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MASTER'S THESIS

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Associative division algebras in field theories and non-commutative geometry

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Abstract: In the first part an approach to field theory using complex quaternions instead of the standard matrix formalism is introduced. This formalism exploits the non-commutativity of quaternions and uses it for a more natural way of performing transformation operations such as Lorentz transformations or parity transformations. The second part builds up the theory of non-commutative geometry using the approach of spectral triples. Gauge groups of spectral triples as well as perturbations of the Dirac operator of these spectral triples are introduced. Then the approach of almost-commutative manifolds is used to derive gauge fields, their transformation properties, and spectral action. Finally, three physically relevant examples of almost-commutative manifolds are presented.

Keywords: division algebras, field theory, non-commutative geometry, spectral triples, gauge fields, spectral action.

I hereby declare that I am the sole author of this thesis; and that I carried out this thesis independently, only with the cited sources, literature and other professional sources.

Munich, December 17, 2018 Juraj Vrábel

to everyone who supported me on this journey

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Preface

The first part of this thesis deals with a non-standard approach to field theories. This approach tries to substitute the classical matrix formalism by mowing towards division algebras, especially complex quaternions. This division algebra perspective has its roots in treating such as [Dixon, 2013, Furey, 2016], and [Greiter and Schuricht, 2003]. This exploits the non-commutativity of quaternions which helps to define Lorentz transformations of left-handed and right-handed spinors by left and right multiplications, respectively. It also yields nicer implementations of some of the operators in field theory, e.g. parity operator which becomes merely a quaternion conjugation.

The second part is dedicated to non-commutative geometry. This relatively new branch of mathematics, which extends the classical Riemannian geometry, was first developed by a French mathematician in [Connes, 1994]. The basic idea of this new geometry relies in a generalization of the famous Gelfand-Naimark theorem, which roughly says that the structure space of the space of continuous function over some topological space is isomorphic to the topological space itself. This shift of focus from topology and geometry to algebra can be extended beyond the commutative algebras of function spaces, e.g. to non-commutative algebras or even to non-associative algebras. The non-commutative case provides a far reaching reinterpretation of the standard model of particle physics [Chamseddine and Connes, 1997, Chamseddine et al., 2007] and [Connes and Marcolli, 2008]. There is also an approach to standard model from non-associative geometry, namely [Farnsworth and Boyle, 2015] and [Farnsworth, 2015]. However, in this thesis I will only consider the non-commutative geometry, and try to show some interesting links between the application of non-commutative geometry in particle physics and division algebras. Concretely, I will exploit the idea of an almost-commutative manifold with the algebra of its finite spectral triple being equal to $\mathbb{C} \oplus \mathbb{Q}$, and $\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{Q}$. This would yield the full electroweak sector of one generation non-commutative standard model with see-saw mechanism for neutrinos, first proposed by [Chamseddine et al., 2007], and a very similar electroweak sector with neutrinos having only Majorana masses, respectively.

What remains to be considered is the non-associative approach of [Farnsworth and Boyle, 2015] applied to the algebra of octonions. In other words it would be interesting to see an almost-commutative manifold with the algebra of the finite spectral triple corresponding to $\mathbb{R}\oplus\mathbb{C}\oplus\mathbb{Q}\oplus\mathbb{O}$.

Organization of the thesis

In the first chapter of this thesis I recall some basic definitions and theorems from division algebras, which will be used throughout this thesis, and Lorentz group and its Lie algebra. Then I move on a mathematical treatment of representations of Lie groups, especially the representation of the proper, orthochronous Lorentz group - spinors and vectors. The following chapter uses the machinery so far developed and introduces a new approach to field theory using complex quaternions. This approach was inspired by [Dixon, 2013, Greiter and Schuricht, 2003], and [Furey, 2016].

The third chapter introduces some basic notions such as Clifford bundles or Dirac operators, and then moves on the development of non-commutative geometry and spectral triples. The penultimate chapter dives into almost-commutative manifolds and their similarities to gauge theories. It also deals with so-called spectral action. In the last chapter I present three examples of paramount importance in the non-commutative standard model. The first deals with massless electrodynamics, and sets up an intuition for the roles played by different parts of real, even, spectral triples. The second application introduces electroweak sector of one generation standard model with see-saw mechanism for neutrinos, first introduced in [Chamseddine et al., 2007]. The last example modifies the previous one in a way that the neutrino masses are Majorana.

Contents

Chapter 1

Basics

Let me start with a recollection of some, for this thesis important, fact about division algebras, Lorentz group, and its Lie algebra. First, I will examine division algebras.

1.1 Division algebras

It turns out that up to isomorphism there are only four normed division algebras, namely real and complex numbers, quaternions, and octonions. Although, the term real numbers was only introduced in 17th century by Descartes, and used to distinguish real roots of polynomials from imaginary ones, the history of real numbers goes far beyond to the Indian, Chinese, and Arabic mathematicians of medieval times. This makes them the oldest and the most popular division algebra. Next on the list are complex numbers which are a bit more showy and much younger (16th century). One could argue that the formalisation of complex numbers as pairs of real numbers by Hamilton in 1835 led him to discover quaternions in 1843, and this subsequently let to the discovery of octonions by Hamilton's friend Graves a couple of months later. I will mostly talk about the first three division algebras since they will be most important for this thesis. So let's start.

In this section I will follow [Baez, 2002] and [Badger, 2006]. The reader is encouraged to go to the mentioned articles for more details on the subject.

1.1.1 Definitions

Definition 1.1.1. A non-zero algebra A is called a division algebra, if the operations of left and right multiplication by any non-zero element are invertible. Equivalently, A is called a division algebra if

$$(a, b \in A) : (ab = 0) \Longrightarrow (a = 0) \vee (b = 0)\,.$$

Definition 1.1.2. An algebra A is called a normed division algebra if it also is a normed vector space with a norm $\|ab\| = \|a\|\|b\|$.

Remark. The previous definition implies that $\|1\| = 1$, and that A is a division algebra.

Definition 1.1.3. Let A be an algebra. Then

i) the commutator is defined as a bilinear map $[\cdot,\cdot] : A \times A \to A$, given by

$$[a, b] = ab - ba\,, \quad \forall a, b \in A\,;$$

ii) the anti-commutator is defined as a bilinear map $\{\cdot,\cdot\} : A \times A \to A$, given by

$$\{a, b\} = ab + ba\,, \quad \forall a, b \in A\,;$$

iii) the associator is defined as a trilinear map $[\cdot,\cdot,\cdot] : A \times A \times A \to A$, given by

$$[a,b,c] = (ab)c - a(bc)\,, \quad \forall a,b,c \in A\,.$$

Definition 1.1.4. An algebra A is said to be

i) unital if it admits unity, i.e.

$$(\exists 1 \in A) : 1a = a1 = a\,, \quad \forall a \in A\,;$$

ii) commutative if the commutator vanishes on any two elements of A, i.e.

$$[a,b] = 0\,, \quad \forall a,b \in A\,;$$

iii) associative if the associator vanishes on any three elements of A, i.e.

$$[a,b,c] = 0\,, \quad \forall a,b,c \in A\,;$$

iv) alternating if the subalgebra generated by any two elements of A is associative, i.e.

$$[a,a,b] = 0 = [a,b,b]\,, \quad \forall a,b \in A\,.$$

Remark. Equivalently, A is alternative if and only if the associator is alternating, i.e. it switches sign whenever any two arguments are exchanged.

Definition 1.1.5. Let A be a non-commutative algebra. Then it makes sense to a define left multiplication $\mathrm{L} : A \to \mathrm{End}(A)$ as

$$\mathrm{L}_a\, b = ab\,, \quad \forall a,b \in A\,;$$

and a right multiplication $\mathrm{R} : A \to \mathrm{End}(A)$ as

$$\mathrm{R}_a\, b = ba\,, \quad \forall a,b \in A\,.$$

Remark. With this definition at hand it is possible to define the associator as

$$[a,b,c] = (ab)c - a(bc) = [\mathrm{R}_c, \mathrm{L}_a]\, b\,, \quad \forall a,b,c \in A\,.$$

Note. I will denote the real numbers by $\mathbb{R}$, the complex numbers by $\mathbb{C}$, the quaternions by $\mathbb{Q}$, and the octonions by $\mathbb{O}$.

Theorem 1.1.1 (Zorn)**.** *If A is an alternating, real division algebra, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ or $\mathbb{O}$.*

Corollary (Hurwitz)**.** *If A is a normed real division algebra, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ or $\mathbb{O}$.*

Note. For a proof of the Zorn's theorem and Hurwitz corollary see *Theorem 4.8* and *Corollary 4.11*, both in [Badger, 2006].

1.1.2 Complex numbers

After the real numbers, the complex numbers, are the second simplest[1] (2-dimensional) real normed division algebra, they are both commutative, and associative. It has two generators, the identity 1, and the imaginary unit $\imath$, such that $\imath^2 = -1$.

Proposition 1.1.2. *The automorphism group[2] of complex numbers $Aut(\mathbb{C})$ is isomorphic to $\mathbb{Z}_2$, with the only non-trivial element being complex conjugation $\cdot^* : \mathbb{C} \to \mathbb{C}$ given by*

$$a + \imath b^* = (a - \imath b)\,, \quad \forall a,b \in \mathbb{R}\,.$$

Remark. The automorphism group of real numbers is trivial, i.e. having only the identity element.

[1] Well, depends on the point of view... In short, $\mathbb{O}$ are one-sided complete, $\mathbb{Q}$ are two-sided complete, and finally $\mathbb{C}$ are neither one- nor two-sided complete and one has to take into account action arising outside of the algebra (complex conjugation) in order to make it complete. For more details see page 39 in [Dixon, 2013].

[2] What is meant here is algebra automorphism, acting trivially on the identity. There is an infinite number of general automorphisms for $\mathbb{C}$. For more see [Ebbinghaus et al., 2013].

1.1.3 Quaternions

Quaternions are the third simplest (4-dimensional) real normed division algebra, they are non-commutative, and associative. The underlying vector space is generated by four basis elements $\{1, \epsilon_x, \epsilon_y, \epsilon_z\}$, where 1 is the identity and the other three elements are subjected to the following relations

$$\epsilon_x^2 = \epsilon_y^2 = \epsilon_z^2 = \epsilon_x\epsilon_y\epsilon_z = -1\,.$$

For notational reasons I will often denote ϵ_x, ϵ_y and ϵ_z by ϵ_1, ϵ_2 and ϵ_3, respectively. Hence, any quaternion can be written as

$$q = q^0 1 + q^i \epsilon_i\,,$$

where the summation convention applies. The directions $1, 2, 3$ or x, y, z will be referred to as spatial directions, the remaining 0 or t (denoting the identity) will be referred to as the time direction.

Any two repeated indices, one upper and the other lower, in a formula indicate summation. The Latin indices from the middle of alphabet (e.g. $i, j, k, \dots$) indicate spatial directions (numbers 1 to 3), and are raised and lowered by the Kronecker delta (δ_{ij}, δ^{ij}).
Greek indices from the middle of alphabet (e.g. $\mu, \nu\,\sigma, \dots$) indicate time and spatial directions (numbers 0 to 3) and are raised and lowered by the "Minkowski eta" ($\eta_{\mu\nu}$, $\eta^{\mu\nu}$) giving minus sign to the spatial direction.

There is a couple of useful relations when it comes to multiplication of quaternions, the first goes as follows[3]

$$\epsilon_i\epsilon_j = -\delta_{ij} + \varepsilon_{ij}{}^k\epsilon_k\,. \tag{1.1}$$

Next, there are commutation, and anti-commutation relations which will be used extensively.

Lemma 1.1.3. *Quaternions satisfy the following commutation, and anti-commutation relations*

$$\left[\frac{\epsilon_i}{2}, \frac{\epsilon_j}{2}\right] = \varepsilon_{ij}{}^k\frac{\epsilon_k}{2}\,, \quad \{\epsilon_i, \epsilon_i\} = -2\delta_{ij}\,. \tag{1.2}$$

For complexified quaternions ($\mathbb{C} \otimes_{\mathbb{R}} \mathbb{Q}$) *one can write*

$$\left[\frac{\iota\epsilon_i}{2}, \frac{\iota\epsilon_j}{2}\right] = \iota\varepsilon_{ij}{}^k\frac{\iota\epsilon_k}{2}\,, \quad \{\iota\epsilon_i, \iota\epsilon_i\} = 2\delta_{ij}\,. \tag{1.3}$$

Proof. Follows directly from the equation (1.1). ☐

Definition 1.1.6. The quaternionic norm $\|\cdot\| : \mathbb{Q} \to \mathbb{R}_+$ is defined to be

$$\|q\|^2 = q\bar{q}\,, \quad \forall q \in \mathbb{Q}\,,$$

where the symbol $\bar{\cdot} : \mathbb{Q} \to \mathbb{Q}$ is quaternionic conjugation given by

$$\overline{q^0 1 + q^i\epsilon_i} = (q^0 1 - q^i\epsilon_i)\,, \quad \forall q \in \mathbb{Q}\,.$$

The scalar product on quaternions $\langle\cdot,\cdot\rangle : \mathbb{Q} \otimes \mathbb{Q} \to \mathbb{R}$ is defined by

$$\langle p, q\rangle = \frac{1}{2}\left(p\overline{q} + q\overline{p}\right)\,, \quad \forall p, q \in \mathbb{Q}\,.$$

The reader can convince himself that $\langle p, q\rangle = p^0 q_0 + p^x q_x + p^y q_y + p^z q_z$. For the case of complex quaternions ($\mathbb{C} \otimes_{\mathbb{R}} \mathbb{Q}$) the situation is slightly more interesting. The norm is no longer positive definite; and the most interesting case where $q' = (q^0 1 + q^i\iota\epsilon_i)$ yields

$$\langle q', q'\rangle = (p^0)^2 - (p^x)^2 - (p^y)^2 - (p^z)^2\,.$$

We will talk about this much later in the next chapter.

[3] This is an immediate consequence of the multiplication table for quaternions, see e.g. Wikipedia.

Remark. Note that for $p, q \in \mathbb{Q}$ being orthogonal with respect to the scalar product is equivalent to $p\overline{q} \in \Im(\mathbb{Q})$, where $\Im$ denotes the imaginary part. The real part will be denoted by $\Re$.
Also note that the inverse of any quaternion q can be written as $q^{-1} = \frac{\overline{q}}{\|q\|^2}$.

Lemma 1.1.4. *The unit quaternions, called* versors, *are isomorphic to the 3-sphere $S^3 \cong SU(2)$.*

Proof. This is quite simple, one just needs to realize that the 3-sphere is a subspace of $\mathbb{C}^2$, namely

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\} ,$$

and that a matrix representation of quaternions is

$$q = q^0 1 + q^i \epsilon_i \mapsto \begin{pmatrix} q^0 + \iota q^3 & -q^2 + \iota q^1 \\ q^2 + \iota q^1 & q^0 - \iota q^3 \end{pmatrix} = Q\,.$$

This map is an injection from $\mathbb{Q}$ to $M_2(\mathbb{C})$ satisfying $\|q\| = \sqrt{\det Q}$. □

Proposition 1.1.5. *The automorphism group of quaternions, $Aut(\mathbb{Q})$, is isomorphic to $SO(3) \cong \mathbb{R}P^3$. Moreover, the automorphism group consists of only inner automorphisms, i.e. automorphisms of the form $q \mapsto pqp^{-1}$ for some $p \in \mathbb{Q}$.*

Proof. The (algebra) automorphism preserves identity and therefore also length. Moreover, from the previous lemma one has that versors are isomorphic to S^3, whose automorphism group is $SO(3)$.
To prove the second part, first note that any versor can be written as

$$\begin{aligned} p =& (\cos(\theta) - p^i \epsilon_i \sin(\theta)) \\ =& \exp(-\theta p^i \epsilon_i)\,, \end{aligned}$$

with an inverse

$$\begin{aligned} \overline{p} =& (\cos(\theta) + p^i \epsilon_i \sin(\theta)) \\ =& \exp(\theta p^i \epsilon_i)\,. \end{aligned}$$

Now, without loss of generality, one can choose our $p = (\cos(\frac{\theta}{2}) - \epsilon_x \sin(\frac{\theta}{2}))$, which is nothing else than a rotation around spatial x-direction by an angle θ

$$\begin{aligned} pqp^{-1} &= \left[\cos\left(\frac{\theta}{2}\right) - \epsilon_x \sin\left(\frac{\theta}{2}\right)\right] q \left[\cos\left(\frac{\theta}{2}\right) + \epsilon_x \sin\left(\frac{\theta}{2}\right)\right] \\ &= \cos^2\left(\frac{\theta}{2}\right) q - \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) (\epsilon_x q - q \epsilon_x) - \sin^2\left(\frac{\theta}{2}\right) \epsilon_x q \epsilon_x \\ &= q^t + q^x \epsilon_x + [\cos(\theta) q^y + \sin(\theta) q^z]\, \epsilon_y + [-\sin(\theta) q^y + \cos(\theta) q^z]\, \epsilon_z\,, \end{aligned} \tag{1.4}$$

where in the last line the trigonometric identities have been used. □

1.2 Lorentz group

The Lorentz group, $O(1, 3)$, is a 6-dimensional, real Lie group of all transformations preserving "Minkowski eta"[4] It is not connected but has four connected components, which are not simply connected. The physically significant component is the one connected to the identity. It is called proper, orthochronous Lorentz group, and I will denoted it by $SO^+(1, 3)$.

The quotient of groups $O(1, 3)/O^+(1, 3)$ consists of the four elements $\{1, P, T, PT\}$, and has again the structure of a group. T is usually called the time-reversal operator, and P is referred to as the parity-reversal operator. From this it is easy to see that 1 and PT are proper, i.e. preserving orientation[5]; 1 and P are orthochronous, i.e. preserving time direction.

[4] I call it "Minkowski eta" because, for now, I am talking about a single point (e.g. I don't have coordinate dependence, yet).

[5] In the matrix representation these have determinant equal to +1.

1.2.1 Lie algebra of Lorentz group

I will denote the Lie algebra of $SO^+(1,3)$ by $so(1,3)$. Its exponentiation gives a universal, spin, double cover of $SO^+(1,3)$, denoted by $SL(2,\mathbb{C}) \cong Spin^+(1,3)$[6].

A general element of $SL(2,\mathbb{C})$ can be written as

$$\Lambda = \exp(\theta^i J_i + \beta^i K_i)\,, \tag{1.5}$$

where $J_i, K_i \in sl(2,\mathbb{C})$ are the rotation and boost generators, respectively.

Remark. A potential point of confusion can arise! The Lie algebra $sl(2,\mathbb{C})$ is a 3-dimensional complex algebra. However, I will look at this algebra, and also at the group it generates, as at a 6-dimensional real algebra, hence six generators J_i, K_i for $i \in \{1,2,3\}$. The reason for this is that it is desired to parametrize boosts and rotations by real angles; the consequences are left- and right-handed spinors, I will talk about this later.

The generators satisfy the following commutation relations

$$\begin{aligned} [J_i, J_j] &= -\varepsilon_{ij}{}^k J_k\,, \\ [J_i, K_j] &= -\varepsilon_{ij}{}^k K_k\,, \\ [K_i, K_j] &= \varepsilon_{ij}{}^k J_k\,. \end{aligned} \tag{1.6}$$

From these relations it is easy to see that

$$K_i = \pm\iota J_i\,. \tag{1.7}$$

I postpone the discussion which sign to pick until the proposition 1.2.6.

In the next definition I will define three inter-related conjugation operators acting on the Lie algebra $sl(2,\mathbb{C})$. Their names are neatly chosen so that they will correspond to the operators of the same name acting on complex quaternions.

Definition 1.2.1. Let J_i, K_i be the generators of $sl(2,\mathbb{C})$, and let X, Y be arbitrary elements of $sl(2,\mathbb{C})$. Then

i) the complex conjugation operator is defined by

$$J_i^* = J_i\,, \quad K_j^* = -K_j\,, \quad XY^* = X^*Y^*\,;$$

ii) the quaternionic conjugation operator is defined by

$$\overline{J_i} = -J_i\,, \quad \overline{K_j} = -K_j\,, \quad \overline{XY} = \overline{Y}\,\overline{X}\,;$$

iii) the complex-quaternionic conjugation operator is defined by combination of the previous two, $(\cdot)^\dagger = (\bar{\cdot})^* = \overline{(\cdot)^*}$, i.e.

$$J_i^\dagger = -J_i\,, \quad K_j^\dagger = K_j\,, \quad XY^\dagger = Y^\dagger X^\dagger\,;$$

The commutation relations from the equation (1.6) exactly determine the form of the anti-commutation relations explored in the following lemma[7].

Lemma 1.2.1. *The generators of $sl(2,\mathbb{C})$ satisfy the following anti-commutation relations*

$$\{J_i, J_j\} = -\frac{1}{2}e\delta_{ij}\,, \quad \{K_i, K_j\} = \frac{1}{2}e\delta_{ij}\,,$$

where e is the identity of $sl(2,\mathbb{C})$.

[6] Note that $Spin(1,3)$ is sometimes in literature defined to be a universal cover of $SO(1,3)$, which is not connected.

[7] Strictly speaking multiplication, and hence anti-commutator, in a Lie algebra are not defined. To make sense out of this, one has to go either to the enveloping algebra or to a (finite-dimensional) representation, which is always isomorphic to some matrix algebra. But since in this thesis I am only concerned by finite-dimensional representations I don't need to worry about this.

Proof. From the commutation relations it is clear that $\left[J^2, J_i\right] = \left[J^2, K_i\right] = 0$ for $J^2 = \delta^{ij} J_i J_j$, and the same holds for K^2. This means that both J^2 and K^2 are proportional to the identity e. The equation

$$ae = K^2 = \delta^{kl} K_k K_l = \frac{1}{2}\delta^{kl} \left\{K_k, K_l\right\} \, ,$$

implies that $\{K_i, K_j\} = \frac{2a}{3} e\delta_{ij}$. Hence, I just need to show that $J^2 = -\frac{3}{4}e$, then the equation (1.7) determine that $K^2 = \frac{3}{4}e$. To this end I need the following messy calculation

$$\begin{aligned}
J^2 \frac{1}{2}\delta^{ij} \left\{J_i, J_j\right\} &= J_i J_j J^j J^i \\
&= \frac{1}{2}\left\{J_i, J_j\right\} J^j J^i + \frac{1}{2}\left[J_i, J_j\right] J^j J^i \\
&= \frac{1}{4}\left\{J_i, J_j\right\}\left\{J^j, J^i\right\} - \frac{1}{4}\epsilon_{ij}{}^{k} J_k \left[J^j, J^i\right] \\
&= \frac{1}{4}\left\{J_i, J_j\right\}\left\{J^j, J^i\right\} + \frac{1}{4}\epsilon_{ij}{}^{k} J_k \epsilon^{jil} J_l \\
&= \frac{1}{4}\left\{J_i, J_j\right\}\left\{J^j, J^i\right\} - \frac{1}{2}\delta^{kl} J_k J_l \\
&= \frac{1}{4}\left\{J_i, J_j\right\}\left\{J^j, J^i\right\} - \frac{1}{4}\delta^{ij}\left\{J_i, J_j\right\} \, ,
\end{aligned}$$

where $J_i J_j = \frac{1}{2}\{J_i, J_j\} + \frac{1}{2}[J_i, J_j]$, anti-symmetry of $\epsilon_{ij}{}^{k}$, and the commutation relations from the equation (1.6) were used extensively. Cancelling the $\{J_i, J_j\}$ in the above calculation, and contracting with δ_{ij} yields

$$3J^2 = J^2 - \frac{3}{2}e \, .$$

□

The generators of $sl(2, \mathbb{C})$ can be combined into a two-tensor $M^{\mu\nu}$ such that

$$\begin{aligned}
M^{0i} &= K^i = K_i \, , \\
M^{ij} &= \epsilon^{ijk} J_k \, ,
\end{aligned} \tag{1.8}$$

and the commutation relations become

$$\left[M^{\mu\nu}, M^{\sigma\rho}\right] = \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma} \, .$$

With this notation a general element of $SL(2, \mathbb{C})$ can be written as

$$\Lambda^V = \exp(\frac{1}{2}\theta_{\mu\nu} M^{\mu\nu}) \, , \tag{1.9}$$

where $\theta_{0i} = \theta_i = \theta^i$, and $\theta_{ij} = \epsilon_{ijk}\beta^k$.

1.2.2 Spinors

In the present subsection I plan to introduce spinor representations of $sl(2, \mathbb{C})$, viewed as a real Lie algebra. But before I delve into this problem, let me first recall the definition of a representation, and some useful results from the theory of representations of $sl(2, \mathbb{C})$.

Definition 1.2.2. A representation of a Lie algebra g is a Lie algebra homomorphism $\rho : g \to \mathrm{End}(V)$, i.e.

$$\rho([X, Y]) = [\rho(X), \rho(Y)] \, , \quad \forall X, Y \in g \, .$$

Theorem 1.2.2. *For each non-negative half-integer j $\left(j = \frac{n}{2}; n \in \mathbb{N}_0\right)$, there is an $2j{+}1$-dimensional irreducible complex representation, V_j, of the complex Lie algebra $sl(2, \mathbb{C})$. Any two irreducible complex representations of $sl(2, \mathbb{C})$ of the same dimension are isomorphic.*

Remark. The very first case, $j = 0$, is a trivial case of 1-dimensional representation. The objects acted upon are called scalars. They don't transform at all, for all three generators are mapped to zero. The representation $j = \frac{1}{2}$ is called fundamental and the objects it acts on are called Weyl spinors.

Theorem 1.2.3. *Let i and j be non-negative half-integers $\left(i = \frac{m}{2}, j = \frac{n}{2}; m, n \in \mathbb{N}_0\right)$. Then for any two complex representations of $sl(2,\mathbb{C})$, V_i and V_j, one has*

$$V_i \otimes V_j \cong V_{i+j} \oplus V_{i+j-2} \oplus \cdots \oplus V_{|i-j|+2} \oplus V_{|i-j|}\,.$$

Note. Proofs of the last two theorems can by found in [Hall, 2015] as Theorem 4.32 and Theorem C.1, respectively.

So far so good, but there is a problem. The theorem 1.2.2 says that all irreducible representations of $sl(2,\mathbb{C})$ of equal dimension are isomorphic. However, from physics it is known that there are two non-isomorphic representations of $sl(2,\mathbb{C})$, namely right-handed and left-handed spinors. The solution to this is in the following proposition.

Proposition 1.2.4. *Let V be a complex vector space, and let g be an n-dimensional complex Lie algebra. If g is viewed as a $2n$-dimensional real Lie algebra, then for each irreducible representation $\rho : g \to \mathrm{End}(V)$ there exists a non-isomorphic, conjugate, irreducible representation $\rho^\dagger : g \to \mathrm{End}\,\overline{V}$ called anti-representation.*

Proof. First, I will show the existence of the anti-representation. If g is viewed as a complex algebra $\rho^\dagger$ is not a representation - it is anti-linear and hence not Lie algebra homomorphism! For g viewed as real, this is not a problem because an anti-linear applied to real numbers is linear.

Second, I will prove that the two representations are non-isomorphic. By definition, two representations are isomorphic if there exists an equivariant isomorphisms between the representation spaces, i.e. it preserves action of the algebra and is linear.
More specifically, two representations $\rho_1 : g \to \mathrm{End}(V_1)$, and $\rho_2 : g \to \mathrm{End}(V_2)$ are isomorphic if there exists a linear isomorphism $\phi : V_1 \to V_2$ such that

$$\phi(\rho_1(X)v) = \rho_2(X)\phi(v)\,, \quad \forall X \in g\,, \forall v \in V_1\,.$$

By choosing $\rho_2 = \rho_1^\dagger$ I force ϕ to be anti-linear which contradicts the assumption of linearity.

$$\phi(\rho_1(X)zv) = \phi(\rho_1(zX)v) = \rho_2(zX)\phi(v) = z^*\rho_2(X)\phi(v)\,, \quad \forall z \in \mathbb{C}\,, \forall X \in g\,, \forall v \in V_1\,.$$

□

The conjugate representation denoted by $\rho^\dagger$ is adjoint of the representation operator ρ; and it depends on the underlying algebra in questions. E.g. for matrix algebra over $\mathbb{C}$ it is the hermitian adjoint i.e. the complex conjugation followed by the transposition; however this does not mean that the operator $\overline{(\cdot)}$ is the transposition operator, since transposition does not fulfil the conditions in the definition 1.2.1!
The case of interest here will be complex quaternions, and the adjoint operation will denote complex conjugation followed by quaternionic conjugation[8].

I also hearten the reader to notice the bar over the underlying space of anti-representation $\overline{V}$. It means that the basis elements of the underlying space of representation and anti-representation are related by this very operation. Moreover, later on in the proposition 1.2.6, I will show that this operation transforms left-handed spinors into right-handed ones and vice versa, but will not flip the spin, i.e. it is the parity operator.

Corollary. *If one views $sl(2,\mathbb{C})$ as a 6-dimensional real algebra, then for $j = \frac{1}{2}$ it has two non-isomorphic representations called fundamental, and anti-fundamental (or conjugate).*

[8]The reason behind this "double conjugation" is perhaps clearer to see from to the way of derivation outlined in the subsection A.1.1 of the appendix A, where I perform complexification of already complex algebra $sl(2,\mathbb{C})$ in order to find the two real spinor representations.

Remark. In physics, elements of V acted upon by the fundamental representation of $sl(2,\mathbb{C})$ are called left-handed Weyl spinors, and the representation is referred to as $\left(\frac{1}{2},0\right)$ with generators J_i^L, K_i^L. On the other hand, elements of $\overline{V}$ corresponding to the anti-fundamental representation are called right-handed Weyl spinors[9], and the representation is denoted by $\left(0,\frac{1}{2}\right)$ with generators $J_i^R = (J_i^L)^\dagger$ and $K_i^R = (K_i^L)^\dagger$. In what follows, it is beneficial to think of V and $\overline{V}$ as, not necessarily orthogonal, subspaces of $sl(2,\mathbb{C})$.

For the next proposition it is necessary to recall two points. The first is that a spinor basis depends on the spatial direction onto which it is being projected, the most common is the projection onto z axis. The second point to recall are the infinitesimal Lorentz transformations of left-handed and right-handed spinors

$$\begin{aligned}
\delta\psi_L &= \frac{1}{2}(\iota\theta^j + \beta^j)\sigma_j\psi_L \\
&= \left(\theta^j J_j^L + \beta^j K_j^L\right)\psi_L\,, \\
\delta\psi_R &= \frac{1}{2}(\iota\theta^j - \beta^j)\sigma_j\psi_R \\
&= \left(\theta^j J_j^R + \beta^j K_j^R\right)\psi_R\,.
\end{aligned} \tag{1.10}$$

The reader should be cautious when comparing these equation with different sources. Some authors, e.g. [Schwartz, 2014], use the opposite notation ($L \leftrightarrow R$), both notations are prevalent in physics literature.

Lemma 1.2.5. *The explicit infinitesimal transformation relations for left-handed and right-handed Weyl spinors go as follows. For the rotations around x, y, and z axes of the left-handed spinors, and for the boosts in those directions one gets*

$$J_x^L\begin{pmatrix}\psi_{L\uparrow}\\ \psi_{L\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\iota\psi_{L\downarrow}\\ \iota\psi_{L\uparrow}\end{pmatrix}\,,\quad J_y^L\begin{pmatrix}\psi_{L\uparrow}\\ \psi_{L\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\psi_{L\downarrow}\\ -\psi_{L\uparrow}\end{pmatrix}\,,\quad J_z^L\begin{pmatrix}\psi_{L\uparrow}\\ \psi_{L\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\iota\psi_{L\uparrow}\\ -\iota\psi_{L\downarrow}\end{pmatrix}\,;$$

$$K_x^L\begin{pmatrix}\psi_{L\uparrow}\\ \psi_{L\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\psi_{L\downarrow}\\ \psi_{L\uparrow}\end{pmatrix}\,,\quad K_y^L\begin{pmatrix}\psi_{L\uparrow}\\ \psi_{L\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\iota\psi_{L\downarrow}\\ \iota\psi_{L\uparrow}\end{pmatrix}\,,\quad K_z^L\begin{pmatrix}\psi_{L\uparrow}\\ \psi_{L\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\psi_{L\uparrow}\\ -\psi_{L\downarrow}\end{pmatrix}\,.$$

For the right-handed spinors rotations agrees to their left-handed counterparts, however boosts pick a minus sign relative to the left-handed spinors

$$J_x^R\begin{pmatrix}\psi_{R\uparrow}\\ \psi_{R\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\iota\psi_{R\downarrow}\\ \iota\psi_{R\uparrow}\end{pmatrix}\,,\quad J_y^R\begin{pmatrix}\psi_{R\uparrow}\\ \psi_{R\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\psi_{R\downarrow}\\ -\psi_{R\uparrow}\end{pmatrix}\,,\quad J_z^R\begin{pmatrix}\psi_{R\uparrow}\\ \psi_{R\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\iota\psi_{R\uparrow}\\ -\iota\psi_{R\downarrow}\end{pmatrix}\,;$$

$$K_x^R\begin{pmatrix}\psi_{R\uparrow}\\ \psi_{R\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\psi_{R\downarrow}\\ -\psi_{R\uparrow}\end{pmatrix}\,,\quad K_y^R\begin{pmatrix}\psi_{R\uparrow}\\ \psi_{R\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\iota\psi_{R\downarrow}\\ -\iota\psi_{R\uparrow}\end{pmatrix}\,,\quad K_z^R\begin{pmatrix}\psi_{R\uparrow}\\ \psi_{R\downarrow}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\psi_{R\uparrow}\\ \psi_{R\downarrow}\end{pmatrix}\,.$$

Proof. Trivially follows from the definition of Pauli matrices and the equation (1.10). □

Proposition 1.2.6. *Let the space of left-handed Weyl spinors, V, be a subspace of $sl(2,\mathbb{C})$; and let $l_\uparrow$ and $l_\downarrow$ be the basis of V projected onto z axis. Then one has two options how to choose a basis, either*

$$\begin{aligned}
l_\uparrow &= \left(\frac{1}{2}e + K_z\right)\,, & r_\uparrow &= \left(\frac{1}{2}e - K_z\right)\,,\\
l_\downarrow &= \left(K_x - \iota K_y\right)\,, & r_\downarrow &= \left(-K_x + \iota K_y\right)\,,
\end{aligned}$$

[9] I would also like to stress that this naming convention has no physical significance, and one could as well have it interchanged.

or

$$l_\uparrow = (K_x + \iota K_y)\ , \qquad r_\uparrow = (-K_x - \iota K_y)\ ,$$
$$l_\downarrow = \left(\frac{1}{2}e - K_z\right)\ , \qquad r_\downarrow = \left(\frac{1}{2}e + K_z\right)\ ,$$

where $r_\uparrow$ *and* $r_\downarrow$ *are the basis elements for the right-handed Weyl spinors.*

Proof. By definition, the basis for right-handed Weyl spinors is related to the left-handed Weyl spinor basis by $r_\uparrow = \overline{l_\uparrow}$ and $r_\downarrow = \overline{l_\downarrow}$. Spinor basis projected onto z spatial direction means that $l_\uparrow$ and $l_\downarrow$ are simultaneous eigenvectors of J_z and K_z. Their eigenvalues, taken from the lemma 1.2.5, are

$$J_z l_\uparrow = +\frac{\iota}{2} l_\uparrow\,, \quad J_z l_\downarrow = -\frac{\iota}{2} l_\downarrow\,,$$
$$K_z l_\uparrow = +\frac{1}{2} l_\uparrow\,, \quad K_z l_\downarrow = -\frac{1}{2} l_\downarrow\,,$$

These equations clarify the question of sign from the equation (1.7) by picking the minus sign, so $K_i = -\iota J_i$. Now, it is easy to guess the possible combinations (e with K_z since they commute with J_z, and K_x with K_y since they anti-commute with J_z)

$$J_z\left(\frac{1}{2}e \pm K_z\right) = \pm\frac{\iota}{2}\left(\frac{1}{2}e \pm K_z\right), \quad K_z\left(\frac{1}{2}e \pm K_z\right) = \pm\frac{1}{2}\left(\frac{1}{2}e \pm K_z\right)\,,$$
$$J_z\left(K_x \pm \iota K_y\right) = \pm\frac{\iota}{2}\left(K_x \pm \iota K_y\right), \quad K_z\left(K_x \pm \iota K_y\right) = \pm\frac{1}{2}\left(K_x \pm \iota K_y\right)\,.$$

Matching rotations and boosts in the remaining two directions of these elements with the demanded rules from the lemma 1.2.5 yields the desired result. □

1.2.3 Vectors

In this section I develop an alternative approach to the classical matrix formalism of spinors and vectors. This will be immensely useful later on when I redefine the whole theory in terms of complex quaternions.

So far I have explored the trivial, fundamental, and anti-fundamental representations of the Lie algebra $sl(2,\mathbb{C})$. The first mentioned has dimension 1 and the other two are both of dimension 2. This gives us scalars, left-handed and right-handed spinors. Now I would like to obtain 4-vectors.

The naive guess would be the representation $(1,0)$ or $(0,1)$ but this turns out to give only a 3-dimensional representation. It can be seen directly from the theorem 1.2.2 which says that complex representation of $sl(2,\mathbb{C})$ with $j = 1$ is 3-dimensional. Furthermore, the theorem A.1.1 (or the remark following the theorem) tells that there exists an isomorphism between real representations of $su(2)$ and complex representations of $su(2)_\mathbb{C} \cong sl(2,\mathbb{C})$; from which one can see that for $j = 1$ the good old 3-vector representation is recovered.

But there is a need for a 4-dimensional representation with Lorentz transformations acting as spatial rotations and boosts. It turns out that this problem is solved quite easily by the next theorem.

Theorem 1.2.7. *The* 4*-dimensional Minkowski space* M *can be interpreted as a subspace of the Lie algebra* $sl(2,\mathbb{C})$ *with orthonormal basis being* $\{e, 2K_1, 2K_2, 2K_3\}$*, where* e *is the identity of the algebra, and* K_i*'s are the boost generators. The scalar product* $\langle\cdot,\cdot\rangle : M \times M \to \mathbb{R}$ *is defined by*

$$\langle u, v\rangle = \frac{1}{2}\left(u\overline{v} + v\overline{u}\right)\ , \quad \forall u, v \in M\,,$$

where $\overline{K_i} = -K_i$ *and* $\overline{e} = e$*. Moreover, the complete set of Lorentz transformations is given by a simultaneous action of the fundamental and the anti-fundamental representations of* $SL(2,\mathbb{C})$*. This is called the* 4-**vector representation***, and is denoted by* $\left(\frac{1}{2},\frac{1}{2}\right)$*.*

Proof. The complex, unital algebra $sl(2,\mathbb{C})$ is 3-dimensional. Hence, taking its three generators together with the identity gives a 4-dimensional complex vector space $M_{\mathbb{C}}$. By taking the identity with the boost generators as a basis, and restricting only to the real span of this basis, one arrives at the Minkowski space M. For $u = u^0 e + u^i 2K_i$, and $v = v^0 e + v^j 2K_j$ the scalar product gives

$$\begin{aligned} 2\langle u, v\rangle &= (u^0 e + u^i 2K_i)(v^0 e - v^j 2K_j) + (v^0 e + v^j 2K_j)(u^0 e - u^i 2K_i) \\ &= 2u^0 v^0 e \pm 2u^0 v^j K_j \pm 2v^0 u^i K_i - 4u^i K_i v^j K_j - 4v^j K_j u^i K_i \\ &= 2u^0 v^0 e - 4u^i v^j \{K_i, K_j\} \\ &= 2(u^0 v^0 - u^i v_i) e\,, \end{aligned}$$

where $\{K_i, K_j\} = \frac{1}{2}\delta_{ij}$, coming from the lemma 1.2.1, has been used. To prove the last statement it is rather easier to work with the infinitesimal transformations.

$$\begin{aligned} \exp(\theta^j J_j^L)\exp(\theta^j J_j^R)v &= \exp(\theta^j J_j) v \exp(-\theta^j J_j) \\ &\approx v^i\theta^j \big(J_j K_i - K_i J_j\big) \\ &= v^i \theta^j \,[J_j, K_i] \\ &= \epsilon_{ij}{}^k v^i \theta^j K_k\,, \\ \exp(\beta^j K_j^L)\exp(\beta^j K_j^R)v &= \exp(\beta^j K_j) v \exp(\beta^j K_j) \\ &\approx v^0 \beta^j 2K_j + 2v^i\beta^j \big(K_j K_i + K_i K_j\big) \\ &= v^0\beta^j 2K_j + 2v^i\beta^j \{K_j, K_i\} \\ &= v^0 \beta^j 2K_j + v^i \beta_i e\,, \end{aligned}$$

where the fundamental representation acts by the left multiplication of J_j or K_j, and that the anti-fundamental representation acts by the right multiplication of $-J_j$ or K_j. The option J_j^L acting by the right multiplication of J_j is not possible because the commutation relations for J_j would pick a minus sign[10].
Furthermore, $[J_i, e] = 0$ and commutation relations from equation (1.6) have been used in the first equation; $K_i^\dagger = K_i$, and $\{K_i, K_j\} = \frac{1}{2}\delta_{ij}$ from the lemma 1.2.1 in the second. The reader should compare this result with the equation (A.4). □

Remark. Another possible choice of the orthonormal basis would be a complex multiple of the chosen one with an appropriately modified scalar product. E.g. by multiplying the basis by ι, the new basis takes the form of $\{\iota e, 2J_1, 2J_2, 2J_3\}$, with a modified scalar product $\langle u, v\rangle = -\frac{1}{2}(u\overline{v} + v\overline{u})$.

At the very end of this section, I would like to provide a proposition, where an alternative way on how to view the relation between spinors and 4-vectors is explored. (This is very close to the representation of 4-vectors using dotted and undotted spinor indices.) It will also prove to be useful in many situation; two worth to mention are building Lorentz invariant Lagrangian, and helping to bridge the difference between the standard matrix formalism of QFT and a the formalism developed in the next chapter.

Proposition 1.2.8. *Let $l_\uparrow$, $l_\downarrow$, $r_\uparrow$, and $r_\downarrow$ be elements of $sl(2,\mathbb{C})$ forming basis for the left-handed spinors and the right-handed spinors, respectively. Then the underlying space of the 4-vector representation $\left(\frac{1}{2},\frac{1}{2}\right)$, i.e. the Minkowski space, takes the form of*

$$\begin{pmatrix} l_\uparrow r_\uparrow^* & l_\uparrow r_\downarrow^* \\ l_\downarrow r_\uparrow^* & l_\downarrow r_\downarrow^* \end{pmatrix}.$$

Moreover, if the spinor basis is projected onto the z axis, the underlying space takes the form of

$$\begin{pmatrix} \frac{e}{2} + K_z & K_x + \iota K_y \\ K_x - \iota K_y & \frac{e}{2} - K_z \end{pmatrix}.$$

[10] I don't consider the option $J_j^L = -\mathrm{R}_{J_j}$ which is of course allowed but it merely interchanges the superscripts $R \leftrightarrow L$ in an unnatural way.

The Lorentz transformations are performed by a simultaneous action of both fundamental and anti-fundamental representations[11]*.*

Proof. The proposition 1.2.4 and the text following it say that the basis for the right-handed spinors is related to the left-handed spinor basis by $r_\uparrow = \overline{l_\uparrow}$ and $r_\downarrow = \overline{l_\downarrow}$, which yields

$$\begin{pmatrix} l_\uparrow r_\uparrow^* & l_\uparrow r_\downarrow^* \\ l_\downarrow r_\uparrow^* & l_\downarrow r_\downarrow^* \end{pmatrix} = \begin{pmatrix} l_\uparrow l_\uparrow^\dagger & l_\uparrow l_\downarrow^\dagger \\ l_\downarrow l_\uparrow^\dagger & l_\downarrow l_\downarrow^\dagger \end{pmatrix} .$$

For a general element $X \in sl(2,\mathbb{C})$ such that $X = ae + b\iota e + c^i K_i + d^i J_i$ for $a,b,c,d \in \mathbb{R}$ one has

$$\begin{aligned} XX^\dagger =& (ae + b\iota e + c^i K_i + d^i J_i)(ae - b\iota e + c^j K_j - d^j J_j) \\ =& + a^2 + b^2 + 2ac^i K_i - 2\iota b d^i J_i \\ &+ \frac{1}{2} c^i c^j \{K_i, K_j\} - \frac{1}{2} d^i d^j \{J_i, J_j\} + c^i d^j (J_j K_i - K_i J_j) \\ =& \left(a^2 + b^2 + \frac{1}{4}(c^2 + d^2) \right) e + \left(ac^i \mp bd^i \right) 2K_i \,, \end{aligned}$$

where the sign in the last equation depends on the sign of the equation (1.7), $K_i = \pm \iota J_i$. This implies that every element of the form $XX^\dagger$ can be written as $V = V^0 e + V^i 2K_i$, for $V^0, V^i \in \mathbb{R}$. In other words every element of the above matrix is an element of the Minkowski space M from the previous theorem 1.2.7! This immediately guarantees that the elements will transform accordingly under Lorentz transformations.

To prove the second part it is enough to employ the proposition 1.2.6. The following multiplications are readily checked assuming $K_i = -\iota J_i$ and the commutation relations from the equation (1.6)

$$\begin{aligned} \left(\frac{1}{2}e + K_z\right) &= l_\uparrow r_\uparrow^* = \left(\frac{1}{2}e + K_z\right)^2 = (K_x + \iota K_y)(K_x - \iota K_y) \,, \\ \left(\frac{1}{2}e - K_z\right) &= l_\downarrow r_\downarrow^* = (K_x - \iota K_y)(K_x + \iota K_y) = \left(\frac{1}{2}e - K_z\right)^2 \,, \\ (K_x + \iota K_y) &= l_\uparrow r_\downarrow^* = \left(\frac{1}{2}e + K_z\right)(K_x + \iota K_y) = (K_x + \iota K_y)\left(\frac{1}{2}e - K_z\right) \,, \\ (K_x - \iota K_y) &= l_\downarrow r_\uparrow^* = (K_x - \iota K_y)\left(\frac{1}{2}e + K_z\right) = \left(\frac{1}{2}e - K_z\right)(K_x - \iota K_y) \,. \end{aligned}$$

□

Remark. The relation between the basis elements of the Minkowski spaces from the theorem 1.2.7 and the proposition 1.2.8 is easily seen to be

$$\begin{aligned} e &= l_\uparrow r_\uparrow^* + l_\downarrow r_\downarrow^* \,, \\ 2K_x &= l_\downarrow r_\uparrow^* + l_\uparrow r_\downarrow^* \,, \\ 2K_y &= \iota \left(l_\downarrow r_\uparrow^* - l_\uparrow r_\downarrow^* \right) \,, \\ 2K_z &= l_\uparrow r_\uparrow^* - l_\downarrow r_\downarrow^* \,. \end{aligned}$$

With the inverse being $\{e, 2K_i\} \mapsto \{eI_{2\times 2}, 2K_x\sigma_x^*, 2K_y\sigma_y^*, 2K_z\sigma_z^*\}$.

Remark. From the theorem 1.2.7, one can truly see that the sign in the equation (1.7) is related to the handedness of the spatial coordinate basis. Had I picked the other sign I would have arrived at equations modified by $K_i \leftrightarrow -K_i$ in the proposition 1.2.6, the mentioned theorem, and in the proposition 1.2.8.

[11] It is understood that the representations are multiplied by $I_{2\times 2}$, the identity matrix

Chapter 2

Field theory in terms of complex quaternions

In this chapter I would like to go through the machinery reviewed and developed in the previous chapter, and apply it to the complex quaternions viewed as a representation of the $sl(2,\mathbb{C})$. I will take advantage of the fact that quaternions are non-commutative algebra, and define the Lorentz transformation for left-handed and right-handed spinors as left and right multiplication by representation of the corresponding elements of $SL(2,\mathbb{C})$. Also the basis for spinors viewed as elements of complex quaternions will be established. Next, I will have a look on the 4-vector representation and develop a way of representing them again as complex quaternions. This representation will strongly resemble the representation of 4-vectors as spinors carrying two spinor indices (one dotted and the other undotted). I will conclude this chapter by derivation of Lorentz invariant Lagrangian, equation of motion and linking this description to the standard matrix one. A similar approach to the one of this thesis was developed by [Furey, 2016] and [Greiter and Schuricht, 2003].

2.1 Complex quaternions as spinors

Theorem 2.1.1. *A linear map $\rho : sl(2,\mathbb{C}) \to \mathrm{End}(\mathbb{C}\otimes\mathbb{Q})$ such that*

$$\rho(J_i) = -\frac{1}{2}\,\mathrm{L}_{\epsilon_i} \equiv J_i^L\,,$$
$$\rho(K_i) = \iota\frac{1}{2}\,\mathrm{L}_{\epsilon_i} \equiv K_i^L\,,$$

defines a fundamental representation $\left(\frac{1}{2},0\right)$.

Note. The operators L_ϵ and R_ϵ are the operators of left and right multiplication, from the definition 1.1.5.

Proof. The only thing to check here are the commutation relations.

$$\begin{aligned}\left[J_i^L, J_j^L\right]\psi &= \frac{1}{4}\left[\mathrm{L}_{\epsilon_i}, \mathrm{L}_{\epsilon_j}\right]\psi \\ &= \frac{1}{4}\left(\epsilon_i\epsilon_j - \epsilon_j\epsilon_i\right)\psi \\ &= \frac{1}{2}{\epsilon_{ij}}^k\,\mathrm{L}_{\epsilon_k}\,\psi \\ &= -{\epsilon_{ij}}^k J_k^L\psi\,,\end{aligned}$$

where the commutation relations of quaternions defined in the lemma 1.1.3, and the definition 1.1.5 of left multiplication have been used. All the other commutation relations follow from the fact that J_i^L and K_i^L satisfy the equation (1.7). □

Note. The complex conjugation is denoted by $(\cdot)^*$, the quaternionic conjugation is denoted by $\overline{(\cdot)}$, and finaly the symbol $(\cdot)^\dagger$ denotes the complex conjugation followed by the quaternionic conjugation, called complex-quaternionic conjugation.

Corollary. *The anti-fundamental representation* $\left(0, \frac{1}{2}\right)$*, conjugate to the above representation* ρ*, is a linear map* $\rho^\dagger : sl(2,\mathbb{C}) \to \mathrm{End}(\overline{\mathbb{C}\otimes\mathbb{Q}})$ *such that*

$$\rho^\dagger(J_i) = \frac{1}{2}\,\mathrm{R}_{\epsilon_i} \equiv J_i^R\,,$$
$$\rho^\dagger(K_i) = \iota\frac{1}{2}\,\mathrm{R}_{\epsilon_i} \equiv K_i^R\,.$$

Proof. This is a bit more tricky than the other proof. First, I need to prove that the representation in question really is conjugate to the fundamental representation; and hence can be called anti-fundamental representation. To this end one writes

$$\begin{aligned}\mathrm{L}_a^\dagger\,\psi &= \left(\mathrm{L}_a\,\psi^\dagger\right)^\dagger \\ &= (a\psi^\dagger)^\dagger \\ &= \psi a^\dagger \\ &= \mathrm{R}_{a^\dagger}\,\psi\,.\end{aligned}$$

Now, the commutation relations can be checked.

$$\begin{aligned}\left[J_i^R, J_j^R\right]\psi &= \frac{1}{4}\left[\mathrm{R}_{\epsilon_i}, \mathrm{R}_{\epsilon_j}\right]\psi \\ &= \frac{1}{4}\psi\left(\epsilon_j\epsilon_i - \epsilon_i\epsilon_j\right) \\ &= \frac{1}{2}\epsilon_{ji}{}^k\psi\epsilon_k \\ &= -\epsilon_{ij}{}^k\,\mathrm{R}_{\epsilon_k}\,\psi\,,\end{aligned}$$

where again the commutation relations of quaternions defined in the lemma 1.1.3, the definition 1.1.5 of the right multiplication, and the anti-symmetry of $\epsilon_{ij}{}^k$ have been used. The other commutation relations follow from the fact that J_i^L and K_i^L satisfy the equation (1.7). □

Remark. Although, I still haven't shown what exactly left-handed and right-handed spinors are, they span 2-dimensional, not necessarily orthogonal, spaces of $\mathbb{C}\otimes\mathbb{Q}$. Only after they are combined into a 4-vector, $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation, they have to span the whole $\mathbb{C}\otimes\mathbb{Q}$.

It would be helpful to know exactly which complex dimensions correspond to the left-handed and which to the right-handed spinors. This turns out to be ambiguous, which makes perfect sense, since spinor basis depends on the spatial direction onto which we are projecting it. The most prevalent choice however, is the projection onto the z direction[1]. In the standard matrix formalism it is easy to see that this is the case, since the generators of z-rotations and z-boosts for spinors are proportional to the third Pauli matrix which is diagonal.

In the case of complex-quaternions, there is no way to tell if the generators are diagonal. There is not even a notion of being diagonal. But as it turns out this is not a problem because as opposed to matrix formalism, where the space acted upon is fixed to be a column matrix, in complex-quaternionic case one is free to pick any basis for $\mathbb{C}\otimes\mathbb{Q}$ one likes. Hence, the desired basis is the one being eigenstate of z-rotation and z-boosts.

It was shown in the proposition 1.2.6 that once the representation of the Lie algebra $sl(2,\mathbb{C})$ acts on itself, i.e. $\rho : sl(2,\mathbb{C}) \to \mathrm{End}\left((sl(2,\mathbb{C})\right)$, the basis fulfilling the conditions of the previous paragraph is not unique. But there are two such equivalent bases. In the next proposition I will explore only one, the other is easily derived from the mentioned proposition of the previous chapter.

[1] Actually, I haven't seen any other choice.

Proposition 2.1.2. *The basis elements for left-handed spinors projected onto the z direction are*

$$l_\uparrow = \frac{1}{2}\left(1 + \iota\epsilon_z\right) , \quad l_\downarrow = \frac{1}{2}\left(\iota\epsilon_x + \epsilon_y\right) ,$$

where $l_\uparrow$ denotes the spin up[2]*, and $l_\downarrow$ denotes spin down. The basis elements for right-handed spinors projected onto the z direction are*

$$r_\uparrow = \frac{1}{2}\left(1 - \iota\epsilon_z\right) , \quad r_\downarrow = \frac{1}{2}\left(-\iota\epsilon_x - \epsilon_y\right) .$$

Proof. By replacing the boost generators K_i in the first basis set of the proposition 1.2.6 with $\iota\epsilon_i$ yields the desired outcome.

$$\begin{aligned} l_\uparrow &= \left(\frac{1}{2}e + K_z\right) = \frac{1}{2}\left(1 + \iota\epsilon_z\right) , & r_\uparrow &= \left(\frac{1}{2}e - K_z\right) = \frac{1}{2}\left(1 - \iota\epsilon_z\right) , \\ l_\downarrow &= \left(K_x - \iota K_y\right) = \frac{1}{2}\left(\iota\epsilon_x + \epsilon_y\right) , & r_\downarrow &= \left(-K_x + \iota K_y\right) = \frac{1}{2}\left(-\iota\epsilon_x - \epsilon_y\right) . \end{aligned}$$

By doing the same procedure on the other set of basis elements one finds out that

$$\begin{aligned} l_\uparrow &= \left(K_x + \iota K_y\right) = \frac{1}{2}\left(\iota\epsilon_x - \epsilon_y\right) , & r_\uparrow &= \left(-K_x - \iota K_y\right) = \frac{1}{2}\left(-\iota\epsilon_x + \epsilon_y\right) , \\ l_\downarrow &= \left(\frac{1}{2}e - K_z\right) = \frac{1}{2}\left(1 - \iota\epsilon_z\right) , & r_\downarrow &= \left(\frac{1}{2}e + K_z\right) = \frac{1}{2}\left(1 + \iota\epsilon_z\right) . \end{aligned}$$

The reader is encouraged to check the correctness of the Lorentz transformations of this basis. While doing this, he might find helpful the subsection B.2.1 in the appendix B. □

2.2 Complex quaternions as vectors

In this section, I would like to outline the construction of Minkowski space.

Theorem 2.2.1. *The choice of the fundamental representation of $sl(2,\mathbb{C})$ made in the theorem 2.1.1 gives rise to a 4-dimensional Minkowski space M with an orthonormal basis $\{1, \iota\epsilon_x, \iota\epsilon_y, \iota\epsilon_z\}$. The scalar product is given by*

$$\langle U, V\rangle = \frac{1}{2}\left(U\overline{V} + V\overline{U}\right) , \quad \forall U, V \in M ,$$

where the operation $\overline{(\cdot)} : \mathbb{C}\otimes\mathbb{Q} \to \mathbb{C}\otimes\mathbb{Q}$ is the quaternionic conjugation. The Lorentz transformation of a 4-vector $V \in M$ take the form

$$\Lambda(\beta^j, \theta^j)V = \exp\left(\frac{1}{2}(-\theta^j + \iota\beta^j)\epsilon_j\right) V \exp\left(\frac{1}{2}(+\theta^j + \iota\beta^j)\epsilon_j\right) .$$

Proof. The whole proof follows directly from the theorem 1.2.7. However, I would like to elaborate more on Lorentz transformations; i.e. I would like to write the explicit transformation relations, since in the mentioned theorem only infinitesimal versions were shown. By kicking off with

$$\begin{aligned} \Lambda(\beta^j, \theta^j)V &= \left[\exp\left(\theta^j J^L_j + \beta^j K^L_j\right) \exp\left(\theta^j (J^L_j)^\dagger + \beta^j (K^L_j)^\dagger\right)\right] V \\ &= \exp\left(\frac{1}{2}(-\theta^j + \iota\beta^j)\epsilon_j\right) V \exp\left(\frac{1}{2}(+\theta^j + \iota\beta^j)\epsilon_j\right) , \end{aligned}$$

[2] This means that the projection onto the z axis points in the positive z direction

one arrives at a general Lorentz transformation. Splitting this further into the rotation part and the boost part should give the known Lorentz transformations. The former yields

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\theta^j\epsilon_j\right)V\exp\left(\frac{1}{2}\theta^j\epsilon_j\right)\\
&\quad=\left[\cos\left(\frac{\theta^j}{2}\right)-\epsilon_j\sin\left(\frac{\theta^j}{2}\right)\right]V\left[\cos\left(\frac{\theta^j}{2}\right)+\epsilon_j\sin\left(\frac{\theta^j}{2}\right)\right]\\
&\quad=\cos^2\left(\frac{\theta^j}{2}\right)V-\sin^2\left(\frac{\theta^j}{2}\right)\epsilon_jV\epsilon_j-\cos\left(\frac{\theta^j}{2}\right)\sin\left(\frac{\theta^j}{2}\right)[\epsilon_j,V]\ .
\end{aligned}$$

Writing out V explicitely, $V=V^0+V^i\iota\epsilon_i$, and evaluating the equation for let's say $j=x$ yields

$$V^t+V^x\iota\epsilon_x+[\cos(\theta^x)V^y+\sin(\theta^x)V^z]\,\iota\epsilon_y+[-\sin(\theta^x)V^y+\cos(\theta^x)V^z]\,\iota\epsilon_z\ ,$$

which proves the assumption about rotations, in this case around x.

For boosts the calculation is similar to the above one. The difference is in a substitution of the commutator by the anti-commutator, and in a replacement of the trigonometric functions by hyperbolic functions

$$\begin{aligned}
&\exp\left(\frac{1}{2}\beta^j\iota\epsilon_j\right)V\exp\left(\frac{1}{2}\beta^j\iota\epsilon_j\right)\\
&\quad=\left[\cosh\left(\frac{\beta^j}{2}\right)+\epsilon_j\sinh\left(\frac{\beta^j}{2}\right)\right]V\left[\cosh\left(\frac{\beta^j}{2}\right)+\epsilon_j\sinh\left(\frac{\beta^j}{2}\right)\right]\\
&\quad=\cosh^2\left(\frac{\beta^j}{2}\right)V+\sinh^2\left(\frac{\beta^j}{2}\right)\epsilon_jV\epsilon_j+\cosh\left(\frac{\beta^j}{2}\right)\sinh\left(\frac{\beta^j}{2}\right)\{\epsilon_j,V\}\ .
\end{aligned}$$

by writing V explicitly and evaluating for a concrete j, e.g. $j=x$, one gets

$$\left[\cosh(\beta^x)V^t+\sinh(\beta^x)V^x\right]+\left[\sinh(\beta^x)V^t+\cosh(\beta^x)V^x\right]\epsilon_x+V^y\epsilon_y+V^z\epsilon_z\ .$$

In other words one gets boosts along x. □

Remark. An astute reader may have already noticed the rotation part in the equation (1.4).

I will now show another way of representing Minkowski space. This representation is still based on complex quaternions but it is slightly closer to the traditional matrix formalism discussed in the appendix A, and hints the intuition behind the connection of the two. It will also help to understand the way how scalars will later be formed, especially when writing down a Lorentz invariant Lagrangian.

Proposition 2.2.2. *The Minkowski space from the previous theorem can be rearranged into a* 2×2 *"Hermitian" matrix*[3]

$$\frac{1}{2}\begin{pmatrix}1+\iota\epsilon_z & \iota\epsilon_x-\epsilon_y\\ \iota\epsilon_x+\epsilon_y & 1-\iota\epsilon_z\end{pmatrix}\ .$$

The Lorentz transformations are performed by a simultaneous action of both fundamental and anti-fundamental representations[4].

Proof. By substituting the basis from the proposition 2.1.2

$$\begin{aligned}
l_\uparrow&=\frac{1}{2}\left(1+\iota\epsilon_z\right)\ , & r_\uparrow&=\frac{1}{2}\left(1-\iota\epsilon_z\right)\ ,\\
l_\downarrow&=\frac{1}{2}\left(\iota\epsilon_x+\epsilon_y\right)\ , & r_\downarrow&=\frac{1}{2}\left(-\iota\epsilon_x-\epsilon_y\right)\ ,
\end{aligned}$$

[3]Hermitian in this context mean complex-quaternionic conjugation followed by the transposition does not change the matrix, i.e. $\left(H^\dagger\right)^T=H$.

[4]It is understood that the representations are multiplied by $I_{2\times2}$, the identity matrix

into the proposition 1.2.8 yields

$$\begin{pmatrix} l_\uparrow r_\uparrow^* & l_\uparrow r_\downarrow^* \\ l_\downarrow r_\uparrow^* & l_\downarrow r_\downarrow^* \end{pmatrix} = \begin{pmatrix} \frac{e}{2} + K_z & K_x + \iota K_y \\ K_x - \iota K_y & \frac{e}{2} - K_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + \iota\epsilon_z & \iota\epsilon_x - \epsilon_y \\ \iota\epsilon_x + \epsilon_y & 1 - \iota\epsilon_z \end{pmatrix} .$$

The reader is encouraged to check these relations with the help of the subsection B.2.2 of the appendix B. Especially, that both choices of basis yield the same matrix.

It is easy to see that the basis of the 4-dimensional Minkowski space from the theorem 1.2.7 and the theorem 2.2.1 can be extracted as follows

$$\begin{aligned} e = \ \ 1 \ &= l_\uparrow r_\uparrow^* + l_\downarrow r_\downarrow^* \,, \\ 2K_1 = \iota\epsilon_x &= l_\downarrow r_\uparrow^* + l_\uparrow r_\downarrow^* \,, \\ 2K_2 = \iota\epsilon_y &= \iota\left(l_\downarrow r_\uparrow^* - l_\uparrow r_\downarrow^*\right) \,, \\ 2K_3 = \iota\epsilon_z &= l_\uparrow r_\uparrow^* - l_\downarrow r_\downarrow^* \,. \end{aligned}$$

From this and the way the fundamental and anti-fundamental representations act it is clear that there is indeed an isomorphism to the Minkowski space from the two previously mentioned theorems, with the isomorphism being $\{e, 2K_i\} \mapsto \{eI_{2\times 2}, 2K_i(\sigma^i)^*\}$ so that

$$V \mapsto \begin{pmatrix} V^t + V^z\iota\epsilon_z & V^x\iota\epsilon_x - V^y\epsilon_y \\ V^x\iota\epsilon_x - V^y\epsilon_y & V^t - V^z\iota\epsilon_z \end{pmatrix} ,$$

where $V = V^0 + V^i\iota\epsilon_i$, and $I_{2\times 2}$ is the 2×2 identity matrix. □

2.3 Lagrangian for complex quaternions

Now, let's have a look on construction of Lagrangians. First thing to realize is that from matrix formalism of QFT it is known that a Lagrangian must be Lorentz invariant and hermitian. In case of complex quaternions this translates to Lorentz invariance, and invariance under complex-quaternionic conjugation.

Theorem 2.3.1. *The Lorentz invariant Lagrangian for massive spinors takes the following form*

$$\mathcal{L} = \iota\left[\psi_L^\dagger \overline{\partial}\psi_L + \psi_R^* \partial \overline{\psi}_R\right] - m\left[\psi_L^\dagger \overline{\psi}_R + \psi_R^* \psi_L\right] ,$$

where $\partial = \partial_0 + \sum_i \iota\epsilon_i\partial_i$ and transforms as a 4-vector. Moreover, every other, physically relevant, Lorentz invariant choice of the Lagrangian is related to this one by complex conjugation, quaternionic conjugation or complex-quaternionic conjugation.

Remark. The above Lagrangian can be put into a more familiar form of

$$\mathcal{L} = \begin{pmatrix} \psi_R^* & \psi_L^\dagger \end{pmatrix} \begin{pmatrix} -m & \iota\partial \\ \iota\overline{\partial} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \overline{\psi}_R \end{pmatrix} .$$

Proof. If one realizes that $\left(\iota\psi_L^\dagger\overline{\partial}\psi_L\right)^\dagger = -\iota\psi_L^\dagger \overleftarrow{\overline{\partial}}\psi_L$ and integrating by parts one obtains $\iota\psi_L^\dagger\overline{\partial}\psi_L$, then it is easy to see that this Lagrangian is invariant under complex-quaternionic conjugation.

To show the second claim one has to realize how spinors and 4-vectors transform under infinitesimal Lorentz transformations. This is done by either left of right multiplication with $\pm\frac{1}{2}\epsilon_i$, for rotations, or with $\pm\frac{1}{2}\iota\epsilon_i$, for boosts. I outline the signs in the next equation, keeping in mind that the upper sign is for rotations and the lower for boosts. In the case they are the same, I use only one

$$\begin{array}{ccc} (\mp)\psi_L & \psi_R(+) & (\mp)V(+) \\ (-)\psi_L^* & \psi_R^*(\pm) & (-)V^*(\pm) \\ \overline{\psi}_L(\pm) & (-)\overline{\psi}_R & (-)\overline{V}(\pm) \\ \psi_L^\dagger(+) & (\mp)\psi_R^\dagger & (\mp)V^\dagger(+)\,. \end{array} \tag{2.1}$$

A moment of thought convinces that the only Lorentz invariant combinations are the one from the theorem and three conjugate versions[5].

The note "physically relevant" in the theorem says that I don't deal with the option where a derivation is sandwiched with one left-handed and one right-handed spinor, since this has no physical relevance. I also leave out the discussion of Majorana fermions. □

Next, I would like to derive the equations of motion from the above Lagrangian. This turns out to be quite easy.

Proposition 2.3.2. *The equations of motion obtained from the Lorentz invariant Lagrangian of the theorem 2.3.1 are*

$$\begin{aligned} \iota\overline{\partial}\psi_L - m\overline{\psi}_R &= 0 \\ \iota\partial\,\overline{\psi}_R - m\psi_L &= 0\,. \end{aligned}$$

Proof. Very easy, just vary with respect to $\psi_L^\dagger$, and ψ_R^* or use the remark and vary with respect to $\begin{pmatrix}\psi_R^* & \psi_L^\dagger\end{pmatrix}$. □

This clearly hints that one should define the complex-quaternionic gamma matrices.

Lemma 2.3.3. *The Clifford algebra corresponding to the Minkowski space from the theorem 2.2.1 in the chiral representation takes the following form*

$$\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\,, \quad \gamma^i = \begin{pmatrix} & \iota\epsilon_i \\ -\iota\epsilon_i & \end{pmatrix}\,.$$

Remark. If one writes $1 = \sigma^0$ and $\iota\epsilon_i = \sigma^i$ the gamma matrices take the exactly same form as those from matrix formalism.

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \overline{\sigma}^\mu & \end{pmatrix}\,.$$

Proof. Here one needs to show that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The case $\mu = \nu = 0$ is trivial, for the case $\mu = 0$ and $\nu = i$ one gets

$$\{\gamma^0, \gamma^i\} = \begin{pmatrix} -\iota\epsilon_i & \\ & \iota\epsilon_i \end{pmatrix} + \begin{pmatrix} \iota\epsilon_i & \\ & -\iota\epsilon_i \end{pmatrix} = 0\,.$$

Finaly, the case $\mu = i$ and $\nu = j$ yields

$$\{\gamma^i, \gamma^j\} = \begin{pmatrix} \{\epsilon_i, \epsilon_j\} & \\ & \{\epsilon_i, \epsilon_j\} \end{pmatrix} = -2\delta^{ij}\,,$$

where the anti-commutation relations of quaternions, defined in the lemma 1.1.3, have been used. □

Definition 2.3.1. The fifth gamma matrix is defined as $\gamma^5 = -\iota\gamma^0\gamma^1\gamma^2\gamma^3$, and in the chiral representation it takes the following form

$$\gamma^5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\,.$$

[5] There is a number of thinks which must be kept in mind, concretely $V^* = \overline{V}$ and $V^\dagger = V$ since the basis of Minkowski space is $\{1, \iota\epsilon_i\}$; ∂ transforms as $\overline{V}$; and that after performing (complex-)quaternionic conjugation one has to integrate by parts to get the derivation act on the right.

By combining the left-handed spinor and quaternionic conjugation of the right-handed spinor into the Dirac spinor

$$\psi = \begin{pmatrix} \psi_L \\ \overline{\psi}_R \end{pmatrix} , \quad \widetilde{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \psi_R^* & \psi_L^\dagger \end{pmatrix} ,$$

the Lagrangian and equation of motion looks precisely like those from matrix formalism.

$$\begin{aligned} \mathcal{L} = \widetilde{\psi}\, (\iota\gamma^\mu \partial_\mu - m)\, \psi \, , \\ (\iota\gamma^\mu \partial_\mu - m)\, \psi = 0 \, . \end{aligned} \tag{2.2}$$

One can also define the slash operator $\partial\!\!\!/ = \gamma^\mu \partial_\mu$.

Chapter 3

Non-commutative geometry

This chapter introduces the notion of non-commutative geometry. The main motivation of this branch of mathematics is the duality between spaces and functions in a commutative setting, namely Gelfand theorem, which will also be discussed. There are many approaches to the non-commutative geometry, one of which is the approach of spectral triples. In this thesis I will adopt spectral triples since they were proved useful in defining non-commutative gauge theories.

The idea behind non-commutative gauge theories resembles the one of Kaluza-Klein. In a standard Kaluza-Klein theory one takes a product of a spin manifold with a circle. This allows to describe electrodynamics in terms of geometry, however one must pay by introducing unobserved dimensions. In the non-commutative setting one has the advantage of keeping the dimensions intact thanks to the product with a finite spaces. But before I delve into the subject I need to define some standard notions such as Clifford bundles and Dirac operators.

3.1 Clifford bundles and Dirac operators

3.1.1 Clifford algebras

Let V be a vector space over a field $\mathbb{K}$ ($= \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$), equipped with a quadratic form $Q : V \to \mathbb{K}$.

Definition 3.1.1. A Clifford algebra for V is a unital algebra $Cl(V, Q)$ over a field $\mathbb{K}$ equipped with a map $\varphi : V \to Cl(V, Q)$ such that $\varphi(v)^2 = Q(v)1$.

In other words, the unital algebra $Cl(V, Q)$ is generated by vectors $v \in V$ subjected to the relation $\varphi(v)^2 = Q(v)1$. Because of this, and the fact that the map φ is injective I will often write v instead of $\varphi(v)$.

Remark. It is easy to show that for a bilinear form $g_Q(u, v) : V \times V \to \mathbb{K}$ associated to the quadratic form Q one has

$$uv + vu = 2g_Q(u, v)\,,$$

where

$$g_Q(u, v) = \frac{1}{2}\big(Q(u + v) - Q(u) - Q(v)\big)\,.$$

I will abbreviate g_Q as g and write $Cl(V, g)$ instead of $Cl(V, Q)$.

The existence of this algebra can be seen in two ways:

i) Let $\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$ be the tensor algebra on V. The first option is to consider the quotient of the tensor algebra

$$Cl(V, Q) = \mathcal{T}(V)/\mathrm{Ideal}\langle \mathrm{u \times v + v \times u - 2g(u, v)1 | u, v \in V} \rangle\,.$$

Since the relations are not homogeneous, the $\mathbb{Z}$-grading is lost, and I only have a $\mathbb{Z}_2$-grading, with the grading χ given by

$$\chi(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k \, .$$

Hence one have $Cl(V,g) = Cl^0(V,g) \oplus Cl^1(V,g)$ for the odd and even parts.

ii) The other option is to define $Cl(V,g)$ as a subalgebra of $\mathrm{End}_{\mathbb{R}}(\wedge^\bullet V)$ generated by all expressions $c(v) = \varepsilon(v) + \iota(v)$ for $v \in V$, where

$$\varepsilon(v) : v_1 \wedge \cdots \wedge v_k \mapsto v \wedge v_1 \wedge \cdots \wedge v_k \, ,$$

and

$$\iota(v) : v_1 \wedge \cdots \wedge v_k \mapsto \sum_{j=1}^{k} (-1)^{j-1} g(v, v_i) v_1 \wedge \cdots \wedge \hat{v_i} \wedge \cdots \wedge v_k \, .$$

Note that for all $u, v \in V$ we have $\varepsilon(v)^2 = 0 = \iota(v)^2$ and $\varepsilon(u)\iota(v) + \iota(v)\varepsilon(u) = g(u,v)1$. Thus

$$c(v)^2 = g(v,v)1 \quad ; c(u)c(v) + c(v)c(u) = 2g(u,v)1 \, .$$

From this it is clear that $Cl(V,g)$ is isomorphic to $\wedge^\bullet V$ as a vector space, however not as algebras. Hence, $\dim Cl(V,g) = \dim \wedge^\bullet V = 2^{\dim V}$.

Lemma 3.1.1. *Any $\mathbb{K}$-linear map $f : V \to A$ that satisfies*

$$f(v)^2 = g(v,v)1_A \, , \quad \forall v \in V \, ,$$

extends to a unique unital $\mathbb{K}$-algebra homomorphism $\tilde{f} : Cl(V,g) \to A$.

Proof. There is really nothing to prove. Since $\tilde{f}(v_1, \ldots, v_r) := f(v_1) \cdots f(v_r)$ gives the uniqueness, provided only that this formula is well-defined. But observe that $\tilde{f}(uv + vu - 2g(u,v)1) = \tilde{f}((u + v)^2 - u^2 - v^2 - 2g(u,v)1) = 0$. □

I will denote the Clifford algebras of vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with the standard metric δ by:

$$Cl_n^+ := Cl(\mathbb{R}^n, \delta) \, ;$$

$$Cl_n^- := Cl(\mathbb{R}^n, -\delta) \, ;$$

$$\mathbb{C}l_n := Cl(\mathbb{C}^n) \, .$$

The Clifford algebra $\mathbb{C}l_n$ is the complexificaiton of both $Cl_n^\pm$.

Definition 3.1.2. The chirality operator γ_{n+1} in $\mathbb{C}l_n$ is defined as

$$\gamma_{n+1} := (-1)^m e_1 \cdots e_n \, ,$$

where $n = 2m$ or $n = 2m + 1$.

By performing a slightly tedious calculation one can find that $\gamma_{n+1} = \gamma_{n+1}^*$, and $\gamma_{n+1}\gamma_{n+1}^* = \gamma_{n+1}^2 = 1$.

Remark. i) If $n = 2m$ is even, then γ_{n+1} is the grading operator. ii) If $n = 2m + 1$ is odd, then γ_{n+1} lies in the odd part, $\mathbb{C}l_n^1$, and the center of $\mathbb{C}l_n$ is generated by 1 and γ_{n+1}.

Proposition 3.1.2. *The even part, $(Cl_{n+1}^-)^0$, is isomorphic to Cl_n^-.*

Proof. Define a map $\Psi : Cl_n^- \to Cl_{n+1}^-$ defined on generators by

$$\Psi(e_i) = e_{n+1}e_i\,.$$

Indeed, for $i, j = 1, \ldots, n$ one has

$$\Psi(e_i)\Psi(e_j) + \Psi(e_j)\Psi(e_i) = -2\delta_{ij} = \Psi(-2\delta_{ij})\,.$$

Thus, Ψ is a homomorphism. Moreover, since it sends generators of one algebra to the generators of the other algebra, and the dimensions of algebras coincide, therefore it is an isomorphism. □

Proposition 3.1.3. *For any $k \geq 1$ one has*

$$Cl_k^+ \otimes_{\mathbb{R}} Cl_2^- \simeq Cl_{k+2}^-\,,$$

$$Cl_k^- \otimes_{\mathbb{R}} Cl_2^+ \simeq Cl_{k+2}^+\,.$$

Proof. The map $\Psi : Cl_{k+2}^- \to Cl_k^+ \otimes_{\mathbb{R}} Cl_2^-$ defined on generators

$$\Psi(e_i) = \left\{ \begin{array}{ll} 1 \otimes e_i & i = 1, 2 \\ e_i \otimes e_1e_2 & i = 3, \ldots, n \end{array} \right\},$$

extends to the desired isomorphism. □

Proposition 3.1.4.

$$Cl_1^+ \simeq \mathbb{R} \oplus \mathbb{R}\,, \quad Cl_1^- \simeq \mathbb{C}\,, \quad Cl_2^+ \simeq M_2(\mathbb{R})\,, \quad Cl_2^- \simeq \mathbb{Q}\,.$$

Proof. Easy. □

From the two proposition one gets the following two relations (known as Bott periodicity[1])

$$Cl_n^+ \otimes_{\mathbb{R}} Cl_4^+ \simeq Cl_{n+4}^{\pm}\,,$$

$$Cl_n^+ \otimes_{\mathbb{R}} Cl_8^+ \simeq Cl_{n+8}^{\pm}\,.$$

For the complex Clifford algebras one has

$$\mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2 \simeq \mathbb{C}l_{n+2}\,,$$

where $\mathbb{C}l_2 \simeq M_2(\mathbb{C})$. From which follows the next proposition.

Proposition 3.1.5. *The irreducible representations of $\mathbb{C}l_n$ are*

$$\begin{array}{ll} \mathbb{C}^{2^m}\,, & (n = 2m)\,; \\ \mathbb{C}^{2^m} \oplus \mathbb{C}^{2^m}\,, & (n = 2m+1)\,. \end{array}$$

Proof. $\mathbb{C}l_{2m} \simeq M_{2^m}(\mathbb{C})$ and $\mathbb{C}l_{2m+1} \simeq M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$. □

Definition 3.1.3. A representation of a Clifford algebra is called a Clifford module.

[1] Originally the Bott periodicity theorem describes periodicity in the homotopy theory of classical groups, this triggered the development of K-theory of stable complex vector bundles.

3.1.2 Clifford Bundles

Let M denote an n-dimensional riemannian manifold, where $n = 2m$ or $n = 2m + 1$.

Definition 3.1.4. The Clifford bundle $Cl^+(T^*M) \to M$ is a fiber bundle whose fiber (at a point $x \in M$) has the structure of Clifford algebra $Cl(T_x^*M, g_x) \simeq Cl_n^+$, and whose local trivializations respect the algebra structure.

Meaning that the action of the transition functions inherited from T^*M, which are given on open $U, V \subset M$ by $t_{UV} : U \cap V \to O(n)$, can be extended fiberwise to Cl_n^+ by

$$\alpha_1 \cdots \alpha_k \mapsto t_{UV}(\alpha_1) \cdots t_{UV}(\alpha_k); \quad (\alpha_1, \ldots, \alpha_k \in T^*M)\,.$$

One can replace the metric g by $-g$ to obtain $Cl^-(T^*M)$.

Finally, the replacement of the fiber $Cl_n^\pm$ by $\mathbb{C}l(T_x^*M) \simeq \mathbb{C}l_n$ in the above definition defines the complex Clifford bundle

$$\mathbb{C}l(T^*M) \simeq Cl^\pm(T^*M) \otimes_\mathbb{R} \mathbb{C}\,.$$

Remark. Note that the definition really means that the Clifford bundle is an associated vector bundle to the orthonormal frame bundle $F(T^*M)$, and the representation $\rho : O(n) \to Aut(Cl_n^+)$:

$$Cl^+(T^*M) = F(T^*M) \times_\rho Cl_n^+\,.$$

From this it is easy to see that Clifford bundles can be easily defined on an arbitrary vector bundle $E \to M$ with a frame bundle $F(E)$ as

$$Cl^+(E) = F(E) \times_\rho Cl^+_{\mathrm{rank}(E)}\,.$$

Definition 3.1.5. A Clifford module bundle $E \to M$ is a vector bundle whose fibers are Clifford modules, i.e. representations of Clifford algebras.

The canonical example is a spinor bundle. In fact, on a spin manifold (defined later), every Clifford module is obtained by twisting the spinor bundle. For now I will be concerned exclusively by complex Clifford module bundles (i.e. their fibers are isomorphic to $\mathbb{C}^{2^m}$, the representations of $\mathbb{C}l_n \simeq M_{2^m}(\mathbb{C})$).

Remark. Sections of s Clifford bundle have a natural action

$$c : \Gamma(\mathbb{C}l(T^*M)) \to \Gamma(\mathrm{End}(E))$$

on the sections of Clifford module bundles, such that

$$(s, c(\kappa)t) = (c(\kappa^*)s, t)\,,$$

where $\kappa \in \Gamma(\mathbb{C}l(T^*M))$ and $(\cdot,\cdot)$ is a hermitian pairing.

Example 3.1.1. Consider a section $s \in M \times \mathbb{C}^{2^m}$, and a one-form $\alpha \in \Omega^1(M)$, then $c(\alpha) \in \Gamma(\mathbb{C}l(T^*M))$. Moreover, the above action is realized by pointwise matrix multiplication (i.e. $c(\alpha_x) \in M_{2^m}(\mathbb{C})$ and $s_x \in \mathbb{C}^{2^m}$).

Definition 3.1.6. A Riemannian manifold is called spin^c manifold if there exists a vector bundle $S \to M$ such that there is an algebra bundle isomorphism

$$\mathbb{C}l(T^*M) \simeq \mathrm{End}(S), \quad n = 2m\,;$$

$$\mathbb{C}l(T^*M)^0 \simeq \mathrm{End}(S), \quad n = 2m + 1\,.$$

The pair (M, S) is called a spin^c structure on M.

Note that when $n = 2m$ the grading operator γ_{n+1} introduces $\mathbb{Z}_2$-grading on $S = S^0 \oplus S^1$.

Definition 3.1.7. A Riemannian spin^c manifold is called spin manifold if there exists an anti-unitary operator $J : \Gamma(S) \to \Gamma(S)$ such that

i) $J(\psi f) = J(\psi) f^*$ for $\psi \in \Gamma(S)$ and $f \in C(M)$;

ii) $J(c(\kappa)\psi) = c(\kappa^*) J(\psi)$ for $\psi \in \Gamma(S)$ and $\kappa \in \Gamma(\mathbb{C}l(T^*M))$;

iii) $J^2 = \pm 1$ whenever M is connected.

In physics, the operator defining spin manifolds is called the charge conjugation.

3.1.3 Connections

Now, I replace sections by smooth sections (with the same notation) and introduce connections on spinor modules. But first, I will recall the notion of a Hermitian connection .

Definition 3.1.8. If a $C(M)$-module $\Gamma(E)$ is equipped with $C(M)$-valued Hermitian pairing, a connection ∇ on $\Gamma(E)$ is called Hermitian connection if

$$(\nabla s, t) + (s, \nabla t) = \mathrm{d}(s,t)\,, \quad s, t \in \Gamma(E)\,.$$

Definition 3.1.9. On a spinor module $\Gamma(S)$, a spin^c connection[2] is any Hermitian connection

$$\nabla^s : \Gamma(S) \to \Omega^1(M) \otimes \Gamma(S)\,,$$

which is compatible with the action of the Clifford bundle $\mathbb{C}l(T^*M)$ in the following way

$$\nabla^S(c(\alpha)s) = c(\nabla\alpha)s + c(\alpha)\nabla^S(s)\,; \quad \forall \alpha \in \Omega^1(M), \forall s \in \Gamma(S)\,,$$

where ∇ is Levi-Civita connection.

If ∇, ∇' are any two connections, then$\nabla - \nabla'$ is a $C(M)$-module map

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')s\,.$$

In the case of spin^c connection this means that locally, over $U \subset M$ for which $S|_U \to U$ is trivial, one can write

$$\nabla^S = \mathrm{d} + \omega\,,$$

where $\omega \in \Omega^1(\mathrm{End}(S|_U))$.

To understand this connection better I need to introduce an isomorphism of Lie algebras $\dot{\mu} : so(T_x^*M) \to spin(T_x^*M)$ defined by[3]

$$\dot{\mu}(A) = \frac{1}{4}\sum_{i,j=1}^{n} g(\mathrm{d}x^i, A\mathrm{d}x^j) c(\mathrm{d}x^i) c(\partial_j)\,.$$

It is the inverse of the adjoint map[4] $ad : spin(T_x^*M) \to so(T_x^*M)$ sending $B \mapsto [B, \cdot]$. Hence, for Christoffel symbols $\Gamma \in \Omega^1(\mathrm{End}(T^*M))$[5] one can write

$$\dot{\mu}(\Gamma) = \frac{1}{4}\Gamma^j_{\bullet i} c(\mathrm{d}x^i) c(\partial_j)\,,$$

and since $\nabla^{LC} = \mathrm{d} - \Gamma$ one has $\omega = -\frac{1}{4}\Gamma^j_{\bullet i} c(\mathrm{d}x^i) c(\partial_j)$. Putting all this together yields a local expression for the connection on $\Gamma(S)$

$$\nabla := \mathrm{d} - \frac{1}{4}\Gamma^j_{\bullet i} c(\mathrm{d}x^i) c(\partial_j)\,.$$

[2]This is just a lift of Levi-Civita connection to the spinor bundle, it will become clearer in local coordinates.
[3]Here I am slightly abusing the notation: $c(\partial_j) = g_{jk} c(\mathrm{d}x^k)$, where g_{jk} is the metric.
[4]Just plug it in and brute-force.
[5]$so(T^*M) \simeq \mathrm{End}(T^*M)$

Remark. In physics the operators $c(\mathrm{d}x^i)$ and $c(\partial_j)$ are called gamma matrices γ^i and γ_j, respectively.

Proposition 3.1.6. *If (S, J) is the data for the spin structure on M, then there is a unique Hermitian spin^c connection $\nabla^S \Gamma(S) \to \Omega^1(M) \otimes \Gamma(S)$ such that*

$$\nabla^S_X J = J \nabla^S_X \,,$$

where X is a real vector field. This is called a spin connection.

Proof. It follows from the fact that $\dot{\mu}(\Gamma)$ is skew-adjoint and it commutes with J. $\square$

For any connection ∇ on a module $\Gamma(E)$ one has $\nabla^2(fs) = f\nabla^2 s$ for $f \in C(M)$ and $s \in \Gamma(E)$. Thus, $\nabla^2 s = Rs$ for a certain 2-form

$$R \in \Omega^2(\mathrm{End}(E)) \,,$$

R is called the curvature of ∇. For the Levi-Civita connection, a local calculation gives

$$\nabla^2 \alpha = (-\mathrm{d}\Gamma + \Gamma \wedge \Gamma)\alpha \,,$$

which yields the local expression of the Riemann curvature tensor

$$R|_U = (-\mathrm{d}\Gamma + \Gamma \wedge \Gamma) \in \Omega^2(so(T^*U)) \,.$$

Likewise, the curvature R^S of a spin connection ∇^S is locally given by

$$R^S = -\mathrm{d}\dot{\mu}(\Gamma) + \dot{\mu}(\Gamma) \wedge \dot{\mu}(\Gamma) \in \Omega^2(\mathrm{End}(S)) \,.$$

3.1.4 Dirac operators

Definition 3.1.10. The Dirac operator is defined as

$$\not{D} := -i\hat{c} \circ \nabla^S \,,$$

where $\hat{c} : \kappa \otimes \psi \mapsto c(\kappa)\psi$ for $\psi \in \Gamma(S)$ and $\kappa \in \Gamma(\mathbb{C}l(T^*M))$.

Remark. The definition of the Dirac operator varies, in some texts it is defined up to a constant factor, i.e.

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{g} \Gamma(TM \otimes S) \xrightarrow{c} \Gamma(S) \,.$$

I will stick with the physicist's notation from the definition.

When using local coordinates, one obtains the following formula

$$\not{D}\psi = -ic(\mathrm{d}x^j)\nabla^S_j \psi = -i\gamma^j \nabla^S_j \psi \,.$$

From this one can easily see that in the case $n = 2m$, the Dirac operator is odd with respect to the $\mathbb{Z}_2$-grading of S.

Lemma 3.1.7. *The adjoint of the connection operator, $\nabla^* : \Gamma(T^*M \otimes S) \to \Gamma(S)$, has the following form*

$$\nabla^* : \alpha \otimes s \mapsto -\operatorname{Tr}\big(\nabla(\alpha \otimes s)\big) \,.$$

Proof. Denote $\alpha \otimes s = A$; locally one gets $\nabla^*(A) = -\sum_i (\nabla_{e_i} A)(e_i)$. I claim that

$$\big(\nabla s, A\big) - \big(s, \nabla^* A\big) = \mathrm{d}^*(\omega) \,,$$

where $\omega = \big(s, A(\cdot)\big)$ is a 1-form, and $\mathrm{d}^*(\alpha) = \mathrm{tr}(\nabla\alpha) = \mathrm{div}\big(\alpha, \cdot\big)$. This is easily shown locally:

$$\begin{aligned}
\mathrm{d}^*\big(s, A(\cdot)\big) &= \sum_i \Big(\nabla_{e_i}\big(s, A\big)\Big)(e_i) \\
&= \sum_i \big(\nabla_{e_i}s, A(e_i)\big) + \sum_i \big(s, (\nabla_{e_i}A)(e_i)\big) \\
&= \big(\nabla s, A\big) - \big(s, \nabla^* A\big)\,.
\end{aligned}$$

Integrating over the manifold and using the Stokes theorem yields the result. □

I need one more definition before everything is ready for Lichnerowicz–Weitzenböck's formula.

Definition 3.1.11. Let e_i be a local orthonormal frame for TM, and let $c : TM \to \mathrm{End}(S)$ be the Clifford action. Then for an $\mathrm{End}(S)$-valued 2-form $K \in \Omega^2(\mathrm{End}(S)$ one can define Clifford contraction of K as follows

$$\mathcal{K} = \sum_{i<j} c(e_i)c(e_j)K(e_i, e_j) \quad \big(= \sum_{i \neq j} \frac{1}{2} c(e_i)c(e_j)K(e_i, e_j)\big)\,.$$

Note that the Clifford contraction belongs to $\mathrm{End}(S)$

Theorem 3.1.8 (Lichnerowicz–Weitzenböck's formula)**.** *The square of the Dirac operator is given by the following formula*

$$\not{D}^2 = \nabla^*\nabla - \mathcal{K}\,,$$

where $\mathcal{K}$ is the Clifford contraction of the curvature of ∇.

Proof. This is easily done in local coordinates

$$\begin{aligned}
\not{D}^2 s &= -\sum_{ij} c(e_j)\nabla_{e_j}(c(e_i)\nabla_{e_i}s) \\
&= -\sum_{ij} c(e_j)c(e_i)\nabla_{e_j}\nabla_{e_i}s \\
&= -\sum_i \nabla^2_{e_i}s - \sum_{j<i} c(e_j)c(e_i)[\nabla_{e_j}, \nabla_{e_i}]s\,,
\end{aligned}$$

where in the second equality I used the property of synchronous orthonormal frames. In the last line, the first term is clearly equal to $-tr(\nabla(\nabla s))$; the second term is the Clifford contraction of the curvature $K(e_i, e_j) = [\nabla_{e_i}, \nabla_{e_j}]$ on S. □

Theorem 3.1.9 (Bochner's theorem)**.** *If the least eigenvalue of $\mathcal{K}$ at each point of a compact manifold M is strictly positive, then there are no non-zero solutions of the equation $\not{D}^2 s = 0$; i.e.* $\ker \not{D} = \{0\}$.

Proof. Since M is compact I end up with $\langle\mathcal{K}s, s\rangle \geq c||s||^2$, for some $c > 0$. But from the previous theorem one has

$$\langle\mathcal{K}s, s\rangle = \langle\not{D}^2 s, s\rangle - ||\nabla s||^2 \leq 0\,.$$

□

Theorem 3.1.10. *Let (M, g) be a compact Riemannian spin manifold without boundary. The Dirac operator $\not{D}$ is essentially self-adjoint on $\Gamma(S)$. Moreover, it has a compact resolvent $(\iota + \not{D})^{-1}$ and*

$$\begin{aligned}
\big[\not{D}, f\big] &= -\iota c(\mathrm{d}f) \\
&= \|f\|_{Lip}\,,
\end{aligned}$$

where $\|\cdot\|_{Lip}$ is a Lipschitz semi-norm.

Proof. I will only show that it is symmetric, e.g. $\langle\psi, \not{D}\phi\rangle = \langle\not{D}\psi, \phi\rangle$. The rest of the proof can be found in [Connes, 1994].

$$\begin{aligned} i(\psi, \not{D}\phi) - i(\not{D}\psi, \phi) &= (\psi, \gamma^i \nabla_i^S \phi) + (\gamma^i \nabla_i^S \psi, \phi) \\ &= (\psi, \nabla_i^S \gamma^i \phi) - (\psi, c(\nabla_i \mathrm{d}x^i)\phi) + (\nabla_i^S \psi, \gamma^i \phi) \\ &= \partial_i(\psi, \gamma^i \phi) - (\psi, c(\nabla_i \mathrm{d}x^i)\phi) \end{aligned}$$

Now, a map $\alpha \mapsto (\psi, c(\alpha)\phi)$ taking one-forms to functions is a vector field $Z_{\psi\phi}$. Hence, the right-hand side becomes

$$\partial_i(\mathrm{d}x^i(Z_{\psi\phi})) - (\nabla_i \mathrm{d}x^i)(Z_{\psi\phi}) = \mathrm{d}x^i(\nabla_i Z_{\psi\phi}) = \mathrm{div} Z_{\psi\phi}\,.$$

The Lipschitz semi-norm is defined as

$$\|f\|_{Lip} = \sup_{x\neq y}\left\{\frac{f(x)-f(y)}{\|x-y\|}\right\}.$$

□

Example 3.1.2. The complexified exterior algebra $\wedge^\bullet T^*M^{\mathbb{C}}$ is a natural $\mathbb{C}l(T^*M)$-bimodule. And its Dirac operator is

$$\begin{aligned} \not{D}\kappa &= c(\mathrm{d}x^i)\nabla_i\kappa \\ &= \varepsilon(\mathrm{d}x^i)\nabla_i\kappa + \iota(\mathrm{d}x^i)\nabla_i\kappa \\ &= (\mathrm{d}+\delta)\kappa\,. \end{aligned}$$

This example gives the motivation behind definition of Dirac operator as a square root of the Laplacian operator, i.e. $\not{D}^2 = (\mathrm{d}+\delta)^2$.

3.1.5 Curvature

Lemma 3.1.11. *Let K be the curvature of ∇^S, and let $c : TM \to \mathrm{End}(S)$ be the Clifford action. Then for any $X, Y, Z \in TM$ one has*

$$[K(X,Y), c(Z)] = c(R(X,Y)Z)\,,$$

where R is the Riemann curvature tensor of the manifold M.

Proof. I will perform the computation in a synchronous orthonormal frame. The connection compatibility yields

$$\begin{aligned} [\nabla^S_{e_i}, \nabla^S_{e_j}](c(e_k)s) &= c([\nabla_{e_i}, \nabla_{e_j}]e_k)s + c(e_k)[\nabla^S_{e_i}, \nabla^S_{e_j}]s \\ K(e_i,e_j)(c(e_k)\cdot s) &= c(R(e_i,e_j)e_k)s + c(e_k)K(e_i,e_j)s\,, \end{aligned}$$

from where the result is immediate. The cross-term vanishes due to the synchronous frame. □

Definition 3.1.12. A Riemann endomorphism, R^S, for a Clifford bundle S is defined to be a $\mathrm{End}(S)$-valued 2-form

$$R^S(X,Y) = \frac{1}{4}\sum_{k,l} c(e_k)c(e_l)\big(R(X,Y)e_k, e_l\big)\,.$$

Remark. Do not confuse this with $\mathcal{K} = \sum_{i<j} c(e_i)c(e_j)K(e_i,e_j)$.

Lemma 3.1.12. *Let $c : TM \to \mathrm{End}(S)$ be the Clifford action. Then for any $X, Y, Z \in TM$*

$$[R^S(X,Y), c(Z)] = c(R(X,Y)Z)\,.$$

Proof. This is just a long and messy calculation. Take $Z = e_k$ plug in the definition of R^S and brute-force.

$$\begin{aligned}
[R^S(X,Y), c(e_k)] &= \frac{1}{4}\sum_{i,j}\big(R(X,Y)e_i, e_j\big)[c(e_i)c(e_j), c(e_k)] \\
&= \frac{1}{2}\sum_{i,j}\big(R(X,Y)e_i, e_j\big)(\delta_{ik}c(e_j) - \delta_{jk}c(e_i)) \\
&= \frac{1}{2}\sum_{j}\Big(\big(R(X,Y)e_k, e_j\big)c(e_j) - \big(R(X,Y)e_j, e_k\big)c(e_j)\Big) \\
&= c\Big(\sum_{j}\big(R(X,Y)e_k, e_j\big)e_j\Big) \\
&= c(R(X,Y)e_k)\,.
\end{aligned}$$

In the second equality I have used $[c(e_i), c(e_k)] = -2\delta_{ik}$, and the third one follows from symmetries of Riemann tensor. □

This just proved the following proposition.

Proposition 3.1.13. *The curvature 2-form K of a Clifford bundle S can be written as*

$$K = R^S + F^S$$

where R^S is the Riemann endomorphism; F^S is called twisting curvature *of S, and commutes with the Clifford algebra action.*

Proposition 3.1.14. *Let $\not{D}$ be the Dirac operator associated to the Clifford bundle S. Then*

$$\not{D}^2 = \nabla^*\nabla - \mathcal{F}^S + \frac{1}{4}\kappa\,,$$

where $\mathcal{F}^S = \sum_{i<j} c(e_i)c(e_j)F^S(e_i, e_j)$ is the Clifford contraction of F^S, and κ is the scalar curvature of the manifold M.

Proof.

$$\begin{aligned}
\sum_{i<j} c(e_i)c(e_j)R^S(e_i, e_j) &= \frac{1}{4}\sum_{i<j,k,l}\big(R(e_i, e_j)e_k, e_l\big)c(e_i)c(e_j)c(e_k)c(e_l) \\
&= \frac{1}{8}\sum_{ijkl}\big(R(e_i, e_j)e_k, e_l\big)c(e_i)c(e_j)c(e_k)c(e_l)\,,
\end{aligned}$$

where in the second equality $R(e_i, e_j) = -R(e_j, e_i)$ was used. One can now skew-symmetrize the first three Clifford elements to obtain

$$c(e_i)c(e_j)c(e_k) = c(e_i \wedge e_j \wedge e_k) + \delta_{jk}c(e_i) - \delta_{ik}c(e_j) + \delta_{ij}c(e_k)\,.$$

Bianchi identity kills the first term, the relation $R(e_i, e_j) = -R(e_j, e_i)$ kills the last term. The other two contribute equally due to symmetries of R. Thus one ends up with

$$\begin{aligned}
\sum_{i<j} c(e_i)c(e_j)R^S(e_i, e_j) &= \frac{1}{4}\sum_{ijl}\big(R(e_i, e_j)e_j, e_l\big)c(e_i)c(e_l) \\
&= -\frac{1}{4}\sum_{i} Ric(e_i, e_i) \\
&= -\frac{1}{4}\kappa\,,
\end{aligned}$$

where the penultimate equality was obtained from the fact that $\sum_j \big(R(e_i, e_j)e_j, e_l\big) = Ric(e_i, e_l)$. □

3.2 Spectral triples

3.2.1 Defining spectral triples

As mentioned, spectral triples will play a crucial part in this thesis. They will be defined shortly but before doing so, one needs some basic definitions of C^*-algebras, and their properties. For more details I encourage the reader to consult [Gracia-Bondia et al., 2013].

Definition 3.2.1. A C^*-algebra, A, is a Banach algebra over $\mathbb{C}$, together with map $* : A \to A$ such that

$$\begin{aligned} (a^*)^* &= a\,, \\ (a+b)^* &= a^* + b^*\,, \\ (ab)^* &= b^* a^*\,, \\ (\lambda a)^* &= \lambda^* a^* \\ \|a^* a\| &= \|a\|\,\|a^*\| \end{aligned}$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

Definition 3.2.2. The structure space $\hat{A}$ of a C^*-algebra A is the set of all unitary equivalence classes of irreducible representations of A.

Theorem 3.2.1. *Let A be a commutative C^*-algebra, then the Gelfand transformation, defined by $A \to C_0(\hat{A})$ such that $\hat{a}(\phi) := \phi(a)$, is an isometric $*$-isomorphism.*

Remark. This theorem is often referred to as Gelfand duality and basically says that any Hausdorff, topological space X is isomorphic to the structure space of the algebra of functions vanishing at infinity, i.e. $X \simeq \widehat{C_0(X)}$.

Note. A proof of the theorem can be found e.g. in [Blackadar, 2006] (Theorem II.2.2.4).

Proposition 3.2.2. *Let M be a Riemannian spin^c-manifold with Driac operator $\not{D}$. One can define the distance between points in $\widehat{C(M)} \equiv M$ as follows*

$$d(x,y) = \sup_{f \in C^\infty(M)} \left\{ \|f(x) - f(y)\| : \left\| [\not{D}, f] \right\| \leq 1 \right\} .$$

This distance coincides with the Riemannian distance d_g.

Proof. The relation $\|f\|_{Lip} \leq 1$, c.f. theorem 3.1.10, ensures that $d \leq d_g$. For the other inequality fix $y \in M$, and take $f_y(z) = d_g(z,y)$. From this follows that $\|f_y\|_{Lip} \leq 1$, and that $d(x,y) \geq \|f_y(x) - f_y(y)\| = d_g(x,y)$. □

Definition 3.2.3. A spectral triple $(A, \mathcal{H}, \not{D})$ consists of:

- a Hilbert space $\mathcal{H}$;
- an involutive algebra A equipped with a faithful representation on $\mathcal{H}$;
- a self-adjoint operator $\not{D}$ on $\mathcal{H}$, with dense domain, such that for all $a \in A$:
  - $a(\mathrm{Dom}\,\not{D}) \subset \mathrm{Dom}\,\not{D}$;
  - the operator $[\not{D}, a]$ defined initially on $\mathrm{Dom}\,\not{D}$, extends to a bounded operator on $\mathcal{H}$;
  - $\not{D}$ has compact resolvent, i.e. $(\not{D} - \lambda)^{-1}$ is compact, when $\lambda \notin \mathrm{sp}\,\not{D}$.

Definition 3.2.4. A spectral triple $(A, \mathcal{H}, \not{D})$ is called real spectral triple if there exist an anti-unitary operator $J : \mathcal{H} \to \mathcal{H}$ such that

$$\begin{aligned} J(\mathrm{Dom}\,\not{D}) &\subset \mathrm{Dom}\,\not{D}\,, \\ \left[a, J b^* J^{-1}\right] &= 0\,, \quad \forall a, b \in A\,. \end{aligned}$$

Definition 3.2.5. A spectral triple is called even spectral triple if there exist a self-adjoint, unitary, $\mathbb{Z}_2$-grading operator γ on $\mathcal{H}$ such that

$$\begin{aligned} \gamma a &= a\gamma\,, \quad \forall a \in A \\ \gamma(\mathrm{Dom}\, D\!\!\!/) &= \mathrm{Dom}\, D\!\!\!/\,, \\ \gamma D\!\!\!/ &= -D\!\!\!/\gamma\,. \end{aligned}$$

Remark. The condition $\left[a, Jb^*J^{-1}\right] = 0$ in the definition of real spectral triple is usually referred to as order zero condition and means that the left and right actions of A on $\mathcal{H}$ commute. The right action is represented as

$$\psi\bar{b} = Jb^*J^{-1}\psi\,, \quad \forall\psi \in \mathcal{H}\,, \forall b \in A\,.$$

The right action will be discussed in more detail in the definition 3.2.14 and the text following it. Usually, for a real spectral triple an additional condition is required, namely that

$$\left[\left[D\!\!\!/, a\right], JbJ^{-1}\right]\,.$$

This is so-called order one condition.

The reality and evenness of spectral triples introduces freedom in choosing signs, represented by ε's, in the following equations

$$\begin{aligned} J^2 &= \varepsilon'\,, \\ JD\!\!\!/ &= \varepsilon'' D\!\!\!/ J\,, \\ J\gamma &= \varepsilon'''\gamma J\,. \end{aligned}$$

It turns out that these signs are uniquely determined by the so-called KO-dimension n of the underlying space[6], and are closely linked to the Bott periodicity ($\mathbb{Z}/\mathbb{Z}_8$) and K-theory.

n	0	1	2	3	4	5	6	7
ε'	1	1	-1	-1	-1	-1	1	1
ε''	1	-1	1	1	1	-1	1	1
ε'''	1		-1		1		-1	

Table: Signs linking KO-dimension and $J^2 = \varepsilon'$, $JD\!\!\!/ = \varepsilon'' D\!\!\!/ J$, $J\gamma = \varepsilon'''\gamma J$.

Definition 3.2.6. The spectral triples $(A_1, \mathcal{H}_1, D\!\!\!/_1)$ and $(A_2, \mathcal{H}_2, D\!\!\!/_2)$ are said to be unitary equivalent if $A_1 = A_2$ and if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a)\,, \quad \forall a \in A\,, \\ UD\!\!\!/_1 U^* &= D\!\!\!/_2\,. \end{aligned}$$

For the purpose of this thesis a product of spectral triples will be important. Let $(A_i, \mathcal{H}_i, D\!\!\!/_i; \gamma_i, J_i)\,, i \in \{1, 2\}$ be two real spectral triples. Then one can form a product spectral triple by

$$\begin{aligned} A &= A_1 \otimes A_2\,, \\ \mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2\,, \\ D\!\!\!/ &= D\!\!\!/_1 \otimes 1 + \gamma_1 \otimes D\!\!\!/_2\,, \\ \gamma &= \gamma_1 \otimes \gamma_2\,, \\ J &= J_1 \otimes J_2\,. \end{aligned} \tag{3.1}$$

Especially, note the occurrence of γ_1 in the third line. This is defined so that the cross term in square vanishes due to $\left\{\gamma_1, D\!\!\!/_1\right\} = 0$, i.e. $D\!\!\!/ = D\!\!\!/_1^2 \otimes 1 + \gamma_1 \otimes D\!\!\!/_2^2$.

[6]One can also think about it the other way around, that the KO-dimension depends on the signs.

Definition 3.2.7. The A-bimodule of Connes' differential one-forms is given by

$$\Omega^1_{\not{D}}(A) := \left\{ \sum_i a_i \left[\not{D}, b_i\right] | a_i, b_i \in A \right\} ,$$

and the corresponding derivation $d : A \to \Omega^1_{\not{D}}(A)$ is given by $d = \left[\not{D}, \cdot\right]$.

Remark. Here a caution is needed. The operator d defined in the above definition is not a differential operator. In order to make it a differential operator, one has to factor out so-called junk forms. For more details see [Landi, 2003] section 6.2.

In case of a real spectral triple one can construct another spectral triple from the original one just by restricting A to

$$A_J := \{a \in A | aJ = Ja^*\} \,. \tag{3.2}$$

Proposition 3.2.3. *Let $(A, \mathcal{H}, \not{D}; J)$ be a real spectral triple. Then*

- *A_J defines an involutive commutative complex sub-algebra of the centre of A.*
- *$(A_J, \mathcal{H}, \not{D}; J)$ is a real spectral triple.*
- *$[a, \omega] = 0$ for all $a \in A$ and for all $\omega \in \Omega^1_{\not{D}}$.*

Proof. First, if $a \in A$ then $a^* \in A$ since $Ja^*J^-1 = (JaJ^{-1})^* = a$ yielding the involution requirement. Moreover, for $a \in A_J$ and $b \in A$ one has $[a, b] = \left[Ja^*J^{-1}, b\right] = 0$ giving that A_J is in the centre of A. The next point, saying that all the conditions for a spectral triple are satisfied, holds true automatically since A_J is a sub-algebra of A. The last point follows from the point one, and the order one condition. □

3.2.2 Gauge groups and algebras

In developing the gauge groups and algebras of non-commutative manifold I borrow some of the definitions from [Van Suijlekom, 2015]. The reader is welcomed to consult the mentioned book and references therein.

Definition 3.2.8. A $*$-automorphism of a $*$-algebra, A, is a linear invertible map $\varphi : A \to A$ such that

$$\varphi(ab) = \varphi(a)\varphi(b)\,, \quad \varphi(a^*) = \varphi(a)^*\,, \quad \forall a, b \in A\,.$$

The group of such automorphisms is denoted by $\mathrm{Aut}(A)$.

I will denote the unitary subgroup of A by $U(A) := \{u \in A | uu^* = u^*u = 1\}$.

Definition 3.2.9. An automorphism φ is called inner automorphism if it has the form $\varphi(a) = uau^*$, for some $u \in U(A)$. The group of inner automorphisms of A is denoted by $\mathrm{Inn}(A)$.
The group of outer automorphisms of A are defined as a group quotient $\mathrm{Out}(A) := \mathrm{Aut}(A)/\mathrm{Inn}(A)$.

From this it is easy to see that $\mathrm{Inn}(A) \simeq U(A)/U(Z(A))$, where $Z(A)$ is the centre of A. Hence, one gets two short exact sequences

$$\begin{aligned} 1 \longrightarrow \mathrm{Inn}(A) \longrightarrow \mathrm{Aut}(A) \longrightarrow \mathrm{Out}(A) \longrightarrow 1\,, \\ 1 \longrightarrow U\big(Z(A)\big) \longrightarrow U(A) \longrightarrow \mathrm{Inn}(A) \longrightarrow 1\,. \end{aligned} \tag{3.3}$$

One should also notice that in case of $A = C^\infty(M)$ one has $\mathrm{Aut}(A) \simeq \mathrm{Diff}(A)$. Here, M is a smooth compact manifold, and $\mathrm{Diff}(M)$ is a group of diffeomorphisms of the manifold.

Proposition 3.2.4. *Let $(A, \mathcal{H}, \not{D}; J, \gamma)$ be a spectral triple, $\pi : A \to B(\mathcal{H})$ be a faithful representation of A on $\mathcal{H}$, and let $U = \pi(u)J\pi(u)J^{-1}$, for $u \in U(A)$. Then the inner automorphism $\varphi_u \in \mathrm{Inn}(A)$ induces a unitary equivalent spectral triple $(A, \mathcal{H}, U\not{D}U^*; J, \gamma)$ with the Dirac operator*

$$U\not{D}U^* = \not{D} + u\left[\not{D}, u^*\right] + \varepsilon'' Ju\left[\not{D}, u^*\right]J^{-1}\,. \tag{3.4}$$

Proof. This is quite straightforward, just use order-zero condition from the definition 3.2.4 and the table with signs above it to show that $U\pi(a)U^* = \pi \circ \varphi_u(a)$, $U\gamma U^* = \gamma$, and that $UJU^* = J$. The Dirac operator is readily checked using order-zero and order-one conditions. □

Definition 3.2.10. The gauge group of a spectral triple $(A, \mathcal{H}, \not{D}; J)$ is defined as

$$G(A, \mathcal{H}; J) := \left\{ U = uJuJ^{-1} | u \in U(A) \right\} .$$

Proposition 3.2.5. *There is a short exact sequence of groups*

$$1 \longrightarrow U(A_J) \longrightarrow U(A) \longrightarrow G(A, \mathcal{H}; J) \longrightarrow 1 .$$

Moreover, there is a surjective map $G(A, \mathcal{H}; J) \to \mathrm{Inn}(A)$.

Proof. One only needs to check that the map $U(A) \to G(A, \mathcal{H}; J)$ is a group homomorphism. To this end use order-zero condition to show that $uJuJ^{-1}vJvJ^{-1} = uvJuvJ^{-1}$. The rest is trivial. □

Remark. Note that if $U(A_J) \simeq U\big(Z(A)\big)$ then the equation (3.3) yields $G(A, \mathcal{H}; J) \simeq \mathrm{Inn}(A)$.

Definition 3.2.11. The gauge algebra of a spectral triple $(A, \mathcal{H}, \not{D}; J)$ is defined as

$$g(A, \mathcal{H}; J) := \left\{ T = X + JXJ^{-1} | X \in u(A) \right\} ,$$

where $u(A)$ are the derivations of $U(A)$.

Remark. The infinitesimal version of the proposition 3.2.5 takes the form of

$$0 \longrightarrow u(A_J) \longrightarrow u(A) \longrightarrow g(A, \mathcal{H}; J) \longrightarrow 0 .$$

3.2.3 Morita equivalence

Morita equivalence turns out to be of immense importance when deriving equations of motion by perturbing the so-called spectral action, which will be defined later. It will also be shown that Morita equivalence is of interest only when dealing with non-commutative rings, since for commutative ones this equivalence relation coincides with isomorphism.

Definition 3.2.12. Let A and B be associative, not necessarily commutative, algebras. If there is an equivalence of categories between A-modules and B-modules, then A and B are said to be Morita equivalent.

Note. I will denote Morita equivalence by $\overset{\mathrm{M}}{\sim}$, and A-module by A-mod.

Proposition 3.2.6. *Commutative algebras are Morita equivalent if and only if they are isomorphic. I.e. for A, B commutative algebras,* $A \overset{M}{\sim} B \iff A \simeq B$.

Proof. The direction $\Leftarrow$ is obvious. Now, suppose that $A \overset{\mathrm{M}}{\sim} B$, then A-mod $\sim$ B-mod which yields that $Z(A\text{-mod}) \simeq Z(B\text{-mod})$. But for associative algebras one has $Z(A\text{-mod}) \simeq Z(A)$. □

Lemma 3.2.7. *Let A be an algebra and let $n \in \mathbb{N}, n > 0$. Then* $\mathrm{Mat}_n(A) \overset{M}{\sim} A$.

Theorem 3.2.8. *Let A and B be two rings, and let $F : A\text{-mod} \to B\text{-mod}$ be an additive right exact functor. Then there exists a (B, A)-bimodule Q, unique up to isomorphism, such that F is isomorphic to the functor*

$$A\text{-mod} \to B\text{-mod} , \quad M \mapsto Q \otimes_A M .$$

The preceding theorem provides an opportunity for an alternative definition of Morita equivalence. Let A and B be two algebras. Then $A \overset{\mathrm{M}}{\sim} B$ if and only if there exists (A, B)-mod, E, and (B, A)-mod, F, such that

$$E \otimes_B F \simeq A , \quad F \otimes_A E \simeq B ,$$

as A and B bi-modules, respectively.

Note. For the proof of the previous lemma and theorem see [Ginzburg, 2005] Lemma 2.2.4 and Theorem 2.3.1.

Definition 3.2.13. A right A-mod, E, is called finitely generated projective module if there exists an idempotent $p \in \mathrm{Mat}_n(A)$ for some $n \in \mathbb{N}$ such that $E = pA^n$.

Lemma 3.2.9. *A right A-mod, E, is finitely generated projective if and only if*

$$\mathrm{End}_A(E) \simeq E \oplus_A \mathrm{Hom}_A(E, A) \,.$$

Note. For the proof I refer the reader to [Van Suijlekom, 2015], Lemma 6.11.

Proposition 3.2.10. *Let A be a $*$-algebra, and E a finite projective right A-mod. Then there exists a hermitian structure on E, that is to say, there is a pairing $\langle \cdot, \cdot \rangle_E : E \times E \to A$ on E.*

Proof. Is a direct consequence of the definition 3.2.13, since there is a hermitian structure on A^n for $n \in \mathbb{N}$, which is preserved by idempotents. □

Theorem 3.2.11. *Let $(A, \mathcal{H}, \not{D})$ be a spectral triple, and let E be a finite projective right A-module; denote $B = \mathrm{End}_A(E)$ and $\mathcal{H}' := E \otimes_A \mathcal{H}$. Then $(B, \mathcal{H}', \not{D}')$ is a spectral triple with the Dirac operator being*

$$\not{D}' = \nabla \otimes 1 + 1 \otimes \not{D} \,,$$

where $\nabla : E \to E \otimes_A \Omega^1_{\not{D}}$ is a hermitian connection.

Proof. First I need to show that $\mathcal{H}'$ is a Hilbert space; this follows directly from the definition 3.2.13, i.e. $\mathcal{H}' \simeq pA^n \otimes_A \mathcal{H} \simeq p\mathcal{H}^n$ for $n \in \mathbb{N}$ and p an idempotent. Moreover, the Hilbert space $\mathcal{H}'$ carries a natural action of B

$$\phi(\eta \otimes_A \psi) = \phi(\eta) \otimes_A \psi \,, \quad \phi \in B, \eta \in E, \psi \in \mathcal{H} \,.$$

The Dirac operator $\not{D}'$ properly generalizes to the ideal, $\eta a \otimes_A \psi - \eta \otimes_A a\psi = 0$, defined by the tensor product over A:

$$\begin{aligned}
\not{D}'(\eta a \otimes_A \psi - \eta \otimes_A a\psi) =& + \big((\nabla \eta) a + \eta \otimes_A \mathrm{d}(a)\big) \otimes_A \psi + \eta a \otimes_A \not{D}(\psi) \\
& - (\nabla \eta) \otimes_A a\psi - \eta \otimes_A \big(\mathrm{d}(a)\psi - a\not{D}(\psi)\big) \\
=& 0 \,.
\end{aligned}$$

The action of B is clearly bounded.
Now I will show the compactness of the resolvent of $\not{D}'$. Any connection can be written as

$$\nabla = p \circ [\not{D}, \cdot] + \omega \,,$$

where $\omega : E \to E \otimes_A \Omega^1_{\not{D}}$, and p an idempotent. This shows that the new Dirac operator $\not{D}'$ can be written as $p\not{D}p + \omega$. Thanks to hermiticity of ∇, ω acts as a bounded, self-adjoint operator. Every self-adjoint operator T satisfies

$$(\iota + T + \omega)^{-1} = (\iota + T)^{-1}\big(1 - \omega(\iota + T + \omega)^{-1}\big) \,.$$

When substituting $T = p\not{D}p$, and realizing that p acts as identity on $\mathcal{H}'$; I only need to show that $(\iota + T)^{-1} = (\iota p + p\not{D}p)^{-1}$ is compact. To this end, take the identity

$$\Big(\iota p + p\not{D}p\Big)\Big(p(\iota + \not{D})^{-1}p\Big) = p\left[\iota + \not{D}, p\right](\iota + \not{D})^{-1}p + p \,,$$

multiply from the left by $(\iota p + p\not{D}p)^{-1}$, and reorder to get

$$(\iota p + p\not{D}p)^{-1} = p(\iota + \not{D})^{-1}p - (\iota p + p\not{D}p)^{-1}p\left[\not{D}, p\right](\iota\not{D})^{-1}p \,,$$

which shows the compactness of $(\iota p + p\not{D}p)^{-1}$ because $(\iota + \not{D})^{-1}$ is compact by definition. □

Definition 3.2.14. Let E be a (B, A)-bimodule. The conjugate module E° is given by the (A, B)-bimodule

$$E^\circ := \{\bar{e} = e \in E\} ,$$

with $a \cdot \bar{e} \cdot b = \overline{b^* \cdot e \cdot a^*}$ for all $a \in A$ and $b \in B$.

I would like to generalize the previous theorem to the case when

$$\mathcal{H}' = E \otimes_A \mathcal{H} \otimes_A E^\circ ,$$

and $\phi \in \big(B = \mathrm{End}_A(E)\big)$ acts as follows

$$\phi(\eta \otimes_A \psi \otimes_A \bar{\xi}) = \psi(\eta) \otimes_A \psi \otimes_A \bar{\xi} , \quad \forall \eta, \xi \in E, \psi \in \mathcal{H} .$$

Theorem 3.2.12. *Let $(A, \mathcal{H}, \not{D}; J, \gamma)$ be an even, real, spectral triple, and let E be a finite projective right A-module. Then $(B, \mathcal{H}', \not{D}'; J', \gamma')$ is a spectral triple with the operators defined by*

$$\begin{aligned}
\not{D}' &= \big(\nabla \otimes 1 \otimes 1\big) + \big(1 \otimes \not{D} \otimes 1\big) + \big(1 \otimes 1 \otimes \overline{\nabla}\big) , \\
J' &= 1 \otimes J \otimes 1 , \\
\gamma' &= 1 \otimes \gamma \otimes 1 .
\end{aligned}$$

Proof. Follows directly from the previous theorem. □

In the above theorem, if one chooses $E = A$, from which follows that $B = \mathrm{End}_A(E) = A$. The Grassmann connection for this choice, $\nabla : A \to \Omega^1_{\not{D}}(A)$, can be written as

$$\begin{aligned}
\nabla &= \mathrm{d} + \omega \\
&= \big[\not{D}, \cdot\big] + \sum_i a_i \big[\not{D}, b_i\big] .
\end{aligned}$$

The action of this connection on the identity element yields

$$\omega = \nabla(1) = \sum_i a_i \big[\not{D}, b_i\big] .$$

The right action of $\overline{\nabla} \bar{a}$ on an element of $\mathcal{H}'$ gives

$$\psi \overline{\nabla} \bar{a} = \big(\varepsilon'' J \mathrm{d} a J^{-1} + \varepsilon'' J \omega a J^{-1}\big)\psi , \quad \psi \in \mathcal{H}', a \in A .$$

Since in this case one has $\mathcal{H} \simeq \mathcal{H}'$ the Dirac operator acts as follows

$$\begin{aligned}
\not{D}' \psi &= \not{D}' (1 \otimes \psi \otimes 1) \\
&= \not{D} \psi + \nabla(1)\psi + \psi \overline{\nabla}(\bar{1}) \\
&= \not{D} \psi + \omega \psi + \varepsilon'' J \omega J^{-1} \psi .
\end{aligned}$$

This is often referred to as the inner perturbation of the Dirac operator by Morita self-equivalence and is denoted by

$$\not{D}_\omega := \not{D} + \omega + \varepsilon'' J \omega J^{-1} , \tag{3.5}$$

where $\omega = \omega^* \in \Omega^1_{\not{D}}(A)$ is called a gauge field or an inner fluctuation of the Dirac operator.

Now it is the right time to put together three separate notions we discussed so far, namely the unitary equivalence of spectral triples from the definition 3.2.6, the gauge group of spectral triples from the definition 3.2.10, and the Morita self-equivalence discussed recently.

Proposition 3.2.13. *Let $(A, \mathcal{H}, \not{D}; J)$ be a real spectral triple, and let $U = uJuJ^{-1}$, where $u \in U(A)$ is unitary, i.e. $U \in G(A, \mathcal{H}; J)$ is a gauge group element. Then the unitary equivalence of the spectral triple by the gauge group element is a special case of Morita self-equivalence in which $\omega = u\big[\not{D}, u^*\big]$.*

Proof. Follows directly from plugging $\omega = u\big[\not{D}, u^*\big]$ into the equation (3.4). □

3.2.4 C*-algebra bundle

It was shown in the proposition 3.2.3, that the sub-algebra A_J defined by the equation (3.2) is in the centre of A, hence commutative. The Gelfand duality theorem 3.2.1 says that there exists a Hausdorff space X such that $A_J \subset C_0(X)$. This space can be interpreted as a background space on which the spectral triple $(A, \mathcal{H}, \not{D}; J)$ describes a non-commutative gauge theory.

Having said this, it follows that the gauge group $G(A, \mathcal{H}; J)$ can be interpreted as a group of vertical or purely non-commutative transformations, hence the name gauge group. In this subsection I would like to outline this structure in a more mathematically rigorous way; hopefully, this will shed some light on the construction of non-commutative gauge theories described later in this thesis. I will mostly follow [Van Suijlekom, 2015] and I urge the reader to consult the book and references therein for more details on the subject.

Definition 3.2.15. An upper semi-continuous C^*-algebra bundle B over a compact topological space X, $\pi : B \to X$, such that each fibre $B_x = \pi^{-1}(x)$ is a C^*-algebra, and the map $a \mapsto \|a\|$ is upper semi-continuous, all the other algebraic operations on B are continuous.

Proposition 3.2.14. *The space $\Gamma(X, B)$ of continuous sections equipped with the norm*

$$\|s\| := \sup_{x \in X} \|s(x)\|_{B_x}$$

forms a C^*-algebra.

In order to construct the C^*-algebra bundle B in the case of interest, one has to first identify $C(X)$ with the completion of A_J. Then define the ideal of A as follows

$$I_x := \{fa | a \in A, f \in C(X), f(x) = 0\}^- .$$

Theorem 3.2.15. *Let I_x be as above and let $B_x := A/I_x$. Then $B := \amalg_{x \in X} B_x$ defines an upper semi-continuous C^*-algebra bundle over X. Moreover, there is a $C(X)$-linear isomorphism of A onto $\Gamma(X, B)$.*

Note. Proofs of the last proposition and theorem can be found in [Kirchberg and Wassermann, 1995].

Now, if one denotes the closure of A as $\overline{A}$ then the closure of a gauge group $G(A, \mathcal{H}; J)$ is the continuous gauge group

$$G(\overline{A}, \mathcal{H}; J) \simeq \frac{U(\overline{A})}{U(\overline{A}_J)} . \tag{3.6}$$

Proposition 3.2.16. *The action of $\alpha \in G(\overline{A}, \mathcal{H}; J)$ on $\overline{A}$ by inner C^*-algebra automorphisms induces an action $\tilde{\alpha} \in G(\overline{A}, \mathcal{H}; J)$ on B by continuous bundle automorphism; i.e.*

$$\pi\big(\tilde{\alpha}(b)\big) = \pi(b) , \quad \forall b \in B .$$

Remark. It should be clear that by the identification made in the theorem 3.2.15 one also has that the induced action $\tilde{\alpha}$ on $\Gamma(X, B)$ coincides with the action of α on $\overline{A}$.

Note. For more details and the proof of the last proposition I refer to [Van Suijlekom, 2015].

Chapter 4

Gauge theories from almost-commutative manifolds

As already mentioned, idea behind the usage of non-commutative gauge theories in particle physics is very similar to that of Kaluza-Klein theories. However, the advantage of the approach of almost-commutative manifolds resides in the fact that the space tensored with a spin manifold is finite; hence no additional dimensions arise, and there is no need for compactification. The finite space can however have a non-trivial structure. In this thesis the structure will be non-commutativity, however one does not need to restrict himself so much. There already exists a very nice generalization to non-associative finite spaces by [Farnsworth, 2015].

4.1 Almost commutative manifolds

In this section I will outline the general construction of almost-commutative manifolds, show their resemblance to standard gauge theories, and lastly compute the so-called spectral action which will provide a Lagrangian.

To define a canonical spectral triple I need to recall the following facts from spin geometry for an even-dimensional Riemannian spin manifold M

- $C^\infty(M)$ is the algebra of smooth functions on M;
- $L^2(S)$ is the Hilbert space of square integrable sections of the spinor bundle $S \to M$;
- $\not{D}_M$ is the Dirac operator associated to the Levi-Civita connection lifted to the spinor bundle (c.f. the definition 3.1.10);
- J_M is the anti-unitary operator defining the spin structure (c.f. the definition 3.1.7);
- γ_M is the chirality operator on the spinor bundle (c.f. the definition 3.1.2 and the definition 3.1.6).

Definition 4.1.1. The spectral triple $(C^\infty(M), L^2(S), \not{D}_M; J_M, \gamma_M)$, corresponding to an even-dimensional, compact, Riemannian spin manifold M, is called canonical spectral triple.

As one might expect, in order to define almost-commutative manifolds the equation (3.1), defining the product of spectral triples, will become handy.

Definition 4.1.2. The almost-commutative manifold $M \times F$ is defined as a spectral triple

$$\left(C^\infty(M, A_F), L^2\big(S \otimes (M \times \mathcal{H}_F)\big), (\not{D}_M \otimes 1 + \gamma_M \otimes \not{D}_F); J_M \otimes J_F, \gamma_M \otimes \gamma_F\right),$$

corresponding to the tensor product of canonical spectral triple and a finite real spectral triple $(A_F, \mathcal{H}_F, \not{D}_F; J_F, \gamma_F)$.

After identifying $U\big(C^\infty(M, A_F)\big)$ with $C^\infty\big(M, U(A_F)\big)$ one can write the gauge group for almost-commutative manifold as

$$G(M \times F) := \Big\{uJuJ^{-1} | u \in C^\infty\big(M, U(\overline{A}_F)\big)\Big\},$$

and the gauge algebra for almost-commutative manifold as

$$g(M \times F) := \Big\{X + JXJ^{-1} | X \in C^\infty\big(M, u(\overline{A}_F)\big)\Big\}.$$

At this point, it should be clear from the discussion in the subsection 3.2.4 that the gauge group of a canonical spectral triple is trivial. The next proposition reveals more details on the gauge group of almost-commutative manifold.

Proposition 4.1.1. *Let M be simply connected. Then the gauge group of an almost-commutative manifold $M \times F$ is given by $G(M \times F) = C^\infty\big(M, G(F)\big)$, where*

$$G(F) = \frac{U(\overline{A}_F)}{U\big((\overline{A}_F)_{J_F}\big)}$$

is the gauge group of a finite space F.

Remark. The gauge algebra of an almost-commutative manifold $M \times F$ is given in a similar fashion, i.e. $g(M \times F) = C^\infty\big(M, g(F)\big)$.

Proof. The proof follows immediately from the following three parts, namely from the equation (3.6), the identification of $U\big(C^\infty(M, A_F)\big)$ with $C^\infty\big(M, U(A_F)\big)$, and from the fact that the map

$$C^\infty\big(M, U(\overline{A}_F)\big) \to C^\infty\big(M, G(F)\big)$$

is surjective since M is simply connected. □

One should appreciate the beauty of this construction, the gauge group of an almost-commutative manifold is the space of sections of a group bundle $M \times G(F)$ which acts on the elements of the Hilbert space $L^2\big(S \otimes (M \times \mathcal{H}_F)\big)$, more precisely it acts fibre-wise on the second factor of the bundle $S \otimes (M \times \mathcal{H}_F)$. This hints that the full symmetry group of an almost-commutative manifold $M \times F$ is a semi-direct product of inner, also called gauge, automorphisms, and outer automorphisms, i.e. diffeomorphisms of M. Putting this together one obtains the symmetry group of $M \times F$ being

$$G(M \times F) \rtimes \mathrm{Diff}(M).$$

It is possible to look at the space $E := S \otimes (M \times \mathcal{H}_F)$ as a twisted spinor bundle also called gauge-multiplet spinor bundle known from gauge theories.

4.2 Gauge fields from inner fluctuations

From now on, I will be concerned only about the 4-dimensional spin manifolds M.

Lemma 4.2.1. *Let $M \times F$ be an almost-commutative manifold, with M being a 4-dimensional manifold. Then the inner perturbation of a Dirac operator by Morita self-equivalence ω is of the form*

$$\omega = \gamma^\mu \otimes A^\mu + \gamma_M \otimes \phi,$$

where $A_\mu := -\iota a \partial_\mu b$, and $\phi := a\left[\not{D}_F, b\right]$ are self-adjoint operators.

Remark. The terms A_μ and ϕ from the above lemma are in the literature usually called gauge field and scalar field, respectively.

Proof. The definition of the Dirac operator, $\not{D} = \not{D}_M \otimes 1 + \gamma_M \otimes \not{D}_F$, on an almost-commutative manifold consists of two terms, one containing the Dirac operator of a spin manifold, $\not{D}_M$, and the other consisting of the finite Dirac operator $\not{D}_F$. This allows for separation of the inner fluctuations also into two terms. The first term is

$$\gamma^\mu \otimes A^\mu := a\left[\not{D}_M \otimes 1, b\right] = -\imath\gamma^\mu \otimes a\partial_\mu b\,,$$

and the other term takes the form of

$$\gamma_M \otimes \phi := a\left[\gamma_M \otimes \not{D}_F, b\right] = \gamma_M \otimes a\left[\not{D}_F, b\right]\,.$$

Combining these two yields the desired result. Moreover, the self-adjoint property is required since the Dirac operator is (essentially) self-adjoint. □

Proposition 4.2.2. *Let $M \times F$ be an almost-commutative manifold, with M being a 4-dimensional manifold. Then the fluctuated Dirac operator on $M \times F$ takes the form of*

$$\not{D}_\omega = -\imath\gamma^\mu \nabla^E_\mu + \gamma_M \otimes \Phi\,,$$

where $\Phi := \not{D}_F + \phi + \varepsilon''_F J_F \phi J_F^{-1}$, and

$$\nabla^E_\mu = \nabla^S_\mu \otimes 1 + \imath 1 \otimes B_\mu$$

is a connection twisted by A_μ on a twisted spinor bundle $E := S \otimes (M \times \mathcal{H}_F)$, i.e. $B_\mu = \big(A^\mu - \varepsilon''_F J_F A_\mu J^{-1}\big)$.

Proof. The fluctuated Dirac operator, $\not{D}_\omega = \not{D} + \omega + \varepsilon'' J\omega J^{-1}$, was defined the equation (3.5). By plugging in ω from the previous lemma and realising the fact that $J_M \gamma^\mu J_M^{-1} = -\gamma^\mu$ for a 4-dimensional spin manifold, the last two terms give

$$\begin{aligned}\omega + \varepsilon'' J\omega J^{-1} &= \gamma^\mu \otimes A_\mu + \gamma_M \otimes \phi + \varepsilon'' J\big(\gamma^\mu \otimes A_\mu + \gamma_M \otimes \phi\big)J^{-1}\\ &= \gamma^\mu \otimes \big(A_\mu - \varepsilon''_F J_F A_\mu J^{-1}\big) + \gamma_M \otimes \big(\phi + \varepsilon''_F J_F \phi J_F^{-1}\big)\,,\end{aligned}$$

where the notation $B_\mu = \big(A_\mu - \varepsilon''_F J_F A_\mu J^{-1}\big)$ is used. Now adding the second term of the Dirac operator $\not{D} = \not{D}_M \otimes 1 + \gamma_M \otimes \not{D}_F$ to the second term in the above expression yields Φ, whereas the first term yields

$$\begin{aligned}-\imath\gamma^\mu \nabla^E_\mu &= \not{D}_M \otimes 1 + \gamma^\mu \otimes \big(A_\mu - \varepsilon''_F J_F A_\mu J_F^{-1}\big)\\ &= \not{D}_M \otimes 1 + \gamma^\mu \otimes B_\mu\,,\end{aligned}$$

with B_μ being a section of the trivial bundle $M \times \mathcal{H}_F$. □

In the proposition 3.2.13 I showed that the unitary equivalence of a spectral triple by a gauge group element is a special case of Morita self-equivalence. The following proposition shows that performing such unitary equivalence transformation on a fluctuated spectral triple gives nothing else than gauge transformation of gauge and scalar fields.

Proposition 4.2.3. *Let M be a 4-dimensional manifold, and let $M \times F$ be an almost-commutative manifold, with a fluctuated Dirac operator $\not{D}_\omega$. Then its unitary equivalence transformation by an element of $U \in G(M \times F)$, such that $U = uJuJ^{-1}$ for $u \in U(A)$, yields gauge transformations of the fields*

$$\begin{aligned}A_\mu &\to uA_\mu u^* - \imath u\partial_\mu u^*\,,\\ \phi &\to u\phi u^* + u\left[\not{D}_F, u^*\right]\,.\end{aligned}$$

Proof. The proposition 3.2.4 determines the unitary transformation of a Dirac operator begin $\not{D}_\omega \to U\not{D}_\omega U^*$. Applying it to the fluctuated Dirac operator form the equation (3.5) one arrives at

$$\begin{aligned} U\not{D}_\omega U^* &= U\not{D}U^* + U\omega U^* + U\varepsilon'' J\omega J^{-1}U^* \\ &= \not{D} + u\left[\not{D}, u^*\right] + \varepsilon'' Ju\left[\not{D}, u^*\right]J^{-1} + u\omega u^* + \varepsilon'' Ju\omega u^* J^{-1} \\ &= \not{D} + \left(u\omega u^* + u\left[\not{D}, u^*\right]\right) + \varepsilon'' J\left(u\omega u^* + u\left[\not{D}, u^*\right]\right)J^{-1}, \end{aligned}$$

where in going from $U = uJuJ^{-1}$ to u is accomplished by using the order one, and order zero conditions. Hence one sees that $U : \omega \mapsto u\omega u^* + u\left[\not{D}, u^*\right]$. The rest of the proof follows directly from applying the following equations

$$\begin{aligned} \omega &= \gamma^\mu \otimes A^\mu + \gamma_M \otimes \phi\,, \\ \not{D} &= -\iota\gamma^\mu \nabla^S_\mu \otimes 1 + \gamma_M \otimes \not{D}_F\,, \end{aligned}$$

and $\left[\nabla^S_\mu, u^*\right] = \partial_\mu u^*$. □

Remark. One can clearly see the connection to the gauge theories and physics. However, there is a small problem. The gauge field A_μ transforms in adjoint representation, as expected. On the other hand the Higgs field ϕ transforms also in the adjoint representation and this is the problem. Since in the standard model Higgs field transforms in the fundamental representation of the gauge group. Fortunately a right choice of the finite space F solves this problem. I refer the reader to [Connes and Marcolli, 2008] and [Van Suijlekom, 2015] for more detailed explanation.

4.3 Spectral action

In this section I only vaguely state the derivation of heat kernel expansion of the spectral action. For more details on the subject I encourage the reader to go to [Chamseddine and Connes, 1997, Connes and Marcolli, 2008, Gracia-Bondia et al., 2013] or [Van Suijlekom, 2015].

The following proposition gives the square of the Dirac operator, see also the theorem 3.1.8.

Proposition 4.3.1 (Generalized Lichnerowicz–Weitzenböck's formula)**.** *Let M be a 4-dimensional manifold, and let $M \times F$ be an almost-commutative manifold, with a fluctuated Dirac operator $\not{D}_\omega$. Then the square of the fluctuated Dirac operator is*

$$(\not{D}_\omega)^2 = \Delta^E - F\,,$$

for $E = S \otimes (M \times \mathcal{H}_F)$ being the twisted spinor bundle, and the endomorphism F is given by

$$F = -\frac{1}{4}s \otimes 1 - 1 \otimes \Phi^2 + \frac{1}{2}\iota\gamma^\mu\gamma^\nu \otimes F_{\mu\nu} - \iota\gamma_M\gamma^\mu \otimes \left[\nabla^E_\mu, \Phi\right]\,,$$

with the following notation

$$\begin{aligned} \Phi :=& \not{D}_F + \phi + \varepsilon''_F J_F \phi J_F^{-1}\,, \\ B_\mu =& A^\mu - \varepsilon''_F J_F A_\mu J^{-1}\,, \\ F_{\mu\nu} :=& \partial_\mu B_\nu - \partial_\nu B_\mu + \iota\left[B_\mu, B_\nu\right]\,, \\ \left[\nabla^E_\mu, \Phi\right] =& \partial_\mu \Phi + \iota\left[B_\mu, \Phi\right]\,, \\ \nabla^E =& \nabla^S_\mu \otimes 1 + \iota 1 \otimes B_\mu\,, \end{aligned}$$

and s being the scalar curvature.

Now comes the definition of the bosonic spectral action. For fermionic part the reader is welcomed to consult the above mentioned literature.

Definition 4.3.1. Let Λ be a real cutoff parameter, and let $f:\mathbb{R}\to\mathbb{R}$ be positive, even, rapidly decaying function such that it makes $f(\not{D}_\omega/\Lambda)$ a traceclass operator. The bosonic spectral action is defined as follows

$$S_{bosonic}[\omega] := \operatorname{Tr} f\Big(\frac{\not{D}_\omega}{\Lambda}\Big)\,.$$

Remark. For the definition of traceclass operators I refer the reader to one of the following two books, [Gracia-Bondia et al., 2013] or [Connes and Marcolli, 2008].

Theorem 4.3.2. *The bosonic spectral action is gauge invariant functional of the gauge field $\omega\in\Omega^1_{\not{D}}(A)$.*

Proposition 4.3.3. *Let M be a 4-dimensional manifold, and let $M\times F$ be an almost-commutative manifold, with a fluctuated Dirac operator $\not{D}_\omega$. Then the asymptotic heat kernel expansion, $\Lambda\to\infty$, of the bosonic spectral action is*

$$\operatorname{Tr}\Big(f\big(\frac{\not{D}_\omega}{\Lambda}\big)\Big)\sim 2f_4\Lambda^4 a_0\big(\not{D}^2_\omega\big)+2f_2\Lambda^2 a_2\big(\not{D}^2_\omega\big)+f(0)a_4\big(\not{D}^2_\omega\big)+\mathcal{O}(\Lambda^{-1})\,.$$

The numbers $f_j=\int_0^\infty f(x)x^{j-1}\mathrm{d}x$, for $j\geq 0$ are moments of f; and a_j are Seeley-DeWitt coefficients of the generalized Laplacian $(\not{D}_\omega)^2=\Delta^E-F$ given by

$$\begin{aligned}
a_0\big(x,(\not{D}_\omega)^2\big) &= (4\pi)^{-\frac{n}{2}}\operatorname{Tr}(\mathrm{id})\,,\\
a_2\big(x,(\not{D}_\omega)^2\big) &= (4\pi)^{-\frac{n}{2}}\operatorname{Tr}\Big(\frac{s}{6}+F\Big)\,,\\
a_4\big(x,(\not{D}_\omega)^2\big) &= (4\pi)^{-\frac{n}{2}}\operatorname{Tr}\Big(-12\Delta s+5s^2-2R^{\mu\nu}R_{\mu\nu}+2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\\
&\qquad\qquad +60sF+180F^2-60\Delta F+30\Omega^E_{\mu\nu}(\Omega^E)^{\mu\nu}\Big)\,.
\end{aligned}$$

Chapter 5

Applications

5.1 The two point space

In this section I will be concerned by a special case of a finite-dimensional spectral triples and their products with canonical spectral triple. This spectral triple is called two point space because as its name hints it describes a manifold consisting of two points.

The algebra of functions on this space is $A_F = \mathbb{C} \oplus \mathbb{C}$, and for now I will choose the Hilbert space to be $\mathcal{H}_F = \mathbb{C} \oplus \mathbb{C}$. Note the obvious representation of the algebra A_F on the Hilbert space

$$\pi : f \mapsto \begin{pmatrix} f(1) & \\ & f(2) \end{pmatrix} , \quad \forall f \in A_F \, .$$

The grading operator

$$\gamma_F = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} ,$$

splits the Hilbert space into two parts, $\mathcal{H}_F^{\pm}$. The Dirac operator can be chosen to be off-diagonal

$$\not{D}_F = \begin{pmatrix} 0 & t \\ t^* & 0 \end{pmatrix} , \quad t \in \mathbb{C} \, ,$$

since any diagonal element would drop out from commutation relations with elements of A_F.

Proposition 5.1.1. *The finite space F_X can have a real structure J_F only if $\not{D}_F = 0$, i.e. $t = 0$. Such a space is of KO-dimension $0, 2$ or 6.*

Note. In order to prove this proposition one needs to introduce Krajewski diagrams for real spectral triples, which play a similar role as Dynkin diagrams for simple Lie algebras. These diagrams give rules for possible relations between $\not{D}_F, J_F$, and $\mathcal{H}_F$. However, in this thesis, I will not explore neither the diagrams, nor classification of finite spectral triples. Therefore, I urge the reader to consult [Krajewski, 1998] for an introduction to Krajewski diagrams and to classification of real spectral triples.

Definition 5.1.1. The 2-dimensional two point space is defined by a finite spectral triple $F_X = (\mathbb{C}^2, \mathbb{C}^2, \not{D}_F; J_F, \gamma_F)$, such that

$$J_F = \begin{pmatrix} & C \\ C & \end{pmatrix} , \qquad \not{D}_F = \begin{pmatrix} 0 & t \\ t^* & 0 \end{pmatrix} , \quad t \in \mathbb{C} \, ,$$

for C being the operator of complex conjugation[1].

[1] The choice of J_F is again based on the Krajewski diagrams and KO-dimension being 6

One could think of performing the product of spectral triples defined in the equation (3.1). Although, this will not give the desired result of electrodynamics it will show interesting properties of the finite Dirac operator, so let's do it anyway. The product of the above two point space with a canonical spectral triple takes the form

$$M \times F_X := \big(C^\infty(M, \mathbb{C}^2), L^2(S) \otimes \mathbb{C}^2, \not{D}_M \otimes 1; J_M \otimes J_F, \gamma_M \otimes \gamma_F\big)\,. \tag{5.1}$$

The Gelfand duality theorem 3.2.1 says that the algebra of the above spectral triple is nothing else than a disjoint union of two copies of the manifold M. For the gauge group one has the following proposition.

Proposition 5.1.2. *Let F_X be the two point space described in the definition 5.1.1, then its gauge group is*

$$G(F_X) = U(1)\,.$$

Proof. Clearly $U(A_F) = U(\mathbb{C}^2) = U(1) \times U(1)$, so I only need to show that $U\big((A_F)_{J_F}\big) = U(1)$ since the equation (3.6) gives

$$G(F) = \frac{U(A_F)}{U\big((A_F)_{J_F}\big)}\,.$$

Now, it is clear that $f \in (A_F)_{J_F}$ if and only if $J_F f^* J_F^{-1} = f$ but this is possible only if $f(1) = f(2)$ since

$$\begin{pmatrix} & C \\ C & \end{pmatrix}\begin{pmatrix} f^*(1) & \\ & f^*(2)\end{pmatrix}\begin{pmatrix} & C \\ C & \end{pmatrix} = \begin{pmatrix} f(2) & \\ & f(1)\end{pmatrix},$$

which completes the proof. □

Corollary. *The gauge group of the product spectral triple $M \times F_X$ described by the equation (5.1) is equal to $C^\infty\big(M, G(F_X)\big)$.*

Proof. This is a direct consequence of the proposition 4.1.1. □

Proposition 5.1.3. *Let $M \times F_X$ be the spectral triple described in the equation (5.1). Then the fluctuated Dirac operator takes the form of*

$$\not{D}_\omega = \not{D} + \gamma^\mu Y_\mu \otimes \gamma_F\,.$$

Moreover, the action of the gauge group $G(M \otimes F_X)$ takes the form of

$$u : Y_\mu \mapsto Y_\mu - \imath u \partial_\mu u^*\,.$$

Proof. The gauge field A_μ is determined by two $U(1)$ gauge fields $X_\mu^1, X_\mu^2 \in C^\infty(M, \mathbb{R})$. From the proof of the proposition 4.2.2 one gets the definition of B_μ to be

$$\begin{aligned} 1 \otimes B_\mu =& A^\mu - J_F A_\mu J_F^{-1} \\ =& \begin{pmatrix} X_\mu^1 & \\ & X_\mu^2 \end{pmatrix} - \begin{pmatrix} X_\mu^2 & \\ & X_\mu^1 \end{pmatrix} \\ =& Y_\mu \otimes \gamma_F\,. \end{aligned}$$

where $\varepsilon_F'' = 1$ thanks to the KO-dimension being 6, and $Y_\mu = X_\mu^1 - X_\mu^2$. The rest follows from the proposition 4.2.3. □

In the next proposition I would like to show what role is played by the finite Dirac operator, and what consequences should one expect by setting it to zero.

Proposition 5.1.4. *The distance between the two copies of M is reciprocally proportional to the mass scale of the theory determined by the Higgs field Ψ which in turn is determined by the finite Dirac operator.*

Proof. The distance formula for a Riemannian spin^c manifold was defined in the proposition 3.2.2. This can easily be generalized to finite space F_X as follows.

$$d_{\not{D}_F}(1,2) = sup\{\|f(1) - f(2)\| \,|f \in A_F, \left\|[\not{D}_F, f]\right\| \leq 1\}\,.$$

Since the finite space in question is two point space, there is only one non-trivial choice. Moreover, an $f \in A_F$ is uniquely determined by a pair of complex numbers $f(1)$ and $f(2)$. Now, the commutator gives

$$\begin{aligned}[\not{D}_F, \pi(f)] =& \left[\begin{pmatrix} & t \\ t^* & \end{pmatrix}, \begin{pmatrix} f(1) & \\ & f(2)\end{pmatrix}\right] \\ =& \big(f(2) - f(1)\big)\begin{pmatrix} & t \\ -t^* & \end{pmatrix}\end{aligned}$$

Hence the norm is $\left\|[\not{D}_F, \pi(f)]\right\| = \|f(2) - f(1)\|\,\|t\|$. Putting everything together yields the distance formula

$$d_{\not{D}_F}(1,2) = \frac{1}{\|t\|}$$

The proposition 5.1.1 implies that in case of real structure J_F the two copies are infinitely apart. The relation to the mass scale follows from the fact that the mass scale of the theory is proportional to the Higgs field, which is contained in the part of the fluctuated Dirac operator $\gamma_M \otimes \Phi$, where $\Phi := \not{D}_F + \phi + \varepsilon''_F J_F \phi J_F^{-1}$, is proportional to $\not{D}_F$. □

Remark. There is a well-known trick to make this theory massive, namely doubling the Hilbert space $\mathcal{H}_F$. In this particular case, fermions will acquire mass without being couplet to the Higgs field.

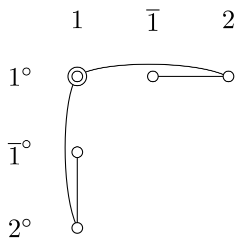
5.2 Electroweak sector with see-saw mechanism

The results presented in this section were first derived in [Chamseddine et al., 2007]. I will take a simplifying approach, meaning I will be concerned only about electroweak sector of one generation standard model.

It is known from the standard model, that the lepton space[2] contains right-handed electron and neutrino as singlets, and a doublet of their left-handed counterparts. The anti-lepton space has the very same structure with the only difference of anti-particles instead of particles. Thus one has that

$$\begin{aligned}\mathcal{H}_{lpt} =& \big\{\nu_R, e_R, (\nu_L, e_L)\big\} \simeq \mathbb{C}^4\,,\\ \overline{\mathcal{H}}_{lpt} =& \big\{\overline{\nu}_R, \overline{e}_R, (\overline{\nu}_L, \overline{e}_L)\big\} \simeq \mathbb{C}^4\,.\end{aligned}$$

Knowing this, one can draw the Krajewski diagram:



Where each vertex, ∘, corresponds to an irreducible representation of a subalgebra of the finite algebra A_{EW}; and each edge, lines connecting vertices, corresponds to an interaction term in the finite Dirac operator $\not{D}_{EW}$. Thus, in the diagram on the left, the horizontal vertices correspond precisely to $\mathcal{H}_{lpt}$, whereas the vertical vertices correspond to $\overline{\mathcal{H}}_{lpt}$. The full Hilbert space $\mathcal{H}_{EW}$ takes the form $\mathcal{H}_{lpt} \oplus \overline{\mathcal{H}}_{lpt} \simeq \mathbb{C}^8$. The finite algebra clearly takes the form of $A_{EW} = \mathbb{C} \oplus \mathbb{Q}$.

In order to find out the action of the algebra A_{EW}, first recall the isomorphism between $\mathbb{Q}$ and the sub-algebra of $M_2(\mathbb{C})$, namely

$$\mathbb{Q} \simeq \left\{\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}\right\}.$$

[2] I am talking here about only one generation.

which maps $q = a + b\epsilon_x + c\epsilon_y + d\epsilon_z$ to the matrix

$$q \mapsto \begin{pmatrix} a + \iota d & b + \iota c \\ -b + \iota c & a - \iota d \end{pmatrix} .$$

I will use the two notation for quaternions interchangeably. Now, one can determine the representation, π, of the algebra A_{EW} restricted to $\mathcal{H}_{lpt} \simeq \mathbb{C}^4$ as follows

$$\pi_{\mathcal{H}_{lpt}} : (\lambda, q) \mapsto \begin{pmatrix} \lambda & 0 & & \\ 0 & \lambda^* & & \\ & & \alpha & \beta \\ & & -\beta^* & \alpha^* \end{pmatrix} , \quad \forall \lambda \in \mathbb{C} , \forall q \in \mathbb{Q} .$$

The restriction of this representation to $\overline{\mathcal{H}}_{lpt} \simeq \mathbb{C}^4$ takes the form

$$\pi_{\overline{\mathcal{H}}_{lpt}} : (\lambda, q) \mapsto \mathrm{diag}(\lambda, \lambda, \lambda, \lambda) , \quad \forall \lambda \in \mathbb{C} , \forall q \in \mathbb{Q} .$$

Putting this together gives the full representation π of A_{EW} on $\mathcal{H}_{EW}$

$$\pi : (\lambda, q) \mapsto \left(\begin{array}{cccc|c} \lambda & 0 & & & \\ 0 & \lambda^* & & & \\ & & \alpha & \beta & \\ & & -\beta^* & \alpha^* & \\ \hline & & & & \lambda I_4 \end{array}\right) , \quad \forall \lambda \in \mathbb{C} , \forall q \in \mathbb{Q} , \tag{5.2}$$

where I_4 is the 4×4 identity matrix. The representation π° of A_{EW}° is determined in a similar manner by using the anti-linear operator J_{EW} defined later. The corresponding Dirac operator $\not{D}_{EW} : \mathcal{H}_{EW} \to \mathcal{H}_{EW}$ takes the form of

$$\not{D}_{EW} = \left(\begin{array}{cccc|cccc} & & Y_\nu^* & 0 & Y_R & 0 & & \\ & & 0 & Y_e^* & 0 & 0 & & \\ Y_\nu & 0 & & & & & & \\ 0 & Y_e & & & & & & \\ \hline Y_R & 0 & & & & & Y_\nu^* & 0 \\ 0 & 0 & & & & & 0 & Y_e^* \\ & & & & Y_\nu & 0 & & \\ & & & & 0 & Y_e & & \end{array}\right) ,$$

where all Y_ν's and Y_e's are the masses of respective particles; and Y_R is the Majorana mass of right-handed neutrion. In the case of three generations the Y's would be 3×3 Yukawa mass matrices and Y_R would be 3×3 Majorana mass matrix.

Eigenvalues of the grading operator γ_{EW} are $+1$ on left-handed particles and -1 on the right-handed particles, i.e.

$$\gamma_{EW} = \mathrm{diag}(-I_2, I_2, -I_2, I_2) .$$

The anti-linear operator J_{EW} maps particles into their corresponding anti-particles, i.e.

$$J_{EW} = \begin{pmatrix} & I_4 \\ I_4 & \end{pmatrix} .$$

Finally, I can state the following proposition.

Proposition 5.2.1. *The finite, real, even, spectral triple*

$$F_{EW} := \left(A_{EW}, \mathcal{H}_{EW}, \not{D}_{EW}; J_{EW}, \gamma_{EW}\right)$$

has KO-dimension 6.

Proof. Finiteness, reality and evenness is clear, to prove that the KO-dimension is 6 one needs to check that $\varepsilon' = \varepsilon'' = 1$ and $\varepsilon''' = -1$. □

Proposition 5.2.2. *The gauge group of the finite spectral triple F_{EW} is given by*

$$G(F_{EW}) = \big(U(1) \times SU(2)\big)/\mathbb{Z}_2 \,.$$

Proof. Clearly $U(A_{EW}) = U(1) \times U(\mathbb{Q})$. From the lemma 1.1.4 one has that $U(\mathbb{Q}) \simeq SU(2)$. For a to be in $(A_{EW})_{J_{EW}}$ it has to satisfy the relation $J_{EW}a = a^* J_{EW}$[3], which means that $\lambda = \lambda^* = q$; and hence $(A_{EW})_{J_{EW}} \simeq \mathbb{R}$. This gives $U\big((A_{EW})_{J_{EW}}\big) \simeq \mathbb{Z}_2$ and the definition 3.2.10 yields the result. □

Corollary. *The gauge group of the almost-commutative manifold $M \times F_{EW}$ takes the form of $C^\infty\big(M, G(F_{EW})\big)$.*

Lemma 5.2.3. *Let $M \times F_{EW}$ be an almost-commutative manifold, with M being a 4-dimensional spin manifold. Then the gauge field, A_μ, from the lemma 4.2.1 takes the form of*

$$A_\mu = \operatorname{diag}\big(\Lambda_\mu, -\Lambda_\mu, Q_\mu, (\Lambda_\mu I_4)\big)\,,$$

where

$$\begin{aligned}\Lambda_\mu &:= -\,\iota\lambda\partial_\mu\lambda'\,, \quad && \lambda,\lambda' \in C^\infty(M,\mathbb{C})\,,\\ Q_\mu &:= -\,\iota q\partial_\mu q'\,, && q,q' \in C^\infty(M,\mathbb{Q})\,,\end{aligned}$$

such that Λ_μ is a real field, and Q_μ is a sum of real fields multiplied by the Pauli matrices.

Proof. From the block diagonal form of the action of A_{EW} on $\mathcal{H}_{EW}$ in the equation (5.2) it is easy to see that the gauge field, $A_\mu = -\iota a\partial_\mu a'$ for some $a, a' \in C^\infty(M, A_{EW})$, splits into three components corresponding to the three diagonal parts, namely $\Lambda_\mu, \tilde{\Lambda}_\mu$, and Q_μ. The first and the last are given in the lemma itself, the remaining is $\tilde{\Lambda}_\mu := -\iota\lambda^*\partial_\mu(\lambda')^*$. However, the hermiticity condition $\Lambda_\mu = (\Lambda_\mu)^*$ forces $\tilde{\Lambda}_\mu = -\Lambda_\mu$ be a real field. The same condition $Q_\mu = \mathbb{Q}_\mu^\dagger$ restricts Q_μ to be the sum of real fields multiplied by the Pauli matrices[4]. □

In the remainder of this chapter I will use the following notation

$$\begin{aligned} Y &= \begin{pmatrix} Y_\nu & \\ & Y_e \end{pmatrix}, \quad & q &= \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},\\ \tilde{Y}_R &= \begin{pmatrix} Y_R & \\ & 0 \end{pmatrix}, & q_\lambda &= \begin{pmatrix} \lambda & \\ & \lambda^* \end{pmatrix}. \end{aligned} \tag{5.3}$$

Lemma 5.2.4. *Let $M \times F_{EW}$ be an almost-commutative manifold, with M being a 4-dimensional spin manifold. Then the scalar field, ϕ, from the lemma 4.2.1 takes the form of*

$$\phi = \left(\begin{array}{cc|c} 0 & Y^\dagger\varphi' & \\ \varphi Y & 0 & \\ \hline & & 0 \end{array}\right),$$

where

$$\varphi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ -\varphi_2^* & \varphi_1^* \end{pmatrix}, \qquad \varphi' = \begin{pmatrix} \varphi_1' & \varphi_2' \\ -(\varphi_2')^* & (\varphi_1')^* \end{pmatrix},$$

for

$$\begin{aligned} \varphi_1 &= \alpha(\lambda' - \alpha') + \beta(\beta')^*\,, \qquad & \varphi_1' &= \lambda(\alpha' - \lambda')\,,\\ \varphi_2 &= -\alpha\beta' + \beta\big((\lambda')^* - (\alpha')^*\big)\,, & \varphi_2' &= \lambda\beta'\,. \end{aligned}$$

[3] Of course, I am using a sloppy notation; one should write $J_{EW}\pi(a) = \pi(a)^* J_{EW}$.
[4] I implicitly assume the embedding of $\mathbb{Q}$ in $M_2(\mathbb{C})$.

Proof. From the above mentioned lemma one has that $\phi := a\left[\not{D}_{EW}, a'\right]$. Plugging in the components yields

$$
\begin{aligned}
a\left[\not{D}_{EW}, a'\right] &= \left(\begin{array}{cc|cc} q_\lambda & & & \\ & q & & \\ \hline & & \lambda & \\ & & & \lambda \end{array}\right)\left[\left(\begin{array}{cc|cc} & Y^\dagger & \tilde{Y}_R & \\ Y & & & \\ \hline \tilde{Y}_R & & & Y^\dagger \\ & & Y & \end{array}\right), \left(\begin{array}{cc|cc} q_{\lambda'} & & & \\ & q' & & \\ \hline & & \lambda' & \\ & & & \lambda' \end{array}\right)\right] \\
&= \left(\begin{array}{cc|c} 0 & q_\lambda(Y^\dagger q' - q_{\lambda'} Y^\dagger) & \\ q(Y q_{\lambda'} - q' Y) & 0 & \\ \hline & & 0 \end{array}\right) \\
&= \left(\begin{array}{cc|c} 0 & Y^\dagger q_\lambda (q' - q_{\lambda'}) & \\ q(q_{\lambda'} - q')Y & 0 & \\ \hline & & 0 \end{array}\right),
\end{aligned}
$$

where in the last line I used the fact that both Y and q_λ are diagonal; hence they commute. After performing the 2×2 matrix multiplications one obtains the equations for φ and φ'. □

Combination of the preceeding two lemmas gives the fluctuated Dirac operator on our almost-commutative manifold.

Proposition 5.2.5. *Let $M \times F_{EW}$ be an almost-commutative manifold, with M being a spin manifold of dimension 4. Then, using the notation from the previous two lemmas, the fluctuated Dirac operator on $M \times F_{EW}$ takes the form of*

$$
\not{D}_\omega = \not{D}_M \otimes 1 + \gamma_M \otimes B_\mu + \gamma_M \otimes \Phi\,,
$$

where

$$
B_\mu = \left(\begin{array}{ccc|ccc} 0 & 0 & & & & \\ 0 & -2\Lambda_\mu & & & & \\ & & (Q_\mu - \Lambda_\mu) & & & \\ \hline & & & 0 & 0 & \\ & & & 0 & 2\Lambda_\mu & \\ & & & & & -(Q_\mu - \Lambda_\mu) \end{array}\right),
$$

and

$$
\Phi = \left(\begin{array}{cc|cc} & Y^\dagger(\varphi'+1) & \begin{matrix} Y_R & 0 \\ 0 & 0 \end{matrix} & \\ (\varphi+1)Y & & & \\ \hline \begin{matrix} Y_R & 0 \\ 0 & 0 \end{matrix} & & & Y^\dagger(\varphi'+1) \\ & & (\varphi+1)Y & \end{array}\right). \tag{5.4}
$$

Proof. The fluctuated Dirac operator is determined in the proposition 4.2.2. Hence I only need to prove the expressions for B_μ and Φ.
For the first expression one can write $B_\mu = A_\mu - J_{EW} A_\mu J_{EW}^{-1}$. After performing the matrix multiplications in the second term one arrives exactly at the desired result.
In the case of the second expression one has $\Phi := \not{D}_{EW} + \phi + J_{EW}\phi J_{EW}^{-1}$, where the second term yields

$$
J_{EW}\phi J_{EW}^{-1} = \left(\begin{array}{c|cc} 0 & & \\ \hline & 0 & Y^\dagger\varphi' \\ & \varphi Y & 0 \end{array}\right).
$$

Putting everything together yields the result. □

Remark. It is interesting to note that the coefficients in front of Λ_μ in the field B_μ correspond to the Hypercharges of corresponding particles, i.e.

ν_R	e_R	ν_L	e_L	$\overline{\nu}_R$	$\overline{e}_R$	$\overline{\nu}_L$	$\overline{e}_L$
0	-2	-1	-1	0	2	1	1

Proposition 5.2.6. *Let $M \times F_{EW}$ be an almost-commutative manifold, described above; and let $u = (\kappa, p) \in U(A_{EW})$ for $\kappa \in C^\infty\big(M, U(1)\big)$ and $p \in C^\infty\big(M, SU(2)\big)$. Then the gauge transformations of the gauge fields defined in the lemma 5.2.3 and the lemma 5.2.4 are*

$$\begin{aligned} \Lambda_\mu &\mapsto \Lambda_\mu - \iota\kappa\partial_\mu\kappa^* , & (\varphi+1) &\mapsto p(\varphi+1)q_\kappa^\dagger , \\ Q_\mu &\mapsto pQ_\mu p^\dagger - \iota p\partial_\mu p^\dagger , & (\varphi'+1) &\mapsto q_\kappa(\varphi'+1)p^\dagger . \end{aligned}$$

Proof. The transformation properties of the gauge field A_μ and the scalar field ϕ are given in the proposition 4.2.3.

In the case of the gauge field, A_μ, first notice that it has a block diagonal form, and that the action of A_{EW} on it has the same structure. Thus, the problem splits into transformations of Λ_μ and Q_μ. Next thing to note is that $\Lambda_\mu \in C^\infty(M, \mathbb{C})$ yielding that $\kappa\Lambda_\mu\kappa^* = \Lambda_\mu$. The rest follows directly from transformation forms of the above mentioned proposition.

In the case of the scalar field, ϕ, one has

$$\begin{aligned} \phi \mapsto & u\phi u^\dagger + u\left[\not{D}_{EW}, u^\dagger\right] \\ =& u(\phi + \not{D}_{EW})u^\dagger - \not{D}_{EW} . \end{aligned}$$

It is clear that this equation has the same structure as the equations in the proof of the lemma 5.2.4. Writing out only the non-zero part acting on $\mathcal{H}_{lpt}$ yields

$$\begin{pmatrix} 0 & Y^\dagger(\varphi'+1) \\ (\varphi+1)Y & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & Y^\dagger q_\kappa(\varphi'+1)p^\dagger \\ p(\varphi+1)q_\kappa^\dagger Y & 0 \end{pmatrix} , \tag{5.5}$$

which completes the proof. ∎

From this last proposition one can see that the gauge fields Λ_μ and Q_μ transform in the adjoint representation, whereas the fields φ and φ' transform in the fundamental representation. Moreover, the lemma 5.2.4 shows that φ and φ' have only two complex degrees of freedom, i.e. are doublets, even though they are written as 2×2 matrices. All this is in agreement with the earlier interpretation of ϕ as the Higgs field.

Now, I would like to show the see-saw mechanism which is built in the Dirac operator $\not{D}_{EW}$. The Dirac operator written in the basis of neutrinos, i.e. omiting every second row and column, can be writen as follows

$$\not{D}_{EW} = \left(\begin{array}{cc|cc} & Y_\nu^* & Y_R & \\ Y_\nu & & & \\ \hline Y_R & & & Y_\nu^* \\ & & Y_\nu & \end{array}\right) .$$

The four eigenvalues of this matrix are

$$\pm\frac{1}{2}\Big(Y_R \pm \sqrt{Y_R^2 + 4Y_\nu^2}\Big) .$$

Assuming the relation $Y_R \gg Y_\nu$ between the Majorana mass, Y_R, and the Dirac mass, Y_ν (c.f. [Schwartz, 2014], section 29.3.4) gives an approximate values for the eigenvalues, $\pm Y_R$ and $\pm\frac{Y_\nu^2}{Y_R}$. This yields the well-known see-saw mechanism. I refer the reader to the above mentioned book for more details on this mechanism.

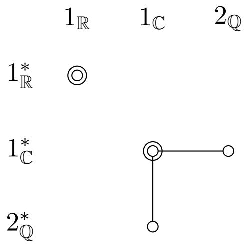
In the next section I will modifie the finite algebra and see what consequences this modification offeres.

5.3 Electroweak sector with Majorana neutrinos

As already mentioned, in the present section I will be concerned with the algebra

$$A_F = \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{Q}\,,$$

which looks much more elegant in a way that it contains all three associative division algebras. This modification will lead to an electroweak sector with only Majorana neutrinos.



The Krajewski diagram gets modified as shown on the left. Note that not only the diagram has less edges, which will force the Dirac operator to have less entries in the diagonal components, it also has two double vertices instead of one. This double vertex allows for giving the electrons a Majorana mass. However, I will not consider this option since it was not observed in experiments.

The Hilbert space stays the same as in the previous example, namely $\mathcal{H}_F = \mathcal{H}_{lpt} \oplus \overline{\mathcal{H}}_{lpt}$, where

$$\begin{aligned} \mathcal{H}_{lpt} &= \big\{\nu_R, e_R, (\nu_L, e_L)\big\} \simeq \mathbb{C}^4\,, \\ \overline{\mathcal{H}}_{lpt} &= \big\{\overline{\nu}_R, \overline{e}_R, (\overline{\nu}_L, \overline{e}_L)\big\} \simeq \mathbb{C}^4\,. \end{aligned}$$

The action of the algebra A_F on the Hilbert space, $\mathcal{H}_F$, is

$$\pi : (r,z,q) \mapsto \left(\begin{array}{cccc|cc} r & 0 & & & & \\ 0 & z & & & & \\ & & \alpha & \beta & & \\ & & -\beta^* & \alpha^* & & \\ \hline & & & & r & \\ & & & & & z^* I_3 \end{array}\right)\,, \qquad \forall r \in \mathbb{R}\,, \forall z \in \mathbb{C}\,, \forall q \in \mathbb{Q}\,. \tag{5.6}$$

The Dirac operator takes the form of

$$\not{D}_F = \left(\begin{array}{cccc|cccc} & & 0 & 0 & Y_R & 0 & & \\ & & 0 & Y_e^* & 0 & 0 & & \\ 0 & 0 & & & & & & \\ 0 & Y_e & & & & & & \\ \hline Y_R & 0 & & & & & 0 & 0 \\ 0 & 0 & & & & & 0 & Y_e^* \\ & & & & 0 & 0 & & \\ & & & & 0 & Y_e & & \end{array}\right)\,.$$

Finally, the grading operator stays the same as in the previous case

$$\gamma_F = \mathrm{diag}(-I_2, I_2, -I_2, I_2)\,,$$

but the anti-unitary operator, J_F, is modified because of the complex-conjugation of the action of A_F on $\overline{\mathcal{H}}_{lpt}$.

$$J_F = C \begin{pmatrix} & I_4 \\ I_4 & \end{pmatrix}\,,$$

where $C : \mathbb{C} \to \mathbb{C}$ is the operator of complex conjugation. I can gather all this in the next proposition.

Proposition 5.3.1. *The finite, real, even, spectral triple*

$$F := \big(A_F, \mathcal{H}_F, \not{D}_F; J_F, \gamma_F\big)$$

has KO-dimension 6.

The gauge group stays the same, however the proof is slightly modified due to the additional component in the finite algebra.

Proposition 5.3.2. *The gauge group of the finite spectral triple F is given by*

$$G(F) = \big(U(1) \times SU(2)\big)/\mathbb{Z}_2\,.$$

Proof. This is analogous to the proof from the proposition 5.2.2, with the only difference being the fact that $(A_F)_{J_F} = \mathbb{R} \oplus \mathbb{R}$ because the relation $J_F a = a^* J_F$ forces $z = z^* = q$ for $a = (r, z, q) \in A_F$. $\square$

The gauge field A_μ and the scalar field ϕ are modified according to the next two lemmas.

Lemma 5.3.3. *The gauge field, A_μ on $M \times F$, defined in the lemma 4.2.1, takes the form of*

$$A_\mu = \operatorname{diag}\big(0, \Lambda_\mu, Q_\mu, 0, (-\Lambda_\mu I_3)\big)\,,$$

where

$$\begin{aligned} \Lambda_\mu &:= -\,\iota z \partial_\mu z'\,, \quad z, z' \in C^\infty(M, \mathbb{C})\,,\\ Q_\mu &:= -\,\iota q \partial_\mu q'\,, \quad q, q' \in C^\infty(M, \mathbb{Q})\,,\end{aligned}$$

such that Λ_μ is a real field, and Q_μ is a sum of real fields multiplied by the Pauli matrices.

Proof. The proof of this proposition is pretty much the same as the one of the lemma 5.2.3 from the previous section. The small difference is in the condition $R_\mu = R_\mu^*$, for $R_\mu = -\iota r \partial_\mu r'$; because $r, r' \in \mathbb{R}$ the condition forces $R_\mu = 0$. $\square$

In the following text I will use a modified notation, originally defined in the equation (5.3), more precisely

$$\begin{aligned} Y &= \begin{pmatrix} 0 & \\ & Y_e \end{pmatrix}\,, \quad q = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}\,,\\ \tilde{Y}_R &= \begin{pmatrix} Y_R & \\ & 0 \end{pmatrix}\,, \quad q_{r,z} = \begin{pmatrix} r & \\ & z \end{pmatrix}\,. \end{aligned} \tag{5.7}$$

Lemma 5.3.4. *The scalar field, ϕ on $M \times F$, defined in the lemma 4.2.1, takes the form of*

$$\phi = \left(\begin{array}{cc|c} 0 & Y^\dagger \varphi' & \\ \varphi Y & 0 & \\ \hline & & 0 \end{array}\right)\,,$$

where

$$\varphi = \begin{pmatrix} \alpha(r' - \alpha') + \beta(\beta')^* & -\alpha\beta' + \beta\big(z' - (\alpha')^*\big) \\ -\beta^*(r' - \alpha') + \alpha^*(\beta')^* & \beta^*\beta' + \alpha^*\big(z' - (\alpha')^*\big) \end{pmatrix}\,, \quad \varphi' = \begin{pmatrix} r(\alpha' - r') & r\beta' \\ -z(\beta')^* & z\big((\alpha')^* - z'\big) \end{pmatrix}\,.$$

Proof. Very same as in the lemma 5.2.4, only substitute q_λ for q_{rz} in the proof. An easier way is to make the substitution $\lambda \mapsto r$ and $\lambda^* \mapsto z$, however one must be careful with complex conjugations! $\square$

Proposition 5.3.5. *The fluctuated Dirac operator on $M \times F$ takes the form of*

$$D\!\!\!/_\omega = D\!\!\!/_M \otimes 1 + \gamma_M \otimes B_\mu + \gamma_M \otimes \Phi\,,$$

where

$$B_\mu = \left(\begin{array}{ccc|ccc} 0 & 0 & & & & \\ 0 & 2Z_\mu & & & & \\ & & (Q_\mu + Z_\mu) & & & \\ \hline & & & 0 & 0 & \\ & & & 0 & -2Z_\mu & \\ & & & & & -(Q_\mu + \Lambda_\mu) \end{array}\right)\,,$$

and Φ is as in the equation (5.4)., with the modified notation from the equation (5.7).

The transformation relations for the gauge and scalar fields are as in the proposition 5.2.6 with the only change being $q_\kappa \mapsto q_{s,w}$ for $s \in R$ and $w \in \mathbb{C}$.

The discussion about the transformation properties, and about the Higgs field, following the proof of the proposition mentioned in the previous paragraph, carries over to the current setting. Moreover, since here $Y = \mathrm{diag}(0, Y_e)$, only the parts of φ and φ' containing z survives. The parts having terms with r are killed by the zero on the diagonal of Y. This restricts φ and φ' to have only two complex degrees of freedom.

The eigenvalues of the Dirac operator, $\not{D}_F$, restricted to the subspace of only neutrinos are $\pm Y_R$ of multiplicity one, and 0 of multiplicity 2, giving that in this setting neutrinos have only Majorana masses.

The link to the hypercharge of particles picks up minus sign, this can be fixed by reinterpreting particles as anti-particles and vice versa.

Appendix A

Physicist's conventions

A.1 Lorentz group representation

This appendix serves the purpose of outlining the standard matrix representations of spinors and 4-vectors.

A.1.1 Spinors as columns

I will kick off the first subsection with some facts about Lie algebras and their complexifications.

Theorem A.1.1. *Let G and H be matrix Lie groups with Lie algebras g and h, respectively; and let $\phi : g \to h$ be a Lie algebra homomorphism. If G is simply connected, there exists a unique Lie group homomorphism $\Phi : G \to H$ such that*

$$\Psi(\exp(X)) = \exp(\psi(X))\,, \quad \forall X \in g\,.$$

Corollary. *Suppose G and H are simply connected matrix Lie groups with Lie algebras g and h, respectively. If g is isomorphic to h, then G is isomorphic to H.*

Remark. This basically mean that there is a bijective correspondence between real representations of a Lie algebra g, and complex representations of its complexification $g_{\mathbb{C}}$. A fact which will be very useful later.

Lemma A.1.2. *The following list of identities holds*

$$\begin{aligned} so(3)_{\mathbb{C}} &\cong su(2)_{\mathbb{C}} \cong sl(2,\mathbb{C})\,,\\ so(4) &\cong su(2) \oplus su(2)\,,\\ so(1,3)_{\mathbb{C}} &\cong sl(2,\mathbb{C})_{\mathbb{C}} \cong so(4;\mathbb{C}) \cong sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})\,. \end{aligned}$$

Note. For the proof of the last lemma and theorem, the reader is encourage to consult the Theorem 5.6 and Proposition 3.38 both in [Hall, 2015].

Physicists usually use a different convention when it comes to exponentiating Lie algebra elements. It goes as follows

$$\Lambda = \exp(\iota\theta^i \tilde{J}_i + \iota\beta^i \tilde{K}_i)\,,$$

where the relation to the generators defined in the equation (1.5) is

$$J_i = \iota\tilde{J}_i\,, \quad K_i = \iota\tilde{K}_i\,.$$

The commutation relations are modified to

$$\begin{aligned} \left[\tilde{J}_i, \tilde{J}_j\right] &= \iota\varepsilon_{ij}{}^k \tilde{J}_k\,,\\ \left[\tilde{J}_i, \tilde{K}_j\right] &= \iota\varepsilon_{ij}{}^k \tilde{K}_k\,,\\ \left[\tilde{K}_i, \tilde{K}_j\right] &= -\iota\varepsilon_{ij}{}^k \tilde{J}_k\,. \end{aligned}$$

However I will stick to the original, mathematician's, definition of the generators from the equation (1.5) satisfying the commutation relation written in the equation (1.6).

At this point a complexification of (already complex, but viewed as a real algebra) $sl(2,\mathbb{C})$ is performed by taking the following complex combinations of generators[1]

$$J_i^+ \equiv \frac{1}{2}(J_i + \iota K_i)\,, \quad J_i^- \equiv \frac{1}{2}(J_i - \iota K_i)\,, \tag{A.1}$$

which simplifies the commutation relations as follows

$$\begin{aligned} \left[J_i^\pm, J_j^\pm\right] &= -\varepsilon_{ij}{}^k J_k^\pm\,, \\ \left[J_i^\pm, J_j^\mp\right] &= 0\,. \end{aligned} \tag{A.2}$$

The algebra generated either by J_i^+ or J_i^- has the same commutation relations as $su(2)$. Due to the exceptional isomorphisms (also called accidental isomorphisms) this algebra has multiple names $su(2) \cong so(3) \cong sl(2,\mathbb{R}) \cong so(1,1)$.

After performing this transformation, often an incorrect claim $so(1,3) \cong su(2) \oplus su(2)$ is made. The problem is easy to spot - the first algebra is non-compact but the second is compact.

The correct way to thing about this is that the other possible complex combination of the generators was forgotten, namely

$$\begin{aligned} K_i^+ &\equiv -\iota J_i^+ = \frac{1}{2}(K_i - \iota J_i)\,, \\ K_i^- &\equiv \iota J_i^- = \frac{1}{2}(K_i + \iota J_i)\,. \end{aligned}$$

After some tedious calculations one can convince himself that we actually got $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$. Therefore the claim holds only after complexification.

$$so(1,3)_\mathbb{C} \cong sl(2,\mathbb{C})_\mathbb{C} \cong su(2)_\mathbb{C} \oplus su(2)_\mathbb{C}\,.$$

But this is exactly what the lemma A.1.2 tells us. So, thanks to the theorem A.1.1 (or rather the remark following it), we are now looking for a complex representation of $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$, which is nothing other than a tensor product of two copies of a complex representation of $sl(2,\mathbb{C})$. These are labelled by a pair of numbers (i,j). Now again, the lemma A.1.2 tells us that $sl(2,\mathbb{C})$ is complexification of $su(2)$, and the theorem A.1.1 tells us that it is sufficient to take real representations of $su(2)$. But we know that the real representations of $su(2)$ are spinors, vectors, etc.

The above discussion yields that there are two inequivalent, meaning non-isomorphic, fundamental representations of $sl(2,\mathbb{C})$. The first one is a complex, 2-dimensional representation denoted by $\left(\frac{1}{2},0\right)$, and called the left-handed, also known as fundamental, with objects of the underlying space called the left-handed Weyl spinors. The other one is again a complex, 2-dimensional representation denoted by $\left(0,\frac{1}{2}\right)$, and called right-handed, also known as anti-fundamental. The objects this second representation acts on are called the right-handed Weyl spinors. The infinitesimal transformations for these representations are

$$\begin{aligned} \delta\psi_L &= \frac{1}{2}(\iota\theta^j + \beta^j)\sigma_j\psi_L \\ &= \left(\theta^j J_j^+ + \beta^j K_j^+\right)\psi_L\,, \\ \delta\psi_R &= \frac{1}{2}(\iota\theta^j - \beta^j)\sigma_j\psi_R \\ &= \left(\theta^j J_j^- + \beta^j K_j^-\right)\psi_R\,, \end{aligned} \tag{A.3}$$

with an obvious notation.

[1] This really is a second complexification because the 3-dimensional complex algebra $sl(2,\mathbb{C})$ is viewed as 6-dimensional real algebra.

What usually follows next in the QTF textbooks are Clifford algebras and γ-matrices. The two representations are joined into $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ called Dirac representation with transformation matrices

$$\Lambda^s = \exp\left(\frac{1}{2}\theta_{\mu\nu}S^{\mu\nu}\right),$$

where $S^{\mu\nu}=\frac{1}{4}\left[\gamma^\mu,\gamma^\nu\right]$. In the chiral basis, gammas are written as

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}.$$

Where σ^0 is the identity matrix and σ^i's are the Pauli matrices; the bar multiplies the spatial Pauli matrices by -1. This representation acts on objects called Dirac spinors, represented in the chiral basis by a column vectors

$$\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}.$$

Let me conclude this subsection by outlining the construction of a Lorentz invariant Lagrangian. For a more self-contained treatment see chapter 10 in [Schwartz, 2014].

By exploiting the transformation relations of left-handed and right-handed spinors, it is realized that the combinations

$$\begin{aligned} V_R^\mu &= (\psi_R^\dagger\psi_R, \psi_R^\dagger\vec{\sigma}\psi_R) = \psi_R^\dagger\sigma^\mu\psi_R\,, \\ V_R^\mu &= (\psi_L^\dagger\psi_L, -\psi_L^\dagger\vec{\sigma}\psi_L) = \psi_L^\dagger\bar{\sigma}^\mu\psi_L\,, \end{aligned}$$

transform as 4-vectors. Here again $\vec{\sigma}$ represents the Pauli matrices. By doing so a Lorentz invariant Lagrangian

$$\begin{aligned} \mathcal{L} &= \iota\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R + \iota\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L - m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R) \\ &= \bar{\psi}(\iota\gamma^\mu\partial_\mu - m)\psi\,, \end{aligned}$$

is defined. Where the bar $\bar{\psi}=\psi^\dagger\gamma^0$ is obviously not the quaternionic conjugation operator from the main text.

A.1.2 Vectors as columns

From quantum field theory it is known that the joint representation $\left(\frac{1}{2},\frac{1}{2}\right)$ is the 4-vector representation. In this subsection, some details of the 4-vector representation will be explored. Here, I will partly follow the already a couple of times mentioned book by [Schwartz, 2014].

The transformation properties of 4-vectors, and of the metric tensor are carried over from special relativity. They take the form of

$$\begin{aligned} X^\mu &\to \Lambda^\mu{}_\nu X^\nu\,, \\ \eta_{\mu\nu} &= (\Lambda^T)_\mu{}^\rho\eta_{\rho\sigma}\Lambda^\sigma{}_\nu \end{aligned}$$

Explicitly, the transformation matrices for rotations around the x, y or z axes take the form

$$\begin{pmatrix}1 &&& \\ & 1 && \\ && \cos(\theta_x) & \sin(\theta_x) \\ && -\sin(\theta_x) & \cos(\theta_x)\end{pmatrix},\ \begin{pmatrix}1 &&& \\ & \cos(\theta_y) && -\sin(\theta_y) \\ && 1 & \\ & \sin(\theta_y) && \cos(\theta_y)\end{pmatrix},\ \begin{pmatrix}1 &&& \\ & \cos(\theta_z) & \sin(\theta_z) & \\ & -\sin(\theta_z) & \cos(\theta_z) & \\ &&& 1\end{pmatrix}.$$

For boosts in the x, y or z directions one gets

$$\begin{pmatrix}\cosh(\beta_x) & \sinh(\beta_x) && \\ \sinh(\beta_x) & \cosh(\beta_x) && \\ &&& \\ &&& \end{pmatrix},\ \begin{pmatrix}\cosh(\beta_y) && \sinh(\beta_y) & \\ & 1 && \\ \sinh(\beta_y) && \cosh(\beta_y) & \\ &&& 1\end{pmatrix},\ \begin{pmatrix}\cosh(\beta_z) &&& \sinh(\beta_z) \\ & 1 && \\ && 1 & \\ \sinh(\beta_z) &&& \cosh(\beta_z)\end{pmatrix}.$$

This can be put into a nice equation for vector transformations

$$\Lambda^V = \exp(\frac{\iota}{2}\theta_{\mu\nu}M^{\mu\nu})\,,$$

see equation (1.8), and equation (1.9).

Infinitesimally, this yields the following transformation rules of a 4-vector X

$$\delta(X^0, X^k) = (\beta_i X^i, \beta^k X^0 + \varepsilon_{ij}{}^k X^i \theta^j)\,, \tag{A.4}$$

where θ^j are the rotation angles and β^i are the boost angles.

Appendix B

Complex Quaternions

B.1 Link between matrix and complex-quaternionic formalism

In this section, I would like to outline the relation between matrix and complex-quaternionic formalism. It is known that the transformation relations for spinors in matrix formalism take the form

$$
\begin{aligned}
\delta\psi_L &= \frac{1}{2}(\iota\theta^j + \beta^j)\sigma_j\psi_L \,, \\
\delta\psi_R &= \frac{1}{2}(\iota\theta^j - \beta^j)\sigma_j\psi_R \,.
\end{aligned}
$$

The left-handed spinors in matrix formalism are good to go, since the Lorentz transformations act by the left multiplication; and the isomorphism is simply $\varphi : \mathrm{L}_{\sigma_i} \mapsto \iota\,\mathrm{L}_{\epsilon_i}$. On generators one gets

$$
\begin{aligned}
J_i^L &: \iota\frac{1}{2}\,\mathrm{L}_{\sigma_i} \overset{\varphi}{\mapsto} -\frac{1}{2}\,\mathrm{L}_{\epsilon_i} \,, \\
K_i^L &: \frac{1}{2}\,\mathrm{L}_{\sigma_i} \overset{\varphi}{\mapsto} \iota\frac{1}{2}\,\mathrm{L}_{\epsilon_i} \,.
\end{aligned}
$$

On the other hand, right handed spinors in the matrix formalism are slightly different. This is because Lorentz transformations act again by the left multiplication, but in complex quaternionic formalism they acts by the right multiplication. Hence, I need a slight modification of the above isomorphism.

$$
\begin{aligned}
J_i^R &: \iota\frac{1}{2}\,\mathrm{L}_{\sigma_i} \overset{\varphi}{\mapsto} -\frac{1}{2}\,\mathrm{L}_{\epsilon_i} \overset{\bar{\cdot}}{\mapsto} \frac{1}{2}\,\mathrm{R}_{\epsilon_i} \,, \\
K_i^R &: -\frac{1}{2}\,\mathrm{L}_{\sigma_i} \overset{\varphi}{\mapsto} -\iota\frac{1}{2}\,\mathrm{L}_{\epsilon_i} \overset{\bar{\cdot}}{\mapsto} \iota\frac{1}{2}\,\mathrm{R}_{\epsilon_i} \,.
\end{aligned}
$$

From which it is clear that the isomorphism for the right handed spinors is $(\bar{\cdot} \circ \varphi) \equiv \overline{\varphi} : \mathrm{L}_{\sigma_i} \mapsto -\iota\,\mathrm{R}_{\epsilon_i}$. Because $*^2 \equiv 1$, and $\dagger$ commutes with φ one can also write

$$
\overline{\varphi} \cong * \circ \varphi \circ \dagger \,.
$$

This yields that using the map $\varphi : \psi_L^{Mat} \to \psi_L^{CQ}$ for left-handed spinors[1], and $\overline{\varphi} : \psi_R^{Mat} \to \overline{\psi}_R^{CQ}$ for right-handed spinors give the desired isomorphisms. The map φ can be also used as a isomorphism map between Clifford algebra representations.

[1] To distinguish between spinors in the two formalism I temporarily use the superscripts Mat and CQ.

I have shown that spinors have the same transformation rules in either representation, c.f. lemma 1.2.5. Now, I would like to connect the 4-vector or $\left(\frac{1}{2},\frac{1}{2}\right)$ representation in complex-quaternionic formalism to its matrix formalism counterpart. To this end recall that in the later formalism $\bar{\psi}\gamma^\mu\psi$ transforms as a 4-vector. The same is true for $\psi_L^\dagger(1+\iota\epsilon_i)\psi_L$ and $\psi_R^*(1-\iota\epsilon_i)\overline{\psi}_L$ (see the equation (2.1)). This gives the connection between 4-vectors of matrix formalism written as 2×2 matrices with two spinorial indices, i.e. $V = V^\mu\sigma_\mu$, where $\sigma^{\dot{\alpha}\beta}$ carries the spinor indices. The corresponding map between them is just the above map φ sending σ_i to $\iota\epsilon_i$ and σ_0 to 1.

It is known that in this formalism the equations of motion take the form

$$(\iota\gamma^\mu\partial_\mu - m)\,\psi = 0\,,$$

which when written out in the chiral basis means

$$\begin{pmatrix} -m & \iota\sigma^\mu\partial_\mu \\ \iota\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0\,.$$

This machinery heavily depends on the Clifford algebra of the underlying Minkowski space. Clearly, the gamma matrices written out in the chiral representation are

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}\,,$$

where σ^0 is a 2×2 identity matrix and σ^i are the Pauli matrices.

This Clifford algebra for complex quaternions can be written out as follows

$$\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\,,\quad \gamma^i = \begin{pmatrix} & \iota\epsilon_i \\ -\iota\epsilon_i & \end{pmatrix}\,.$$

Using this and the combination of the left-handed spinor and quaternionic conjugation of the right-handed spinor into the Dirac spinor yields the desired outcome for equation of motion, precisely like in the equation (2.2)

$$\begin{pmatrix} -m & \iota(\partial_0 + \sum_i \iota\epsilon_i\partial_i) \\ \iota(\partial_0 - \sum_i \iota\epsilon_i\partial_i) & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \overline{\psi}_R\end{pmatrix} = 0\,.$$

B.2 Calculations with complex-quaternions

B.2.1 Action of Lorentz generators on complex-quaternions

For the basis

$$l_\uparrow = \frac{1}{2}(1+\iota\epsilon_z)\,, \qquad \overline{l_\uparrow} = r_\uparrow = \frac{1}{2}(1-\iota\epsilon_z)\,,$$

$$l_\downarrow = \frac{1}{2}(\iota\epsilon_x+\epsilon_y)\,, \qquad \overline{l_\downarrow} = r_\downarrow = \frac{1}{2}(-\iota\epsilon_x-\epsilon_y)\,;$$

one has the following Lorentz transformations

rotations of $l_\uparrow$

$$\begin{aligned}
\mathrm{L}_{-\epsilon_x} &: (1+\iota\epsilon_z) \mapsto \iota(\iota\epsilon_x+\epsilon_y)\,,\\
\mathrm{L}_{-\epsilon_y} &: (1+\iota\epsilon_z) \mapsto -(\iota\epsilon_x+\epsilon_y)\,,\\
\mathrm{L}_{-\epsilon_z} &: (1+\iota\epsilon_z) \mapsto \iota(1+\iota\epsilon_z)\,,
\end{aligned}$$

boosts of $l_\uparrow$

$$\begin{aligned}
\mathrm{L}_{\iota\epsilon_x} &: (1+\iota\epsilon_z) \mapsto (\iota\epsilon_x+\epsilon_y)\,;\\
\mathrm{L}_{\iota\epsilon_y} &: (1+\iota\epsilon_z) \mapsto \iota(\iota\epsilon_x+\epsilon_y)\,;\\
\mathrm{L}_{\iota\epsilon_z} &: (1+\iota\epsilon_z) \mapsto (1+\iota\epsilon_z)\,.
\end{aligned}$$

rotations of $l_\downarrow$

$$\begin{aligned}
\mathrm{L}_{-\epsilon_x} &: (\iota\epsilon_x+\epsilon_y) \mapsto \iota(1+\iota\epsilon_z)\,,\\
\mathrm{L}_{-\epsilon_y} &: (\iota\epsilon_x+\epsilon_y) \mapsto (1+\iota\epsilon_z)\,,\\
\mathrm{L}_{-\epsilon_z} &: (\iota\epsilon_x+\epsilon_y) \mapsto -\iota(\iota\epsilon_x+\epsilon_y)\,,
\end{aligned}$$

boosts of $l_\downarrow$

$$\begin{aligned}
\mathrm{L}_{\iota\epsilon_x} &: (\iota\epsilon_x+\epsilon_y) \mapsto (1+\iota\epsilon_z)\,;\\
\mathrm{L}_{\iota\epsilon_y} &: (\iota\epsilon_x+\epsilon_y) \mapsto -\iota(1+\iota\epsilon_z)\,;\\
\mathrm{L}_{\iota\epsilon_z} &: (\iota\epsilon_x+\epsilon_y) \mapsto -(\iota\epsilon_x+\epsilon_y)\,.
\end{aligned}$$

rotations of $r_\uparrow$

$$\begin{aligned}
\mathrm{R}_{\epsilon_x} &: (1-\iota\epsilon_z) \mapsto \iota(-\iota\epsilon_x-\epsilon_y)\,,\\
\mathrm{R}_{\epsilon_y} &: (1-\iota\epsilon_z) \mapsto -(-\iota\epsilon_x-\epsilon_y)\,,\\
\mathrm{R}_{\epsilon_z} &: (1-\iota\epsilon_z) \mapsto \iota(1-\iota\epsilon_z)\,,
\end{aligned}$$

boosts of $r_\uparrow$

$$\begin{aligned}
\mathrm{R}_{\iota\epsilon_x} &: (1-\iota\epsilon_z) \mapsto -(-\iota\epsilon_x-\epsilon_y)\,;\\
\mathrm{R}_{\iota\epsilon_y} &: (1-\iota\epsilon_z) \mapsto -\iota(-\iota\epsilon_x-\epsilon_y)\,;\\
\mathrm{R}_{\iota\epsilon_z} &: (1-\iota\epsilon_z) \mapsto -(1-\iota\epsilon_z)\,;
\end{aligned}$$

rotations of $r_\downarrow$

$$\begin{aligned}
\mathrm{R}_{\epsilon_x} &: (-\iota\epsilon_x-\epsilon_y) \mapsto \iota(1-\iota\epsilon_z)\,,\\
\mathrm{R}_{\epsilon_y} &: (-\iota\epsilon_x-\epsilon_y) \mapsto (1-\iota\epsilon_z)\,,\\
\mathrm{R}_{\epsilon_z} &: (-\iota\epsilon_x-\epsilon_y) \mapsto -\iota(-\iota\epsilon_x-\epsilon_y)\,,
\end{aligned}$$

boosts of $r_\downarrow$

$$\begin{aligned}
\mathrm{R}_{\iota\epsilon_x} &: (-\iota\epsilon_x-\epsilon_y) \mapsto -(1-\iota\epsilon_z)\,;\\
\mathrm{R}_{\iota\epsilon_y} &: (-\iota\epsilon_x-\epsilon_y) \mapsto \iota(1-\iota\epsilon_z)\,;\\
\mathrm{R}_{\iota\epsilon_z} &: (-\iota\epsilon_x-\epsilon_y) \mapsto (-\iota\epsilon_x-\epsilon_y)\,.
\end{aligned}$$

In the case of the basis

$$l_\uparrow = \frac{1}{2}(\iota\epsilon_x-\epsilon_y)\,, \qquad \overline{l_\uparrow} = r_\uparrow = \frac{1}{2}(-\iota\epsilon_x+\epsilon_y)\,,$$

$$l_\downarrow = \frac{1}{2}(1-\iota\epsilon_z)\,, \qquad \overline{l_\downarrow} = r_\downarrow = \frac{1}{2}(1+\iota\epsilon_z)\,.$$

the transformations take the from of

rotations of $l_\uparrow$

$$\begin{aligned}
\mathrm{L}_{-\epsilon_x} &: (\iota\epsilon_x-\epsilon_y) \mapsto \iota(1-\iota\epsilon_z)\,,\\
\mathrm{L}_{-\epsilon_y} &: (\iota\epsilon_x-\epsilon_y) \mapsto -(1-\iota\epsilon_z)\,,\\
\mathrm{L}_{-\epsilon_z} &: (\iota\epsilon_x-\epsilon_y) \mapsto \iota(\iota\epsilon_x-\epsilon_y)\,,
\end{aligned}$$

boosts of $l_\uparrow$

$$\begin{aligned}
\mathrm{L}_{\iota\epsilon_x} &: (\iota\epsilon_x-\epsilon_y) \mapsto (1-\iota\epsilon_z)\,;\\
\mathrm{L}_{\iota\epsilon_y} &: (\iota\epsilon_x-\epsilon_y) \mapsto \iota(1-\iota\epsilon_z)\,;\\
\mathrm{L}_{\iota\epsilon_z} &: (\iota\epsilon_x-\epsilon_y) \mapsto (\iota\epsilon_x-\epsilon_y)\,.
\end{aligned}$$

rotations of $l_\downarrow$

$$\begin{aligned}
\mathrm{L}_{-\epsilon_x} &: (1-\iota\epsilon_z) \mapsto \iota(\iota\epsilon_x-\epsilon_y)\,,\\
\mathrm{L}_{-\epsilon_y} &: (1-\iota\epsilon_z) \mapsto (\iota\epsilon_x-\epsilon_y)\,,\\
\mathrm{L}_{-\epsilon_z} &: (1-\iota\epsilon_z) \mapsto -\iota(1-\iota\epsilon_z)\,,
\end{aligned}$$

boosts of $l_\downarrow$

$$\begin{aligned}
\mathrm{L}_{\iota\epsilon_x} &: (1-\iota\epsilon_z) \mapsto (\iota\epsilon_x-\epsilon_y)\,;\\
\mathrm{L}_{\iota\epsilon_y} &: (1-\iota\epsilon_z) \mapsto -\iota(\iota\epsilon_x-\epsilon_y)\,;\\
\mathrm{L}_{\iota\epsilon_z} &: (1-\iota\epsilon_z) \mapsto -(1-\iota\epsilon_z)\,.
\end{aligned}$$

rotations of $r_\uparrow$

$$\begin{aligned}
\mathrm{R}_{\epsilon_x} &: (-\iota\epsilon_x+\epsilon_y) \mapsto \iota(1+\iota\epsilon_z)\,,\\
\mathrm{R}_{\epsilon_y} &: (-\iota\epsilon_x+\epsilon_y) \mapsto -(1+\iota\epsilon_z)\,,\\
\mathrm{R}_{\epsilon_z} &: (-\iota\epsilon_x+\epsilon_y) \mapsto \iota(-\iota\epsilon_x+\epsilon_y)\,,
\end{aligned}$$

boosts of $r_\uparrow$

$$\begin{aligned}
\mathrm{R}_{\iota\epsilon_x} &: (-\iota\epsilon_x+\epsilon_y) \mapsto -(1+\iota\epsilon_z)\,;\\
\mathrm{R}_{\iota\epsilon_y} &: (-\iota\epsilon_x+\epsilon_y) \mapsto -\iota(1+\iota\epsilon_z)\,;\\
\mathrm{R}_{\iota\epsilon_z} &: (-\iota\epsilon_x+\epsilon_y) \mapsto -(-\iota\epsilon_x+\epsilon_y)\,;
\end{aligned}$$

rotations of $r_\downarrow$

$$\begin{aligned}
\mathrm{R}_{\epsilon_x} &: (1+\iota\epsilon_z) \mapsto \iota(-\iota\epsilon_x+\epsilon_y)\,,\\
\mathrm{R}_{\epsilon_y} &: (1+\iota\epsilon_z) \mapsto (-\iota\epsilon_x+\epsilon_y)\,,\\
\mathrm{R}_{\epsilon_z} &: (1+\iota\epsilon_z) \mapsto -\iota(1+\iota\epsilon_z)\,,
\end{aligned}$$

boosts of $r_\downarrow$

$$\begin{aligned}
\mathrm{R}_{\iota\epsilon_x} &: (1+\iota\epsilon_z) \mapsto -(-\iota\epsilon_x+\epsilon_y)\,;\\
\mathrm{R}_{\iota\epsilon_y} &: (1+\iota\epsilon_z) \mapsto \iota(-\iota\epsilon_x+\epsilon_y)\,;\\
\mathrm{R}_{\iota\epsilon_z} &: (1+\iota\epsilon_z) \mapsto (1+\iota\epsilon_z)\,.
\end{aligned}$$

B.2.2 Multiplication tables for spinor bases

$$\begin{aligned}
(1+\iota\epsilon_z)(1+\iota\epsilon_z) &= 2(1+\iota\epsilon_z) & (1-\iota\epsilon_z)(1+\iota\epsilon_z) &= 0\\
(1+\iota\epsilon_z)(1-\iota\epsilon_z) &= 0 & (1-\iota\epsilon_z)(1-\iota\epsilon_z) &= 2(1-\iota\epsilon_z)\\
(1+\iota\epsilon_z)(\iota\epsilon_x+\epsilon_y) &= 0 & (1-\iota\epsilon_z)(\iota\epsilon_x+\epsilon_y) &= 2(\iota\epsilon_x+\epsilon_y)\\
(1+\iota\epsilon_z)(\iota\epsilon_x-\epsilon_y) &= 2(\iota\epsilon_x-\epsilon_y) & (1-\iota\epsilon_z)(\iota\epsilon_x-\epsilon_y) &= 0
\end{aligned}$$

$$
\begin{aligned}
(\iota\epsilon_x + \epsilon_y)(1 + \iota\epsilon_z) &= 2(\iota\epsilon_x + \epsilon_y) & (\iota\epsilon_x - \epsilon_y)(1 + \iota\epsilon_z) &= 0 \\
(\iota\epsilon_x + \epsilon_y)(1 - \iota\epsilon_z) &= 0 & (\iota\epsilon_x - \epsilon_y)(1 - \iota\epsilon_z) &= 2(\iota\epsilon_x - \epsilon_y) \\
(\iota\epsilon_x + \epsilon_y)(\iota\epsilon_x + \epsilon_y) &= 0 & (\iota\epsilon_x - \epsilon_y)(\iota\epsilon_x + \epsilon_y) &= 2(1 + \iota\epsilon_z) \\
(\iota\epsilon_x + \epsilon_y)(\iota\epsilon_x - \epsilon_y) &= 2(1 - \iota\epsilon_z) & (\iota\epsilon_x - \epsilon_y)(\iota\epsilon_x - \epsilon_y) &= 0
\end{aligned}
$$

$$
\begin{aligned}
2(1 + \iota\epsilon_z) &= (1 + \iota\epsilon_z)(1 + \iota\epsilon_z) = (\iota\epsilon_x - \epsilon_y)(\iota\epsilon_x + \epsilon_y) \\
2(1 - \iota\epsilon_z) &= (\iota\epsilon_x + \epsilon_y)(\iota\epsilon_x - \epsilon_y) = (1 - \iota\epsilon_z)(1 - \iota\epsilon_z) \\
2(\iota\epsilon_x + \epsilon_y) &= (\iota\epsilon_x + \epsilon_y)(1 + \iota\epsilon_z) = (1 - \iota\epsilon_z)(\iota\epsilon_x + \epsilon_y) \\
2(\iota\epsilon_x - \epsilon_y) &= (1 + \iota\epsilon_z)(\iota\epsilon_x - \epsilon_y) = (\iota\epsilon_x - \epsilon_y)(1 - \iota\epsilon_z)
\end{aligned}
$$

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