schwartz space to square integrable functions.

Definition 5.1. For $\delta > 0$ we define:

$$U_\delta = \{e \in S^2, k \in \mathbb{R}^3; |e\cdot k| \geq \delta|k|\} \subset S^2 \times \mathbb{R}^3 \tag{5.3}$$

For fixed $k \in \mathbb{R}^3$, we call

$$U_\delta(k) = \{e \in S^2; |e\cdot k| \geq \delta|k|\} \subset S^2 \tag{5.4}$$

and for fixed $e \in S^2$:

$$U_\delta(e) = \{k \in \mathbb{R}^3; |e\cdot k| \geq \delta|k|\} \subset \mathbb{R}^3 \tag{5.5}$$

Theorem 5.2. *For fixed $k \in \mathbb{R}^3\backslash 0$, any $g \in C^1(S^2,\mathbb{R})$ such that $\int_{S^2} g = 1$ and $g \geq 0$ and any $\delta > 0$, we can choose a distribution of the smearing vectors $e \in S^2$ such that*

$$\sum_{j=1}^{n}\frac{1}{n}\frac{e_{j,i}}{e_j\cdot k}\chi(U_\delta(e_j))(k) \longrightarrow \int_{U_\delta(k)} d\Omega(e)\frac{e_i}{e\cdot k}g(e), \tag{5.6}$$

where $\chi : \mathbb{R}^3 \to \{0,1\}$ denotes the characteristic function.

Proof. For fixed $k \in \mathbb{R}^3\backslash 0$, we restrict the symbol to a domain of the smearing vectors where it is well defined. For any $\delta > 0$ the symbol

$$\sum_{j=1}^{n}\frac{1}{n}\frac{e_{j,i}}{e_j\cdot k}\chi(U_\delta(e_j))(k) = \sum_{j=1}^{n}\frac{1}{n}\frac{e_{j,i}}{e_j\cdot k}\Bigg|_{e_j\in U_\delta(k)} \tag{5.7}$$

is continuous on $U_\delta(e_j)$ and bounded and hence well defined.
The idea is to write (5.7) in a form that it is similar to a Riemann sum (see Appendix A) which allows us to construct a distribution of smearing vectors such that we can investigate the convergence behaviour.
Let $S_j^n, j \in \{1,\dots,n\}$ be disjoint measurable subsets with non-zero measure such that $e_j \in S_j^n$ and call $c_j^n := \frac{1}{n\cdot\mu(S_j^n)}$. Note that since $\mu(S^2) = 4\pi$ every measurable subset has finite measure. We complete the system of sets $\{S_j^n, 1 \leq j \leq n\}$ to a finite decomposition $\mathfrak{I}^n$ of S^2. For each $S_{j'}^n \in \mathfrak{I}^n\backslash\{S_j^n; j \in \{1,\dots,n\}\}$ we pick one vector $e_{j'} \in \mathfrak{I}_{j'}^n$ and set

$c^n_{j'} = 0$.
Then, we formally have:

$$\sum_{j=1}^{n} \frac{1}{n} \frac{e_{j,i}}{e_j \cdot k} = \sum_{j=1}^{N} c^n_j \frac{e_{j,i}}{e_j \cdot k} \mu(S^n_j) \tag{5.8}$$

where N is the number of subset of the decomposition. Note that $\lim\limits_{n\to\infty} N = \infty$. Additionally, we assume that we have chosen the decomposition such that $|d^n| \to 0$ where $d^n = \sup\limits_j (diam(S^n_j))$.
Furthermore, we assume that the $\lim\limits_{n\to\infty} c^n_j$ exists for every j and set $\lim\limits_{n\to\infty} c^n_j = c_j$. Let $g : S^2 \to \mathbb{R}$ be a function such that

$$e_j \mapsto g(e_j) = c_j \tag{5.9}$$

Note that for any $c \in \mathbb{R}$, we can choose the distribution of the smearing vectors and the decomposition of U_δ such that $\lim\limits_{n\to\infty} c^n_j = c$.
Since $\mathfrak{I}^n$ is a decomposition of S^2 it is clear that $\mathfrak{I}^n|_{U_\delta(k)} := \{S^n_j \cap U_\delta; 1 \le j \le N\}$ is a decomposition of $U_\delta(k)$ for every n. Since $diam(S^n_j\big|_{U_\delta(k)}) \le diam(S^n_j)$ for all n, j, we have $|\, d^n|_{U_\delta(k)}\,| \to 0$. In the following computations we will drop $|_{U_\delta(k)}$ for sake of clarity:

$$|\int_{U_\delta(k)} d^3e \; \frac{e_i}{e\cdot k} g(e) - \lim_{n\to\infty} \sum_{j=1}^{N} c^n_j \frac{e_{j,i}}{e_j \cdot k} \chi(U_\delta(e_j))(k) \mu(S^n_j\big|_{U_\delta(k)})| \tag{5.10}$$

$$\le \lim_{n\to\infty} \sum_{j=1}^{N} \sup_{e\in S^n_j} |\frac{g(e)e_i}{e\cdot k} - \frac{c^n_j e_{j,i}}{e_j\cdot k}| \mu(S^n_j) \tag{5.11}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{N} \sup_{e\in S^n_j} |\frac{g(e)e_i(e_j\cdot k) - c^n_j e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}| \mu(S^n_j) \tag{5.12}$$

$$\le \lim_{n\to\infty} \sum_{j=1}^{N} \sup_{e\in S^n_j} \left(|g(e) \frac{e_i(e_j\cdot k) - e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}| + |\frac{(c^n_j - g(e))e_{j,i}}{e_j\cdot k}| \right) \mu(S^n_j) \tag{5.13}$$

$$\le \lim_{n\to\infty} \sum_{j=1}^{N} \left(\sup_{e\in S^n_j} |g(e) \frac{e_i(e_j\cdot k) - e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}| + \sup_{e\in S^n_j} |\frac{(c^n_j - g(e))e_{j,i}}{e_j\cdot k}| \right) \mu(S^n_j) \tag{5.14}$$

$$\le \lim_{n\to\infty} \sum_{j=1}^{N} \left(\sup_{e\in S^n_j} |g(e) \frac{e_i(e_j\cdot k) - e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}| + \sup_{e\in S^n_j} |(c^n_j - g(e))| \sup_{e\in S^n_j} |\frac{e_{j,i}}{e_j\cdot k}| \right) \mu(S^n_j) \tag{5.15}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{N} \left(\sup_{e\in S^n_j} |g(e) \frac{e_i(e_j\cdot k) - e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}| + |(c^n_j - g(\tilde{e}^n_j))| \sup_{e\in S^n_j} |\frac{e_{j,i}}{e_j\cdot k}| \right) \mu(S^n_j) \tag{5.16}$$

for some $\tilde{e}_j^n \in \overline{S_j^n}$. We have $\lim_{n\to\infty}(g(e_j) - g(\tilde{e}_j^n)) = g(e_j) - \lim_{n\to\infty} g(\tilde{e}_j^n)$. Using that g is continuous we can swap g with the limit: $g(e_j) - \lim_{n\to\infty} g(\tilde{e}_j^n) = g(e_j) - g(\lim_{n\to\infty} \tilde{e}_j^n) = 0$. Let $|c_j^n - g(e_j)| < \epsilon_j''^n$ and $g(e_j) - g(\tilde{e}_j^n) < \epsilon'^n$ for some null series $\left(\epsilon_j''^n\right)_{n\in\mathbb{N}}$ and $(\epsilon'^n)_{n\in\mathbb{N}}$ and let $\frac{\epsilon^n}{2} = \max(\sup_j(\epsilon_j''^n), \epsilon'^n)$ which obviously is a null series again.

$$(5.16) \leq \lim_{n\to\infty} \sum_{j=1}^{N} \Big(\sup_{e\in S_j^n} |g(e)\frac{e_i(e_j\cdot k) - e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}| + \big(|(c_j^n - g(e_j))| \tag{5.17}$$

$$+ |g(e_j) - g(\tilde{e}_j^n)|\big) \sup_{e\in S_j^n} |\frac{e_{j,i}}{e_j\cdot k}|\Big)\mu(S_j^n) \tag{5.18}$$

$$\leq \lim_{n\to\infty} \sum_{j=1}^{N} \left(\sup_{e\in S_j^n} |g(e)\frac{e_i(e_j\cdot k) - e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}| + \epsilon^n \sup_{e\in S_j^n} |\frac{e_{j,i}}{e_j\cdot k}| \right) \mu(S_j^n) \tag{5.19}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{N} \left(\sup_{e\in S_j^n} |g(e)\frac{e_i(e_j\cdot k) - e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}| \right) \mu(S_j^n) \tag{5.20}$$

In the last step we used:

$$\lim_{n\to\infty} \epsilon^n \sum_{j=1}^{N} \sup_{e\in S_j^n} |\frac{e_{j,i}}{e_j\cdot k}|\mu(S_j^n) \tag{5.21}$$

$$\leq \lim_{n\to\infty} \epsilon^n \sum_{j=1}^{N} \frac{1}{\delta|k|}\mu(S_j^n) \tag{5.22}$$

$$= \lim_{n\to\infty} \frac{\epsilon^n}{\delta|k|} V = 0 \tag{5.23}$$

Hence, we are left with:

$$\lim_{n\to\infty} \sum_{j=1}^{N} \sup_{e\in S_j^n} |g(e)\frac{e_i(e_j\cdot k0) - e_{j,i}(e\cdot k)}{(e\cdot k)(e_j\cdot k)}|\mu(S_j^n) \tag{5.24}$$

$$\leq \lim_{n\to\infty} \sum_{j=1}^{N} \sup_{e\in S_j^n} |g(e)| \left(\sup_{e\in S_j^n} |\frac{e_i(e_j - e)\cdot k}{(e\cdot k)(e_j\cdot k)}| + |\frac{diam(S_j^n)}{e_j\cdot k}| \right) \mu(S_j^n) \tag{5.25}$$

$$\leq \frac{1}{|k|} \lim_{n\to\infty} \sum_{j=1}^{N} \sup_{e\in S_j^n} |g(e)| \left(\frac{diam(S_j^n)}{\delta^2} + \frac{diam(S_j^n)}{\delta} \right) \mu(S_j^n) \tag{5.26}$$

Since g is continuous and S^2 is bounded g is also bounded on S^2. Let $c \in \mathbb{R}$ such that

$\sup_{e \in S_j^n} |g(e)| \leq c.$

$$\leq c \frac{1}{|k|} \lim_{n\to\infty} d^n \left(\frac{1}{\delta^2} + \frac{1}{\delta}\right) \sum_{j=1}^{N} \mu(S_j^n) \tag{5.27}$$

$$= \frac{c}{|k|} \lim_{n\to\infty} d^n \left(\frac{1}{\delta^2} + \frac{1}{\delta}\right) V = 0 \tag{5.28}$$

To show that the limit is well defined, we need to show that it is independent of the choice of the decomposition. Let us assume that the distribution of smearing vectors is fixed and that $\mathfrak{I}^n$ and $\mathfrak{I}'^n$ are two families decompositions of $U_\delta(k)$ with above mentioned properties and call their subsets S_j^n and $S_j'^n$ respectively. Then, the difference of the Riemann sums is:

$$\sum_{j=1}^{N} c_j^n \frac{e_{j,i}}{e_j \cdot k} \mu(S_j^n) - \sum_{j=1}^{N'} c_j'^n \frac{e_{j,i}}{e_j \cdot k} \mu(S_j'^n) = \sum_{j=1}^{N} (c_j^n \mu(S_j^n) - c_j'^n \mu(S_j'^n)) \frac{e_{j,i}}{e_j \cdot k} \tag{5.29}$$

$$= \sum_{j=1}^{N} (\frac{1}{n} - \frac{1}{n}) \frac{e_{j,i}}{e_j \cdot k} = 0 \tag{5.30}$$

From this result we can also conclude that for a fixed distribution of smearing vectors, the limit is independent of the choice of δ in the sense that the smearing function g in the integrand is independent of $\delta > 0$.
The reason is that for $\delta' > \delta$, we have $U_{\delta'}(k) \subset U_\delta(k)$. Assume that the smearing function on $U_\delta(k)$ and $U_{\delta'}(k)$ are called g and g' respectively. Since, the limit of the Riemann-type sum is independent of the choice of the decomposition $\mathfrak{I}$ of U_δ, we can choose one such that there is a decomposition $\mathfrak{I}' \subset \mathfrak{I}$ of $U_{\delta'}$. Because of this the restriction of the integral is

$$\left(\int_{U_\delta(k)} d\Omega(e)\ \frac{e_i}{e \cdot k} g(e) \right)\Bigg|_{U_{\delta'}(k)} = \int_{U_{\delta'}(k)} d\Omega(e)\ \frac{e_i}{e \cdot k} g'(e) \tag{5.31}$$

and in particular, it is true that $g|_{U_{\delta'}(k)} = g'$.
Furthermore, we note:

$$|\int d\Omega(e)\ g(e) - \lim_{n\to\infty} \sum_{j=1}^{N} c_j^n \mu(S_j^n)| \tag{5.32}$$

$$\leq \lim_{n\to\infty} \sum_{j=1}^{N} \sup_{e \in S_j^n} |g(e) - c_j^n| \mu(S_j^n) \tag{5.33}$$

$$\leq \lim_{n\to\infty} \sum_{j=1}^{n} \epsilon_j''^n \mu(S_j^n) \tag{5.34}$$

$$\leq \lim_{n\to\infty} \sup_j \epsilon''^n_j \sum_{j=1}^{N} \mu(S^n_j) \tag{5.35}$$

$$= \lim_{n\to\infty} \sup_j \epsilon''^n_j V \tag{5.36}$$

$$=0 \tag{5.37}$$

Anyway, we recall from the definition of c^n_j

$$\sum_{j=1}^{N} c^n_j \mu(S^n_j) = \sum_{j=1}^{n} \frac{1}{n} = 1 \tag{5.38}$$

and hence:

$$\int_{S^2} d\Omega(e)\, g(e) = 1 \tag{5.39}$$

□

The next step is to define the symbol smeared over the whole sphere. This will make it possible for us to define the smeared symbol independent of the choice of $k \in \mathbb{R}^3$. The idea is to make use of $\lim_{\delta\to 0} U_\delta(k) = S^2$ to define the limit $\lim_{\delta\to 0}\left(\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{n}\frac{e_{j,i}}{e_j\cdot k}\chi(U_\delta(k))\right)$. First, we prove two lemmas that will help us to make sense of this expression.

Lemma 5.3. *Let $k \in \mathbb{R}^3$ and $V_\delta(k) := \{e \in S^2; |e\cdot k| \leq \delta\} = S^2\backslash U_\delta(k)$. Then:*

$$\mu(V_\delta(k)) = 4\pi\frac{\delta}{|k|} \tag{5.40}$$

for $\delta < |k|$ and

$$\mu(V_\delta(k)) = 4\pi \tag{5.41}$$

for $\delta \geq |k|$.

Proof. Note: $V_\delta(k) = \{e \in S^2; |(R_k e)\cdot(R_k\hat{k})|k|| < \delta\} = \{e \in S^2; |(R_k e)_3| < \delta\} = \{e \in S^2; |(R_k e)_3| < \frac{\delta}{|k|}\} = \{e \in S^2; |\cos(\theta_k)| < \frac{\delta}{|k|}\}$ where $R_k \in SO(3)$ is the rotation matrix that sends $k \mapsto R_k k = |k|e_3$ and θ_k is the polar angle in the rotated system. Note that cos is invertible on the interval $(0,\pi)$. Assume $\delta < |k|$, otherwise $V_\delta = S^2$ and we know $\mu(S^2) = 4\pi$.
Furthermore, cos is anti-symmetric around $\theta_k = \frac{\pi}{2}$ with $\cos(\frac{\pi}{2}) = 0$. Hence $V_\delta(k) = \{e \in S^2; \cos(\theta_k) \in (-\frac{\delta}{|k|}, \frac{\delta}{|k|})\}$. Since cos is strictly decreasing on $(0,\pi)$, we get $V_\delta =$

$\{e \in S^2; \theta_k \in (\arccos(\frac{\delta}{|k|}), \arccos(-\frac{\delta}{|k|}))\}$.
This allows us to compute the measure of V_δ:

$$\mu(V_\delta(k)) = \int_{V_\delta(k)} d^3e \; 1 \tag{5.42}$$

$$= \int_0^{2\pi} d\phi_k \int_{\arccos(\frac{\delta}{|k|})}^{\arccos(-\frac{\delta}{|k|})} d\theta_k \;\; \sin(\theta_k) \tag{5.43}$$

$$= 2\pi[-\cos(\theta_k)]_{\arccos(\frac{\delta}{|k|})}^{\arccos(-\frac{\delta}{|k|})} \tag{5.44}$$

$$= 4\pi \frac{\delta}{|k|} \tag{5.45}$$

□

Lemma 5.4.

$$|(x - \frac{\pi}{2}) \tan(x)| \leq \frac{\pi}{2} \quad \forall \, x \in \left[\frac{\pi}{2}, \pi\right] \tag{5.46}$$

Proof. Since $\cos(x) \leq 0$ on $[\frac{\pi}{2}, \pi]$ and $\cos(x)$ is concave on that interval, we have:

$$-\cos(x) \geq 1 - \frac{2}{\pi}x \tag{5.47}$$

Since $|\sin| \leq 1$ and $\tan(x)$ is neagtive on $\left[\frac{\pi}{2}, \pi\right]$, we have

$$|\tan(x)| = -\tan(x) \leq \frac{1}{1 - \frac{2}{\pi}x} \tag{5.48}$$

and hence:

$$|(x - \frac{\pi}{2}) \tan(x)| \leq \frac{\frac{\pi}{2} - x}{1 - \frac{2}{\pi}x} \tag{5.49}$$

$$= \frac{\pi}{2} \tag{5.50}$$

□

Theorem 5.5. *Let $k \in \mathbb{R}^3 \backslash \{0\}$ be fixed, then:*

$$\lim_{\delta \to 0} \int_{U_\delta(k)} d^3e \; \frac{e_i}{e \cdot k} g(e) = PV - \int_{S^2} d^3e \; \frac{e_i}{e \cdot k} g(e) \tag{5.51}$$

Proof.

$$|PV - \int_{S^2} d\Omega(e) \; \frac{e_i}{e \cdot k} g(e) - \int_{U_\delta(k)} d^3e \; \frac{e_i}{e \cdot k} g(e)| \tag{5.52}$$

$$= |PV - \int_{V_\delta(k)} d\Omega(e) \; \frac{e_i}{e \cdot k} g(e)| \tag{5.53}$$

Lemma 5.3 tells us in particular: $\lim_{\delta\to 0}\mu(V_\delta(k)) = 0$.

$$|PV - \int_{V_\delta(k)} d\Omega(e)\ \frac{e_i}{e\cdot k} g(e)| \tag{5.54}$$

$$=|PV - \int_{V_\delta(k)} d\Omega(e)\ \frac{k_i\frac{e\cdot k}{|k|^2} + (e\cdot\epsilon^n)\epsilon^n_i}{e\cdot k} g(e)| \tag{5.55}$$

$$\leq \frac{|k_i|}{|k|^2}|\int_{V_\delta(k)} g(e)| + |\epsilon^n_i||PV - \int_{V_\delta(k)} d\Omega(e)\frac{e\cdot\epsilon^n}{e\cdot k} g(e)| \tag{5.56}$$

$$\leq c\frac{|k_i|}{|k|^2}\mu(V_\delta(k)) + \frac{|\epsilon^n_i|}{|k|}|PV - \int_{R_k^{-1}(V_\delta)} d\Omega(e)\ \frac{e_n}{e_3} g(R_p^{-1}e)| \tag{5.57}$$

$$=4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon^n_i|}{|k|}|PV - \int_{\arccos(\frac{\delta}{|k|})}^{\arccos(-\frac{\delta}{|k|})} d\theta_k\ \tan(\theta_k)\sin(\theta_k)\int_0^{2\pi} d\phi_k \begin{pmatrix}\cos(\phi_k)\\ \sin(\phi_k)\end{pmatrix} g(R_k^{-1}e)| \tag{5.58}$$

$$=4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon^n_i|}{|k|}|PV - \int_{\arccos(\frac{\delta}{|k|})}^{\arccos(-\frac{\delta}{|k|})} d\theta_k\ \tan(\theta_k)\sin(\theta_k)\int_0^{2\pi} d\phi_k \begin{pmatrix}\cos(\phi_k)\\ \sin(\phi_k)\end{pmatrix} g_k(\theta_k,\phi_k)| \tag{5.59}$$

In the last step we express $g(R_k^{-1}e)$ as function of θ_k and ϕ_k. We furthermore assume that $g \in C^1(S^2,\mathbb{R})$ which automatically implies that g' is bounded and hence $\mathcal{L}^1$ and g is C^1 in both θ_k and ϕ_k. In particular, this means that g_k has the same properties. The theorem of differentiability of parameter depending integrals now tells us that $\int_0^{2\pi} d\phi_k \begin{pmatrix}\cos(\phi_k)\\ \sin(\phi_k)\end{pmatrix} g_k(\theta_k,\phi_k)\sin(\theta_k) := f_k(\theta_k)$ is C^1 in θ_k.

$$4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon^n_i|}{|k|}|PV - \int_{\arccos(\frac{\delta}{|k|})}^{\arccos(-\frac{\delta}{|k|})} d\theta_k\ \tan(\theta_k)\sin(\theta_k)\int_0^{2\pi} d\phi_k \begin{pmatrix}\cos(\phi_k)\\ \sin(\phi_k)\end{pmatrix} g_k(\theta_k,\phi_k)| \tag{5.60}$$

$$=4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon^n_i|}{|k|}|PV - \int_{\arccos(\frac{\delta}{|k|})}^{\arccos(-\frac{\delta}{|k|})} d\theta_k f_k(\theta_k)\tan(\theta_k)| \tag{5.61}$$

At this point, we note that the integrand has one singularity at $\frac{\pi}{2} \in [\arccos(\delta), \arccos(-\delta)]$. Inserting the definition of the principal value gives:

$$=4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon^n_i|}{|k|}|\lim_{\epsilon\to 0^+}\int_{\frac{\pi}{2}+\epsilon}^{\arccos(-\frac{\delta}{|k|})} d\theta_k f_k(\theta_k)\tan(\theta_k) + \int_{\arccos(\frac{\delta}{|k|})}^{\frac{\pi}{2}-\epsilon} d\theta_k f_k(\theta_k)\tan(\theta_k)| \tag{5.62}$$

$$=4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon^n_i|}{|k|}|\lim_{\epsilon\to 0^+}\int_{\frac{\pi}{2}+\epsilon}^{\arccos(-\frac{\delta}{|k|})} d\theta_k \tan(\theta_k)\left(f_k(\theta_k) - f_k(\pi-\theta_k)\right)| \tag{5.63}$$

$$\leq 4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon_i^n|}{|k|}\lim_{\epsilon\to 0^+}\int_{\frac{\pi}{2}+\epsilon}^{\arccos(-\frac{\delta}{|k|})} d\theta_k(-\tan(\theta_k))|f_k(\theta_k) - f_k(\pi-\theta_k)| \tag{5.64}$$

Since f_k is C^1 in θ_k and we assumed the derivation to be bounded, we can apply the mean value theorem which tells us $|f_k(\theta_k) - f_k(\pi-\theta_k)| \leq c'|2\theta_k - \pi| = 2c'(\frac{\pi}{2} - \theta_k)$ where $c' := \sup_{x\in S^2}|f'(x)| \in \mathbb{R}^+$ is the bound of the derivative of f_k.

$$\leq 4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon_i^n|}{|k|}\lim_{\epsilon\to 0^+}\int_{\frac{\pi}{2}+\epsilon}^{\arccos(-\frac{\delta}{|k|})} d\theta_k(-\tan(\theta_k))2c'(\frac{\pi}{2} - \theta_k) \tag{5.65}$$

Applying Lemma 5.4 gives us

$$\begin{aligned}
&4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon_i^n|}{|k|}\lim_{\epsilon\to 0^+}\int_{\frac{\pi}{2}+\epsilon}^{\arccos(-\frac{\delta}{|k|})} d\theta_k(-\tan(\theta_p))2c'(\frac{\pi}{2} - \theta_k) && (5.66)\\
\leq &4\pi c\frac{|k_i|}{|k|^3}\delta + \frac{|\epsilon_i^n|}{|k|}\lim_{\epsilon\to 0^+}\int_{\frac{\pi}{2}+\epsilon}^{\arccos(-\frac{\delta}{|k|})} d\theta_k\; 2c'\frac{\pi}{2} && (5.67)\\
= &4\pi c\frac{|k_i|}{|k|^3}\delta + \pi c'\frac{|\epsilon_i^n|}{|k|}\left[\arccos(-\frac{\delta}{|k|}) - \frac{\pi}{2}\right] && (5.68)
\end{aligned}$$

In total, we have:

$$\begin{aligned}
&\lim_{\delta\to 0}|PV - \int_{S^2} d\Omega(e)\;\frac{e_i}{e\cdot k}g(e) - \int_{U_\delta(k)} d\Omega(e)\;\frac{e_i}{e\cdot k}g(e)| && (5.69)\\
&\leq \lim_{\delta\to 0}\left(4\pi c\frac{|k_i|}{|k|^3}\delta + \pi c'\frac{|\epsilon_i^n|}{|k|}\left[\arccos(-\frac{\delta}{|k|}) - \frac{\pi}{2}\right]\right) && (5.70)\\
&= 0 && (5.71)
\end{aligned}$$

□

5.2 Smeared Observables

In the last section, we discussed the convergence of the "inverse" differential operator appearing in the Axial gauge. We will use these results to elaborate on the convergence of the observables.
As we have discussed, the symbol

$$\frac{1}{n}\sum_{j=1}^{n}\frac{e_{j,i}}{e_j\cdot k} \tag{5.72}$$

is not well-defined due to the severe singularities in the denominator and thus it can not map the Schwartz space to the space of square integrable functions.

Proposition 5.6. *For every $f \in \mathcal{S}(\mathbb{R}^3)$ every $\delta > 0$ there is a symbol $S_{\delta,i}$ depending on $n \in \mathbb{N}$ such that formally:*

$$\lim_{\delta \to 0} S_{\delta,i} = \frac{1}{n} \sum_{j=1}^{n} \frac{e_{j,i}}{e \cdot k} \tag{5.73}$$

$$S_{\delta,i} f \in L^2 \tag{5.74}$$

$$\lim_{n \to \infty} \left\| (S_{\delta,i} - \int_{U_\delta(k)} \frac{e_i}{e \cdot k} g(e)) f \right\| = 0 \tag{5.75}$$

$$\lim_{\delta \to 0} \lim_{n \to \infty} \|S_{\delta,i} f\|_{L^2} = \left\| PV - \int_{S^2} d\Omega(e) \, \frac{e_i}{e \cdot k} g(e) f \right\|_{L^2} \tag{5.76}$$

Proof. We will construct the series of symbols $S_{\delta,i}$ in a way that we can use the results of the previous section to proof Proposition 5.6.
As we already discussed, every addend of $\frac{1}{n} \sum\limits_{j=1}^{n} \frac{e_{j,i}}{e_j \cdot k}$ has severe singularities coming from the scalar product in the denominator. Restricting every term to a domain where the symbol is bounded and continuous removes those singularities and we can investigate the convergence of the restricted symbol acting on a Schwartz function. Hence, we define:

$$S_{\delta,i} := \frac{1}{n} \sum_{j=1}^{n} \frac{e_{j,i} \chi(U^j)(k)}{e_j \cdot k} \tag{5.77}$$

with $U_\delta(e_j) := U^j$ for any $\delta > 0$.
The first statement of the proposition follows immediately from the fact $\lim\limits_{\delta \to 0} U^j(k) = S^2$.
Assume $f \in \mathcal{S}(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3} d^3k \left| \frac{1}{n} \sum_{j=1}^{n} \frac{e_{j,i} \chi(U^j)(k)}{e_j \cdot k} \hat{f}(k) \right|^2 \leq \frac{1}{n^2} \int_{\mathbb{R}^3} d^3k \left| \sum_{j=1}^{n} \frac{e_{j,i}}{\delta |k|} \hat{f}(k) \right|^2 \tag{5.78}$$

$$= \sum_{j,k=1}^{n} \frac{e_j \cdot e_k}{\delta^2 n^2} \left\| \frac{f}{\Delta^{\frac{1}{2}}} \right\|_{L^2}^2 \tag{5.79}$$

which is finite for $\delta > 0$ since $\Delta^{-\frac{1}{2}} : \mathcal{S}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ (see Appendix B), which proves the second statement.
To prove the third statement we use the results from the last section and in particular

(5.28) to get:

$$\int_{\mathbb{R}^3} d^3k \left| \int_{U_\delta} \frac{e_i}{e\cdot k} g(e)\hat{f}(k) - \frac{1}{n}\sum_{j=1}^{n} \frac{e_{j,i}\chi(U^j)(p)}{e_j\cdot k}\hat{f}(k)\right|^2 \tag{5.80}$$

$$= \int_{\mathbb{R}^3} d^3k \left| \frac{1}{n}\sum_{j=1}^{n} \Big(\int_{U_\delta(k)} \frac{e_i}{e\cdot k} g(e) - \frac{e_{j,i}\chi(U^j)(k)}{e_j\cdot k}\Big)\right|^2 \left|\hat{f}(k)\right|^2 \tag{5.81}$$

$$\leq \int_{\mathbb{R}^3} d^3k \left| \frac{c}{|k|} d^n \left(\frac{1}{\delta^2}+\frac{1}{\delta}\right) V\right|^2 \left|\hat{f}(k)\right|^2 \tag{5.82}$$

$$= \left| cVd^n \left(\frac{1}{\delta}+\frac{1}{\delta^2}\right)\right|^2 \int_{\mathbb{R}^3} d^3k \left|\frac{\hat{f}(k)}{|k|}\right|^2 \tag{5.83}$$

$$= \left| cVd^n \left(\frac{1}{\delta}+\frac{1}{\delta^2}\right)\right|^2 \left\| \frac{f}{\Delta^{\frac{1}{2}}}\right\|_{L^2}^2 \tag{5.84}$$

Since the decomposition is chosen such that $\lim\limits_{n\to\infty} d^n = 0$, this estimates gives:

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} d^3k \left| \int_{U_\delta(k)} \frac{e_i}{e\cdot k} g(e)\hat{f}(k) - \frac{1}{n}\sum_{j=1}^{n} \frac{e_{j,i}\chi(U^j)(k)}{e_j\cdot k}\hat{f}(k)\right|^2 = 0 \tag{5.85}$$

For the proof of the last statement, we use Theorem 5.5 and (5.70):

$$\left\| \int_{U_\delta} d^3k \frac{e_i}{e\cdot k} g(e) f(k) - PV - \int_{S^2} d^3k \frac{e_i}{e\cdot k} g(e) f(k)\right\|_{L^2} \tag{5.86}$$

$$= \left\| PV - \int_{V_\delta(p)} d^3k \frac{e_i}{e\cdot k} g(e) f(k)\right\|_{L^2} \tag{5.87}$$

$$\leq \left\| \left(4\pi c \frac{|k_i|}{|k|^2}\delta + \pi c' \frac{|\epsilon_i^n|}{|k|}\left[\arccos(-\delta) - \frac{\pi}{2}\right]\right) f\right\|_{L^2} \tag{5.88}$$

$$\leq 4\pi c\delta \left\| \frac{f}{|k|}\right\|_{L^2} + \pi c' \left[\arccos(-\delta) - \frac{\pi}{2}\right] \left\| \frac{f}{|k|}\right\|_{L^2} \tag{5.89}$$

and hence:

$$\lim_{\delta\to 0} \left\| \int_{U_\delta} d^3k \frac{e_i}{e\cdot k} g(e) f(p) - PV - \int_{S^2} d^3k \frac{e_i}{e\cdot k} g(e) f(k)\right\|_{L^2} = 0 \tag{5.90}$$

$\square$

Lemma 5.7. *The limit* $\lim\limits_{\delta\to 0}\lim\limits_{n\to\infty} S_{\delta,i} f = PV - \int_{S^2} d\Omega(e)\ \frac{e_i}{e\cdot k} g(e) f(k) \in L^2(\mathbb{R}^3)$ *for all* $f \in \mathcal{S}(\mathbb{R}^3)$.

Proof.

$$PV - \int_{S^2} d^2e \, \frac{e_i}{e \cdot k} g(e) f(k) = \frac{k_i}{|k|^2} f(k) + \frac{\epsilon_i^n(k)}{|k|} f(k) PV - \int_{S^2} d\Omega(e) \, \frac{e_n}{e_3} g(R_k^{-1} e) \quad (5.91)$$

In the proof of Theorem 5.5, we showed that the latter integral is bounded measurable function in k, since it only depends on $\frac{k}{|k|}$.
Thus, we can use the results from Appendix B and know that $\frac{f}{|k|} \in L^2(\mathbb{R}^3)$. Moreover, the integral expression maps $L^2 \to L^2$ as bounded function. □

Statement 5.8. *Using the continuity of the fields (Theorem 2.65), we have the representation of the canonical fields and observables in the smeared Axial gauge for $f, g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$:*

$$A^{ax}(f) = \Phi_S(\omega^{-\frac{1}{2}} P_T(\hat{f})) - \Phi_S(\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e) P_T(e \frac{k \cdot \hat{f}}{e \cdot k}) g(e)) \quad (5.92)$$

$$\pi^{ax}(g) = \Phi_S(i\omega^{\frac{1}{2}} P_T(\hat{g})) - i \left\langle \int_{S^2} d\Omega(e) \frac{e}{e \cdot k} \widehat{j_0} g(e), \hat{g} \right\rangle_{L^2(\mathbb{R}^3, \mathbb{R})} \quad (5.93)$$

$$B(f) = \Phi_S(\omega^{-\frac{1}{2}} (\widehat{curl(f)})) \quad (5.94)$$

Remark 5.9. *The representation of the observables is similar to the one given in [MSY05], page 34. The authors, however, use a different notation. In their notation, the choice of the functions f is the analogue to the choice of the smearing function g in our formulation.*

Remark 5.10. *In Statement 5.8 and subsequently, we omit the indication that the integral is a principal value integral.*

5.3 Longitudinal Fields and Examples

In this section, we will discuss properties of the representation of the observables in the smeared Axial gauge that are valid for all possible choices of a smearing function. In particular, we will see that the representation of the longitudinal share of the electric field in the smeared Axial gauge manifestly coincides with the longitudinal share of the electric field in the Coulomb gauge.
Moreover, we will compute the inverse differential operators appearing in the smeared Axial gauge for two particular choices of smearing functions.
It turns out that it its possible and useful to expand g in terms of so called spherical harmonics.

Theorem 5.11. *Stone-Weierstraß theorem Let X be a compact metric space and and $R \subset C(X, \mathbb{R})$ a subalgebra that separates points. Then $R \subset C(X, \mathbb{R})$ is dense in the norm topology.*

Proof. See [Rud91], Theorem 5.7 or [HS71], Anhang I,6. □

Remark 5.12. *It is well known that S^2 is compact under the Euclidean topology inherited from R^3 which makes it a compact metric space meaning we can apply the Weierstraß theorem to $g \in C(S^2, \mathbb{R})$.*
By the Stone-Weierstraß theorem the restriction of the set of polynomials in $\mathbb{R}^3$ to the unit sphere S^2 is dense in $C(S^2)$. Therefore, considering the density of $C(S^2)$ in $L^2(S^2)$ the polynomials on S^2 are dense in $L^2(S^2)$.
Anyway, the generating set of the polynomials on S^2 is neither irreducible, nor orthogonal.

However, the set of spherical harmonics is an irreducible set of orthogonal functions that is dense $L^2(S^2)$. Before discussing the aforementioned properties, we define the spherical harmonics.

Definition 5.13. The spherical harmonics are C^∞ functions $Y^{s,c}_{lm} : S^2 \to \mathbb{R}; l \in \mathbb{N}, m \in [0, l]$. In terms of the polar and azimuthal angle $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, they are defined as:

$$Y^s_{lm}(\theta, \phi) = \frac{1}{\sqrt{2\pi}} N_{lm} P_{lm}(cos(\theta)) \sin(m\phi) \tag{5.95}$$

$$Y^c_{lm}(\theta, \phi) = \frac{1}{\sqrt{2\pi}} N_{lm} P_{lm}(cos(\theta)) \cos(m\phi) \tag{5.96}$$

where $N_{lm} \in \mathbb{R}$ is a constant and P_{lm} are associated Legendre polynomials which are

$$P_{lm}(x) = (1 - x^2)^{\frac{|m|}{2}} \left(\frac{\partial}{\partial x}\right)^{|m|} P_l(x) \tag{5.97}$$

$$P_l(x) = -\frac{1}{2^l} \sum_{k=0}^{[\frac{l}{2}]} (-1)^k \frac{(2l - 2k)!}{k!(l - k)!(l - 2k)!} x^{l-2k} \tag{5.98}$$

$$=: \sum_{k=0}^{[\frac{l}{2}]} \alpha_{kl} x^{l-2k} \tag{5.99}$$

with $[\frac{l}{2}]$ meaning rounding $\frac{l}{2}$ to the next smaller integer.

Proposition 5.14. *The algebra generated by the spherical harmonics is dense in $C(S^2)$.*

Proof. See [AH12], Theorem 2.31 . □

Due to the density of $C(S^2)$ in $L^2(S^2)$, the spherical harmonics are also dense in $L^2(S^2)$. The reason that spherical harmonics are useful is

Proposition 5.15. *The Hilbert space $L^2(S^2)$ admits an orthogonal direct sum decomposition of the form:*

$$L^2(S^2) = \bigoplus_{l=0}^{\infty} Y_l \tag{5.100}$$

where Y_l denotes the space of spherical harmonics spanned by $\{Y^{s,c}_{lm}, m \in [0, l]\}$.

Proof. See [Tah15], Theorem 8.24 . □

This tells us that $\int_{S^2} d^3x\ Y_{lm} = c\ \delta_{l,0}\delta_{m,0}$. According to this we can easily satisfy the condition $\int_{S^2} d\Omega(e)\ g(e) = 1$ by expanding g in terms of spherical harmonics. g has the expansion $g = \frac{1}{c} Y_{0,0} + \sum_{l=1}^{\infty} \sum_{m=0}^{l} c^s_{lm} Y^s_{lm} + c^c_{lm} Y^c_{lm}$.

Lemma 5.16. *For any admissible smearing function g, i.e. $g \in C^1(S^2)$, $g \geq 0$ and g is normed, we have:*

$$\int_{S^2} d\Omega(e)\ \frac{e_i}{e \cdot k} g(e) = \frac{k_i}{|k|^2} + \epsilon^n_i(k) \int_{S^2} d\Omega(e)\ \frac{e \cdot \epsilon^n}{e \cdot k} g(e) \tag{5.101}$$

$$= \frac{k_i}{|k|^2} + \frac{\epsilon^n_i}{|k|} \int_{S^2} d\Omega(e)\ \frac{e_n}{e_3} g(R_k^{-1} e) \tag{5.102}$$

where R_k is a rotation matrix $R_k \in SO(3)$ that sends $R_k : k \mapsto e_3$.

Proof. We start by extending $\frac{k}{|k|}$ to an orthonormal basis for $\mathbb{R}^3$ with the standard polarization vectors $\epsilon^n(p)$ and decompose $e \in S^2 \subset \mathbb{R}^3$ in this basis:

$$e = \frac{e \cdot k}{|k|^2} k + (e \cdot \epsilon^1)\epsilon^1 + (e \cdot \epsilon^2)\epsilon^2 \tag{5.103}$$

Inserting this to Equation 5.101 we get

$$\int_{S^2} d\Omega(e)\ \frac{e_i}{e \cdot k} g(e) = \frac{k_i}{|k|^2} \int_{S^2} g(e) + \epsilon^n_i(p) \int_{S^2} d\Omega(e)\ \frac{e \cdot \epsilon^n}{e \cdot k} g(e) \tag{5.104}$$

$$= \frac{k_i}{|k|^2} + \epsilon^n_i(k) \int_{S^2} d\Omega(e)\ \frac{e \cdot \epsilon^n}{e \cdot k} g(e) \tag{5.105}$$

where in the last term summation over $n \in \{1, 2\}$ is understood.
Since $R_k \in SO(3)$, it leaves the Euclidean scalar product on $\mathbb{R}^3$ invariant. Using this, we can substitute:

$$\frac{k_i}{|k|^2} + \epsilon^n_i(k) \int_{S^2} d\Omega(e)\ \frac{e \cdot \epsilon^n}{e \cdot k} g(e) \tag{5.106}$$

$$= \frac{k_i}{|k|^2} + \frac{\epsilon^n_i}{|k|} \int_{S^2} d\Omega(e)\ \frac{e_n}{e_3} g(R_k^{-1} e) \tag{5.107}$$

□

Note that R_k is not uniquely determined by the assumption that $R_k : k \mapsto e_3$. Thus, we need to fix R_k for explicit computations. It is straightforward to check that

$$R_k = \begin{pmatrix} \cos(\theta_k)\cos(\phi_k) & -\sin(\phi_k) & \cos(\phi_k)\sin(\theta_k) \\ \cos(\theta_k)\sin(\phi_k) & \cos(\phi_k) & \sin(\theta_k)\sin(\phi_k) \\ -\sin(\theta_k) & 0 & \cos(\theta_k) \end{pmatrix} \tag{5.108}$$

is such a rotation matrix if we assume that θ_k and ϕ_k are the polar and azimuthal angles of k respectively.
The inverse being the transpose (since it is an orthogonal matrix) then reads

$$R_k^{-1} = \begin{pmatrix} \cos(\theta_k)\cos(\phi_k) & \cos(\theta_k)\sin(\phi_k) & -\sin(\theta_k) \\ -\sin(\theta_k) & \cos(\phi_k) & 0 \\ \cos(\phi_k)\sin(\theta_k) & \sin(\theta_k)\sin(\phi_k) & \cos(\theta_k) \end{pmatrix} \tag{5.109}$$

which maps:

$$R_k^{-1} : S^2 \to S^2 \tag{5.110}$$

$$\begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \mapsto \tag{5.111}$$

$$\begin{pmatrix} \cos(\theta)\cos(\phi_k)\sin(\theta_k) + \sin(\theta)\left(\cos(\theta_k)\cos(\phi_k)\cos(\phi) - \sin(\phi)\sin(\phi_k)\right) \\ \cos(\theta)\sin(\theta_k)\sin(\phi_k) + \sin(\theta)\left(\cos(\theta_k)\cos(\phi)\sin(\phi_k) + \cos(\phi_k)\sin(\phi)\right) \\ \cos(\theta)\cos(\theta_k) - \cos(\phi)\sin(\theta_k)\sin(\theta) \end{pmatrix} \tag{5.112}$$

5.3.1 Example of constant smearing

In the following, we will discuss two different examples for the smeared Axial gauge. That, in particular, means that we will compute the smearing symbol for two different admissible smearing functions g. The most intuitive option is choosing g to be constant. In order for g to be normed, we have to set $g(e) = \frac{1}{4\pi}$.

Proposition 5.17. *Assume that the axial gauge is constantly smeared out over the sphere S^2, i.e. that the smearing function $g = \frac{1}{4\pi}$, then the representation of the electric field coincide with its representation in the Coulomb gauge.*

Proof. Due to the symmetry of sin and cos, we find:

$$\int_{S^2} d\Omega(e) \frac{e \cdot \epsilon^n}{e \cdot k} \frac{1}{4\pi} = \frac{1}{4\pi |k|} \int_0^{2\pi} d\phi \begin{pmatrix} \sin(\phi) \\ \cos(\phi) \end{pmatrix} \int_0^{\pi} d\theta \ \tan(\theta)\sin(\theta) \tag{5.113}$$

$$= 0 \tag{5.114}$$

Hence, the smeared canonical momentum for a test function $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$ is:

$$\pi^{ax}(f) = \pi^f(f) - i \langle \frac{k}{|k|^2} \widehat{j_0}, \hat{f} \rangle \tag{5.115}$$

$$= \pi^C(f) \tag{5.116}$$

□

Remark 5.18. *As we have discussed, the constant contribution from the expansion of the smearing function in terms of the spherical harmonics is required for g to be normed. This computations shows that the constant contribution reproduces the representation of the electric field in the Coulomb gauge.*
Additionally, the previous computations show that spherical harmonics of higher order only contribute to the transversal electric field.
That means that the longitudinal shares of the observables in the Coulomb and smeared Axial gauge are manifestly equivalent.
Hence, possible inequivalences of the representations of the observables can only arise from the transversal share of the electric field in the respective representations. Thus, for the investigation of the equivalence of the Coulomb and smeared Axial gauge, it is sufficient to only study the respective representations of the transversal Weyl algebra.

5.3.2 Another important Example

We will study a second example of an admissible smearing function g to show that the smeared Axial gauge and the Coulomb gauge do not manifestly coincide.
It is noteworthy, that we will use this example of a smearing function later to construct a counter-example for unitary equivalence of the two gauges under consideration.

Proposition 5.19. *There exist admissible smearing functions $g \in C^1(S^2)$ such that the smeared Axial gauge and the Coulomb gauge are not manifestly equivalent.*

Proof. In order to prove this proposition, we will explicitly state an example of such a smearing function. Choose:

$$g(e) = \frac{1}{4\pi} + \sqrt{\frac{9}{60\pi^3}}(Y_{2,1}(e) + Y_{2,-1}(e)) \tag{5.117}$$

$$= \frac{1}{4\pi} + \frac{3}{4\pi}\cos(\theta_e)\sin(\theta_e)\left(\cos(\phi_e) - \sin(\phi_e)\right) \tag{5.118}$$

$$= \frac{1}{4\pi} + \frac{3}{4\pi}e_3(e_2 - e_1) \tag{5.119}$$

From the last example, we know that the constant term in the expansion of g reproduces the Coulomb term. Lemma 5.16 tells us that we only need to compute the longitudinal share of the integral.
The behaviour of g und R_k^{-1} can easily be read off Equation 5.110 and inserted to the integral in Lemma 5.16:

$$\frac{\epsilon^n}{|k|}\int_{S^2} d\Omega(e)\ \frac{e_n}{e_3} g(R_k^{-1}e) \tag{5.120}$$

$$= \frac{1}{|k|}\Big(\epsilon^1(k)\cos(2\theta_k)(\cos(\phi_k) - \sin(\phi_k)) - \epsilon^2(k)\cos(\theta_k)(\cos(\phi_k) + \sin(\phi_k))\Big) \tag{5.121}$$

The integral was calculated with Mathematica. □

This tells us that the resulting electric field in the smeared Axial gauge with smearing function g from (5.117) has, opposed to the Coulomb gauge, some non-vanishing longitudinal contribution coming from the external current j_0 given by (5.121).
Consequently, the representations of the electric fields do not coincide manifestly. However, this is a bit too much to expect since for equivalence of different gauges it is sufficient if there exists a unitary map that intertwines between the observables in the different gauges.

5.4 Transversal Fields

As explained in Remark 5.18, we only need to consider the transversal share of the electric field to investigate the unitary equivalence. This has the advantage that the representation of the electric and magnetic field is irreducible in the transversal Fock Space $\Gamma_s(\mathfrak{h}_{\mathfrak{T}})$. We will moreover adjust the test function space to the space L that we defined and discussed in subsection 2.6.4
In this section, we will only work with the quantized fields.

Observation 5.20. *The representation of the transversal share of the magnetic field in both gauges for $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$ is:*

$$B^T(f) = \Phi_S(\omega^{-\frac{1}{2}}(\widehat{curl(f)})) \tag{5.122}$$

The representation of the transversal share of the canonical momenta in the two gauges under consideration for $g \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)$ is:

$$\pi^{C,T}(g) = \Phi_S(i\omega^{\frac{1}{2}} P_T(\hat{g})) \tag{5.123}$$

$$\pi^{ax,T}(g) = \Phi_S(i\omega^{\frac{1}{2}} P_T(\hat{g})) - i \left\langle \int_{S^2} d\Omega(e) \frac{P_T(e)}{e \cdot k} \widehat{j_0 g}(e), \hat{g} \right\rangle_{L^2(\mathbb{R}^3, \mathbb{R})} \tag{5.124}$$

$$= \Phi_S(i\omega^{\frac{1}{2}} P_T(\hat{g})) - \left\langle \omega^{\frac{1}{2}} \int_{S^2} d\Omega(e) \frac{P_T(e)}{e \cdot k} \widehat{j_0 g}(e), i\omega^{\frac{1}{2}} P_T(\hat{g}) \right\rangle_{\mathfrak{h}_T} \tag{5.125}$$

Remark 5.21. *For the representation of the observables, we obviously only need the transversal share of the test functions. Moreover, for the real valued test function, we always have a factor of $\omega^{-\frac{1}{2}}$ while for the complex valued, we always have a factor of $\omega^{\frac{1}{2}}$. That means, we can equivalently work with the test function L for the transversal fields.*
As we have explained in subsection 2.6.4 $(L, Im(\langle \cdot, \cdot \rangle))$ is a symplectic space and dense in $\mathfrak{h}_T$ which means that the Fock representation of the Weyl algebra $\mathcal{W}(L, Im(\langle \cdot, \cdot \rangle))$ is irreducible.

Corollary 5.22. *Let Π_F be the Fock representation of the symplectic space $(L, Im(\langle \cdot, \cdot \rangle))$, then we have*

$$\Pi_F(W(f)) = e^{i\left(B((1+C)f) + \pi^C((1-C)f)\right)} \tag{5.126}$$

for every $f \in L$.

For the definition of the representation of the transversal fields in the smeared Axial gauge, it is useful to adopt the notion of a coherent state representation from [Roe70]. A similar notation for the massless scalar field can be found in [Kun98].

Definition 5.23. Let ω_0 denote the Fock vacuum state, L' be the real, algebraic dual of L and $F \in L'$. Then

$$\omega_F(f) = \omega_0(f) e^{iF(f)}, \qquad f \in L \tag{5.127}$$

is called the to F associated *coherent state*.
The corresponding automorphism of $\mathcal{W}(L, Im(\langle\cdot,\cdot\rangle))$:

$$\mathfrak{F} : \mathcal{W}(L, Im(\langle\cdot,\cdot\rangle)) \to \mathcal{W}(L, Im(\langle\cdot,\cdot\rangle)) \tag{5.128}$$

$$W(f) \mapsto e^{iF(f)} W(f) \tag{5.129}$$

is called the to F associated *coherent automorphism*.
We will call the representation $\Pi^F_{coh} = \Pi_F \circ \mathfrak{F}$ the to $F \in L'$ associated *coherent representation*.

Note that formally a coherent automorphism amounts to shifts: $a^\dagger(k) \mapsto a^\dagger(k) + g(k), a(k) \mapsto a(k) + \overline{g}(k)$ for some function g.

Observation 5.24. *The exponentiated canonical momentum operators in the smeared Axial gauge have the form for $f \in (1-C)L_\pi$*

$$e^{i\pi^{ax}(f)} = e^{-i\langle \omega^{\frac{1}{2}} \int_{S^2} d\Omega(e) \frac{P_T(e)}{e\cdot k} g(e) \widehat{j_0}, f\rangle} e^{i\Phi_S(f)} \tag{5.130}$$

$$= e^{iF^{ax}(f)} \Pi_F(f) \tag{5.131}$$

with $F^{ax}(h) = Im(\langle i\omega^{\frac{1}{2}} \int_{S^2} d\Omega(e) \frac{P_T(e)}{e\cdot k} g(e) \widehat{j_0}, h\rangle)$ for $g \in L$. We note

$$i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot(-k)} g(e) \widehat{j_0}(-k) = \overline{i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot k} g(e) \widehat{j_0}(k)} \tag{5.132}$$

due to the realness of j_0 and hence $i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot(k)} g(e) \widehat{j_0}(k) \in (1+C)\mathfrak{h}_T$.
Using Plancherel's formula, we have

$$\langle i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot k} g(e) \widehat{j_0}, h\rangle \in \mathbb{R} \qquad h \in (1+C)L_\phi \tag{5.133}$$

$$\langle i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot k} g(e) \widehat{j_0}(k), h\rangle \in i\mathbb{R} \qquad h \in (1-C)L_\pi \tag{5.134}$$

That means:

$$\Pi^F_{coh}(W(h)) = \Pi_F(W(h)) \qquad h \in (1+C)L_\phi \tag{5.135}$$

$$\Pi^F_{coh}(W(h)) = e^{-i\langle \omega^{\frac{1}{2}} \int_{S^2} d\Omega(e) \frac{P_T(e)}{e\cdot k} g(e) \widehat{j_0}, h\rangle} \Pi_f(W(h)) \qquad h \in (1-C)L_\pi \tag{5.136}$$

The question of unitary equivalence of the two gauges under consideration can be reformulated in terms of representations to the question whether the two representations Π_F and Π^F_{coh} of $\mathcal{W}(L, Im(\langle\cdot,\cdot\rangle))$ are unitary equivalent.

6 Unitary equivalence for vanishing total charge

In this section, we will address the main question of this thesis of the unitary equivalence of the Coulomb and smeared Axial gauge.
As we have discussed in the last section, the longitudinal shares of the observables in both gauges coincide. Hence, a potential unitary inequivalence can only come from the transversal parts.
Moreover, we have discussed the representations of the transversal Weyl algebra that correspond to the observables in the respective gauge.
The goal of this section is to show that the representations Π_F and Π^F_{coh} are unitary equivalent if the total charge carried by the external current vanishes $\widehat{j_0}(0) = Q = 0$ (Recall the definition of the total charge from Statement 3.3. Then it is obvious that $Q = \widehat{j_0}(0)$). We will prove the equivalence by explicitly stating a unitary operator on $\Gamma_s(\mathfrak{h}_T)$ that intertwines between the representations.

Remark 6.1. *Subsequently, we will denote the transversal share of the fields simply by $\pi^{C/ax}$ and $B^{C/ax}$ respectively to avoid an overload of notation.*
The transversal fields are operator valued distributions on $\Gamma_s(\mathfrak{h}_T)$.

Theorem 6.2 (Taylor's formula)**.** *Let $U \subset \mathbb{R}^n$ be open and, $x \in U$ and $\xi \in \mathbb{R}^n$ such that $x + t\xi \in U\ \forall t \in [0,1]$. Let further $f \in C^{k+1}(U, \mathbb{R}^n)$. Then there is a $\theta \in [0,1]$ such that:*

$$f(x+\xi) = \sum_{|\alpha| \le k} \frac{D^\alpha f(x)}{\alpha!}\xi^\alpha + \sum_{|\alpha|=k+1} \frac{D^\alpha f(x+\theta\xi)}{\alpha!}\xi^\alpha \tag{6.1}$$

Proof. [For08] □

Lemma 6.3. *Let $f \in \mathcal{S}(\mathbb{R}^n)$, then we can write:*

$$f(x) = f(0)\rho(x) + \sum_{j=1}^{n} x_j \Psi_j(x) \tag{6.2}$$

with $\Psi_j, \rho \in \mathcal{S}(\mathbb{R}^n)$ and $\rho|_{B_{\frac{1}{2}(0)}} = 1$ and $\rho(x) = 0$ for all $|x| \ge \frac{3}{4}$.

Proof. This proof is taken from [Mel97].
We start by writing $f = \rho f + (1-\rho)f$. We note that the first addend is supported

inside of $B_{\frac{3}{4}}(0)$ and the second addend outside of $B_{\frac{3}{4}}(0)$ due to the choice of ρ.
On $B_1(0)$ we can write, according to Taylor's formula (Theorem 6.2):

$$f(x) = f(0) + \sum_{j=1}^{n} x_j \zeta_j, \ \zeta_j \in C^\infty \tag{6.3}$$

Since the product $\rho\phi$ is supported in $B_1(0)$, we have

$$\rho(x) f(x) = f(0)\rho(x) + \sum_{j=1}^{n} x_j \rho \zeta_j(x) \tag{6.4}$$

with $\rho\zeta_j \in \mathcal{S}(\mathbb{R}^n)$.
Since the second addend of the above decomposition $(1-\rho)f$ is supported outside of $B_{\frac{1}{2}}(0)$ and $(1-\rho)f$ due to the closeness of $\mathcal{S}(\mathbb{R}^n)$ under multiplication and addition, we have $\zeta := |x|^{-2}(1-\rho)f \in \mathcal{S}(\mathbb{R}^n)$ and since $\mathcal{S}(\mathbb{R}^n)$ is also closed under multiplication with polynomials, we have

$$(1-\rho)f = \sum_{j=1}^{n} x_j (x_j \zeta) \in \mathcal{S}(\mathbb{R}^n) \tag{6.5}$$

with $x_j\zeta \in \mathcal{S}(\mathbb{R}^n)$.
All this results together give:

$$f(x) = f(0)\rho(x) + \sum_{j=1}^{n} x_j \left(\rho\zeta_j(x) + x_j \zeta\right) \tag{6.6}$$

Calling $\Psi_j := \rho\zeta_j(x) + x_j\zeta \in \mathcal{S}(\mathbb{R}^n)$ gives the desired result. □

Lemma 6.4. *Let $g \in C^1(S^2)$, then:*

$$PV - \int_{S^2} d\Omega(e) \ \frac{P_T(e)}{e \cdot k} g(e) = \frac{1}{|k|} c(\frac{k}{|k|}) \tag{6.7}$$

with $c \in L^\infty(S^2)$.

Proof.

$$|k| \cdot PV - \int_{S^2} d\Omega(e) \ \frac{P_T(e)}{e \cdot k} g(e) \tag{6.8}$$

$$= \lim_{\epsilon \to 0} \left(\int_0^{\frac{\pi}{2}-\epsilon} + \int_{\frac{\pi}{2}+\epsilon}^{\pi} \right) d\theta \int_0^{2\pi} d\phi \ \tan(\theta)\sin(\theta)\sin/\cos(\phi) g(R_k^{-1} e) \tag{6.9}$$

$$= \lim_{\epsilon \to 0} \int_0^{\frac{\pi}{2}-\epsilon} \tan(\theta)\sin(\theta) \left(\tilde{g}(\theta, \hat{k}) - \tilde{g}(\pi - \theta, \hat{k} \right) \tag{6.10}$$

In the last step, we performed the substitution $e \to R_k e$ and rewrote

$$\int_0^{2\pi} d\phi \ \tan(\theta)\sin(\theta)\sin/\cos(\phi) g(R_k^{-1}e) := \tilde{g}(\theta, \hat{k}) \tag{6.11}$$

where $\hat{k} := \frac{k}{|k|}$ and $\tilde{g} \in C^1$ due to the differentiability of parameter dependent integrals. Furthermore, we performed a substitution $\theta \to \pi - \theta$ in the second integral.
Since $\tilde{g} \in C^1$, we can make use of the mean value theorem and write $\tilde{g}(\theta, \hat{k}) - \tilde{g}(\pi - \theta, \hat{k}) \le 2c(\hat{k})\left(\frac{\pi}{2} - \theta\right)$.
Using Lemma 5.4 and repeating the arguments in the Proof of Theorem 5.5 gives the result. □

Lemma 6.5. *Let $\hat{f} \in \mathcal{S}(\mathbb{R}^3)$, then:*

$$\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e) \frac{e \cdot \epsilon^n}{e \cdot k} g(e) \widehat{\overline{f}} \in \mathfrak{h} \Leftrightarrow \widehat{f}(0) = 0 \tag{6.12}$$

Proof. Lemma 6.4 gives us:

$$\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e) \frac{e \cdot \epsilon^n}{e \cdot k} g(e) \widehat{\overline{f}} = \omega^{-\frac{3}{2}} \int_{S^2} d\Omega(e) \frac{e_n}{e_3} g(R_k^{-1}e) \widehat{f} \tag{6.13}$$

$$= \omega^{-\frac{3}{2}} \widehat{f} c(\frac{k}{|k|}) \tag{6.14}$$

We use Lemma 6.3 to rewrite:

$$\omega^{-\frac{3}{2}} \widehat{f} c(\frac{k}{|k|}) = \omega^{-\frac{3}{2}} c(\hat{k}) \left(\hat{f}(0)\rho(k) + \sum_{j=1}^{n} k_j \Psi_j(k) \right) \tag{6.15}$$

Now, we write $\omega^{-\frac{3}{2}} k_j \Psi_j(k) := \omega^{-\frac{1}{2}} \tilde{\Psi}_j(k)$. Since $\omega^{-1} k_j = \tilde{k}_j$ is a bounded function $\tilde{\Psi}_j \in \mathcal{S}(\mathbb{R}^n)$ and hence:

$$\omega^{-\frac{3}{2}} \widehat{f} c(\frac{k}{|k|}) = c(\tilde{k}) \left(\omega^{-\frac{3}{2}} \hat{f}(0)\rho(k) + \omega^{-\frac{1}{2}} \sum_{j=1}^{n} \tilde{\Psi}_j(k) \right) \tag{6.16}$$

As we have shown in Lemma B.18, the second addend is square-integrable. The first addend, however, is not square-integrable unless $\hat{f}(0) = 0$, in which case it is evidently square-integrable. □

Corollary 6.6. *Assume $\widehat{j_0}(0) = 0$, then the operator:*

$$U := e^{i\Phi_S(-i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e) \ \frac{P_T(e)}{e \cdot k} g(e) \widehat{j_0})} \tag{6.17}$$

is unitary on $\Gamma_s(\mathfrak{h}_T)$ and intertwines between Π_F and Π_{coh}^F:

$$U \Pi_F(W(F)) U^\dagger = \Pi_{coh}^F(W(f)) \tag{6.18}$$

for all $f \in L$.

Proof. From Lemma 6.5, we know that $-i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot k} g(e)\widehat{j_0} \in \mathfrak{h}_T \Leftrightarrow \widehat{j_0}(0) = 0$. The unitarity of U then follows immediately from Theorem 2.65.
Since

$$-i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot(-k)} g(e)\widehat{j_0}(-k) = \overline{-i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot k} g(e)\widehat{j_0}(k)} \tag{6.19}$$

due to the realness of j_0, we know that $\mathcal{F}^{-1}\left(-i \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot k} g(e)\widehat{j_0}(k)\right) \in L^2_{\mathbb{R}}(\mathbb{R}^3)$. Hence, U commutes with every operator of the form $e^{i\phi(h)}, h \in (1+C)L_\phi$ due to Theorem 2.68.
We need to check the commutation relation for $h \in (1-C)L_\pi$. From the Weyl relations (Theorem 2.68, 3.) follows with

$$\alpha(h) := \langle -i\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e)\, \frac{P_T(e)}{e\cdot k} g(e)\widehat{j_0}, h\rangle \tag{6.20}$$

the commutation relation:

$$U\Pi_F(W(h))U^\dagger = e^{-i\alpha(h)}\Pi_F(h) \tag{6.21}$$

Since $\alpha(h) = -F^{ax}(h)$ for all $h \in (1+C)L_\pi$, we have proved the equivalence of the two different representations. □

Remark 6.7. *Similar results already appeared in the works of [HLL94] and [NTO94] where the authors treated the problem of the unitary equivalence of the Coulomb and Axial gauge on a formal level. The authors of both papers, however, did not elaborate on the issue of well-definiteness of U.*

7 Unitary inequivalence for non-vanishing total charge

In the last chapter, we proved that the representations of the observables in the Coulomb and smeared Axial gauge are unitarily equivalent if $\widehat{j_0}(0) = 0$.
The intention of this chapter is to discuss the case when the total external charge does not vanish $\widehat{j_0}(0) \neq 0$. It will turn out that there does not exist a unitary transformation that intertwines between the representations Π_F and Π^F_{coh} in this case.
The strategy of the presented proof is taken from [Kun98] where the author showed that certain representation are inequivalent by constructing a so-called central sequence. The sequence of the Weyl operators corresponding to the central sequence was proven to converge to multiple of the identity. For certain representations, the author proved that the respective limits are different, which allowed for concluding that the representations are not unitary equivalent.
We, however, have a slightly different situation than the author in [Kun98] and hence need to change some technical details. The idea behind of this proof remains unchanged.

Theorem 7.1. *Assume $\widehat{j_0}(0) \neq 0$, then there does not exist a unitary operator $U : \Gamma(\mathfrak{h}_T) \to \Gamma(\mathfrak{h}_T)$ such that*

$$\langle \Psi, U\Pi^F_{coh}(W)U^\dagger\Psi\rangle = \langle \Psi, \Pi_F(W)\Psi\rangle \tag{7.1}$$

for all $\Psi \in \Gamma(\mathfrak{h}_T)$ and $W \in \mathcal{W}(L, Im(\langle\cdot,\cdot\rangle))$.

For sake of clarity, we will split the proof of this theorem into several lemmas.

Lemma 7.2. *For every $\epsilon > 0$, there is a family $(f_\lambda)_{\lambda\in\mathbb{R}} \subset L_\pi$ such that there for all $g \in L$, there is $\lambda' \in \mathbb{R}$ with*

$$\|[W(g), W(f_\lambda)]\| \leq \epsilon \tag{7.2}$$

for all $\lambda \geq \lambda'$.

Proof. Let $f \in (1 - C)L_\pi$. Then we can write $f = i\omega^{\frac{1}{2}}\widehat{f'}$, $f' \in P_T(\mathcal{S}(\mathbb{R}^3, \mathbb{R}^3))$. Furthermore, choose $\widehat{f'}$ and hence f to have compact support and define:

$$\widehat{f'_\lambda(\cdot)} := \lambda^2\widehat{f'}(\lambda\cdot) \tag{7.3}$$

$$f_\lambda := i\omega^{\frac{1}{2}}\widehat{f'_\lambda} \tag{7.4}$$

It is obvious that we have:

$$f_\lambda(\cdot) = \lambda^{\frac{3}{2}} f(\lambda \cdot) \tag{7.5}$$

Since $f \in (1-C)L_\pi$, it is clear that:

$$[W(g), W(f_\lambda)] = 0 \;\forall g \in (1-C)L_\pi \tag{7.6}$$

So, it is sufficient to verify Equation 7.2 for $g \in (1+C)L_\phi$. The Weyl relations give:

$$[W(g), W(f_\lambda)] = \left(e^{-\frac{i}{2}\langle \widehat{g'}, \widehat{f'_\lambda}\rangle} - e^{-\frac{i}{2}\langle \widehat{g'}, \widehat{f'_\lambda}\rangle}\right) W(g+f_\lambda) \tag{7.7}$$

$$= 2i\sin(\frac{\langle \widehat{g'}, \widehat{f'_\lambda}\rangle}{2}) W(g+f_\lambda) \tag{7.8}$$

with $g' \in P_T(\mathcal{S}(\mathbb{R}^3) \otimes \mathbb{R}^3)$ and we have the relation $g = \omega^{\frac{1}{2}}\widehat{g'}$.
The scalar product can be rewritten as:

$$\langle \widehat{g'}, \widehat{f'_\lambda}\rangle = \langle \widehat{g'}, \lambda^2 \widehat{f'}(\lambda \cdot)\rangle \tag{7.9}$$

$$= \lambda^{-1} \langle \widehat{g'}(\frac{\cdot}{\lambda}), \widehat{f'}\rangle \tag{7.10}$$

$$= \lambda^{-1} \langle \widehat{g'}(\frac{\cdot}{\lambda}) 1_{supp(f)}, \widehat{f'}\rangle \tag{7.11}$$

where $1_{supp(f)}$denotes the characteristic function of the support of f. Now, it is obvious that $\widehat{g'}(\frac{\cdot}{\lambda})1_{supp(f)} \in \mathfrak{h}_T$ with

$$\left\|\widehat{g'}(\frac{\cdot}{\lambda})1_{supp(f)}\right\|_2 \le \left\|\widehat{g'}\right\|_\infty (\mu(supp(f)))^{\frac{1}{2}} < \infty \tag{7.12}$$

since $\widehat{g'} \in P_T(\mathcal{S}(\mathbb{R}^3) \otimes \mathbb{C}^3)$ and $\mu(supp(f)) < \infty$.
Hence:

$$|\langle \widehat{g'}, \widehat{f'_\lambda}\rangle| \le \lambda^{-1} \left\|\widehat{g'}\right\|_\infty (\mu(supp(f)))^{\frac{1}{2}} \left\|\widehat{f'}\right\|_2 \tag{7.13}$$

Hence, for all $\lambda \ge \frac{\|\hat{g}\|_{L^\infty} \mu(supp(f))^{\frac{1}{2}} \|\hat{f}\|_{L^2}}{\epsilon}$, we have:

$$|\langle \widehat{g'}, \widehat{f'_\lambda}\rangle| \le \epsilon \tag{7.14}$$

Due to the anti-symmetry of sin around 0 and the concavity on $[0, \frac{\pi}{2}]$, we have for all such $\lambda \in \mathbb{R}$:

$$\|[W(g), W(f_\lambda)]\| = |\sin(|\frac{\langle \hat{g}, \omega^{\frac{1}{2}} \hat{f}_{\lambda'}\rangle}{2}|) \tag{7.15}$$

$$\le \epsilon \tag{7.16}$$

□

Corollary 7.3. *For every finite sum* $\tilde{A} = \sum_{i=1}^{N} c_i W(h_i)$ *of Weyl operators* $W(h_i), h_i \in L$, *and every* $\delta > 0$, *there is* $\tilde{\lambda} \in \mathbb{R}$ *such that:*

$$\left\|[\tilde{A}, W(f_\lambda)]\right\| \leq \delta \tag{7.17}$$

Proof.

$$\left\|[\tilde{A}, W(f_\lambda)]\right\| \leq \sum_{i=1}^{N} |c_i| \, \|[W(h_i), W(f_\lambda)]\| \tag{7.18}$$

According to Lemma 7.2, for every $\epsilon > 0$ and every h_i, there is $\lambda_i \in \mathbb{R}$ such that:

$$\|[W(h_i), W(f_\lambda)]\| \leq \epsilon \tag{7.19}$$

for all $\lambda \geq \lambda_i$.
Choose $\tilde{\lambda} := \max_i \lambda_i = \frac{\max_i \|h_i\|_\infty}{\epsilon} \mu(supp(f)) \left\|\hat{f}\right\|_2$, then:

$$\left\|[\tilde{A}, W(f_\lambda)]\right\| \leq \sum_{i=1}^{N} |c_i| \epsilon := \delta \tag{7.20}$$

□

Corollary 7.4. *Let* $A \in W(L, \sigma)$. *For every* $\alpha > 0$, *there exists* $\lambda_0 \in \mathbb{R}$ *such that:*

$$\|[A, W(f_\lambda)]\| \leq \alpha \tag{7.21}$$

for all $\lambda \geq \lambda_0$.

Proof. For every $A \in W(L, \sigma)$ there is a sequence $(\tilde{A}_n)_{n \in \mathbb{N}} \subset W(L, \sigma)$ of finite sums of Weyl operators that approximate A in norm.
That means for every $\alpha > 0$ there is $N \in \mathbb{N}$ such that:

$$\left\|A - \tilde{A}_m\right\| \leq \frac{\alpha}{3} \tag{7.22}$$

for all $m \geq N$. Assume that we have chosen m sufficiently large such that (7.22) is satisfied. According to Corollary 7.3, we can choose λ such that $\|[A_n, W(f_\lambda)]\| \leq \frac{\alpha}{3}$. Then, we have:

$$\begin{aligned}
\|[A, W(f_\lambda)]\| &= \|[A - A_n + A_n, W(f_\lambda)]\| && (7.23)\\
&\leq \|[A - A_n, W(f_\lambda)]\| + \|[A_n, W(f_\lambda)]\| && (7.24)\\
&\leq 2\frac{\alpha}{2} + \frac{\alpha}{3} = \alpha && (7.25)
\end{aligned}$$

□

Lemma 7.5. *Assume that the Fock representation Π_F and the coherent representation Π^F_{coh} of $\mathcal{W}(L,\sigma)$ were unitary equivalent, i.e. there is $U \in Aut(\Gamma_s(\mathfrak{h}_T))$ such that for every $f \in (1+C)L_\phi$:*

$$\Pi^F_{coh}(W(f)) = U^{-1}\Pi_F(W(f))U \tag{7.26}$$

Then for every $\epsilon > 0$ there is $\lambda_f \in \mathbb{R}$ such that

$$|\left\langle \Omega, \left(\Pi^F_{coh}(W(f_\lambda)) - \Pi_F(W(f_\lambda))\right)\Omega\right\rangle| \leq \epsilon \tag{7.27}$$

for all $\lambda \geq \lambda_f$.

Proof. It is well known that for every $\Psi \in \mathfrak{h}_T, \|\Psi\| = 1$, there is a unitary $V \in Aut(\Gamma_s(\mathfrak{h}_T))$ such that $V\Psi = \Omega$. According to Theorem 2.41, there is an $A \in W(L,\sigma)$ such that $\Pi_F(A)\Psi = \Omega$ since Π_F is an irreducible representation.
That means:

$$\begin{aligned}\langle \Psi, \Pi_F(W(f_\lambda))\Psi\rangle &= \langle \Omega, \Pi_F(A^*)\Pi_F(W(f'_\lambda))\Pi_F(A)\Omega\rangle &(7.28)\\ &= \langle \Omega, [\Pi_F(A^*), \Pi_F(W(f_\lambda))]\Pi_F(A)\Omega\rangle + \langle \Omega, \Pi_F(W(f_\lambda))\Omega\rangle &(7.29)\\ &= c_\lambda + \langle \Omega, \Pi_F(W(f_\lambda))\Omega\rangle &(7.30)\end{aligned}$$

with $c_\lambda := \langle \Omega, [\Pi_F(A^*), \Pi_F(W(f_\lambda))]\Pi_F(A)\Omega\rangle$.
Now, let $\Psi = U\Omega$ with U from the assumptions. Then we have:

$$\begin{aligned}\langle \Psi, \Pi^F_{coh}(W(f_\lambda))\Psi\rangle &= \langle \Omega, U^{-1}\Pi^F_{coh}(W(f_\lambda))U\Omega\rangle &(7.31)\\ &= \langle \Omega, \Pi^F_{coh}(W(f_\lambda))\Omega\rangle &(7.32)\end{aligned}$$

Together, this gives:

$$\begin{aligned}|\left\langle \Omega, \left(\Pi_F(W(f_\lambda)) - \Pi^F_{coh}(W(f_\lambda))\right)\Omega\right\rangle| &= |c_\lambda| &(7.33)\\ &= |\langle \Omega, [\Pi_F(A^*), \Pi_F(W(f_\lambda))]\Pi_F(A)\Omega\rangle| &(7.34)\\ &\leq \|[\Pi_F(A^*), \Pi_F(W(f_\lambda))]\| &(7.35)\\ &\leq \|\Pi_F([A^*, W(f_\lambda)])\| &(7.36)\\ &= \|[A^*, W(f_\lambda)]\| &(7.37)\end{aligned}$$

Applying Corollary 7.4 gives the result. □

Proposition 7.6. *Choosing the smeared Axial gauge as in subsection 5.3.2 with the smearing function $g(e) = \frac{1}{4\pi} + \frac{3}{4\pi}e_3(e_2 - e_1)$ and the test function from Lemma 7.2 via $\widehat{f'} = i\vec{\epsilon}_1(k)k_2\zeta(|k|)$ where $\zeta \in C_0^\infty(\mathbb{R}_+)$- Then all assumptions from Lemma 7.2 are satisfied.*
In this case (7.27) from Lemma 7.5 can not be satisfied for all test functions if $\widehat{j_0}(0) \neq 0$.

Proof. To start, let us concentrate only on the scalar product appearing in the smeared Axial gauge in the definition of F^{ax} for general f and g:

$$\langle \hat{f}'_\lambda, \int_{S^2} d\Omega(e)\; i\frac{P_T(e)}{e\cdot p} g(e)\widehat{j_0}\rangle \tag{7.38}$$

Note that the estimate from the Proof of Lemma 7.2 can no longer be applied since $\int_{S^2} d\Omega(e) i\frac{P_T(e)}{e\cdot k} g(e)$ does not map $\mathcal{S}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.
Moreover, the scalar product is real since $f'_\lambda \in P_T\left(\mathcal{S}_\mathbb{R}(\mathbb{R}^3)\otimes\mathbb{R}^3\right)$ is real and as discussed in (6.19) also $\mathcal{F}^{-1}\left(-i\int_{S^2} d\Omega(e)\; \frac{P_T(e)}{e\cdot k} g(e)\widehat{j_0}(k)\right) \in P_T\left(L^2_\mathbb{R}(\mathbb{R}^3)\otimes\mathbb{R}^3\right)$ is real .

$$\begin{aligned}
\langle \hat{f}'_\lambda, \int_{S^2} d\Omega(e)\; i\frac{P_T(e)}{e\cdot k} g(e)\widehat{j_0}\rangle_{\mathfrak{h}_T} &= \langle -i\int_{S^2} d\Omega(e)\; \frac{P_T(e)}{e\cdot k} g(e)\hat{f}'_\lambda, \widehat{j_0}\rangle_{L^2} &(7.39)\\
&= \langle -i\int_{S^2} d\Omega(e)\; \frac{P_T(e)}{e\cdot k} g(e)\hat{f}', \widehat{j_0}(\frac{\cdot}{\lambda})\rangle &(7.40)\\
&= \langle -i\int_{S^2} d\Omega(e)\; \frac{P_T(e)}{e\cdot k} g(e)\hat{f}', \widehat{j_0}(\frac{\cdot}{\lambda})1_{supp(f)}\rangle &(7.41)
\end{aligned}$$

At this point we need the fact that f is compactly supported and $\widehat{j_0}$ is a Schwartz function to use the theorem of dominated convergence which gives us:

$$\lim_{\lambda\to\infty}\langle \int_{S^2} d\Omega(e)\; \frac{P_T(e)}{e\cdot k} g(e)\hat{f}', \widehat{j_0}(\frac{\cdot}{\lambda})1_f\rangle = \widehat{j_0}(0)\,\langle \int_{S^2} d\Omega(e)\; \frac{P_T(e)}{e\cdot k} g(e)\hat{f}', 1_f\rangle \tag{7.42}$$

Before we evaluate the integral (7.42) for our particular choice of f and g, we need to make sure that f and g satisfy the assumptions that we made so far.
In subsection 5.3.2 we explained that g is an admissible choice for a smearing function. In particular, it is obvious that $g \in C^\infty(S^2)$.
As product of C^∞-functions $\widehat{f'}$ itself is C^∞ and since we assumed that ζ is compactly supported, it is obvious that $\widehat{f'}$ is compactly supported as well. So, we only need to show that $\widehat{f'}$ is the Fourier transformation of real-valued function.
It is easy to check that the inverse Fourier transformation of a function h is real if $\overline{h}(\cdot) = h(-\cdot)$. Now, it is straightforward to check that f has this symmetry property and hence is the Fourier transformation of a real-valued Schwartz function.
Recall from (5.121) that for the particular choice of g and f, we have:

$$\begin{aligned}
\langle -i\int_{S^2} d\Omega(e)\frac{P_T(e)\cdot\widehat{f'}}{e\cdot k} g(e), 1_f\rangle &= \int_{\mathbb{R}^3} d^3k\; k_2\frac{\zeta(|k|)}{|k|}\cos(2\theta_k)(\cos(\phi_k)-\sin(\phi_k)) &(7.43)\\
&= \int_0^\infty drr^2\zeta(r)\int_0^\pi d\theta\sin^2(\theta)\cos(2\theta)\times &(7.44)\\
&\quad \int_0^{2\pi} d\phi\sin(\phi)(\sin(\phi)-cos(\phi)) &(7.45)\\
&= \frac{\pi^2}{4}\int_0^\infty drr^2\zeta(r) &(7.46)
\end{aligned}$$

This tells us

$$\langle \Omega, \left(\Pi^F_{coh}(W(f_\lambda)) - \Pi_F(W(f_\lambda)) \right) \Omega \rangle = \left(e^{-i\langle \hat{f}'_\lambda, \int_{S^2} d\Omega(e) \ i\frac{P_T(e)}{e \cdot k} g(e) \widehat{j_0} \rangle_{\mathfrak{h}_T}} - 1 \right) e^{-\frac{1}{4}\|f_\lambda\|^2} \tag{7.47}$$

We chose f_λ such that $\|f_\lambda\| = \|f\|$ independently of λ such that:

$$\langle \Omega, \left(\Pi^F_{coh}(W(f_\lambda)) - \Pi_F(W(f_\lambda)) \right) \Omega \rangle = \left(e^{-i\langle \hat{f}'_\lambda, \int_{S^2} d\Omega(e) \ i\frac{P_T(e)}{e \cdot k} g(e) \widehat{j_0} \rangle_{\mathfrak{h}_T}} - 1 \right) e^{-\frac{1}{4}\|f\|^2} \tag{7.48}$$

One easily verifies:

$$\|f\|^2 = \frac{4\pi}{3} \int_{\mathbb{R}_+} dr \ r^4 |\zeta(r)|^2 < \infty \tag{7.49}$$

Hence, (7.48) is bounded for all $\lambda \in \mathbb{R}$. Moreover, we know from (7.42) that:

$$\lim_{\lambda \to \infty} \langle \Omega, \left(\Pi^F_{coh}(W(f_\lambda)) - \Pi_F(W(f_\lambda)) \right) \Omega \rangle = \left(e^{-\widehat{j_0}(0) \frac{\pi^2}{4} \int_{\mathbb{R}_+} dr r^2 \zeta(r)} - 1 \right) e^{-\frac{1}{4}\|f\|^2} \tag{7.50}$$

For an appropriate choice of ζ the integral exponent can be any multiple of $\widehat{j_0}(0)$. That implies that we can choose f such that (7.50) does not converge to 0 if and only if $\widehat{j_0}(0) \neq 0$.
If (7.50) does not converge to 0, it is obvious that Lemma 7.5 cannot be satisfied. □

Remark 7.7. *Theorem 7.1 now follows as a corollary because (7.27) needs to be satisfied whenever a unitary as in Theorem 7.1 exists. In Proposition 7.6, we showed that (7.27) can not be satisfied for all test function if* $\widehat{j_0}(0) \neq 0$ *which implies the non-existence of a unitary intertwining between the observables in the respective gauges.*

8 Conclusion and Open Questions

In this thesis, we addressed the problem of unitary equivalence of different gauges of the electromagnetic field coupled to an external current on a mathematically rigorous level.
However, we have seen in Chapter 3 that for physically admissible gauges, there may appear problems in the well-definiteness of the observables from a mathematical point of view.
While this issue did not occur for the Coulomb gauge, there were severe singularities in the formal definition of the representation corresponding to the Axial gauge, which prevented us from dealing with the unitary equivalence of the representations on a rigorous level.
In the Chapters 4 and 5, we justified the YSM-type smearing of the Axial gauge by artificially extending the degrees of freedom and imposing the Axial gauge condition several times. Repeating this procedure allowed for defining a regularized representation of the Axial gauge without any pathologies, in the course of which a mathematically thorough investigation of the gauge equivalence was possible.
In particular, in the Chapters 6 and 7 we showed, that the representation of the smeared Axial gauge is unitarily equivalent to the representation of the Coulomb gauge if and only if the total electric charge vanishes.
To be precise, we showed that there is a smearing function g such that the corresponding representation Π^F_{coh} is not unitarily equivalent to the Fock representation Π_F on the transversal Fock Space. In Section 5.4, we explained that the representations Π^F_{coh} and Π_F describe the transversal observables in the smeared Axial and the Coulomb gauge respectively.
Anyway, in Section 5.3 we have worked out that the representations Π^F_{coh} for the constant smearing function is manifestly equivalent to the Fock representation Π_F.
Hence, together with the results of Chapter 7, we conclude that there are unitarily inequivalent representations among the smeared Axial gauge representations depending on the choice of g. Thus, we should add this dependence to the notation $\Pi^F_{coh}(g)$.
An open question, that we did not address in this thesis, is the classification of smearing functions such that the representations $\Pi^F_{coh}(g')$ are unitarily equivalent for all g' in the same class.

Acknowledgements

I would especially like to thank my supervisor Wojciech Dybalski for all the discussions and new ideas, all the helpful advice for the thesis and applications and the encouragement to apply as a speaker for the 42nd LQP Workshop.
My family and friends deserve a special gratitude for supporting me all the time not just during my thesis but for the whole time of my studies.

Appendix A

Riemann Integration

The purpose of this chapter is to recall the construction of the Riemann integral and the main results of the convergence of Riemann sums. This results of this chapter can be found in most standard textbooks for Analysis. As main references we chose [Kö04],[Roc13] and [Fri13].

A.1 Riemann integration in one dimension

We start this chapter by recalling the construction of the Riemann integral over an interval in one dimension. The Riemann integral was introduced as a measure of the area between a bounded function $f : I = [a, b] \to \mathbb{R}$ and the x-axis.

Definition A.1. For this, let $\mathfrak{J} = \{x_0, \ldots, x_n\} \subset I$ with $a = x_0 < x_1 < \cdots < x_n = b$ be a decomposition of I and denote by $|I_i| = x_{i+1} - x_i$ the length of the i-th subinterval and call $|\mathfrak{J}| := \max\limits_{1<i<n} |I_i|$ the *fineness* of the decomposition $\mathfrak{J}$.
Let $\mathfrak{J}'$ be another decomposition of I that results from $\mathfrak{J}$ by adding points, then we call $\mathfrak{J}'$ a *refinement* of $\mathfrak{J}$ and have $|\mathfrak{J}'| \leq |\mathfrak{J}|$.
$\xi = (\xi_1, \ldots, \xi_n)$ is called an *intermediate vector* of $\mathfrak{J}$ if $x_i \leq \xi_i \leq x_{i+1}$.

Definition A.2. Let $\mathfrak{J}$ be a decomposition of I and ξ an appendant intermediate vector, then:

$$S(\mathfrak{J}, \xi, f) := \sum_{i=1}^{n-1} f(\xi_i)|I_i| \tag{A.1}$$

is called the *Riemann sum* of f associated to $\mathfrak{J}$ and ξ.
Let $(\mathfrak{J}_n)_{n=1}^{\infty}$ be a series of decompositions of I such that $\lim\limits_{n\to\infty} |\mathfrak{J}_n| = 0$ and let $\xi^{(n)}$ be an intermediate vector for $\mathfrak{J}_n$. Then, the series of Riemann sums $S(\mathfrak{J}_n, \xi^{(n)}, f)$ is called a *Riemann series* for f.

Definition A.3. A function $f : I \to \mathbb{R}$ is called *Riemann integrable* if every Riemann series $S(\mathfrak{J}_n, \xi^{(n)}, f)$ converges.

Proposition A.4. *Let $f : I \to \mathbb{R}$ be a Riemann integrable function, then the limit of Riemann series $S(\mathfrak{I}_n, \xi^{(n)}, f)$ is unique and in particular independent of the choice of $(\mathfrak{I}_n)_{n=1}^{\infty}$ and $\xi^{(n)}$.*

Proof. Let S^1 and S^2 be two Riemann series for f. Then $S = (S_1^1, S_1^2, S_2^1, S_2^2, \dots)$ is also a Riemann series which converges due to the assumption that f is Riemann integrable. With S being a convergent series all subseries of S also converge to the same limit. This is in particular true for S^1 and S^2. □

Definition A.5. Let $f : I \to \mathbb{R}$ be a Riemann integrable function, then we call the unique limit $\lim\limits_{n\to\infty} S(\mathfrak{I}_n, \xi^{(n)}, f)$ the *Riemann integral* of f and denote it by $\int_a^b f(x)dx$.

There is a different way to construct the Riemann sum that is more helpful to prove useful properties of the Riemann-integral.

Definition A.6. Let $f : I \to \mathbb{R}$ be a function and $\mathfrak{I}$ a decomposition just like in the previous definitions. Define:

$$m_i := \inf_{x_{i-1}<x<x_i} f(x) \tag{A.2}$$

$$M_i := \sup_{x_{i-1}<x<x_i} f(x) \tag{A.3}$$

Then, one calls:

$$L(f, \mathfrak{I}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \tag{A.4}$$

$$U(f, \mathfrak{I}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \tag{A.5}$$

the lower and the upper sum of f with respect to $\mathfrak{I}$ respectively.
One calls $I_*(f) := \sup\limits_{\mathfrak{I}:\text{decomposition of } I} L(f, \mathfrak{I})$ the lower integral and $I^*(f) := \inf\limits_{\mathfrak{I}:\text{decomposition of } I} U(f, \mathfrak{I})$ the upper integral of f.
If $I_*(f) = I_*(f)$, one calls f Darboux-integrable over I.

Proposition A.7. *A function $f : I \to \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable and bounded.*

Proof. See [Roc13], Satz 8.8 . □

Proposition A.8. *Every continuous function $f : I \to \mathbb{R}$ is Riemann integrable.*

Proof. See [Fri13], Folgerung 2.3.5 . □

A.2 Riemann integration in $\mathbb{R}^n$

In this chapter, we will generalize the idea of the Riemann integration over an interval in one dimension to arbitrary many dimensions. First, we need to clarify the analogue of an interval in $\mathbb{R}^n$.
Under a *closed interval* in $\mathbb{R}^n$ we understand a product

$$[a_1, b_1] \times \cdots \times [a_n, b_n] = \{x \in \mathbb{R}^n; a_i \leq x_i \leq b_i\} \tag{A.6}$$

and analogously under an *open interval* in $\mathbb{R}^n$ we understand a product:

$$(a_1, b_1) \times \cdots \times (a_n, b_n) = \{x \in \mathbb{R}^n; a_i < x_i < b_i\} \tag{A.7}$$

Let $I \subset \mathbb{R}^n$ be an open or closed interval, then we set its content to:

$$|I| := \prod_{i=1}^{n} (b_i - a_i) \tag{A.8}$$

which gives us for $n = 1$ intervals and their length, $n = 2$ rectangles and their are, $n = 3$ cuboids and their volumes and so on.

Definition A.9. Let $I = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ be an interval. A product of decompositions

$$\mathfrak{J} = \mathfrak{J}_1 \times \cdots \times \mathfrak{J}_n \tag{A.9}$$

where $\mathfrak{J}_i$ is a decomposition of I_i is called a *decomposition* of I. The set of subintervals of $\mathfrak{J}$ is the collection of all possible products $\mathfrak{T}_1 \times \cdots \times \mathfrak{T}_n$ where $\mathfrak{T}_i$ is a subset of the decomposition $\mathfrak{J}_i$.
A decomposition $\mathfrak{J}'$ is called a *refinement* of $\mathfrak{J}$ if every $\mathfrak{J}'_i$ is a refinement of $\mathfrak{J}_i$ in the sense of Definition A.1.
The *fineness* of a decomposition $\mathfrak{J}$ is defined via:

$$|\mathfrak{J}| = \max_i |\mathfrak{J}_i| \tag{A.10}$$

The definition of an intermediate vector in $\mathbb{R}^n$ is analogue to the definition in one dimension, see Definition A.1.

Definition A.10. Let $f : \mathbb{R}^n \supset I \to \mathbb{R}$ be a function and $\mathfrak{J}$ a decomposition of I with subintervals $I_i, i \in \{1, \ldots, r\}$ and ξ an intermediate vector of $\mathfrak{J}$, then:

$$S(\mathfrak{J}, \xi, f) = \sum_{i=1}^{r} f(\xi_i)|I_i| \tag{A.11}$$

is called the *Riemann sum* of f associated to $\mathfrak{J}$ and ξ. *Riemann series* are defined analogously to the one dimensional case.

Definition A.11. A function $f : \mathbb{R}^n \supset I \to \mathbb{R}$ is called *Riemann integrable* if every Riemann series converges.

Remark A.12. *If f is Riemann integrable, then the limit is unique. The modification of the proof of Proposition A.4 is straightforward.*

Remark A.13. *With this preparation, the definition of Darboux integrability in $\mathbb{R}^n$ is analogue to the definition in one dimension.*

Proposition A.14. *A function $f : I \to \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable and bounded.*

Proof. See [Roc13], Satz 13.8 . □

Proposition A.15. *Every continuous function $f : I \to R$ is Riemann-integrable.*

Proof. See [Fri13], Satz 2.3.5 . □

In [Kö04] there is a result about the convergence of Riemann series over arbitrary subsets $A \in \mathbb{R}^n$. To define a Riemann sum, we first need to generalize the definitions of Definition A.1 to an arbitrary set $A \in \mathbb{R}^n$.

Definition A.16. Under a decomposition of *fineness* $\delta > 0$ of a subset $A \subset \mathbb{R}^n$ we understand a collection $\mathfrak{I}$ of subsets $(A_i)_{i \in J}$ for some index set J such that:

1. $\bigcup\limits_{i \in J} A_i = A$
2. $A_i \cap I_k$ is a zero set for all $i \neq k$
3. $diam(A_i) \leq \delta$ for all $i \in J$

Let $\mathfrak{I}'$ and $\mathfrak{I}$ be two decompositions of A such that for every $A_i \in \mathfrak{I}$ there is a collection $\{A'_{i,k} \in \mathfrak{I}'\}$ such that $\bigcup\limits_k A'_{i,k} = A_i$, then $\mathfrak{I}'$ is called a *refinement* of $\mathfrak{I}$ and we have $|\mathfrak{I}'| \leq |\mathfrak{I}|$.
$\xi = (\xi_1, \ldots, \xi_n)$ is called an *intermediate vector* of $\mathfrak{I}$ if $\xi_i \in A_i$.
The definition of a *Riemann sum* and a *Riemann series* are analogue to the previous ones.

Proposition A.17. *Let $A \in \mathbb{R}^n$ be a compact set and $f : A \to \mathbb{R}$ be continuous, then every Riemann series converges and we call the (unique) limit*

$$\int_A f(x)dx \tag{A.12}$$

Proof. See [Kö04], Chapter 7.8, Satz 16. □

Remark A.18. *The proof idea of Proposition A.4 can also be applied in this case to prove the uniqueness of the limit.*
In [Kö04] the author even proves that $\int_A f(x)dx$ coincides with the Lebesgue integral.

Appendix B

Inverse differential operators

The idea of finding inverse differential operators lead to the study of pseudo-differential operators. The leading idea is to reduce the theory to the so-called symbols which are defined via the Fourier transformation. The results allow to make sense of some inverse differential operators.
An essential role in this formalism plays the Fourier transformation. To define and state the basic properties of the Fourier transformation, we recall fundamental function spaces:

Definition B.1.

$$\mathcal{S}(\mathbb{R}^n) := \{\phi \in C^\infty(\mathbb{R}^n); \forall \alpha, \beta \in \mathbb{N}_0^n : \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi| < \infty\} \tag{B.1}$$

is called the *Schwartz space* of rapidly decreasing functions.
The topological dual space of $\mathcal{S}(\mathbb{R}^n)$

$$\{f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}; f \text{ linear and continuous}\} \tag{B.2}$$

is called the space of *tempered distributions*.

Definition B.2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\mathbb{K}$ some field and $0 < p < \infty$. Then define

$$\mathcal{L}^p(\Omega) := \{f : \Omega \to \mathbb{K}, f \text{ is measurable and } \int_\Omega |f|^p d\mu < \infty\}. \tag{B.3}$$

with the half norm for $p \geq 1$:

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}. \tag{B.4}$$

Then the quotient space $L^p(\Omega) = \mathcal{L}^p(\Omega) \mod \ker \|\cdot\|_p$ is a Banach space and one refer to it as L^p-space.

Remark B.3. *For every $p \geq 1$, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ with the Lebesgue measure with respect to the norm $\|\cdot\|_p$. For a proof see e.g. [For08].*

Definition B.4. The Fourier transformation is an isometric automorphism $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ defined via

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} \frac{d^n x}{(2\pi)^{\frac{n}{2}}} f(x) e^{ix\cdot\xi} \tag{B.5}$$

where $\cdot$ indicates the standard Euclidean scalar product on $\mathbb{R}^n$.

Remark B.5. *Due to the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, the Fourier transformation can be extended to an isometric isomorphism on $L^2(\mathbb{R}^n)$.*

Remark B.6. *The Fourier transformation on $\mathcal{S}'(\mathbb{R}^n)$ is defined via the composition, i.e. let $\phi \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$:*

$$\mathcal{F}[\phi](f) = \phi(\mathcal{F}[f]) \tag{B.6}$$

On $\mathcal{S}'(\mathbb{R}^n)$ the Fourier transformation is a linear bijection.

Lemma B.7 (Basic properties of the Fourier transformation)**.** *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then:*

1. $\mathcal{F}[\partial_j f] = i\xi_j \mathcal{F}[f] = i\xi_j \hat{f}$
2. $\partial_{\xi_j} \hat{f} = \mathcal{F}[-ix_j f]$
3. $\mathcal{F}[f \star g] = \hat{f}\hat{g}$*, where $\star$ denotes the convolution*
4. *Let $(\rho_\epsilon f)(x) := f(\epsilon x)$ denote the dilaton, then:*

$$\mathcal{F}[\rho_\epsilon f] = \epsilon^{-n} \rho_{\epsilon^{-1}} \hat{f} \tag{B.7}$$

Proof. The proof can be found in many textbooks for Analysis or Partial Differential Equations, e.g. [Abe12]. □

Lemma B.8. *Let $f \in \mathcal{S}(\mathbb{R}^n)$, then there is a constant $C_k \in \mathbb{R}$ such that*

$$|f(x)| \leq \frac{C_k}{(1 + |x|)^k} \tag{B.8}$$

for every $k \in \mathbb{N}$.

Proof. By definition, we have that with $f \in \mathcal{S}(\mathbb{R}^n)$ also $x^\alpha f(x) \in \mathcal{S}(\mathbb{R}^n)$ and hence $f(x) \leq c_0$ and $x^\alpha f(x) \leq c_\alpha$ for some constants $c_\alpha \in \mathbb{R}$. This gives us

$$(1 + |x|) f(x) = f(x) + \left(\sum_{j=1}^{n} x_j^2 f(x)^2 \right)^{\frac{1}{2}} \tag{B.9}$$

$$\leq c_0 + \left(\sum_{j=1}^{n} c_j^2 \right)^{\frac{1}{2}} \leq C_1 \tag{B.10}$$

for an appropriate $C_1 \in \mathbb{R}$.
This tells us:

$$f(x) \leq \frac{C_1}{1+|x|} \tag{B.11}$$

This proof can be generalized to arbitrary orders $k \in \mathbb{N}$ in the same since $x^\alpha f$ is bounded for all $\alpha \in \mathbb{N}_0^n$. □

B.1 Fractional Laplacians

Using the basic properties of the Fourier transformation, it is evident that a differential operator $P(D) = \sum_\alpha c_\alpha D^\alpha$ ($D^\alpha = i\partial^\alpha$) is mapped the multiplication operator with its symbol $p(\xi) = \sum_\alpha c_\alpha \xi^\alpha$.
The idea is to link properties of the Differential operator $P(D)$ to properties of its symbol $p(\xi)$.

Example B.9. *The symbol of the Laplacian* $\Delta = \sum_{j=1}^{n} \partial_j \partial_j$ *is* $p(\xi) = -|\xi|^2$.

This suggests that the inversion of a differential operator P corresponds to the multiplication with the inverse symbol $\frac{1}{p(\xi)}$ on the level of the Fourier multiplication. Formally, we define

$$Qf = \mathcal{F}^{-1}[\frac{1}{p(\xi)}\hat{f}(\xi)] \tag{B.12}$$

and have:

$$PQf = \mathcal{F}^{-1}[p(\xi)\mathcal{F}[Qf]] = \mathcal{F}^{-1}[\frac{p(\xi)}{p(\xi)}\hat{f}(\xi)] = f \tag{B.13}$$

, i.e. Q is the inverse of P.
This discussion is only true on a formal level. One needs to assure that Q is a well-defined operator. If it is, Q is not a differential operator but belongs to the class of *pseudodifferential operators*.
First of all, we notice that for a polynomial differential operator there are obstacles in the definition of the inverse Q due to the singularities. However, if the singularity turns out to be integrable, then the definition makes sense and we will discuss an example of such operators subsequently.
If, however, the singularities are not integrable, e.g. for $p(\xi) = 0$ for some $\xi \neq 0$, we can modify the domain to make sense of Q as an inverse. The following definitions and theorems are taken from [Uma15].

Definition B.10. Let $G \subset \mathbb{R}^n$ be an open subset. Denote by $\Psi_{G,p}$ the set of functions $\phi \in L^p(\mathbb{R}^n)$ such that $supp(\phi) \Subset G$.

Proposition B.11. *The space $\Psi_{G,p}$ is invariant under $A(D)$ for every differential operator with symbol $A(\xi) \in C^\infty(G)$. Moreover, $A(D) : \Psi_{G,p} \to \Psi_{G,p}$ is a continuous mapping.*

Proposition B.12. *Let $A(\xi) \in C^\infty(G)$. If $\frac{1}{A(\xi)} \in C^\infty(G)$, then the operator $A^{-1}(D)$ corresponding to the symbol $\frac{1}{A(\xi)}$ is the inverse of $A(D)$ on $\Psi_{G,p}$.*

In the following, we will discuss a certain class of examples of operators such that the symbol vanishes at $\xi = 0$, but the inverse can still be defined as in (B.12).

Definition B.13. Let $\alpha \in 2\mathbb{N}$, then we call $(-\Delta)^{\frac{\alpha}{2}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ defined via:

$$(-\Delta)^{\frac{\alpha}{2}} f = \mathcal{F}^{-1}[|\xi|^\alpha \hat{f}(\xi)] \tag{B.14}$$

powers of the Laplacian of order $\frac{\alpha}{2}$.

Remark B.14. *Note that the inverse Fourier transformation is well defined since the Schwartz space is closed under multiplication with polynomials. The definition of the powers coincides with the usual composition of operators.*

Remark B.15. *Let $0 < \alpha < n$, then $|\xi|^{-\alpha} \in L^1_{loc}(\mathbb{R}^n)$. In particular, this means, that the singularity at $\xi = 0$ is removed by the integration measure. Moreover, it is obvious that $|\xi|^{-\alpha}$ is bounded at $\mathbb{R}^3 \backslash K$ for any K containing $0 \in \mathbb{R}^3$. Thus, $|\xi|^{-\alpha}$ can be viewed as element in $\mathcal{S}'(\mathbb{R}^n)$ [AH99].*

Definition B.16. Let $\alpha < n$, then the operators

$$I_\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \tag{B.15}$$

$$f \mapsto \mathcal{F}^{-1}[|\xi|^{-\alpha} \hat{f}] \tag{B.16}$$

are called *Riesz potential operators*.
The Riesz potential operators can equivalently be defined as convolution operators with the kernel

$$K_\alpha = \frac{\gamma_{\alpha,n}}{|x|^{n-\alpha}} \tag{B.17}$$

with $\gamma_{\alpha,n} \in \mathbb{R}$ being some constant only depending on α and n [AH99]. In formulas, we then have:

$$I_\alpha f = K_\alpha \star f \tag{B.18}$$

Proposition B.17. *Let $n \geq 3$, then I_2 is the fundamental solution of the negative of the Laplacian, i.e. $-\Delta I_2 = \delta_0$ in a distributional sense.*

Proof.

$$(-\Delta I_2) f = -\Delta \mathcal{F}^{-1}[|\xi|^2 \hat{f}] = \mathcal{F}^{-1}[\frac{|\xi|^2}{|\xi|^2} \hat{f}] = f \tag{B.19}$$

in the sense of distributions.
See e.g. [Hor90] for a different proof. ☐

The idea of *fractional Laplacians* is to generalize Definition B.13 to non-integer power $\alpha \in \mathbb{R}$. Unfortunately, we can not just replace the integer power by a a fractional since the inverse Fourier transformation does not exist as $|\xi|^\alpha : \mathcal{S}(\mathbb{R}^n) \not\to \mathcal{S}(\mathbb{R}^n)$ for $\alpha \notin 2\mathbb{N}$ as a multiplication operator. However, we have:

Lemma B.18. *Let $\alpha \geq -\frac{n-1}{2}$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then:*

$$|x|^\alpha f \in L^2(\mathbb{R}^n) \tag{B.20}$$

Proof. Using Lemma B.8, we have:

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^2 |x|^{2\alpha} dx &\leq C_k \int_{\mathbb{R}^n} \frac{|x|^{2\alpha}}{(1+|x|)^{2k}} dx && \text{(B.21)}\\
&= C_k \Xi(n) \int_{\mathbb{R}_+} \rho^{2\alpha+n-1} (1+\rho)^{-2k} d\rho && \text{(B.22)}\\
&\leq C_k \Xi(n) \int_{\mathbb{R}_+} (1+\rho)^{2\alpha+n-1-2k} d\rho && \text{(B.23)}\\
&= C_k \Xi(n) \int_1^\infty \rho^{2\alpha+n-1-2k} d\rho && \text{(B.24)}
\end{aligned}$$

Choosing k appropriately big gives the result. Note that the second estimate is only true if the exponent is positive, which restricts $\alpha \geq -\frac{n-1}{2}$. By $\Xi(n)$ we denote the measure of the $n-1$-dimensional sphere. □

Since the Fourier transformation extends to an isomorphism on $L^2(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ respectively, Lemma B.18 and Definition B.16 allow us to can extend the definition of Definition B.13 to $\alpha > -n$.

Definition B.19. Operators $\Delta^{\frac{\alpha}{2}} : \mathcal{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for $\alpha \geq 0$ defined via:

$$(\Delta^{\frac{\alpha}{2}} f)(x) = \mathcal{F}^{-1}[|\xi|^\alpha \hat{f}] \tag{B.25}$$

are called *fractional Laplacians*.

Proposition B.20. *Let $\alpha_1, \alpha_2 > -n$ such that $\alpha_1 + \alpha_2 > -n$, then:*

$$\Delta^{\frac{\alpha_1}{2}} \circ \Delta^{\frac{\alpha_2}{2}} = \Delta^{\frac{\alpha_1+\alpha_2}{2}} \tag{B.26}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$, then:

$$\begin{aligned}
(\Delta^{\frac{\alpha_1}{2}} \circ \Delta^{\frac{\alpha_2}{2}}) f &= \Delta^{\frac{\alpha_1}{2}} \mathcal{F}^{-1}[|\xi|^{\alpha_2} \hat{f}] && \text{(B.27)}\\
&= \mathcal{F}^{-1} \circ |\xi|^{\alpha_1} \circ \mathcal{F} \circ \mathcal{F}^{-1}[|\xi|^{\alpha_2} \hat{f}] && \text{(B.28)}\\
&= \mathcal{F}^{-1} \circ |\xi|^{\alpha_1+\alpha_2} \hat{f} && \text{(B.29)}
\end{aligned}$$

Note that for this computation, we silently assumed that the domain of $\Delta^{\frac{\alpha_1}{2}}$ is enlarged to all objects such that the definition makes sense. □

Corollary B.21. *I_α is the fundamental solution of $\Delta^{\frac{\alpha}{2}}$ for $0 \leq \alpha < n$.*

B.2 Fourier Multiplier

In this section, we will mainly follow the exhibitions of Fourier multipliers in [Abe12]. The main result that we will use is:

Theorem B.22. *Let $m : \mathbb{R}^n \to \mathbb{C}$ be a measurable function. Then:*

$$m(D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \tag{B.30}$$

$$f \mapsto \mathcal{F}^{-1}[m(\xi)\hat{f}] \tag{B.31}$$

is a well-defined bounded operator $m(D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty$.
In this case it is true that $\|m(D)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \|m\|_\infty$.

Proof. See [Abe12], Thm. 2.13. □

Definition B.23. Since $m_j(\xi) := \frac{\xi_j}{|\xi|} \leq 1$, it is evident $m_j \in L^\infty$. The bounded operators $R_j \cdot := \mathcal{F}^{-1}[m_j \hat{\cdot}]$ on $L^2(\mathbb{R}^n)$ are called *Riesz operators*.
Therefore

$$\partial_i \partial_j (\Delta)^{-1} f = R_i R_j f = \mathcal{F}^{-1}[\frac{\xi_i \xi_j}{|\xi|^2} \hat{f}] \tag{B.32}$$

is well defined on $L^2(\mathbb{R}^n)$ and if $f \in L^2(\mathbb{R}^n)$, then $\partial_i \partial_j (\Delta)^{-1} f \in L^2(\mathbb{R}^n)$. In the case that the derivatives do not exist in the classical sense, they are meant in the distributional sense.

Bibliography

[Abe12] H. Abels. *Pseudodifferential and Singular Integral Operators*. De Gruyter graduate lectures. De Gruyter, 2012.

[AH99] D. Adams and L. Hedberg. *Function Spaces and Potential Theory*. Grundlehren der mathematischen Wissenschaften. Springer, 2 edition, 1999.

[AH12] K. Atkinson and W. Han. *Spherical Harmonics and Approximations on the Unit Sphere: An Introduction*. Lecture Notes in Mathematics 2044. Springer-Verlag Berlin Heidelberg, 1 edition, 2012.

[Ara18] A. Arai. *Analysis on Fock spaces and mathematical theory of quantum fields*. World Scientific, 1 edition, 2018.

[Att] S. Attal. Lectures in quantum noise theory. available under `http://math.univ-lyon1.fr/~attal/Fock_Spaces.pdf`.

[BJ87] D. Buchholz and P Jacobi. On the nuclearity condition for massless fields. *Letters in Mathematical Physics*, 13:313–323, 1987.

[BR10] E. Binz and A. Rieckers. Symplectic geometry of maxwell theory and the photon concept. *Journal of Physics Conference Series*, 237, 06 2010.

[DEF+99] P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J. W. Morgan, D.R. Morrison, and E. Witten. *Quantum Fields and Strings: A Course for Mathematicians*, volume Volume 1. American Mathematical Society, 1st edition, 1999.

[Dir30] P. Dirac. *The Principles of Quantum Mechanics*. Oxford University Press, 1930.

[Dir64] P. Dirac. *Lectures on Qunatum Mechanics*. Yeshiva University Press, 1964.

[Dyb17] W. Dybalski. Algebraic quantum field theory, July 2017. available under `http://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/WojciechDybalski/AQFT11.pdf`.

[For08] O. Forster. *Analysis 2*. Vieweg+Teubner Verlag, 2008.

[Frä11] C. Fräßdorf. Quantization of singular systems in canonical formalism, 2011. available under `http://users.physik.fu-berlin.de/~pelster/Bachelor/fraessdorf.pdf`.

[Fri13] K. Fritzsche. *Grundkurs Analysis 2: Differentiation und Integration in mehreren Veränderlichen*. Springer Spektrum, 2 edition, 2013.

[Giu03] D. Giulini. That Strange Procedure Called Quantisation. In D. Giulini, C. Kiefer, and C. Laemmerzahl, editors, *Quantum Gravity: From Theory to Experimental Search*, volume 631 of *Lecture Notes in Physics, Berlin Springer Verlag*, pages 17–40, 2003.

[Gri78] V. Gribov. Quantization of non-abelian gauge theories. *Nucl. Phys.*, B139, 1978.

[Gro46] H. Groenewold. On the principles of elementary quantum mechanics. *Physica*, 12, 1946.

[HJKM03] H.-J-Kowalsky and G. O. Michler. *Lineare Algebra*. de Gruyter Lehrbuch. Walter de Gruyter, 2003.

[HLL94] K. Haller and E. Lim-Lombridas. Quantum gauge equivalence in qed. *Found. Phys*, 24, 1994.

[Hor90] Lars Hormander. *The Analysis of Linear Partial Differential Operators. I, Distribution Theory and Fourier Analysis*. Grundlehren Der Mathematischen Wissenschaften. Springer, 2 edition, 1990.

[HR15] R. Honegger and A. Rieckers. *Photons in Fock Space and Beyond*, volume 1. World Scientific Publishing Co, 2015.

[HS71] F. Hirzebruch and W. Scharlau. *Einführung in die Funktional-analysis*. Hochschultascenbucher 296. Bibliographisches Institut, 1971.

[Kö04] K. Königsberger. *Analysis 2*. Springer, 5 edition, 2004.

[KR86] R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras*, volume 2 of *Pure and applied mathematics 100-100, 2*. Academic Press, 1986.

[Kun98] Walter Kunhardt. On infravacua and the localization of sectors. *Journal of Mathematical Physics*, 39, 1998.

[Mel97] R. Melrose. Introduction to microlocal analysis, 1997. available under `https://math.mit.edu/~rbm/Lecture_notes.html`.

[Mey01] C. D. Meyer. *Matrix analysis and applied linear algebra*. SIAM: Society for Industrial and Applied Mathematics, 2001.

[MN76] T. Maskawa and H. Nakajima. Singular lagrangians and dirac-fadeev method. *Progress of Theoretical Physics*, 56, 1976.

[MSY05] J. Mund, B. Schroer, and J. Yngvarsson. String-localized quantum fields and modular localization. *Commun. Math. Phys.*, 268, 2005.

[NTO94] Y. Nakawaki, A. Tanaka, and K. Ozaki. Verification on equivalence of the axial gauge to the coulomb gauge in qed by embedding in the indefinite metric hilbert space. *Progress of Theoretical Physics*, 91, 1994.

[Roc13] S. Roch. Vorlesung analysis ii, 2013. available under `http://www3.mathematik.tu-darmstadt.de/index.php?eID=tx_nawsecuredl&u=0&g=0&t=1527095448&hash=c8bc4c9126c9bc565518d9fb6cae2e4e69a50fd2&file=fileadmin/home/users/186/Skripte_Roch/analysis_II_ss13.pdf`.

[Roe70] G. Roepstorff. Coherent photon states and spectral condition. *Communications in Mathematical Physics*, 19, 1970.

[Rot09] J. Rotman. *An introduction to Homological Algebra*. Springer, 2 edition, 2009.

[RR10] H. Rothe and K. Rothe. Operator quantization of second class systems. In *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, World Scientific Lecture Notes in Physics - Vol. 81, pages 154–163. World Scientific Publishing, 2010.

[RS75] M. Reed and B. Simon. *Methods of mathematical physics. Fourier analysis, Self-adjointness*, volume 2 of *Methods of Modern Mathematical Physics, Vol. 2*. Academic Press, 1975.

[Rud91] W. Rudin. *Functional analysis*. McGraw-Hill, 2 edition, 1991.

[Spo04] H. Spohn. *Dynamics of charged particles and their radiation field*. Cambridge University Press, 2004.

[Sun82] K. Sundermeyer. *Constrained Dynamics*. Lecture Notes in Physics 169. Springer-Verlag Berlin Heidelberg, 1 edition, 1982.

[Tah15] A. Taheri. *Function Spaces and Partial Differential Equations*, volume 1. Oxford University Press, 1 edition, 2015.

[Thi07] T. Thiemann. *Modern canonical quantum general relativity*. Cambridge monographs on mathematical physics. Cambridge University Press, 1 edition, 2007.

[Uma15] S. Umarov. *Introduction to Fractional and Pseudo-Differential Equations with Singular Symbols*. Developments in Mathematics 41. Springer International Publishing, 1 edition, 2015.

[Wei95] S. Weinberg. *The quantum theory of fields*. Cambridge University Press, 1995.

Declaration of Autorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise. No other person's work has been used without due acknowledgement in this thesis. All references and verbatim extracts have been quoted, and all sources of information have been specifically acknowledged

Place, Date:

Signature: