On the emergence of Bekenstein entropy from spherical symmetry and quantum criticality

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Abstract

Since its introduction by Jacob Bekenstein, the origin of the entropy of a black hole has remained an open problem in physics. Inspired by the description of black holes as Bose-Einstein condensates of gravitons, it has been suggested that certain properties of black holes may emerge as well in certain non-gravitational condensates. In this master thesis we address the emergence of Bekenstein entropy from quantum criticality and spherical symmetry, deepening this connection between black holes and condensed matter physics. This emergence is possible due to the appearance of gapless modes and the scaling of their number with the area. First, we review the connection between black holes and condensates and reproduce its main results. Next, we consider a particular model with spherical symmetry and coupling proportional to the momentum, which has already been solved with periodic boundary conditions. From a combination of analytical and numerical computations we argue that, if the condensate is put inside a 2-ball (therefore with non-periodic boundary conditions) it still can reach quantum criticality and, in a certain regime, Bekenstein entropy. The topology of the 2-ball is of particular interest, for it allows more direct analogies with the black hole case.

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Contents

Chapter 1

Introduction and motivation

1.1 What are black holes?

Black holes are regarded as one of the most fascinating physical objects since they were first discovered as a solution of Einstein's field equations. The idea of a gravitating object that keeps a region of space-time out of reach was even first considered already before General Relativity (GR) and the notion of gravity as a theory of space-time, when it was conjectured what would happen if an object were so massive that the scape velocity would reach the speed of light at some close distance. The escape velocity for a particle sitting in a Newtonian gravitational field created by for instance point-like mass M is given by:

$$v_e = \sqrt{\frac{2G_N M}{r}} \tag{1.1}$$

Where r is the distance to the point-like mass G_N is Newton's gravitational constant. By setting $v_e = c$ this gives:

$$r = \frac{2G_N M}{c^2} \tag{1.2}$$

Which is the so called Schwarzschild or gravitational radius r_S, i.e. the distance from the point-like mass beyond which light cannot escape the gravitational pull. Of course this argument is not rigorous: in Newtonian mechanics the speed of light does not hold any special status as an observer-independent speed nor as an upper bound on any physical velocity. But it serves as a first approximation to the problem.

In GR the notion of gravitational force is replaced by a refinement in the definition of an inertial or free observer: gravity manifests itself as the dynamics of space-time, which determines the geodesic trajectories of free-falling objects. GR as the theory of space-time is tightly related to and fulfills two key physical principles: the Equivalence Principle, that states that the laws of physics reduce locally to those of Special Relativity (SR), and the Principle of General Covariance, that states that the laws of physics are invariant under any coordinate transformation [30]. This means that, in GR, gravity is not treated as a force but rather as the interaction that determines the nature of the trajectories of free falling objects.

For any static and spherically symmetric source, in 1916 Karl Schwarzschild determined its metric to take the following form (set as usual $c \equiv 1$ from now on):

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{1.3}$$

For any object of size $R < r_S$ this metric describes a black hole. The 2-sphere of radius r_S is called the event horizon. Loosely speaking, any geodesic trajectory that crosses it from the outside region can never cross it back and any geodesic trajectory starting in the interior region can never cross the event horizon. It is remarkable that the naive Newtonian computation gives the same result as in GR. This is due to the character of r_S as fundamental length-scale of the system. Indeed it is the simplest length that one can build using solely G_N, the coupling constant of the theory, and M, the parameter that fully determines the physical system.

1.2 The problem with black holes

The mere existence of black holes poses a threat *a priori* to our knowledge from another field of physics. The second law of thermodynamics states that any physical process taking place in an isolated system has as a result an increase in the entropy of such system:

$$dS \geq 0 \tag{1.4}$$

Hence, the loss of entropy due to the fall of an object into a black hole could be seen as a direct violation of the second law.

Certainly it can be first argued that the laws of thermodynamics are not rigorous fundamental laws but rather an empirical result which is rigorously proven to be statistically favoured, but can also be violated in certain cases. However, the entropy loss due to the fall into a black hole is due to the very nature of the physical phenomenon and not some statistical fluctuation that may or may not happen. Therefore, the statistical argument cannot be summoned here.

One must either disregard the second law or extend it to somehow include phenomena related to the black hole itself. Indeed the most reasonable approach would be to assign an entropy to the black hole. We will go deeper into this in the next chapter, but we advance that this black hole entropy or Bekenstein entropy (for it was first introduced by Jacob Bekenstein in 1972 [3]) is found to be proportional to the area of the event horizon of the black hole. More precisely:

$$S_{BH} = k_B \frac{A}{4L_P^2} \tag{1.5}$$

Where $L_P^2 = \hbar G_N$ is the Planck length squared and k_B is the Boltzmann constant. Oftentimes the Boltzmann constant is simply set to 1 so that the entropy is dimensionless and, correspondingly, temperature carries units of energy. We will use this convention from now on. If black holes do have entropy, do they also have temperature? If we accept black holes to be some sort of thermodynamic systems, indeed entropy can be always defined, regardless of whether the system is in thermodynamic equilibrium or not. This is not true for temperature, which can only be defined for systems in thermodynamic equilibrium. However, even if we take a stationary black hole, like the Schwarzschild black hole, thermal objects emit thermal radiation. This is inconsistent with the very definition of black holes, which emit nothing. We could accept that the black hole has vanishing temperature but then the well-established relation $T = \partial E/\partial S$ could not possibly hold.

A quantum treatment is required in order to overcome this. Stephen Hawking found in 1975 [20] that black holes do emit radiation with a thermal spectrum and the so called Hawking temperature (for the case of a Schwarzschild black hole):

$$T_H = \frac{\hbar}{8\pi G_N M} \tag{1.6}$$

A full quantum description is however necessary in order to completely understand black hole phenomena. There is no consensus around the solution to this problem. In this master thesis we explore a particular line of research: the link between black holes and condensed matter physics.

This master thesis is organized as follows. In chapter 2 we go into the details of the well-known black hole physics, the laws that govern their mechanics and the arguments that lead to their extension to thermodynamic laws, reviewing the work of Bekenstein, Hawking and others. In chapter 3 we review the Quantum-N portrait of black holes, a proposal by Gia Dvali and César Gómez to understand black holes from the quantum point of view as a Bose-Einstein condensate of gravitons. Following this concept, in chapter 4 we review a condensed matter model studied by Gia Dvali from which Bekenstein entropy emerges due to spherical symmetry and quantum criticality. Until this point all contents are reviews with own comments of the author. Moreover, detailed computations are reproduced and added where deemed necessary. In chapter 5 we extend the model of chapter 4 to a topology more suitable or relevant for the description of real black holes and argue by a combination of analytic and numerical methods that, in a regime where full spherical symmetry is approximately restored, quantum criticality leads to the emergence of Bekenstein entropy as well.

Chapter 2

Black hole mechanics and thermodynamics

In this chapter we review the mechanics and thermodynamics of black holes, i.e. what parameters their physics depends on and how they are related to each other. This behaviour was first encoded as laws of black hole mechanics [20]. Their striking similarities with thermodynamics allowed for the introduction of the concepts of entropy [3] and temperature [20] of a black hole. This motivates for yet another link: between black holes and quantum information, as well as the concept of holography [26]. Besides the original literature, two standard references on General Relativity [30] and Black Holes [28] are used throughout this chapter.

2.1 The properties of space-time

The study of black holes in classical gravity, i.e. in GR, is equivalent to the study of their space-time geometry. Minkowski (or simply flat) space-time equipped with the well-known Minkowski metric $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ in inertial coordinates is replaced by a general (not necessarily flat) space-time equipped with a metric tensor with Lorentzian signature.

$$\left(\mathbb{R}^4, \eta\right) \to (M, g) \tag{2.1}$$

The Lorentzian signature is required so that causality is well-defined by the existence of causal curves (with light-like and time-like tangent vectors) and non-causal curves (with space-like tangent vectors). Furthermore, this description admits arbitrary coordinate transformations, i.e. it is generally covariant. It is always possible to pick coordinates such that the metric reduces locally to that of Minkowski, i.e. this description also fulfills the Equivalence Principle. The trajectory of a free particle (a geodesic curve) must be adapted accordingly. In particular, for a trajectory with tangent vector u^μ the following replacement takes place:

$$u^\mu \partial_\mu u^\nu = 0 \quad \to \quad u^\mu \nabla_\mu u^\nu = 0 \tag{2.2}$$

Where ∇_μ is a covariant derivative which, unlike the usual partial derivative ∂_μ, is invariant under general coordinate transformations. Its precise form in components depends on the object on which it acts. For instance, it is equal to the partial derivative when it acts on a scalar field, but it takes the following form when acting on a vector:

$$\nabla_\mu a^\nu = \partial_\mu a^\nu + \Gamma^\nu_{\mu\alpha} a^\alpha \tag{2.3}$$

Where $\Gamma^\nu_{\mu\alpha}$ are called Christopher symbols and depend on the metric tensor. As a general rule, Lorentz invariant expressions can be turned into generally covariant ones by replacing partial derivatives by covariant derivatives.

In order to describe the space-time geometry of a black hole it is first necessary to consider it to be isolated, i.e. independent of the interaction with other gravitational sources. Such a system should have a vanishing gravitational field or, in terms of geometry, should become flat at distances far enough from the black hole. This is the idea behind asymptotic flatness. Loosely speaking, an asymptotically flat space-time admits a set of coordinates (x^0, x^1, x^2, x^3), where x^0 is a time-like coordinate and x^i are space-like coordinates, such that:

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(1/r) \quad \text{for} \quad r \to \infty \quad \text{with} \quad r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \tag{2.4}$$

We would also like to exploit the symmetries that the space-time of a black hole may have. Recall from classical mechanics as well as classical and quantum field theory that symmetries in the Lagrangian come with conserved quantities by means of Noether's theorem. An analogous procedure is possible in GR. Symmetries of the space-time metric in GR are described with the so called Killing vectors, which satisfy that the Lie derivative of the metric along them vanishes, or simply the following condition in local coordinates:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{2.5}$$

One of the main applications of Killing vector fields is that they allow us to construct conserved quantities along geodesics. For a curve parametrised by the affine parameter λ with tangent vector u^μ the following conservation law holds:

$$\frac{d}{d\lambda}\left(g_{\mu\nu} u^\mu \xi^\nu\right) = 0 \tag{2.6}$$

As an example, in Minkowski space-time $g_{\mu\nu} = \eta_{\mu\nu}$ the four coordinate vectors ∂_μ are Killing vectors. This has a result the conservation of the 4-momentum $p^\mu = m u^\mu$ for any particle moving along a geodesic curve. Likewise, there are two relevant Killing vectors for the study of black holes:

- If the space-time has a time-like Killing vector, then the space-time is said to be stationary.
- If the space-time has a space-like Killing vector whose integral curves are closed, then the space-time is said to be axis-symmetric.

Equally important is the construction of quantities associated with the whole space-time and not a particular geodesic. To every Killing vector field ξ we can associate a so called Komar integral, which can be defined on the boundary of a space-like hypersurface Σ:

$$K_\xi(\Sigma) = C \int_{\partial\Sigma} dS_{\mu\nu} \nabla^\mu \xi^\nu \tag{2.7}$$

Where C is some constant. By computing this integral at infinity for an asymptotically flat space-time it becomes a well-defined property of space-time.

Komar integrals are computed at infinity, but there is still one more quantity which is more tightly related to the event horizon instead. The event horizon is a trapping surface in the sense that causal curves cannot cross it back. Therefore, it is also a light-like surface: a surface whose normal vector is light-like. For stationary black holes, there is always a Killing vector field which is normal to the event horizon, and so $\xi^\mu \xi_\nu = 0$. Therefore there exists a function κ which is constant on the horizon and fulfills:

$$\nabla^\mu \left(\xi^\nu \xi_\nu\right) = -2\kappa\xi^\mu \tag{2.8}$$

This function κ is called the surface gravity and is a well-defined property of the black hole.

When we considered the Schwarzschild metric of a spherically symmetric static black hole, it was by assumption time independent. It describes a so called eternal black hole. However, black holes are expected to evolve in time. As a starting point, they must have been created by some physical process. The paradigmatic case is the gravitational collapse of a heavy star, even though one can think of other processes such as the gravitational collapse of density perturbations in the early universe or an ultra-planckian particle scattering. Furthermore, if an object falls into a black hole, a process allowed by the geodesic structure of the Schwarzschild space-time, this should lead to a change in the properties of the black hole itself.

The behaviour of black holes under these changes is described by the so called laws of black hole mechanics, which we will review in the following sections. We will not however deal with dynamic metric tensors, but rather study the behaviour of the parameters that determine an stationary metric. In the following section we consider the most general stationary black hole metric, the Kerr-Newmann metric.

2.2 The Kerr-Newmann metric

The Kerr-Newmann metric describes an stationary, charged, rotating black hole. Hence, it is parametrized by solely three quantities: its mass M, its angular momentum J and its electric charge Q. It is given by:

$$ds^2 = \left(\frac{\Delta - a^2sin^2\theta}{\Sigma}\right)dt^2 + \frac{2asin^2\theta\left(r^2+a^2-\Delta\right)}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left[\frac{\left(r^2+a^2\right)^2 - \Delta a^2 sin^2\theta}{\Sigma}\right]sin^2\theta d\phi^2 \tag{2.9}$$

With:

$$\Sigma = r^2 + a^2cos^2\theta \quad , \quad \Delta = r^2 + a^2 + Q^2 - 2Mr \quad \text{and} \quad a = \frac{J}{M} \tag{2.10}$$

Due to its charge it is accompanied by an electromagnetic field described by the potential:

$$A_\mu = \frac{Qr}{\Sigma}\left(1, 0, 0, asin^2\theta\right) \tag{2.11}$$

It is established by several "no-hair theorems" that the metric of a black hole cannot depend on any additional parameter. It is however legitimate to ask ourselves what these parameters really mean physically, how they can be computed or what happens when their values are perturbed. In order to do this we need them to be more than mere parameters in an expression. Since mass and angular momentum arise from symmetry, while charge arises from the coupling to another field, we will use Komar integrals and Gauss' law to redefine them.

Let us start with the electric charge Q enclosed in a certain space-like hypersurface Σ with boundary $\partial\Sigma$:

$$Q(\Sigma) = \int_\Sigma dS_\mu j^\mu \tag{2.12}$$

Here j^μ is the electric 4-current and dS_μ is the hypersurface-element 1-form. We can use the equation of motion of the electromagnetic field in order to express the 4-current in terms of the electromagnetic field-strength tensor $F^{\mu\nu}$ as well as Stokes theorem in order to obtain the final expression:

$$Q(\Sigma) = \int_\Sigma dS_\mu \nabla_\nu F^{\mu\nu} = \frac{1}{2}\int_{\partial\Sigma} dS_{\mu\nu}F^{\mu\nu} \tag{2.13}$$

If one considers a stationary space-time, then $\xi = \partial_t$ is a Killing vector field. Let Σ be an arbitrarily large ball surrounding the black hole. Then the Komar mass of the black hole is defined as:

$$M(\Sigma) = -\frac{1}{8\pi G}\int_{\partial\Sigma} dS_{\mu\nu}\nabla^\mu(\partial_t)^\nu \tag{2.14}$$

For non-charged black holes, M is independent of the chosen Σ as long as it contains the event horizon and the inner region. For a charged black hole, such as the Kerr-Newmann black hole, Σ should be picked big enough so that all the electromagnetic field is included.

Similarly, if one considers an axis-symmetric space-time, then $\xi = \partial_\theta$ is a Killing vector field and one defines analogously the Komar angular momentum of a black hole:

$$J(\Sigma) = \frac{1}{16\pi G} \int_{\partial\Sigma} dS_{\mu\nu} \nabla^\mu (\partial_\theta)^\nu \tag{2.15}$$

Both quantities can be computed at infinity for asymptotically flat space-times in order to obtain the mass and angular momentum for the whole black hole, which turn out to be equivalent to the parameters M and J considered in the Kerr-Newmann metric already discussed.

For a particular linear combination of the two Killing vector fields of the Kerr-Newmann space-time, one obtains a Killing vector field which is normal to the event horizon of the black hole. This linear combination is:

$$\xi = \partial_t + \Omega_H \partial_\theta \tag{2.16}$$

Where Ω_H is called the angular velocity of the horizon. Therefore, the surface gravity κ can be consistently defined for the Kerr-Newmann space-time. Two other important properties linked to the event horizon are its area A and the electric potential on it Φ_H. Now that the parameters that fully determine the black hole dynamics are well-defined, we can study the laws that they obey.

2.3 The laws of black hole mechanics

Now we are ready to state the laws of black hole mechanics, introduced in [2]:

- Zeroth law:

  For any stationary black hole one can define the surface gravity. Its value is:

$$\kappa = \frac{\sqrt{M^2 - a^2 - Q^2}}{2M\left(M + \sqrt{M^2 - a^2 - Q^2}\right) - Q^2} \tag{2.17}$$

  Its proof can be found in [2] [28] [30].

- First law:

  The variation of the black hole mass after any physical process is a function of the variation of its event horizon area, its angular momentum and its charge:

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ \tag{2.18}$$

  Its proof can be found in [2] [28] [30].

- Second law:

  The variation of the black hole area after any physical process is always non-negative:

  $$dA \geq 0 \tag{2.19}$$

  Its proof can be found in [19] [28] [30].

- Third law:

  No finite amount of physical processes can result in $\kappa = 0$.

  Its proof can be found in [22].

The similarities between these laws and the laws of thermodynamics are striking. This fact suggests that black holes should be treated as thermodynamical systems. Let us focus on the second law, which is the most relevant one for this thesis. If we attempt to establish an analogy between black hole mechanics and thermodynamics, it seems that the area of the event horizon behaves as some sort of entropy or somehow the black hole entropy scales as the area. This is still one of the most puzzling properties of black holes and lacks yet a definite explanation. The concept of entropy of a physical system requires a microscopic description for it, a matter that is far from consensus regarding black holes. From the quantum point of view, the entropy of a physical system is given by the von Neumann formula:

$$S = -Tr\left[\rho log(\rho)\right] \tag{2.20}$$

Where ρ is the density matrix describing the physical system. In the case of a mixed state composed of a set of pure orthogonal states $|\psi_i\rangle$ with probabilities p_i, the entropy reduces to:

$$S = -\sum_i p_i log(p_i) \tag{2.21}$$

Compare the von Neumann entropy with Shannon's information formula [3]:

$$I = -\sum_i p_i log(p_i) \tag{2.22}$$

This brings a correspondence between entropy and information. Therefore, physical systems with large entropy can be regarded as physical systems storing a large amount of information which is unknown to an external observer. From this point of view, it is quite intuitive to think of a black hole as an object with entropy, since due to the "no hair theorems" an external observer has no information on the properties of the interior or the fate of the infalling matter, besides the total mass, angular momentum and electric charge.

2.4 Area as entropy: the Bekenstein argument

The idea of identifying the area of the event horizon as entropy or recognizing the scaling of the entropy with the area of the event horizon was suggested by Jacob Bekenstein [3] [4] [5]. As he argued, in the event of a creation of a black hole, any information an observer had about its constituents is lost, with the exception of its total mass, angular momentum and charge, which become the sole properties of the black hole that can be proved from its exterior. Therefore, this loss of information must be compensated by the black hole carrying entropy or, correspondingly, storing information.

Not only do black holes store information, but also they set un upper bound on the information that can be compressed in a region of space. This is the so called Bekenstein bound [6]:

$$S \leq \frac{2\pi RM}{\hbar} \tag{2.23}$$

Let us further think about the meaning of this area-scaling entropy. By dimensional analysis this area should be expressed in units of the Planck-length squared, which is the only fundamental length-scale available in gravity:

$$S_{BH} \sim \frac{A}{L_P^2} \tag{2.24}$$

Since $L_P^2 = \hbar G_N$ this introduces quantum effects into the game. Indeed it is no surprise that that any underlying description of a physical system should ultimately be quantum. This is only a proof of the importance of such quantum properties.

For $\mathcal{N}$ equiprobable states the von Neumann entropy in eq (2.21) reduces to:

$$S = -\mathcal{N}\frac{1}{\mathcal{N}}log\left(\frac{1}{\mathcal{N}}\right) = log(\mathcal{N}) \tag{2.25}$$

Hence, we can imagine the Bekenstein entropy as follows: consider a 2-sphere of area A covered with area elements of size L_P^2, being all of them distinguishable by some label or quantum number. Assume further that each of them can be found in a discrete number of quantum states n. Then the number of possible states of the 2-sphere is given by:

$$\mathcal{N} = n^{\frac{A}{L_P^2}} \tag{2.26}$$

The logarithm of this expression scales as the area A of the sphere. However, this large number of states is not per se enough if they are not degenerate. It is not trivial to see how these independent degrees of freedom could be gapless. Indeed, for a degree of freedom localized in a box of size L_P one would expect an enormous energy gap of $\Delta E \simeq \frac{\hbar}{L_P} = M_P$. This question should be successfully addressed by any realistic model.

Therefore, the two key ingredients that any quantum description should satisfy in order to exhibit Bekenstein entropy can be summarized as 1) the appearance of gapless modes and 2) the scaling of the number of these gapless modes with the area of the system.

Only then a quantum system would exhibit Bekenstein entropy. The possibility of such mechanism taking place in a black hole is thrilling. For it entails the storage of the information of a 3-dimensional volume into a 2-dimensional surface, one speaks of storing information in holographic degrees of freedom. The emergence of this holographic degrees of freedom is still an open problem. We will deal later with one of the current proposals in the framework of the Quantum N-portrait.

2.5 Hawking radiation: from mechanics to thermodynamics

The very same concept of black hole in classical GR prevents it from emitting any radiation and therefore from having any kind of temperature. This picture changes if we take quantum effects into account. Consider for instance the action of a free scalar field in Minkowski space-time:

$$S = \int d^4x \partial^\mu \phi \partial_\mu \phi \tag{2.27}$$

This can be extended to arbitrary geometric backgrounds defined by their metric tensor $g_{\mu\nu}$. These backgrounds are taken to be classical solutions of Einstein's field equations.

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{2.28}$$

The study of such actions is part of Quantum Field Theory in curved space-time. [23] Recall that in ordinary QFT in flat space-time, the vacuum state of the theory is chosen as such state that is invariant under Lorentz transformations, which also turns out to be the state of minimum energy (eigenstate of the Hamiltonian with lowest eigenvalue). This is not trivially extendible to arbitrary space-time geometries. However, when the space-time is stationary, then it is possible to pick as basis of eigenfunctions with defined and positive frequency [28]. The vacuum state of the corresponding quantum modes turns out to be the state of minimum energy and therefore a legitimate vacuum state.

Consider now the evolution of a stationary space-time into another stationary space-time, with some non-stationary space-time in between. The preferred choice of basis for the solution of the generalized Klein-Gordon equation need not be the same for both stationary space-times. For instance, the scalar field could admit the following mode expansion in the first space-time:

$$\phi = \sum_i \left(a_i \psi_i + a_i^\dagger \psi_i^* \right) \tag{2.29}$$

Whereas in the second space-time the eigenfunctions and ladder operators could be much different:

$$\phi = \sum_i \left(a'_i \psi'_i + a'^{\dagger}_i \psi'^{*}_i \right) \tag{2.30}$$

In order for both expansions to be consistent, for the scalar field is one and only and spans over the whole space-time, two conditions must be fulfilled. The first simply relates both basis of eigenvectors by a linear transformation:

$$\psi'_i = \sum_j \left(A_{ij} \psi_j + B_{ij} \psi^{*}_j \right) \tag{2.31}$$

In addition, the ladder operators must also transform:

$$a'_j = \sum_i \left(a_i A_{ij} + a^{\dagger}_i B^{*}_{ij} \right) \tag{2.32}$$

This a so called Bogoliubov transformation. Such transformations will play a crucial role in this master thesis. As a consequence, the vacua of both stationary space-times will not only be different, but also will be eigenstates of eigenvalue 0 of different number operators. When the expected value of the number operator of the second space-time is evaluated in the vacuum state of the first space-time, one finds:

$$\langle N'_i \rangle = \left(B^{\dagger} B \right)_{ii} \tag{2.33}$$

And hence particle creation can occur during the non-stationary transition. For a transition to a black hole, the following result was found by Stephen Hawking [20]:

$$\langle N'_i \rangle = \left(B^{\dagger} B \right)_{ii} = \frac{1}{e^{\frac{2\pi\omega_i}{\kappa}} - 1} \tag{2.34}$$

This corresponds to a thermal spectrum, i.e. a Bose-Einstein distribution of temperature:

$$T_H = \frac{\kappa}{2\pi} \tag{2.35}$$

Which is the so called Hawking radiation.

Now the whole picture is more clear. The analogy between the laws of black hole mechanics and thermodynamics is justified by the thermal nature of the Hawking radiation emitted by the black hole. Furthermore, the factor that should relate entropy and area is now fixed and equal to 1/4 in order for the temperature-entropy term in the first law to be consistent. Still, even though it is possible to trace the quantum origin of the black hole temperature with the techniques provided by QFT in curved space-time, the same cannot be achieved for the entropy. Motivated by this, in the next chapter we consider a candidate to a full quantum description of black holes.

Chapter 3

Quantum N-portrait of black holes

The existence of Bekenstein entropy comes with the need of finding a quantum description of black holes. Even classically, the entropy of a physical system emerges from the degeneracy of a macro-state, i.e. from the existence of a large number of micro-states, with different properties, which however lead to a same macro-state. But if we accept that any fundamental theory of Nature must be quantum, as our knowledge from the other fundamental interactions strongly suggests, then we should aim for a quantum microscopic description of black holes as well.

This chapter is mainly a review of the Quantum N-portrait of black holes, a model introduced by Gia Dvali and César Gómez in [15] and extended in [13] [14] [16] [17] among other papers. The main idea is to view classical solutions of Einstein's field equations as Bose-Einstein condensates of soft gravitons with a large occupation number N. In order to achieve this, a quantum treatment of gravity is required. This is possible when it is regarded as an effective field theory [8] [7] [9] [11]. Furthermore the model establishes a link between black hole physics and condensed matter theory.

3.1 Gravity as a quantum field theory

It is sometimes claimed that there exists a deep fundamental incompatibility between gravity and quantum physics. This statement is an exaggeration. From the point of view of quantum field theory, gravity can be formulated as the theory of a symmetric rank-2 massless tensor field. This theory, or rather this point of view of GR, has two key features:

- It is an effective field theory, only descriptive up to the Planck scale, at which the perturbative expansion breaks down and a still to be found UV-completion is needed.
- It departs from classical GR via quantum (loop) corrections only reasonably near the Planck scale. This means that the theory is as powerful as classical GR in terms of making predictions for experiments at our reach. However, it is much more cumbersome computationally since vertexes and propagators always carry a complicated index structure.

GR as a quantum field theory is therefore mostly useful conceptually: it reconciles gravity and quantum physics and allows us to effectively treat gravitons and their interactions in the low energy regime. Consider the Einstein-Hilbert action, which describes full GR:

$$S_{EH} = \frac{2}{\kappa^2} \int d^4x \sqrt{-g} R \tag{3.1}$$

Where $\kappa^2 = 32\pi G_N$. Then one performs an expansion around the Minkowski metric, which is a vacuum solution for vanishing cosmological constant:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{3.2}$$

Keeping the expansion to linear order only, GR reduces to the so called linear gravity, with the following lagrangian:

$$\mathcal{L}_{LIN} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h^\lambda_\mu \partial_\nu h^{\mu\nu} - \partial^\mu h_{\mu\nu}\partial^\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \tag{3.3}$$

Or, more compactly:

$$\mathcal{L}_{LIN} = \frac{1}{2} h^{\mu\nu} \left(\mathcal{E} h\right)_{\mu\nu} \tag{3.4}$$

Were we wrote the full differential operator as $\mathcal{E}$. However, this lagrangian is purely kinetic. Graviton-graviton interactions arise only when additional terms are taken into account:

$$\mathcal{L} = \mathcal{L}_{LIN} + \kappa \mathcal{O}\left((\partial h)^3\right) + \kappa^2 \mathcal{O}\left((\partial h)^4\right) \tag{3.5}$$

One could of course add yet additional terms and, in fact, an infinite amount of them is needed to recover the Einstein-Hilbert action. Nevertheless, we restrict ourselves to this point, where gravitons interact through 3-point and 4-point vertices, i.e. via graviton exchange or one vertex interaction. Then the coupling strength given by these processes is:

$$\alpha = \frac{L_P^2}{L^2} \tag{3.6}$$

This a dimensionless quantity that relates both length-scales of the graviton-scattering process: the Planck length $L_P = \hbar G_N$ and the characteristic wavelength L of the interacting gravitons. Since the graviton propagator scales as $\sim 1/p^2$, the corresponding potential of the long-range interaction takes the familiar expression of the Newtonian potential:

$$V(r) = -\hbar \frac{\alpha}{r} \tag{3.7}$$

In the low energy or, equivalently, large wave-length regime, α is small, but otherwise the interaction strength would include further powers of α. Here one could also recover

Newton's potential by computing α with the Comptom wavelength of some massive particle. The expansion breaks down at $\alpha \sim 1$, which happens when a scattering process approaches the Planck length. Computations beyond this point require a yet to be found UV-completion of the theory. That is why GR as a quantum field theory is actually an effective field theory. The existence of an energy cut-off is a success of the quantum approach to GR.

3.2 Minkowski space-time as a coherent state of gravitons

Staying in the low energy regime, we could ask ourselves about the quantum structure of classical solutions of GR. In the same way that every-day classical objects possess and underlying quantum structure, one could expect any classical solution of a physical theory to be some sort of effective description of a quantum solution. Before applying this idea to black holes, let us explore how Minkowski space-time can be realized as a coherent state in linearized gravity [17]. Consider the linearized gravity lagrangian with the addition of a mass term:

$$\mathcal{L}_{FP} = \frac{1}{2} h^{\mu\nu} \left(\mathcal{E}h\right)_{\mu\nu} - \frac{1}{2} m^2 h^{\mu\nu} \left(h_{\mu\nu} - \eta_{\mu\nu} h\right) \tag{3.8}$$

This is the so called Fierz-Pauli lagrangian. The equation of motion for this massive graviton is:

$$\left(\mathcal{E}h\right)_{\mu\nu} = m^2 (h_{\mu\nu} - \eta_{\mu\nu} h) \tag{3.9}$$

Given that the divergence of $\left(\mathcal{E}h\right)_{\mu\nu}$ is identically zero, the divergence of the whole equation of motion gives the so called Fierz-Pauli constraint:

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = 0 \tag{3.10}$$

The massive and the massless theories are of course related. One must however be careful before taking the limit $m \to 0$, for the massive theory has 5 propagating degrees of freedom while the massless one has only 2. These 5 degrees of freedom can be decomposed by the Stückelberg mechanism as follows:

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + \frac{1}{6} \eta_{\mu\nu} \phi + \frac{1}{3} \frac{\partial_\mu \partial_\nu}{m^2} \phi \tag{3.11}$$

Where $\tilde{h}_{\mu\nu}$ is a massless graviton, A_μ is a massless vector field and ϕ is a real scalar field. In this language, the lagrangian is invariant under the gauge symmetry:

$$\tilde{h}_{\mu\nu} \to \tilde{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad A_\mu \to A_\mu - \xi_\mu \tag{3.12}$$

Which can be used to set $A_\mu = 0$ by gauge fixing. In the massless case, the lagrangian has furthermore a conserved current due to the gauge symmetry:

$$J_{\gamma(\mu\nu)} = \partial_\gamma h_{\mu\nu} - \eta_{\mu\nu}\partial_\gamma h - (\partial_\mu h_{\gamma\nu} + \partial_\nu h_{\gamma\mu}) + \frac{1}{2}(\eta_{\gamma\mu}\partial_\nu + \eta_{\gamma\nu}\partial_\nu)h + \eta_{\mu\nu}\partial^\beta h_{\gamma\beta} \tag{3.13}$$

This current is conserved thanks to the equation of motion:

$$\partial^\gamma J_{\gamma(\mu\nu)} = (\mathcal{E}h)_{\mu\nu} = 0 \tag{3.14}$$

In the massive case, however, $(\mathcal{E}h)_{\mu\nu} = m^2(h_{\mu\nu} - \eta_{\mu\nu}h)$ and therefore the current is not conserved. Still, the trace of the current vanishes due to the Fierz-Pauli constraint:

$$J_\gamma = \eta^{\mu\nu} J_{\gamma(\mu\nu)} = 2(\partial^\nu h_{\gamma\nu} - \partial_\gamma h) = 0 \tag{3.15}$$

This defines a new current that is indeed conserved for the massive Fierz-Pauli field and can therefore be used to construct a conserved (vanishing) charge:

$$Q := -\int d^3x J_0 = \int d^3x 2\left(\partial_t \tilde{h} - \partial^\nu \tilde{h}_{t\nu}\right) + \int d^3x \partial_t \phi \equiv Q_{\tilde{h}} + Q_\phi \tag{3.16}$$

The Stückelberg decomposition was used in order to split the charge Q into two contributions from the massless graviton and the scalar field. The would-be contribution from the vector field vanishes off-shell because it is a boundary term. In contrast, Q is meant to vanish on-shell.

$$\begin{aligned} Q_A &= \int d^3x 2\left[2\partial_0\partial^\mu A_\mu - \partial^\mu\left(\partial_0 A_\mu + \partial_\mu A_0\right)\right] = \int d^3x 2\partial^\mu\left(\partial_\mu A_0 - \partial_0 A_\mu\right) \\ &= \int d^3x 2\partial^i\left(\partial_i A_0 - \partial_0 A_i\right) = 0 \end{aligned} \tag{3.17}$$

At the quantum level, this classically vanishing charge Q defines an operator that annihilates any physical state of the graviton field. In the massless limit (i.e. taking directly $m \to 0$ but keeping the five, decoupled, degrees of freedom), the action on the states of the scalar field is exactly the opposite to the action on the states of the massless graviton. Let us expand the scalar field in the usual way:

$$\phi(x) = \int \frac{d^3k}{\sqrt{\omega_k}}\left(e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}a_k + e^{+i(\omega_k t - \vec{k}\cdot\vec{x})}a_k^\dagger\right) \tag{3.18}$$

Then the part of the charge corresponding to the scalar field is:

$$\begin{aligned} Q_\phi &= \int d^3x \partial_t \phi = \int \frac{d^3x d^3k}{\sqrt{\omega_k}}(-i\omega_k)\left(e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}a_k + e^{+i(\omega_k t - \vec{k}\cdot\vec{x})}a_k^\dagger\right) \\ &= \int \frac{d^3k}{\sqrt{\omega_k}}(-i\omega_k)\left(e^{-i\omega_k t}a_k - e^{i\omega_k t}a_k^\dagger\right)\delta^{(3)}(\vec{k}) = -i\sqrt{m}\left(e^{-imt}a_0 - e^{imt}a_0^\dagger\right) \end{aligned} \tag{3.19}$$

Remarkably enough, this is the generator of a displacement operator. Hence, its exponential $e^{-i\nu Q_\phi}$ generates a coherent state of zero-momentum particles with shift parameter ν and occupation number $N = m\nu^2$. The expectation value of the field operator for such a state is given simply by the shift parameter, i.e. $\langle\nu|\,\phi\,|\nu\rangle = \nu$. Therefore we can also find the expected value for the massless graviton field:

$$\langle\nu|\,\tilde{h}_{\mu\nu}\,|\nu\rangle = -\frac{1}{6}\eta_{\mu\nu}\,\langle\nu|\,\phi\,|\nu\rangle = -\frac{1}{6}\eta_{\mu\nu}\nu \tag{3.20}$$

Therefore Minkowski spacetime corresponds not only to the vacuum state of the theory but also to a collection of coherent states, characterized by their shift parameter ν or, corrrespondingly, to their occupation number. This realization opens the door to general interpretations of classical solutions of Einstein's equations as some kind of quantum state of gravitons. In the case of Minkowski spacetime this was quite trivial, since the Fierz-Pauli gravitons are added to a classical background, which is Minkowski metric itsef, and one would think that the addition of zero-energy and zero-momentum modes should not change the background metric. However, we need not stop here: is it possible to build other classical (or appropximately classical) configurations by means of graviton states built on top of Minkowski spacetime? This idea, applied to black holes, is the key of the quantum N-portrait for black holes.

3.3 Black holes as Bose-Einstein condensates of gravitons

After checking the addition of zero-energy and zero-momentum modes to Minkowski background, let us apply this idea to the addition of other kind of modes. Recall that for very long wave-lengths, the coupling strength α between gravitons is very small, and we can reduce the interaction to only first order in it. For macroscopic wave-lengths, this coupling strenght is extremely small. However, due to the fact that the graviton field is bosonic, it can form condensates, i.e. quantum states of very large occupation number N, so that the individual coupling strength between a pair of gravitons α is almost negligible, but the collective coupling strength αN is not.

Consider the total energy of a graviton located in such a condensate. Its kinetic energy is given by $E_k = \hbar/L$, whereas its potential energy is given $V = -\alpha N\hbar/L$. Let us impose the sum of them to be zero:

$$E_k + V = (1-\alpha N)\frac{\hbar}{L} = 0 \tag{3.21}$$

This is possible only if the occupation number reaches a critical value given by:

$$N = N_C = 1/\alpha \tag{3.22}$$

Then the total energy of the gravitons located in the condensate vanishes or, equivalently, the kinetic energy is compensated by the potential energy and the condensate becomes self-sustained. This means that the condensate can exist without coupling to any other source but the graviton field itself. In this sense, the graviton field is unlike other fields such as the photon field: its self-coupling allows for its self-sourcing and ultimately for self-sustainability. The kinetic energy of the graviton becomes its effective mass in the bound state and is given by:

$$m = E_k = \frac{\hbar}{L} = \frac{\hbar}{\sqrt{N}L_P} = \frac{M_P}{\sqrt{N}} \tag{3.23}$$

And the total mass of the system is simply:

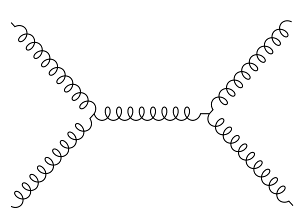
$$M = Nm = \sqrt{N}M_P \tag{3.24}$$

We can still think of such condensate even if the occupation number is lower than the critical point, i.e. out of the self-sustainability condition. Suppose that we are given a condensate of N gravitons with mass $m = M_P/\sqrt{N}$ in a region of size $R > L$. Then the corresponding coupling strength is $\alpha = L_P^2/R^2 < L_P^2/L^2$. Since $\alpha < 1/N$ the condensate is not self-sustained and requires some external source to be maintained. This could describe any gravitational field created by a source which is not a black hole. But if the condensate shrinks to the size L, then it becomes self-sustained. L is nothing but $L = L_P M/M_P$ and therefore corresponds to the Schwarzschild radius of the system.

Self-sustainability is also linked to maximal packing. This means, that at the critical point, all defining characteristics of the condensate can be expressed in terms of N. To sum this up:

- Characteristic wavelength and size of the condensate: $L = \sqrt{N}L_P$.
- Coupling strength $\alpha = 1/N$.
- Mass or energy of the condensate: $M = \sqrt{N}\frac{1}{L_P}$

The quantum N-portrait pictures Hawking radiation as emission of gravitons from the condensate due to graviton-graviton scattering. In particular, consider a $2 \to 2$ scattering process. If one of the scattered gravitons increases its energy above the escape energy of the gravitational potential, then it will leave the condensate. The rate of such a process can be estimated as follows:



$$\Gamma_{2\to 2} = \frac{1}{N^2}N^2\frac{1}{\sqrt{N}L_P} = \frac{1}{\sqrt{N}L_P} \tag{3.25}$$

Some comments on this:

- The factor $1/N^2 = \alpha^2$ comes from the two vertices, each of them delivering a factor equal to the coupling strength α to the scattering amplitude.
- The factor $N^2 \simeq N(N-1)$ comes from the combinatorics of choosing two gravitons among N indistinguishable ones.
- The factor $\frac{\hbar}{\sqrt{N}L_P}$ is simply the characteristic energy of the process.
- Corrections to this rate are $1/N$-suppressed. The reason is that the combinatoric factors are fixed for the process but the addition of the next to leading order term in the perturbative expansion of the amplitude leads to three vertices instead of two and therefore an additional overall factor of $1/N$. Similarly, further corrections scale as larger inverse powers of N.

The number of gravitons N changes as:

$$\frac{dN}{dt} = -\Gamma_{2\to 2} = -\frac{1}{\sqrt{N}L_P} \tag{3.26}$$

And the total mass of the condensate M changes accordingly:

$$\frac{dM}{dt} = \frac{dM}{dN}\frac{dN}{dt} = -\frac{M_P}{\sqrt{N}}\frac{1}{\sqrt{N}L_P} = -\frac{M_P}{NL_P} = -\frac{M_P^3}{M^2 L_P} \tag{3.27}$$

Compare this with the total power emitted by a thermal source (black-body radiation):

$$P = \sigma A T^4 \tag{3.28}$$

Where $\sigma \sim \frac{k_B^4}{\hbar^3}$ is the Stefan-Boltzmann constant and k_B is the Boltzmann constant, A is the area of the radiating body and T is its temperature. Here we kept k_B explicit to keep better track of the units. For the graviton condensate we know that $A \sim L^2 = L_P^2 M^2/M_P^2$. By direct comparison we get the following approximate result for the temperature of the emitted radiation:

$$k_B T_H \sim \frac{M_P^2}{M} \tag{3.29}$$

Which agrees with the Hawking temperature up to a numerical factor. This result is exact when terms suppressed by inverse powers of N are negligible, i.e. in the limit $N \to \infty$. A more detailed discussion of the various physical limits of the graviton condensate will be considered in a later section.

3.4 Black holes as Bose-Einstein condensates at quantum phase transition

One can go further into the study of the quantum N-portrait of black holes by establishing direct analogies with actual, solvable Bose-Einstein condensates. In particular, properties that arise at the critical point of a phase transition can be linked to the quantum N-portrait. In order to illustrate this, let us consider a simple and well known model: a bosonic field with a quartic self-interaction:

$$H = -\hbar L_0 \int_V d^3x \psi(x) \nabla^2 \psi(x) - g \int_V d^3x \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) \tag{3.30}$$

Such a Hamiltonian with attractive interaction has been studied for instance in [24]. The field $\psi(x)$ is such that the particle number density is given by the correlator $n(x) = \langle \psi(x)^\dagger \psi(x) \rangle$. Therefore the field $\psi(x)$ has dimension $[\text{length}]^{-3/2}$. The model comes then with two parameters, a characteristic length L_0 and a coupling constant g of dimension $[\text{length}]^3[\text{mass}]$. By comparison with the usual kinetic term $-\hbar^2 \nabla^2 / 2m$, it is clear that L_0 is nothing but the Compton wave-length of the bosons and therefore corresponds to the range of their interactions.

As in the quantum N-portrait, we wish to localize the particles in a certain region of space. For simplicity let us pick it to be a finite cubic box of size $2\pi R$ with periodic boundary conditions. Moreover, let us fix the total number of particles to N:

$$\int d^3x \psi^\dagger(x) \psi(x) = N \tag{3.31}$$

The eigenfunctions of the Laplace operator in the finite size box are nothing but the plane waves:

$$\psi(x) = \frac{1}{\sqrt{V}} e^{i \frac{\vec{k}\vec{x}}{R}} \tag{3.32}$$

Due to the boundary conditions the wave vector $\vec{k}$ is quantized such that its components are restricted to be integers. Knowing this, let us expand the field as a series of ladder operators:

$$\psi(x) = \sum_{\vec{k}} \frac{a_{\vec{k}}}{\sqrt{V}} e^{i \frac{\vec{k}\vec{x}}{R}} \quad \text{and} \quad \psi^\dagger(x) = \sum_{\vec{k}} \frac{a^\dagger_{\vec{k}}}{\sqrt{V}} e^{-i \frac{\vec{k}\vec{x}}{R}} \tag{3.33}$$

As usual, the ladder operators obey the canonical commutation relation:

$$[a_{\vec{k}}, a^\dagger_{\vec{k}}] = \delta_{\vec{k}\vec{k}'} \tag{3.34}$$

By plugin this in the Hamiltonian it becomes:

$$H = \sum_{\vec{k}} \vec{k}^2 \frac{\hbar L_0}{R^2} a^\dagger_{\vec{k}} a_{\vec{k}} - \sum_{\vec{k},\vec{k}',\vec{p}} \frac{g}{V} a^\dagger_{\vec{k}+\vec{p}} a^\dagger_{\vec{k}'-\vec{p}} a_{\vec{k}} a_{\vec{k}'} \tag{3.35}$$

Where momentum conservation arises as a consequence of the orthogonality of the plane waves:

$$\int_V d^3x \frac{1}{\sqrt{V}} e^{i\frac{\left(\vec{a}+\vec{b}\right)\cdot\vec{x}}{R}} = \delta^{(3)}\left(\vec{a}+\vec{b}\right) \tag{3.36}$$

Introducing a re-scaling of the Hamiltonian and the coupling constant:

$$\mathcal{H} \equiv \frac{R^2}{\hbar L_0} H \quad \text{and} \quad \alpha \equiv \frac{4gR^2}{\hbar V L_0} \tag{3.37}$$

We can rewrite the Hamiltonian in a simpler and dimension-less form:

$$\mathcal{H} = \sum_{\vec{k}} \vec{k}^2 a_{\vec{k}}^\dagger a_{\vec{k}} - \frac{1}{4}\alpha \sum_{\vec{k},\vec{k}',\vec{p}} a_{\vec{k}+\vec{p}}^\dagger a_{\vec{k}'-\vec{p}}^\dagger a_{\vec{k}} a_{\vec{k}'} \tag{3.38}$$

Since we are interested in the formation of a Bose-Einstein condensate and the quantum fluctuations on top of it, we assume that the lowest mode $k = 0$ has a large occupation number $N_0 \gg 1$ associated to it. In this situation the ladder operators corresponding to the $k = 0$ mode act on the N_0-particle state as follows:

$$a_0^\dagger \left|N_0\right\rangle = \sqrt{N_0+1}\left|N_0+1\right\rangle \simeq \sqrt{N_0}\left|N_0\right\rangle \quad \text{and} \quad a_0\left|N_0\right\rangle = \sqrt{N_0}\left|N_0-1\right\rangle \simeq \sqrt{N_0}\left|N_0\right\rangle \tag{3.39}$$

On this grounds it is physically justified to perform the so called Bogoliubov approximation or replacement, namely the replacement of the ladder operators of the $k = 0$ mode by c-numbers:

$$a_0^\dagger = a_0 = \sqrt{N_0} \tag{3.40}$$

Let us plug this in the Hamiltonian:

$$\begin{aligned}\mathcal{H} =& \sum_{\vec{k}\neq 0} \vec{k}^2 a_{\vec{k}}^\dagger a_{\vec{k}} - \frac{1}{4}\alpha \sum_{\vec{k},\vec{k}'} \sqrt{N_0} a_{\vec{k}'+\vec{k}}^\dagger a_{\vec{k}} a_{\vec{k}'} \cdot 2 - \frac{1}{4}\alpha \sum_{\vec{k}',\vec{p}} a_{\vec{p}}^\dagger a_{\vec{k}'-\vec{p}}^\dagger \sqrt{N_0} a_{k'} \cdot 2 \\ & - \frac{1}{4}\alpha \sum_{\vec{p}} N_0 a_{-\vec{p}} a_{\vec{p}} - \frac{1}{4}\alpha \sum_{\vec{p}} N_0 a_{\vec{p}}^\dagger a_{-\vec{p}}^\dagger - \frac{1}{4}\alpha \sum_{\vec{p}} N_0 a_{\vec{p}}^\dagger a_{\vec{p}} \cdot 4 - \frac{1}{4}\alpha N_0^2\end{aligned} \tag{3.41}$$

Recall that N is a conserved quantity, given by the previously established normalization condition for the field operator. Therefore, we need to express the Hamiltonian in terms of N and not N_0. The normalization becomes after the Bogoliubov replacement:

$$N = N_0 + \sum_{\vec{k}\neq 0} a_{\vec{k}}^\dagger a_{\vec{k}} \tag{3.42}$$

With this, the Hamiltonian becomes:

$$
\begin{aligned}
\mathcal{H} = &\sum_{\vec{k}\neq 0} \vec{k}^2 a^\dagger_{\vec{k}} a_{\vec{k}} - \frac{1}{4}\alpha \sum_{\vec{k},\vec{k}'} \left(\sqrt{N - \sum_{\vec{k}\neq 0} a^\dagger_{\vec{k}} a_{\vec{k}}} \right) a^\dagger_{\vec{k}'+\vec{k}} a_{\vec{k}} a_{\vec{k}'} \cdot 2 \\
&- \frac{1}{4}\alpha \sum_{\vec{k}',\vec{p}} a^\dagger_{\vec{p}} a^\dagger_{\vec{k}'-\vec{p}} \left(\sqrt{N - \sum_{\vec{k}\neq 0} a^\dagger_{\vec{k}} a_{\vec{k}}} \right) a_{k'} \cdot 2 \\
&- \frac{1}{4}\alpha \sum_{\vec{p}} \left(N - \sum_{\vec{k}\neq 0} a^\dagger_{\vec{k}} a_{\vec{k}} \right) a_{-\vec{p}} a_{\vec{p}} - \frac{1}{4}\alpha \sum_{\vec{p}} \left(N - \sum_{\vec{k}\neq 0} a^\dagger_{\vec{k}} a_{\vec{k}} \right) a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}} \\
&- \frac{1}{4}\alpha \sum_{\vec{p}} \left(N - \sum_{\vec{k}\neq 0} a^\dagger_{\vec{k}} a_{\vec{k}} \right) a^\dagger_{\vec{p}} a_{\vec{p}} \cdot 4 - \frac{1}{4}\alpha \left(N - \sum_{\vec{k}\neq 0} a^\dagger_{\vec{k}} a_{\vec{k}} \right)^2
\end{aligned}
\tag{3.43}
$$

Since we are mainly interested in the physics of the condensate we neglect the interactions between the non-condensed bosons, i.e. those terms that are cubic or quartic in the remaining ladder operators. Then we get (up to unimportant 0-point energy terms):

$$
\mathcal{H} = \sum_{\vec{k}\neq 0} \left[\left(\vec{k}^2 - \frac{\alpha N}{2} \right) a^\dagger_{\vec{k}} a_{\vec{k}} - \frac{1}{4}\alpha N \left(a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} + a_{\vec{k}} a_{-\vec{k}} \right) \right] \tag{3.44}
$$

This Hamiltonian is bilinear but is not diagonal, so it is not possible to read off from it the energy levels on top of the condensate. In order to do that, it must be diagonalized. This can be achieved by means of a Bogoliubov transformation, which is a linear transformation of the ladder operators. Such transformations were already discussed in the context of Hawking radiation and are equally important here.

$$
a_{\vec{k}} = u_{\vec{k}} b_{\vec{k}} + v^*_{-\vec{k}} b^\dagger_{-\vec{k}} \quad \text{and} \quad a^\dagger_{\vec{k}} = u^*_{\vec{k}} b^\dagger_{\vec{k}} + v_{-\vec{k}} b_{-\vec{k}} \tag{3.45}
$$

The only restriction to its coefficients is that the new ladder operators b and $b^\dagger$ must fulfill the canonical commutation relations. This implies:

$$
\begin{aligned}
1 = [a_{\vec{k}}, a^\dagger_{\vec{k}}] &= [u_{\vec{k}} b_{\vec{k}} + v^*_{-\vec{k}} b^\dagger_{-\vec{k}}, u^*_{\vec{k}} b^\dagger_{\vec{k}} + v_{-\vec{k}} b_{-\vec{k}}] \\
&= |u_{\vec{k}}|^2 [b_{\vec{k}}, b^\dagger_{\vec{k}}] + u_{\vec{k}} v_{-\vec{k}} [b_{\vec{k}}, b_{-\vec{k}}] + v^*_{\vec{k}} u^*_{\vec{k}} [b^\dagger_{-\vec{k}}, b^\dagger_{\vec{k}}] + |v_{-\vec{k}}|^2 [b^\dagger_{-\vec{k}}, b_{-\vec{k}}] \\
&= |u_{\vec{k}}|^2 - |v_{-\vec{k}}|^2
\end{aligned}
\tag{3.46}
$$

We will make furthermore two Ansätze:

- The Bogoliubov coefficients are equal for $\vec{k}$ and $-\vec{k}$ i.e. $u_{\vec{k}} = u_{-\vec{k}}$ and $v_{\vec{k}} = v_{-\vec{k}}$. This is reasonable on physical grounds, since the Hamiltonian has rotational symmetry. For this very same reason, the coefficients should furthermore only depend on $\vec{k}^2$.

We will see that that is indeed the case, but for now we will not assume it. Still, we obtain the following interesting restriction:

$$|u_{\vec{k}}|^2 - |v_{\vec{k}}|^2 = 1 \tag{3.47}$$

Which suggests that we can parametrize the Bogoliubov coefficients by means of hyperbolic functions, since $cosh^2(x) - sinh^2(x) = 1$ and therefore we express them in terms of one single momentum-dependent angle:

$$u_{\vec{k}} = cosh\left(\theta_{\vec{k}}\right) \quad \text{and} \quad v_{\vec{k}} = sinh\left(\theta_{\vec{k}}\right) \tag{3.48}$$

- The Bogoliubov coefficients are real. In principle this is not necessarily true, but as we will see, for this model real solutions are obtainable.

Now, in order to obtain this coefficients, we just have to plug in the transformation in the Hamiltonian and impose that any non-diagonal terms vanish. This will allow us also to obtain the energy spectrum:

$$\begin{aligned} \mathcal{H} =& \sum_{\vec{k}\neq 0} \left(\vec{k}^2 - \frac{\alpha N}{2}\right) \left(cosh\left(\theta_{\vec{k}}\right) b^\dagger_{\vec{k}} + sinh\left(\theta_{\vec{k}}\right) b_{-\vec{k}}\right) \left(cosh\left(\theta_{\vec{k}}\right) b_{\vec{k}} + sinh\left(\theta_{-\vec{k}}\right) b^\dagger_{-\vec{k}}\right) \\ & - \frac{1}{4}\alpha N \sum_{\vec{k}\neq 0} [\left(cosh\left(\theta_{\vec{k}}\right) b^\dagger_{\vec{k}} + sinh\left(\theta_{-\vec{k}}\right) b_{-\vec{k}}\right) \left(cosh\left(\theta_{-\vec{k}}\right) b^\dagger_{-\vec{k}} + sinh\left(\theta_{\vec{k}}\right) b_{\vec{k}}\right) \\ & \cdot \left(cosh\left(\theta_{\vec{k}}\right) b_{\vec{k}} + sinh\left(\theta_{-\vec{k}}\right) b^\dagger_{-\vec{k}}\right) \left(cosh\left(\theta_{-\vec{k}}\right) b_{-\vec{k}} + sinh\left(\theta_{\vec{k}}\right) b^\dagger_{\vec{k}}\right)] \end{aligned} \tag{3.49}$$

Regroup the different terms by factoring out the operators:

$$\begin{aligned} \mathcal{H} =& \sum_{\vec{k}\neq 0} b^\dagger_{\vec{k}} b_{\vec{k}} \left[\left(\vec{k}^2 - \frac{\alpha N}{2}\right) \left(cosh^2\left(\theta_{\vec{k}}\right) + sinh^2\left(\theta_{\vec{k}}\right)\right) - \frac{1}{4}\alpha N \cdot 4 sinh\left(\theta_{\vec{k}}\right) cosh\left(\theta_{\vec{k}}\right)\right] \\ & + b^\dagger_{\vec{k}} b^\dagger_{-\vec{k}} \left[\left(\vec{k}^2 - \frac{\alpha N}{2}\right) cosh\left(\theta_{\vec{k}}\right) sinh\left(\theta_{\vec{k}}\right) - \frac{1}{4}\alpha N \left(cosh^2\left(\theta_{\vec{k}}\right) + sinh^2\left(\theta_{\vec{k}}\right)\right)\right] \\ & + b_{\vec{k}} b_{-\vec{k}} \left[\left(\vec{k}^2 - \frac{\alpha N}{2}\right) cosh\left(\theta_{\vec{k}}\right) sinh\left(\theta_{\vec{k}}\right) - \frac{1}{4}\alpha N \left(cosh^2\left(\theta_{\vec{k}}\right) + sinh^2\left(\theta_{\vec{k}}\right)\right)\right] \end{aligned} \tag{3.50}$$

In order for the Hamiltonian to be diagonal we simply need the coefficients of terms of the form bb or $b^\dagger b^\dagger$ to vanish. Using the known hyperbolic identities $cosh(x)sinh(x) = \frac{1}{2}sinh(2x)$ and $cosh^2(x) + sinh^2(x) = cosh(2x)$, the expression left is:

$$\left(\vec{k}^2 - \frac{\alpha N}{2}\right) \frac{1}{2} sinh\left(2\theta_{\vec{k}}\right) - \frac{1}{4}\alpha N \cosh\left(2\theta_{\vec{k}}\right) = 0 \tag{3.51}$$

Which simply leads to :

$$tanh\left(2\theta_{\vec{k}}\right) = \frac{\alpha N}{2\left(\vec{k}^2 - \frac{\alpha N}{2}\right)} \tag{3.52}$$

From this expression we can derive the Bogoliubov coefficients $u_{\vec{k}} = sinh\left(\theta_{\vec{k}}\right)$ and $v_{\vec{k}} = cosh\left(\theta_{\vec{k}}\right)$.

Once the diagonalization condition is fulfilled, the hamiltonian looks like this:

$$\mathcal{H} = \sum_{\vec{k}\neq 0} \epsilon_{\vec{k}} b^{\dagger}_{\vec{k}} b_{\vec{k}} \tag{3.53}$$

And the energy spectrum of the excitations on top of the condensate looks like this:

$$\epsilon_{\vec{k}} = \left(\vec{k}^2 - \frac{\alpha N}{2}\right) cosh\left(2\theta_{\vec{k}}\right) - \frac{1}{2}\alpha N sinh\left(2\theta_{\vec{k}}\right) \tag{3.54}$$

By squaring this and subtracting four times the diagonalization condition 3.51 squared (which is equal to 0), we get:

$$\begin{aligned}\epsilon^2_{\vec{k}} = &\left(\vec{k}^2 - \frac{\alpha N}{2}\right)^2 cosh^2\left(2\theta_{\vec{k}}\right) + \frac{1}{4}\alpha^2 N^2 sinh^2\left(2\theta_{\vec{k}}\right) - \left(\vec{k}^2 - \frac{\alpha N}{2}\right)\alpha N cosh\left(2\theta_{\vec{k}}\right) sinh\left(2\theta_{\vec{k}}\right)\\ &- \left(\vec{k}^2 - \frac{\alpha N}{2}\right)^2 sinh\left(2\theta_{\vec{k}}\right) - \frac{\alpha^2 N^2}{4} cosh^2\left(\theta_{\vec{k}}\right) + \alpha N cosh\left(2\theta_{\vec{k}}\right) sinh\left(2\theta_{\vec{k}}\right)\left(\vec{k}^2 - \frac{\alpha N}{2}\right)\end{aligned} \tag{3.55}$$

Using the hyperbolic identity $cosh^2(x) - sinh^2(x) = 1$ this leads to:

$$\epsilon^2_{\vec{k}} = \left(\vec{k}^2 - \frac{\alpha N}{2}\right)^2 - \frac{\alpha^2 N^2}{4} = \vec{k}^2\left(\vec{k}^2 - \alpha N\right) \tag{3.56}$$

And finally the energy spectrum is (we choose the positive sign of the square root for consistency with the result for $\alpha = 0$, which can be directly read off from the non-transformed Hamiltonian):

$$\epsilon_{\vec{k}} = \sqrt{\vec{k}^2\left(\vec{k}^2 - \alpha N\right)} \tag{3.57}$$

The energy gap of the first Bogoliubov level (those modes with $\vec{k}^2 = 1$) vanishes for:

$$N = N_C = \frac{1}{\alpha} \tag{3.58}$$

Then one of the criterion for the emergence of Bekenstein entropy is fulfilled, namely the appearance of gapless modes. Remarkably enough, the condition that relates N and α coincides with the self-sustainability condition for the quantum N-portrait of black holes. For the condensate, this implies an energy degeneracy that leads to the emergence of a macroscopic entropy. For $\alpha > 1/N$, these modes acquire a negative energy and become tachyonic. In this regime the uniform Bose-Einstein condensate is no longer the ground state of the theory and therefore the description in terms of the Bogoliubov approximation breaks down. This means that at $\alpha = 1/N$ a phase transition takes place.

Recall that we neglected the interactions of the bosonic excitations on top of the condensate among themselves. These interactions are suppressed with a factor of $1/N$ with respect to the condensate - excitations interactions and therefore, for finite N, the Hamiltonian is not exactly diagonal and there can be transitions from the Bose-Einstein condensate background to the first Bogoliubov level. These excitations are energetically very cheap. Furthermore, there are some a-particles which do not belong to the condensate. This phenomenon is called quantum depletion. The number density of depleted a-particles to each $\vec{k}$-level is given by:

$$\begin{aligned} n_{\vec{k}} &= \langle a^\dagger_{\vec{k}} a_{\vec{k}} \rangle = \langle \left(u_{\vec{k}} b^\dagger_{\vec{k}} + v_{\vec{k}} b_{-\vec{k}} \right) \left(u_{\vec{k}} b_{\vec{k}} + v_{\vec{k}} b^\dagger_{-\vec{k}} \right) \rangle \\ &= u^2_{\vec{k}} \langle b^\dagger_{\vec{k}} b_{\vec{k}} \rangle + u_{\vec{k}} v_{\vec{k}} \langle b_{-\vec{k}} b_{\vec{k}} \rangle + v_{\vec{k}} u_{\vec{k}} \langle b^\dagger_{\vec{k}} b^\dagger_{-\vec{k}} \rangle + v^2_{\vec{k}} \langle b_{-\vec{k}} b^\dagger_{-\vec{k}} \rangle \\ &= v^2_{\vec{k}} \end{aligned} \tag{3.59}$$

It was assumed here that the Bose-Einstein condensate is at 0 temperature, so that the number density of b-particles for $\vec{k} \neq 0$ is 0. So, we need to know explicitly the Bogoliubov coefficients in order to compute the number density of depleted particles. Use now the following algebraic expressions for hyperbolic functions:

$$sinh^2(x) = \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) \quad \text{and} \quad tahn^{-1}(x) = \frac{1}{2} log \left(\frac{1+x}{1-x} \right) \tag{3.60}$$

Therefore:

$$\begin{aligned} v^2_{\vec{k}} &= sinh^2 \left(\frac{1}{4} log \left(\frac{1 + \frac{\alpha N}{2\left(\vec{k}^2 - \frac{\alpha N}{2}\right)}}{1 - \frac{\alpha N}{2\left(\vec{k}^2 - \frac{\alpha N}{2}\right)}} \right) \right) = sinh^2 \left(\frac{1}{4} log \left(\frac{\vec{k}^2}{\vec{k}^2 - \alpha N} \right) \right) \\ &= \frac{1}{4} \left(\sqrt{\frac{\vec{k}^2}{\vec{k}^2 - \alpha N}} + \sqrt{\frac{\vec{k}^2 - \alpha N}{\vec{k}^2}} - 2 \right) = \frac{1}{2} \left(\frac{\vec{k}^2 - \frac{\alpha N}{2}}{\epsilon_{\vec{k}}} - 1 \right) \end{aligned} \tag{3.61}$$

At the critical point $\alpha = 1/N$ this quantity diverges for $\vec{k}^2 = 1$, which indicates a very fast quantum depletion and a phase transition. Near the critical point, the coefficients are suppressed for large $\vec{k}^2$ and hence it is enough to consider only the first Bogoliubov level in order to compute the total number density of depleted particles:

$$\Delta N \simeq n_1 = \frac{1}{2} \left(\frac{1 - \frac{\alpha N}{2}}{\sqrt{1 - \alpha N}} - 1 \right) \sim \sqrt{N} \tag{3.62}$$

The characteristic time-scale of the interaction that leads to depletion is given by:

$$\Delta t \sim L_0 \tag{3.63}$$

And therefore the time-scale for $\sqrt{N}$ particle scatterings is:

$$\Delta T = \sqrt{N}\delta t \sim \sqrt{N}L_0 \tag{3.64}$$

$$\frac{\Delta N}{\Delta T} \sim \frac{\sqrt{N}}{L_0} \tag{3.65}$$

One can recover from this model the physics of the quantum N-portrait by setting the size of the condensate equal to the characteristic length of the interaction $R = L_0 \equiv L$ and its coupling constant proportional to Newton's gravitational constant or, equivalently, the Planck length squared: $g = \hbar L_P^2$. For this choice of the parameters the interaction strength becomes $\alpha \sim \frac{L_P^2}{L^2}$. Then every parameter can be expressed in terms of the occupation number N and the Planck length as was the case for the quantum N-portrait, and the system satisfies the maximal packing condition. The rate of depletion is then:

$$\frac{\Delta N}{\Delta T} = \frac{1}{\sqrt{N}L_P} \tag{3.66}$$

In agreement with the Quantum N-portrait. The modes of the first Bogoliubov play the role of the holographic degrees of freedom of the black hole and are responsible for both the emergence of macroscopic Bekenstein entropy and Hawking radiation, via quantum degeneracy at the critical point and quantum depletion, respectively.

Actually, this model does not exhibit Bekenstein entropy itself, since it fulfills the criterion of appearance of gapless modes but not the scaling of their number with the area of the condensate. Instead, we see that the number of degenerate states scales as $\sim N$, since we can define up to N states with some of the constituent bosons in the first Bogoliubov level. This alone does not lead to an entropy proportional to N. In addition, it is required some additional discrete characteristic with n possible values so that the number of degenerate states becomes n^N and the entropy $S \sim N$. This role could be played for instance by helicity in the case of a Bose-Einstein condensate of gravitons.

3.5 The classical and semi-classical limits

The quantum N-portrait introduces a full quantum description of the black hole phenomena, albeit it still breaks down at the Planck scale. After all, gravity as a quantum field theory is an effective one. In order to proper recover the black hole physics known in GR or in QFT in curved space-time, two physical limits are introduced:

- The classical limit. It corresponds to setting:

$$\hbar \to 0 \tag{3.67}$$

This way the all quantum effects are neglected. In particular, the black hole does not emit Hawking radiation.

- The semi-classical limit. It corresponds to setting:

$$G_N \to 0 \quad , \quad L = MG_N = \text{fixed}, \quad \hbar = \text{fixed} \tag{3.68}$$

This way quantum effects are preserved, but their back-reaction to the geometric background of the black hole is neglected. In particular, the spectrum of the Hawking radiation emitted by the black hole is purely thermal, with temperature given by $T_H = \hbar/L$.

In both limits the entropy diverges. This is expected since in both limits the Planck length becomes $L_P \to 0$. In the framework of the quantum N-portrait, the information stored by the condensate becomes infinite, but it is also infinitely hard to recover this information from the exterior. In the classical limit there is no radiation emitted at all, whereas in the semi-classical limit the only emitted radiation is thermal and carries no information besides the value of its own temperature.

3.6 Physics of condensates: general Bogoliubov transformations

We have seen how the quantum N-portrait describes black holes as Bose-Einstein condensates of gravitons, which are furthermore at the critical point of quantum phase transition from a condensate phase to some other phase. In the analogous Bose-Einstein condensate discussed in the previous section this other phase was a bright soliton [24]. Near this critical point, the energy gap to the lowest quantum excitations on top of the condensate vanishes, so that the quantum state becomes degenerate (Bekenstein entropy emerges) and the condensate undergoes quantum depletion (Hawking radiation occurs). All this information is obtained by diagonalizing the Hamiltonian of this quantum excitations by means of a Bogoliubov transformation. Let us finish the review of the quantum N-portrait by studying the general properties of these transformations. The relevant original literature can be found in the papers by Constantino Tsallis [29] and Yoel Tikochinsky [27].

Consider a general Hamiltonian in second quantized form:

$$H = \sum_{ij} \left(\omega_{ij} a_i^\dagger a_j + \omega_{ij}^* a_i a_j^\dagger + \nu_{ij} a_i a_j + \nu_{ij}^* a_i^\dagger a_j^\dagger \right) \tag{3.69}$$

Where i, j are understood as multi-indeces. Introduce the notation:

$$\vec{a} = \begin{bmatrix} a_1 \\ \dots \\ a_N \\ a_1^\dagger \\ \dots \\ a_N^\dagger \end{bmatrix} \vec{a}^\dagger = \begin{bmatrix} a_1^\dagger & \dots & a_N^\dagger & a_1 & \dots & a_N \end{bmatrix} \tag{3.70}$$

So the Hamiltonian becomes:

$$H = \vec{a}^\dagger \begin{bmatrix} \omega & \nu^* \\ \nu & \omega^* \end{bmatrix} \vec{a} = \vec{a}^\dagger \mathcal{H} \vec{a} \tag{3.71}$$

In this formalism the canonical commutation relation is simply given by:

$$\vec{a}\vec{a}^\dagger - \left(\vec{a}^{\dagger T}\vec{a}^T\right)^T = I_- = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{3.72}$$

Now consider a transformation matrix T which acts on the ladder operators $\vec{b} = T\vec{a}$. For the canonical commutation relation to be respected this matrix has to satisfy $TI_-T^\dagger = I_-$:

$$\begin{aligned} \vec{b}\vec{b}^\dagger - \left(\vec{b}^{\dagger T}\vec{b}^T\right)^T = & T\vec{a}\vec{a}^\dagger T^\dagger - \left(T^{\dagger T}\vec{a}^{\dagger T}\vec{a}^T T^T\right)^T = T\vec{a}\vec{a}^\dagger T^\dagger - T\left(\vec{a}^{\dagger T}\vec{a}^T\right)^T T^\dagger \\ & T\left(\vec{a}\vec{a}^\dagger - \left(\vec{a}^{\dagger T}\vec{a}^T\right)\right)^T T^\dagger = TI_-T^\dagger \end{aligned} \tag{3.73}$$

Furthermore, the Hamiltonian is diagonalized if:

$$[H, \vec{b}] = -2I_-\mathcal{H}_D\vec{b} \tag{3.74}$$

Where $\mathcal{H}_D$ is the coefficient matrix of the diagonal hamiltonian. Notice that the LHS can also be written as follows:

$$[H, \vec{b}] = [H, T\vec{a}] = T[H, \vec{a}] = -2TI_-\mathcal{H}\vec{a} \tag{3.75}$$

Where $\mathcal{H}$ is the coefficient matrix of the original hamiltonian H. Hence:

$$TI_-\mathcal{H}T^{-1} = I_-\mathcal{H}_D \tag{3.76}$$

Therefore, the eigenenergies of the system are given by the absolute values of the eigenvalues of the matrix $I_-\mathcal{H}$. The Bogoliubov coefficients can be obtained from the transformation matrix T. One could argue that in general $I_-\mathcal{H}$ need not be an hermitian matrix and therefore there is no guarantee that it is diagonalizable. However, take a look again at the property $TI_-T^\dagger$, which means that $T^{-1} = I_-T^\dagger I_-$. Hence:

$$TI_-\mathcal{H}I_-T^\dagger = \mathcal{H}_D \tag{3.77}$$

Since $\mathcal{H}$ is hermitian, $I_-\mathcal{H}I_-$ is hermitian as well and always admits a diagonalization by a unitary transformation. However, T is not unitary and therefore it is not useful to obtain the eigenvalues of $I_-\mathcal{H}I_-$. Despite this, starting from the fact that such a unitary diagonalization matrix exists, we reproduce the proof that also the matrix T with the desired properties exists, namely that it diagonalizes the non-hermitian matrix $I_-\mathcal{H}$ and preserves $TI_-T^\dagger = I_-$. It is important for the proof to assume that $\mathcal{H}$ is not only hermitian but also positive definite, in which case $I_-\mathcal{H}I_-$ is also hermitian an positive definite. Let us name U the unitary matrix that diagonalizes $I_-\mathcal{H}I_-$ and let us call E the resulting diagonal matrix.

$$U^\dagger I_-\mathcal{H}I_-U = E \tag{3.78}$$

Let us introduce the regular (but not unitary) matrix $V = UE^{-1/2}$, so that:

$$V^\dagger I_-\mathcal{H}I_-V = I \tag{3.79}$$

Now, the matrix $V^\dagger I_-V$ is hermitian by construction and therefore can be diagonalized by the unitary matrix P:

$$P^\dagger V^\dagger I_-VP = \eta \tag{3.80}$$

Then the matrix $W = VP$ conserves the number of positive eigenvalues of I_- and $I_-\mathcal{H}I_-$ because it performs a congruence transformation:

$$W^\dagger I_-W = \eta \quad \text{and} \quad W^\dagger I_-\mathcal{H}I_-W = I \tag{3.81}$$

Notice as a check that indeed $I_-\mathcal{H}I_-$ is positive definite. On the other hand, half of the entries of the diagonal matrix η are positive and the other half are negative. Let us build a new matrix θ containing the absolute value of the entries of η, i.e. $\theta_{ij} = |\eta_{ij}|$. Then let us introduce the matrix $\zeta = W\theta^{-1/2}$, whose column vectors we denote as ζ_i for convenience. These column vectors have the property that they rearrange the positive and negative entries of I_- as they are in η:

$$\zeta_i^\dagger I_-\zeta_j = \frac{\eta_{ij}}{|\eta_{ij}|} \tag{3.82}$$

And furthermore:

$$\zeta_i^\dagger I_-\mathcal{H}I_-\zeta_j = |\eta_{ij}|^{-1} \tag{3.83}$$

Now let us relabel the column vectors ζ_i in two groups, z_i and $\hat{z}_i$ in such a way that:

$$z_i^\dagger I_-z_j = \delta_{ij}I \quad \text{and} \quad \hat{z}_i^\dagger I_-\hat{z}_j = -\delta_{ij}I \tag{3.84}$$

Therefore, the matrix resulting of putting together first the column vectors z_i and then $\hat{z}_i$ satisfies both:

$$Z^\dagger I_- Z = I_- \quad \text{and} \quad Z^\dagger I_- \mathcal{H} I_- Z = X \tag{3.85}$$

Where X is diagonal. Then we see that $Z = T^\dagger$ is a solution to the diagonalization problem and the diagonal Hamiltonian is $\mathcal{H}_D = X$. This statement completes the proof. Hence, it is possible to obtain the true energy levels of a Hamiltonian, as long as its coefficient matrix $\mathcal{H}$ is positive definite, by diagonalizing $I_- \mathcal{H}$ with the usual procedures, which can in particular be implemented numerically. From a physical point of view, this result makes perfectly sense: if the Hamiltonian describes for instance the quantum excitations on top of a Bose-Einstein condensate, one would suspect that something goes wrong if the system is allowed to take energies below the condensate itself. This generalizes as well one of the results considered in the previous section, in particular how the condensate depletes and undergoes a quantum phase transition at the point where the energy of the first Bogoliubov level vanishes.

Chapter 4

Attractive Bose-Einstein gas with periodic boundary conditions

In chapter 3 we saw how the Quantum N-portrait of black holes explains successfully one of the key properties of semi-classical black holes, namely Hawking radiation. It goes even further in suggesting a first order quantum correction to the perfect thermal spectrum of Hawking radiation that scales as N^{-1} and therefore is only relevant for black holes of small mass. It also introduces a framework in which the other key semi-classical property of black holes, Bekenstein entropy, can be explained and unterstood. However, its precise computation is left.

It is often argued that it is a fundamental property of gravity that a gravity theory in d-dimensions can be described by degrees of freedom of a non-gravity theory in $(d-1)$-dimensions. This of course matches with the picture that represents an area-scaling entropy as corresponding to a set of labeled minimal area elements with one degree of freedom each that can take a discrete number of states. Another realization of this fundamental property is the conjectured AdS/CFT correspondence. These ideas are linked under the concept of holography.

It is legitimate then to ask ourselves what is the origin of holography, i.e. does holography only exist in gravity or is it linked to defining properties of gravity that nevertheless can be found in other physical theories? With the introduction of the quantum N-portrait of black holes, one reasonable first step is to check whether holography can be a property of other, possibly simpler, Bose-Einstein condensates at the critical point of phase transition. In this chapter we review a model introduced by Gia Dvali in [12], which allows an explicit and exact computation of an area-scaling entropy. This motivates the further pursue of the study of black holes and other classical gravitational configurations as graviton condensates and deepens the connection between black hole physics and condensed matter theory.

4.1 The model

We consider therefore the following model for a scalar field ψ that encodes the distribution of the particle number density on a d-dimensional sphere of radius R and total volume Ω, i.e. $n(x) = \langle \psi^\dagger(x)\psi(x)\rangle$. It is described by the Hamiltonian:

$$H = \int_{S_d} d^d\Omega \left(\psi^\dagger \left(-\frac{\hbar^2}{2mR^2}\Delta \right) \psi - g\Omega \left(\psi^\dagger \left(-\frac{\hbar^2}{2mR^2}\Delta \right) \psi^\dagger \right) \left(\psi \left(-\frac{\hbar^2}{2mR^2}\Delta \right) \psi \right) \right) \tag{4.1}$$

This model has two key properties shared with gravity:

- It is attractive. Notice that the kinetic term expressed is positive defined, whereas the interaction term carries an opposite sign.
- The interactions are derivative-coupled, which means that the corresponding interaction strength is proportional to the momentum.

However, it departs from the graviton field since it lacks the following two key properties:

- The spin-2 nature of the fundamental field.
- The relativistic (Lorentz) invariance.

Note that here the Laplacian is given by derivatives on the angular coordinates of the sphere and therefore is dimension-less. For convenience, let us rescale the Laplacian and correspondingly the Hamiltonian as follows:

$$\frac{\hbar^2}{2mR^2}\Delta \to \Delta \tag{4.2}$$

So the Hamiltonian becomes:

$$H = \int_{S_d} d^d\Omega \left(\psi^\dagger \left(-\Delta \right) \psi - g\Omega \left(\psi^\dagger \left(-\Delta \right) \psi^\dagger \right) \left(\psi \left(-\Delta \right) \psi \right) \right) \tag{4.3}$$

This way, the Hamiltonian and the Laplacian operators have dimensions of $[energy]$, whereas the field ψ is dimensionless. The coupling g, however, carries dimension of $[energy]^{-1}$. We advance now, due to the dimensionality of the coupling that, contrary to the model studied in section 3.4., the critical point will not be achieved at $gN = 1$ but rather must be related to some fundamental or characteristic scale of the system.

4.2 The free Hamiltonian

Let us first consider the free case, i.e. when $g = 0$:

$$H = \int_{S_d} d^d\Omega \left(\psi^\dagger \left(-\Delta\right) \psi\right) \tag{4.4}$$

The eigenfunctions of the Laplacian on a d-sphere are the well-known spherical harmonics:

$$\Delta Y_k = -\epsilon_k Y_k \quad \text{with} \quad \epsilon_k = \frac{\hbar^2}{2mR^2} k_d \left(k_d + d - 1\right) \tag{4.5}$$

The label k or generalized angular momenta is a multi-index composed solely of integers:

$$k \equiv \left(k_1, ..., k_d\right) \tag{4.6}$$

That satisfy the property:

$$|k_1| \leq k_2 \leq ... \leq k_d = 0, 1, ..., \infty \tag{4.7}$$

For $d = 2$ this reduces to the familiar quantum numbers l and m describing angular momentum, where $l \geq 0$ (but is otherwise arbitrary) and $|m| \leq l$.

The spherical harmonics form a complete set of orthonormal functions on the sphere:

$$\int_{S_d} d^d\Omega Y_k^* Y_{k'} = \delta_{kk'} \tag{4.8}$$

Note that the eigenenergy ϵ_k depends only on k_d and therefore will be degenerate in the other quantum numbers. In particular the number of modes with the same eigenenergy scales as follows:

$$\mathcal{N} = \sum_{k_{d-1}=0}^{k_d} \sum_{k_{d-2}=0}^{k_{d-1}} ... \sum_{k_1=-k_2}^{k_2} \sim \left(k_d\right)^{d-1} \tag{4.9}$$

We will see later how this is related to the emergence of Bekenstein-like entropy.

4.3 The interacting Hamiltonian

Let us know recover the full interacting Hamiltonian:

$$H = \int_{S_d} d^d\Omega \left(\psi^\dagger \left(-\Delta\right) \psi - g\Omega \left(\psi^\dagger \left(-\Delta\right) \psi^\dagger\right) \left(\psi \left(-\Delta\right) \psi\right)\right) \tag{4.10}$$

Knowing the eigenfunctions for the free model, let us represent the field operator as an infinite sum over the annihilation operators for the modes with different generalized angular momenta:

$$\psi = \sum_k Y_k(\theta_a) a_k \quad \text{and correspondingly} \quad \psi^\dagger = \sum_k Y_k^*(\theta_a) a_k^\dagger \tag{4.11}$$

This is the typical expansion in non-relativistic field theories, where an expansion over both creation and destruction operators is unnecessary since causality is not an issue. The creation and annihilation operators satisfy the algebra of the canonical commutation relation:

$$[a_j, a_k^\dagger] = \delta_{jk} \quad [a_j, a_k] = 0 = [a_j^\dagger, a_k^\dagger] \tag{4.12}$$

After inserting this we get the following Hamiltonian in second quantization form:

$$H = \sum_k \epsilon_k a_k^\dagger a_k - g\Omega \sum_{s,k,q,r} C_{skqr} \epsilon_k \epsilon_r a_s^\dagger a_k^\dagger a_q a_r \tag{4.13}$$

In the kinetic term due to the orthonormality property of the spherical harmonics we can get rid of one sum. In the interacting term this is not possible since we have an integral of four spherical harmonics, for which we introduce the notation:

$$C_{skqr} := \int_{S_d} d^d\Omega Y_s^* Y_k^* Y_q Y_r \tag{4.14}$$

This Hamiltonian looks like, and is indeed, unbounded from below. This can be avoided by the addition of higher interacting terms which are repulsive. As a result, the total energy will decrease for higher and higher angular momentum until it reaches a minimum, after which the higher order, repulsive terms become dominating. The existence of this minimum or lower bound can be mimicked by simply setting an angular momentum cut-off at $k_d = k_*$, which corresponds to an eigenergy of the free Hamiltonian of:

$$\epsilon_* = \frac{\hbar^2}{2mR^2} k_* (k_* + d - 1) \tag{4.15}$$

At this point it is simply necessary to assume that $k_* \gg 1$ or, equivalently, that $\epsilon_* \gg \frac{\hbar^2}{2mR^2}$.

In the next step we consider that the field is in the Bose-Einstein condensate phase, i.e. is in a state where the ground state mode (the mode with quantum number $k = 0$) is said to be macroscopically occupied, which means that the occupation number of this mode is very large, whereas the occupation numbers of the other modes ($k \neq 0$) are restricted to be much smaller, but not necessarily of order 0 or 1:

$$\langle a_0^\dagger a_0 \rangle = N_0 \gg 1 \quad \text{and} \quad \langle a_0^\dagger a_0 \rangle \ll N_0 \tag{4.16}$$

In this regime it is allowed to perform the so called Bogoliubov approximation or replacement, as we did in section 3.3. After the Bogoliubov replacement the Hamiltonian looks like this:

$$H = \sum_{k\neq 0}^{k_*} \epsilon_k a_k^\dagger a_k - gN_0\Omega \sum_{k\neq 0}^{k_*} \epsilon_k^2 C_{0k0r} a_k^\dagger a_k - g\sqrt{N_0}\Omega \sum_{s,k,r\neq 0}^{k_*} C_{sk0r}\epsilon_k\epsilon_r a_s^\dagger a_k^\dagger a_r$$
$$- g\sqrt{N_0}\Omega \sum_{k,q,r\neq 0}^{k_*} C_{0kqr}\epsilon_k\epsilon_r a^\dagger a_q a_r - g\Omega \sum_{s,k,q,r\neq 0}^{k_*} C_{skqr}\epsilon_k\epsilon_r a_s^\dagger a_k^\dagger a_q a_r \quad (4.17)$$

However, the conserved quantity is the total number of particles N and not the total number of condensed particles N_0. We proceed then to use the restriction:

$$N = N_0 + \sum_{k\neq 0}^{k_*} a_k^\dagger a_k \quad (4.18)$$

And so the Hamiltonian becomes:

$$H = \sum_{k\neq 0}^{k_*} \epsilon_k a_k^\dagger a_k - g\left(N - \sum_{k\neq 0}^{k_*} a_k^\dagger a_k\right)\Omega \sum_{k\neq 0}^{k_*} \epsilon_k^2 C_{0k0r} a_k^\dagger a_k$$
$$- g\sqrt{\left(N - \sum_{k\neq 0}^{k_*} a_k^\dagger a_k\right)}\Omega \sum_{s,k,r\neq 0}^{k_*} C_{sk0r}\epsilon_k\epsilon_r a_s^\dagger a_k^\dagger a_r$$
$$- g\sqrt{\left(N - \sum_{k\neq 0}^{k_*} a_k^\dagger a_k\right)}\Omega \sum_{k,q,r\neq 0}^{k_*} C_{0kqr}\epsilon_k\epsilon_r a^\dagger a_q a_r$$
$$- g\Omega \sum_{s,k,q,r\neq 0}^{k_*} C_{skqr}\epsilon_k\epsilon_r a_s^\dagger a_k^\dagger a_q a_r \quad (4.19)$$

Taking the cut-off k_* to be fixed, the model still keeps two control parameters, the total particle number N and the coupling strength g. Since N is very large due to the macroscopical occupation of the ground state we can take the limit to infinite. However, this makes most terms in the Hamiltonian infinite as well unless we take also the limit of vanishing coupling. We can combine them in the so called double-scaling limit, which keeps only the collective coupling as relevant interaction strength:

$$N \to \infty \quad g \to 0 \quad gN = finite \quad (4.20)$$

After doing this, we are left with only one control parameter, the effective collective coupling gN and furthermore those terms of Hamiltonian proportional to $g\sqrt{N}$ or simply g vanish, leaving only:

$$H = \sum_{k\neq 0}^{k_*} \epsilon_k a_k^\dagger a_k - gN\Omega \sum_{k\neq 0}^{k_*} \epsilon_k^2 C_{0k0r} a_k^\dagger a_k \quad (4.21)$$

In this case the integral of the four spherical harmonics takes a simple form:

$$C_{0k0r} = \int_{S_d} d^d\Omega |Y_o|^2 Y_k^* Y_r = \frac{1}{\Omega}\int_{S_d} d^d\Omega Y_k^* Y_r = \frac{\delta_{kr}}{\Omega} \tag{4.22}$$

So that:

$$H = \sum_{k\neq 0}^{k_*} \epsilon_k \left(1 - gN\epsilon_k\right) a_k^\dagger a_k \tag{4.23}$$

Remarkably enough, this Hamiltonian is perfectly diagonal, so we can read off its corresponding energy levels directly. But recall that the effective collective coupling gN remains as a control parameter of the system. Then we can give it a particular value such that the eigenenergy of some of the modes vanishes and they become gapless with respect to the ground state. This happens precisely if we take the limit:

$$gN \to \frac{1}{\epsilon_*} \tag{4.24}$$

Finally, the Hamiltonian becomes:

$$H = \sum_{k\neq 0}^{k_*} \epsilon_k \left(1 - \frac{\epsilon_k}{\epsilon_*}\right) a_k^\dagger a_k \tag{4.25}$$

And we obtain a one-parameter family of theories labeled by the cut-off energy ϵ_*.

4.4 The emergence of Bekenstein-like entropy

As mentioned already, all modes with momentum corresponding to the cut-off momentum, i.e. with $k = k_*$ become gapless. This means now that there are many states with the energy of the ground state and therefore entropy emerges. This entropy is related to the degeneracy of the eigenenergy ϵ_*, given by the number of modes with $k_d = k_*$, which is $\mathcal{N}_{k_*} \sim k_*^{d-1}$ as it was mentioned earlier. But how should we interpret this scaling?

The physical cut-off is given by the energy ϵ_* and not the mode k_*. Therefore, let us express the mode in terms of the energy:

$$k_* \approx \frac{2mR^2\epsilon_*}{\hbar^2} \equiv \frac{R^2}{L_*^2} \tag{4.26}$$

This approximation is very good as long as $k_* \gg 1$. We introduced $L_* \equiv \frac{\hbar}{\sqrt{2m\epsilon_*}}$, which in some sense is the fundamental or characteristic quantum length-scale of the model, because any sphere of radius $R < L_*$ would have a cut-off energy smaller than the first eigenenergy and therefore has no quantum fluctuations on top of the Bose-Einstein condensate. Taking this into account the degeneracy scales as the volume of a $(d-1)$-dimensional sphere of radius R:

$$\mathcal{N}_{k_*^2} \sim \left(\frac{R}{L_*}\right)^{d-1} \tag{4.27}$$

Now recall that we restricted the occupation number of all non-ground-state modes to be much smaller than N. Let us be more precise and label as n the maximum occupation number that any of this modes can achieve. Then the number of quantum states corresponding to gapless modes are:

$$N_{states} = (n+1)^{\mathcal{N}_{k_*}} \tag{4.28}$$

And the corresponding entropy is:

$$S = log\left(N_{States}\right) \sim log(n+1)\left(\frac{R}{L_*}\right)^{d-1} \tag{4.29}$$

And we obtain the so called area law for the entropy or Bekenstein-like entropy.

This models shows in a very clear way a realization of one key idea of the quantum N-portrait of black holes, namely that entropy scaling as the area can emerge as a property of Bose-Einstein condensates at the critical point of phase transition. To understand why a phase transition occurs here, notice that for $gN > \frac{1}{\epsilon_*}$ the lowest quantum excitations have negative energy and therefore the description of the physical system as a Bose-Einstein condensate and the subsequent Bogoliubov replacement breaks down.

In this case the degenaracy is due to both quantum criticality and spherical symmetry. The former makes the higher modes effectively gapless, so they have the same energy as the macroscopical state, which can then become degenerate. The latter stablishes the precise form of the degeneracy. Of course it is no surprise that symmetries lead to degeneracies in energy, since their generators (in these case generalized angular momentum operators) commute with the Hamilton operator.

If this mechanism is also taking place in black holes, as the quantum N-portrait suggests, the Planck length L_P will assume the same role as the fundamental or characteristic length-scale of the Bose-Einstein condensate L_*. Of course, even though this model resembles gravity, as already mentioned, due to its attractive interaction with derivative coupling, it lacks Lorentz invariance and the fundamental field is massive and has spin 0 instead of spin 2. One could expect the graviton to become effectively massive in some situations, such as in a bound state, but not to change its spin. Nevertheless, the key point is that some properties of black holes as gravitational systems might be irrelevant for Bekenstein entropy to emerge from them. Or, equivalently, those relevant properties might be more related at the quantum level with properties of the condensate rather than geometric concepts such as the event horizon. It is indeed also possible that the emergence of Bekenstein-like entropy in this model is purely accidental, in the sense that it might be unrelated to the way Bekenstein entropy emerges in black holes.

From the physical point of view, since the entropy of a black hole scales as the area of its event horizon, which has the topology of S_2, the most interesting case of the attractive Bose-Einstein condensate would be the one defined on S_3, since an entropy proportional to

R^2 emerges from it. However, black holes do not have the topology of S_3. If we consider their interior as well, they have a topology of the 2-ball $B_2 = S_2 \times [0, R]$. By setting non-periodic boundary conditions on one of the angular coordinates, S_3 can be deformed into B_2 at the cost of breaking partially its spherical symmetry. Other condensates with non-periodic boundary conditions have been studied already. A recent example was considered by Gia Dvali, Marco Michel and Sebastian Zell in [18], in particular in one spatial dimension only and with a different interacting term. They found that indeed quantum criticality does occur.

The next chapter of this thesis will be hence devoted to the study of the model reviewed in this chapter but with non-periodic boundary conditions. We find that quantum criticality also takes place despite them. In light of this work as well as the work developed in [18], it seems that non-periodic boundary conditions are not an obstacle for quantum criticality to be achieved. Furthermore, we will argue how in a certain regime the full Bekenstein-like entropy can be recovered by means of the correct degeneracy of the emergent gapless modes at the critical point.

Chapter 5

Attractive Bose-Einstein gas with non-periodic boundary conditions

We continue exploring the link between black holes and Bose-Einstein condensates at the critical point of quantum phase transition by extending the model introduced in chapter 4 to the case of non-periodic boundary conditions. This allows us to study the bosonic field living on a 2-ball, which is the topology of the event horizon of a black hole and its interior. With a combination of analytic and numerical computations, we argue that criticality can be achieved and as a result Bekenstein-like entropy can emerge.

5.1 The free Hamiltonian

We consider first the free Hamiltonian:

$$H = -\frac{\hbar^2}{2m} \int_{B_2} d^3x \psi^\dagger \Delta \psi \tag{5.1}$$

One can here expand the field operators in terms of a complete set of eigenfunctions of the quantum-mechanical Hamiltonian, namely functions satisfying the eigenvalue equation:

$$-\Delta f = E f \tag{5.2}$$

Since the problem has spherical symmetry it is appropriate to rewrite the Laplacian in spherical coordinates:

$$\Delta = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\Delta_{S_2} \tag{5.3}$$

With Δ_{S_2} being the laplacian on the 2-sphere, which we do not write explicitly for only its eigenfunctions, the spherical harmonics, will be needed. The form of the whole Laplacian suggests the use of the following separation ansatz:

$$f(x) = R(r) Y_{lm}(\theta, \phi) \tag{5.4}$$

After insertion this leaves:

$$-R''(r)Y_{lm}(\theta,\phi)-\frac{2}{r}R'(r)Y_{lm}(\theta,\phi)+\frac{1}{r^2}l(l+1)R(r)Y_{lm}(\theta,\phi)=ER(r)Y_{lm}(\theta,\phi) \quad (5.5)$$

We can cancel the spherical harmonics in this equation in order to obtain a purely radial equation. Let us multiply by r^2 as well:

$$r^2R''(r)+2rR'(r)+(-l(l+1)+r^2E)R(r)=0 \quad (5.6)$$

This looks almost like the Bessel equation. It can be achieved upon defining $S(r)=r^{1/2}R(r)$. Then:

$$\begin{aligned} R'(r) =& r^{-1/2}S'(r)-\frac{1}{2}r^{-3/2}S(r) \\ R''(r) =& r^{-1/2}S''(r)-\frac{1}{2}r^{-3/2}S'(r)+\frac{3}{4}r^{-5/2}S(r)-\frac{1}{2}r^{-3/2}S'(r) \\ =& r^{-1/2}S''(r)-r^{3/2}S'(r)+\frac{3}{4}r^{-5/2}S(r) \end{aligned} \quad (5.7)$$

The equation is left as follows:

$$r^2S''(r)+rS'(r)+(-l(l+1)+r^2E-1/4)S(r)=0 \quad (5.8)$$

Rewriting $-l(l+1)-1/4=-(l+1/2)^2$ and rescaling $s=E^{1/2}r$ this leaves:

$$s^2S''(s)+sS'(s)+(s^2-(l+1/2)^2)S(s)=0 \quad (5.9)$$

Its solutions are the Bessel functions of first and second kind: $J_{l+1/2}(s)$ and $Y_{l+1/2}(s)$

Boundary conditions are non-trivial. For a given l, Dirichlet boundary conditions for $f=0$ at $r=R$ can only be set if the l-th Bessel function of first o second kind vanishes at $s=E^{1/2}r=R$. This gives a quantization conditions for E, namely $E^{1/2}r$ must be a zero of $J_{l+1/2}$, which we can label with n. If the $(l+1/2)$-th Bessel function of the first kind achieves its n-th zero at the boundary s_{nl} corresponding to the physical boundary $r=R$, then $s_{nl}=RE_{nl}^{1/2}$.

Hence, the energy is quantized $E_{nl}=\left(\frac{s_{nk}}{R}\right)^2$. In particular for $l=0$ this means $E_{0n}=\left(\frac{n\pi}{R}\right)^2$. For a general l the corresponding zeros must be computed numerically. There are however many helpful approximate relations, bounds and estimates for them (see for instance [10] [21]).

Taking all this into account, the solutions of the eigenvalue problem are just:

$$F_{nlm}(x)=r^{-1/2}J_{l+1/2}(E_{nl}^{1/2}r)Y_{lm}(\theta,\phi) \quad (5.10)$$

They will have eigenenergy E_{nl}, which will depend on the size of the system. Actually this eigenvalue doesn't have the dimensions of energy, but instead the dimensions of the

Laplacian, i.e. $[length]^{-2}$. Upon rescaling of the Laplacian in order to make it of dimension of energy $-\frac{\hbar^2}{2m}\Delta \to -\Delta$, the eigenenergy becomes:

$$E_{nl} = \frac{\hbar^2 s_{nl}^2}{2mR^2} \tag{5.11}$$

And its therefore dependent on the ratio of the two length scales of the system: R and m^{-1}. Notice, however, that the argument of the Bessel function must be dimensionless, so it will stay as $\frac{s_{nl}}{R}r$, regardless of the rescaling of the Laplacian.

Furthermore, as opposed to the problem on the sphere, 0 is not an eigenvalue. This will have consequences when we study the interacting Hamiltonian later.

Note that these solutions are orthogonal but are not normalized, because the angular part is normalized but the radial part is not. The corresponding orthogonality relation is [21]:

$$\begin{aligned} \int_0^R r J_{l+1/2}\left(\frac{s_{nl}}{R}r\right) J_{l+1/2}\left(\frac{s_{n'l}}{R}r\right) dr &= 0 && \text{for} \quad n \neq n' \\ &= \frac{1}{2}R^2 J_{l+3/2}^2(s_{nl}) && \text{for} \quad n = n' \end{aligned} \tag{5.12}$$

From now on it will be understood that the eigenfunctions are always normalized and as such will be denoted by $f_{nlm}(x)$.

5.2 The interacting Hamiltonian

Consider now the addition of a quartic interacting term to the Hamiltonian:

$$H = \int_{B_2} d^3x \left[\psi^\dagger \left(-\frac{\hbar^2}{2m}\Delta\right)\psi - g\Omega R^3 \left(\psi^\dagger \left(-\frac{\hbar^2}{2m}\Delta\right)\psi^\dagger\right)\left(\psi\left(-\frac{\hbar^2}{2m}\Delta\right)\psi\right)\right] \tag{5.13}$$

Where Ω is simply the angular volume i.e. 4π and R is the radius of the ball. Note that the fields ψ have dimension $[length]^{-3/2}$ and the coupling g has dimension $[energy]^{-1}$.

We can again rescale the Laplacian for convenience:

$$H = \int_{B_2} d^3x \left[\psi^\dagger (-\Delta)\psi - g\Omega R^3 \left(\psi^\dagger \Delta \psi^\dagger\right)(\psi\Delta\psi)\right] \tag{5.14}$$

Let us expand the field operator ψ as an infinite sum over the destruction operators of the modes with different quantum numbers:

$$\psi = \sum_{n,l,m} f_{nlm}(x) a_{nlm} \tag{5.15}$$

Where n runs from 1 to ∞, l runs from 0 to ∞ and m runs from $-l$ to l, all being integers. Then we get:

$$H = \int_{B_2} d^3x \Big[\sum_{n,n',l,l',m,m'} E_{n'l'} f^*_{nlm}(x) f_{n'l'm'}(x) a^\dagger_{nlm} a_{n'l'm'} - \\ - g\Omega R^3 \sum_{n_i,l_i,m_i} E_{n_2l_2} E_{n_4l_4} a^\dagger_{n_1l_1m_1} a^\dagger_{n_2l_2m_2} a_{n_3l_3m_3} a_{n_4l_4m_4} \cdot \\ \cdot f^*_{n_1l_1m_1} f^*_{n_2l_2m_2} f_{n_3l_3m_3} f_{n_4l_4m_4} \Big] \tag{5.16}$$

These sums get simplified after integrating over the 2-ball. In particular, in the kinetic term the orthonormality leads to a single sum $\sum_{nlm}$ of number operators multiplied by the eigenenergy corresponding to each energy level.

In order to further simplify the Hamiltonian we can put ourselves in a situation in which one of the energy levels is macroscopically occupied, i.e. it has a very large occupation number $N_0 = \langle a^\dagger a \rangle \gg 1$. We assume this to be the ground states, i.e. the states corresponding to the quantum numbers $l = 0$, $n = 1$ and m arbitrary.

In this situation it is legitimate to perform the so called Bogoliubov replacement: $a_{100} = N_0^{1/2}$ and $a^\dagger_{100} = N_0^{1/2}$. Then we have to consider those terms in the sum that contain the ladder operators corresponding to the ground state separately.

Some terms vanish or get simplified because of the orthogonality of the spherical harmonics. Physically this property is related to the fact that the Hamiltonian must not carry any overall angular momentum:

$$\int_{S_2} d\Omega Y^*_{lm} Y_{l'm'} = \delta_{ll'} \delta_{mm'} \tag{5.17}$$

By noticing that $Y_{00} = \frac{1}{\sqrt{4\pi}}$, let us see more precisely how selection rules are implemented due to the orthogonality relation:

- Terms with no ground state operators are not affected.
- Terms with just one ground state operator contain integrals of the form:

$$\frac{1}{\sqrt{4\pi}} \int_{S_2} d\Omega Y^*_{l_1m_1} Y_{l_2m_2} Y_{l_3m_3} \tag{5.18}$$

There are special relations for this kind of integrals, but we will not need them since these terms will become irrelevant shortly.

- Terms with two ground state operators get the following simplification:

$$\frac{1}{4\pi} \int_{S_2} d\Omega Y^*_{lm} Y_{l'm'} = \frac{\delta_{ll'} \delta_{mm'}}{4\pi} \tag{5.19}$$

$$\frac{1}{4\pi} \int_{S_2} d\Omega Y_{lm} Y_{l'm'} = \frac{1}{4\pi} \int_{S_2} d\Omega Y^*_{lm} Y^*_{l'm'} = (-1)^m \frac{\delta_{ll'} \delta_{m,-m'}}{4\pi} \tag{5.20}$$

Where orthogonality was used as well as the known property:

$$Y^*_{lm} = (-1)^m Y_{l,-m} \tag{5.21}$$

- Terms with three ground state operators get simply the integral of one spherical harmonic, which is always zero except for Y_{00}.

For the next step let us take a look at all the terms in the Hamiltonian. For convenience, let us group them in powers of the occupation number N_0:

- Terms proportional to N_0^0:

$$\begin{aligned} H \supset & \sum_{n,l,m\neq 1,0,0} E_{nl} a^\dagger_{nlm} a_{n'l'm'} \\ & - g\Omega R^3 \int_{B_2} d^3x [\sum_{n_i,l_i,m_i\neq 0} E_{n_2l_2} E_{n_4l_4} a^\dagger_{n_1l_1m_1} a^\dagger_{n_2l_2m_2} a_{n_3l_3m_3} a_{n_4l_4m_4} \cdot \\ & \cdot f_{n_1l_1m_1} f_{n_2l_2m_2} f_{n_3l_3m_3} f_{n_4l_4m_4}] \end{aligned} \tag{5.22}$$

- Terms proportional to $N_0^{1/2}$:

$$\begin{aligned} H \supset & - g\Omega R^3 \int_{B_2} d^3x [\sum_{n_i,l_i,m_i\neq 0} \\ & + E_{n_2l_2} E_{n_4l_4} \sqrt{N_0} a^\dagger_{n_2l_2m_2} a_{n_3l_3m_3} a_{n_4l_4m_4} f^*_{100} f^*_{n_2l_2m_2} f_{n_3l_3m_3} f_{n_4l_4m_4} \\ & + E_{10} E_{n_4l_4} \sqrt{N_0} a^\dagger_{n_1l_1m_1} a_{n_3l_3m_3} a_{n_4l_4m_4} f^*_{n_1l_1m_1} f^*_{100} f_{n_3l_3m_3} f_{n_4l_4m_4} \\ & + E_{n_2l_2} E_{n_4l_4} \sqrt{N_0} a^\dagger_{n_1l_1m_1} a^\dagger_{n_2l_2m_2} a_{n_4l_4m_4} f^*_{n_1l_1m_1} f^*_{n_2l_2m_2} f_{100} f_{n_4l_4m_4} \\ & + E_{n_2l_2} E_{10} \sqrt{N_0} a^\dagger_{n_1l_1m_1} a^\dagger_{n_2l_2m_2} a_{n_3l_3m_3} f^*_{n_1l_1m_1} f^*_{n_2l_2m_2} f_{n_3l_3m_3} f_{100} \end{aligned} \tag{5.23}$$

- Terms proportional to N_0:

$$\begin{aligned} H \supset & E_{10} N_0 - g\Omega R^3 \int_{B_2} d^3x [\sum_{n_i,l_i,m_i} \\ & E_{10} E_{n_4l} N_0 a_{n_3lm} a_{n_4l,-m} f^*_{100} f^*_{100} f_{n_3lm} f_{n_4l,-m} \\ & + E_{n_2l} E_{n_4l} N_0 a^\dagger_{n_2lm} a_{n_4lm} f^*_{100} f^*_{n_2lm} f_{100} f_{n_4lm} \\ & + E_{n_2l} E_{10} N_0 a^\dagger_{n_2lm} a_{n_3lm} f^*_{100} f^*_{n_2lm} f_{n_3lm} f_{100} \\ & + E_{10} E_{n_4l} N_0 a^\dagger_{n_1lm} a_{n_4lm} f^*_{n_1lm} f^*_{100} f_{100} f_{n_4lm} \\ & + E_{10} E_{10} N_0 a^\dagger_{n_1lm} a_{n_3lm} f^*_{n_1lm} f^*_{100} f_{n_3lm} f_{100} \\ & + E_{n_2l} E_{10} N_0 a^\dagger_{n_1lm} a^\dagger_{n_2l,-m} f^*_{n_1l,-m} f^*_{n_2lm} f_{100} f_{100}] \end{aligned} \tag{5.24}$$

- Terms proportional to $N_0^{3/2}$:

$$\begin{aligned} H \supset -g\Omega R^3 \int_{B_2} d^3x[\sum_{n_i,l_i,m_i} \\ E_{10}^2 N_0^{3/2} a_{n00}^\dagger f_{n00}^* f_{100}^* f_{100} f_{100} \\ + E_{n0} E_{10} N_0^{3/2} a_{n00}^\dagger f_{100}^* f_{n00}^* f_{100} \\ + E_{10}^2 N_0^{3/2} a_{n00} f_{100}^* f_{100}^* f_{n00} f_{100} \\ + E_{10} E_{n0} N_0^{3/2} a_{n00} f_{100}^* f_{100}^* f_{100} f_{n00} \end{aligned} \tag{5.25}$$

- Term proportional to N_0^2:

$$H \supset -\, g\Omega R^3 \int_{B_2} d^3x E_{10}^2 N_0^2 |f_{100}|^4] \tag{5.26}$$

However, N_0 is not a fixed quantity of the system. Instead, the total particle number N is conserved. As usual, they are related by the total particle number conservation:

$$N = N_0 + \sum_{n,l,m\neq 0} a_{nlm}^\dagger a_{nlm} \tag{5.27}$$

This is actually only relevant for the term proportional to N^2, for the corrections to the rest will be neglected in the next step. At this point we can get rid of some of the terms by taking the so called double scaling limit, i.e. $N \to \infty$ and $g \to 0$ while keeping gN fixed, as it was introduced already in chapter 4. This has the following effects:

- Terms proportional to g only vanish because of the limit $g \to 0$. This leaves the kinetic diagonal terms as the only ones proportional to N^0.
- Terms proportional to $gN^{1/2}$ vanish because the limit $g \to 0$ is faster than the limit $N^{1/2} \to \infty$.
- Terms proportional to gN stay untouched and will be crucial to the analysis of the problem in this limit.
- Terms proportional to $gN^{3/2}$ become infinitely big because the limit $N^{3/2} \to \infty$ is faster than the limit $g \to 0$. They correspond to terms linear in ladder operators. At first glance it seems that they pose a threat to the validity of the ongoing manipulations but, as we we will see later, they simply affect the Hilbert space in a way that the eigenvalues of the problem remain untouched.

  These terms need perhaps a more detailed explanation regarding the replacement (5.27). For large N, one can perform a Puiseux expansion around $N = \infty$ (which was obtained with the software Mathematica):

$$\left(N - \sum_{n,l,m\neq 0} a^\dagger_{nlm} a_{nlm}\right)^{3/2} \simeq N^{3/2} - \frac{3\sqrt{N}}{2} \sum_{n,l,m\neq 0} a^\dagger_{nlm} a_{nlm} + \mathcal{O}\left(N^{-1/2}\right) \tag{5.28}$$

Therefore, from the terms proportional to $N_0^{3/2}$, only those actually proportional to $N^{3/2}$ survive. As mentioned above, those correspond to terms linear in ladder operators.

- Terms proportional to gN^2 become infinitely big for the same reason as before. They correspond to terms proportional to the identity operator, and therefore they only provide a sort of zero-point energy which is irrelevant for dynamics of the problem.

After applying the particle number conservation restriction, most of the terms stay the same as if we applied the double-scaling limit to gN_0 instead of gN. The reason for this is that the terms with additional ladder operators tend faster to zero and vanish in most of the cases in these limit. The only exception is a term which was originally proportional to N_0^2:

$$\begin{aligned} H \supset & -g\Omega R^3 \int_{B_2} d^3x E_{10}^2 N_0^2 |f_{100}|^4] \\ &= -g\Omega R^3 \int_{B_2} d^3x E_{10}^2 \Big(N^2 - 2N \sum_{n,l,m\neq 0} a^\dagger_{n,l,m} a_{n,l,m} \\ &+ \sum_{n_i,l_i,m_i} a^\dagger_{n_1,l_1,m_1} a_{n_1,l_1,m_1} a^\dagger_{n_2,l_2,m_2} a_{n_2,l_2,m_2}\Big) |f_{100}|^4] \end{aligned} \tag{5.29}$$

As mentioned already, the term proportional to N^2 is irrelevant for the dynamics of the system. The term with four ladder operators is equally unimportant, for vanishes in the double-scaling limit. Only the term proportional to N and two ladder operators must be taken into account: it establishes non-negligible corrections to the diagonal elements of the coefficient matrix.

After all, the Hamiltonian becomes:

$$\begin{aligned} H = & \sum_{n,l,m\neq 1,0,0} E_{nl} a^\dagger_{nlm} a_{nlm} \\ & - g\Omega R^3 \int_{B_2} d^3x \Big[\sum_{n_i,l_i,m_i\neq 0} \\ & + (E_{10} + E_{n_1 l})(E_{10} + E_{n_2 l}) N a^\dagger_{n_1 lm} a_{n_2 lm} |f_{100}|^2 f^*_{n_1 lm} f_{n_2 lm} \\ & - 2E_{10}^2 N a^\dagger_{nlm} a_{nlm} |f_{100}|^4 \\ & + E_{10} E_{nl} N a_{nlm} a_{n'l,-m} f^*_{100} f^*_{100} f_{nlm} f_{n'l,-m} \\ & + E_{nl} E_{10} N a^\dagger_{nlm} a^\dagger_{n'l,-m} f^*_{nl,-m} f^*_{n'lm} f_{100} f_{100} \Big] \end{aligned} \tag{5.30}$$

Following the same procedure as for the Bose gas on the sphere, we establish an energy cut-off E^*. This is necessary in order to guarantee that the Hamiltonian is bounded from below and can be seen as the effect of higher derivative self-interactions with positive sign that should eventually be included in the Hamiltonian. We choose this cut-off to correspond to $E^* = E_{1l^*}$, where l^* acts also as an angular momentum cut-off, since any Bessel function of higher order will have its first zero further away and therefore its corresponding eigenenergy will also be greater than the cut-off [10]:

$$s_{1l} < s_{1,l+1} < s_{2l} < s_{2,l+1} < s_{3l} < \ldots \tag{5.31}$$

This mode is a candidate to become critical for the smallest possible value of the effective coupling gN since it has the highest possible eigenergy. As we will see shortly, it is also the only mode that can be diagonalized analytically. Other candidates to become critical are modes with $l < l^*$ but $n > 1$ with eigenenergies $E_{nl} \lesssim E^*$. We will analize them numerically later.

5.3 Analitic Bogoliubov transformation

Let us take a look at the terms in the Hamiltonian corresponding to the $l = l^*$ modes, which all have an energy equal to the cut-off $E_{1l^*} = E^*$.

$$\begin{aligned} H = &\sum_{m=-l^*}^{l^*} E^* a^\dagger_{1l^*m} a_{1l^*m} \\ &- gNR^3\Omega \int_{B_2} d^3x \Big[\sum_{m=-l^*}^{l^*} (E_{10}+E^*)^2 a^\dagger_{1l^*m} a_{1l^*m} |f_{100}|^2 |f_{1l^*m}|^2 \\ &- 2E_{10}^2 a^\dagger_{1l^*m} a_{1l^*m} |f_{100}|^4 + E_{10}E^*(a_{1l^*m}a_{1l^*,-m} + a^\dagger_{1l^*m} a^\dagger_{1l^*,-m}) f_{100}^2 |f_{1l^*m}|^2 \Big] \end{aligned} \tag{5.32}$$

Now let us perform the integral over the spatial coordinates. The angular part of the integral gives simply the inverse of the angular volume Ω, so it is canceled. For the radial part, the value of the integral depends of course of l^* and, in general, must be computed numerically. Nevertheless, let us introduce for convenience the notation:

$$\begin{aligned} C^* &\equiv \int_0^R dr J^2_{l^*+1/2}(E^{*1/2} r) J^2_{1/2}(E_{10}^{1/2} r) \\ C^0 &\equiv \int_0^R dr J^4_{1/2}(E_{10}^{1/2} r) \end{aligned} \tag{5.33}$$

This way the Hamiltonian looks like this:

$$H \supset \sum_{m=-l^*}^{l^*} E^* a^\dagger_{1l^*m} a_{1l^*m} - gNR^3 \big[C^*(E_{10}+E^*)^2 a^\dagger_{1l^*m} a_{1l^*m} \\ - 2E_{10}^2 a^\dagger_{1l^*m} a_{1l^*m} C^0 + (-1)^m C^* E_{10} E^* (a_{1l^*m} a_{1l^*,-m} + a^\dagger_{1l^*m} a^\dagger_{1l^*,-m}) \big] \tag{5.34}$$

In order to inspect the true energy level corresponding to the cut-off mode, we should now diagonalize the Hamiltonian by means of a Bogoliubov transformation. For the sake of a more clear notation, let us in the following omit the label $1l^*$, for it is included in all the operators, and leave only m. As introduced in chapter 3, the Bogoliubov transformation is a linear transformation of the ladder operators of the following form:

$$\begin{aligned} a_m &= u_m b_m + v^*_{-m} b^\dagger_{-m} \\ a^\dagger_m &= u^*_m b^\dagger_m + v_{-m} b_{-m} \end{aligned} \tag{5.35}$$

Where, in general, u_m and v_m are a set of complex coefficients. Insert this in the Hamiltonian:

$$\begin{aligned} H \supset \sum_{m=-l^*}^{l^*} & E^*(|u_m|^2 b^\dagger_m b_m + u^*_m v^*_{-m} b^\dagger_m b^\dagger_{-m} + v_{-m} u_m b_{-m} b_m + |v_{-m}|^2 b_{-m} b^\dagger_{-m}) \\ & - gNR^3 \big[\left(C^*(E_{100}+E^*)^2 - 2E_{10}^2 C^0 \right) \cdot \\ & \cdot (|u_m|^2 b^\dagger_m b_m + u^*_m v^*_{-m} b^\dagger_m b^\dagger_{-m} + v_{-m} u_m b_{-m} b_m + |v_{-m}|^2 b_{-m} b^\dagger_{-m}) \\ & + (-1)^m C^* E_{10} E^* (u_m u_{-m} b_m b_{-m} + u_m v^*_m b_m b^\dagger_m + v^*_{-m} u_{-m} b^\dagger_{-m} b_{-m} + v^*_{-m} v^*_m b^\dagger_{-m} b^\dagger_m) \\ & + (-1)^m C^* E_{10} E^* (u^*_m u^*_{-m} b^\dagger_m b^\dagger_{-m} + u^*_m v_m b^\dagger_m b_m + v_{-m} u^*_{-m} b_{-m} b^\dagger_{-m} + v_{-m} v_m b_{-m} b_m) \big] \end{aligned} \tag{5.36}$$

For the diagonalization to take place we need to impose that the coefficients in front of the terms bb and $b^\dagger b^\dagger$ vanish, i.e.:

$$E^* u^*_m v^*_{-m} - gNR^3 [\left(C^*(E_{100}+E^*)^2 - 2E_{10}^2 C^0 \right) u^*_m v^*_{-m} + (-1)^m C^* E_{10} E^* \left(v^*_{-m} v^*_m + u_m u_{-m} \right)] = 0 \tag{5.37}$$

With the Ansatz that the Bogoliubov coefficients are real and equal for m and $-m$, this equation suffices to determine them. In particular, one can introduce a useful parametrization in terms of hyperbolic functions:

$$u_m = cosh(\theta_m) \quad v_m = sinh(\theta_m) \tag{5.38}$$

This reduces the number of parameters on which the Bogoliubov transformation depends because it incorporates an additional restriction, namely that the new ladder operators obey the canonical commutation relations. Then the diagonalization condition can be rewritten as:

$$\begin{aligned}&\left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]cosh(\theta_m)sinh(\theta_m)\\&-gNR^3C^*(-1)^mE_{10}E^*\left(sinh^2(\theta_m)+cosh^2(\theta_m)\right)=0\end{aligned} \tag{5.39}$$

Now we can use these well-known identities for hyperbolic functions:

$$\begin{aligned}cosh(x)sinh(x) &= \frac{1}{2}sinh(2x)\\ cosh^2(x)+sinh^2(x) &= cosh(2x)\end{aligned} \tag{5.40}$$

Which gives:

$$\left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]\frac{1}{2}sinh(2\theta_m) - gNR^3C^*(-1)^mE_{10}E^*cosh(2\theta_m) = 0 \tag{5.41}$$

And then divide the whole equation by $cosh(2\theta_m)$ in order to find:

$$tanh(2\theta_m) = \frac{2(-1)^m gNR^3C^*E_{10}E^*}{E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)} \tag{5.42}$$

In the limit $E_{10} \to 0$ then $tanh(2\theta_m) \to 0$ and so $u_m = 1$ and $v_m = 0$, which means that the system does not need to be diagonalized in this limit. This is important for consistency because it connects these results with those of the theory on the sphere (i.e. the theory with periodic boundary conditions): from the point of view of the energy eigenvalues, the main difference between both models is the disappearance of modes with eigenenergy 0.

From the obtained expression it is clear as well that θ_m has no dependence whatsoever on m, except for a relative sign between even and odd m. This is due to the symmetry of the function: $tanh(-x) = -tanh(x)$. The other basic hyperbolic functions are symmetric as well: $sinh(-x) = -sinh(x)$ and $cosh(-x) = cosh(x)$. These relative minus signs must be carried around carefully and should cancel in the expression of any physical quantity such as the Bogoliubov energies or the density of depleted particles. This is consistent with the spherical symmetry of the problem.

Once diagonalized, the Hamiltonian takes the form:

$$H \supset \sum_{m=-l^*}^{l^*} \epsilon_m b_m^\dagger b_m \tag{5.43}$$

With Bogoliubov energies given by the coefficients of the $b^\dagger b$ terms in the Hamiltonian:

$$\begin{aligned}\epsilon_m =& \left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]\left(cosh^2\theta_m + sin^2\theta_m)\right)\\&-4(-1)^m gNR^3C^*E_{10}E^*\left(cosh\theta_m \cdot sinh\theta_m)\right)\end{aligned} \tag{5.44}$$

Or, using again the identities of the hyperbolic functions:

$$\epsilon_m = \left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]cosh(2\theta_m) - 2(-1)^mC^*E_{10}E^*sinh(2\theta_m) \tag{5.45}$$

In order to obtain a neat expression for the energy ϵ_m, let us write equation (5.45) squared minus eight times equation (5.41) squared:

$$\begin{aligned}\epsilon_m^2 =&\left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]^2 cosh^2(2\theta_m) + 4\left(gNR^3C^*E_{10}E^*sinh(2\theta_m)\right)^2\\ &-4(-1)^m\left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]gNR^3C^*E_{10}E^*sinh(\theta_m)cosh(2\theta_m)\\ &-\left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]^2 sinh^2(2\theta_m) - 4\left(gNR^3C^*E_{10}E^*cosh(2\theta_m)\right)^2\\ &+4(-1)^m\left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]gNR^3C^*E_{10}E^*sinh(2\theta_m)cosh(2\theta_m)\end{aligned} \tag{5.46}$$

The two terms proportional to $cosh(2\theta_m)sinh(2\theta_m)$ cancel out and furthermore we can use the hyperbolic identity:

$$cosh^2(x) - sinh^2(x) = 1 \tag{5.47}$$

So that we obtain the following simplified expression:

$$\epsilon^2 = \left[E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right)\right]^2 - 4\left(gNR^3C^*E_{10}E^*\right)^2 \tag{5.48}$$

Where the subscript m was omitted, for any dependence on m in the expression has been removed. This is nothing but the energy of the modes corresponding to $l = l^*$ and arbitrary m which are built on top of the condensate. Effectively the fact that the Hamiltonian needed diagonalization brings a correction to the eigenenergy of the free Hamiltonian E^* in the interacting theory, which now we have computed in the Bogoliubov approximation. Since this level has the highest eigenenergy in the free theory, it is a clear candidate to become zero for the smallest value of the effective coupling gN.

Let us see when this energy vanishes. The difference of squared expressions can be conveniently factorized so that $\epsilon^2 = 0$ implies either:

$$\left(E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right) + 2gNR^3C^*E_{10}E^*\right) = 0 \tag{5.49}$$

Or

$$\left(E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2E_{10}^2C^0\right) - 2gNR^3C^*E_{10}E^*\right) = 0 \tag{5.50}$$

And therefore $\epsilon^2 = 0$ is satisfied for:

$$gNR^3 = \frac{E^*}{C^*(E_{10}^2 + E^{*2}) - 2C^0E_{10}^2} \quad \text{or} \quad gNR^3 = \frac{E^*}{C^*(E_{10}^2 + E^{*2} + 4E_{10}E^*) - 2C^0E_{10}^2} \tag{5.51}$$

For these both values the $l = l^*$ modes become gapless. We want gN to be kept as small as possible, and therefore we pick the second solution of the equation. For the case of a very large cut-off $E^* \gg E_{10}$ this reduces to:

$$gNR^3C^* \to \frac{1}{E^*} \tag{5.52}$$

This is a second indicator that the model on the 2-ball reduces to model on the 3-sphere, i.e. that for high energy cut-offs the effect of the non-periodic boundary conditions is suppressed. Indeed, this scaling looks very similar to the one encountered in chapter 4. However, we should be still a bit careful at this point, since C^* is a non-trivial integral of Bessel functions. It carries actually dimension $[length]^{-3}$ and furthermore scales as R^{-3}. Let us check this:

$$\begin{aligned} C^* &= \frac{\int_0^R J_{1/2}^2\left(\frac{s_{10}}{R}r\right) J_{l^*+1/2}^2\left(\frac{s_{1l^*}}{R}r\right) dr}{\int_0^R J_{1/2}^2\left(\frac{s_{10}}{R}r\right) rdr \int_0^R J_{l^*+1/2}^2\left(\frac{s_{1l^*}}{R}r\right) rdr} \\ &= \frac{R\int_0^1 J_{1/2}^2\left(s_{10}r\right) J_{l^*+1/2}^2\left(s_{1l^*}r\right) dr}{R^4\int_0^1 J_{1/2}^2\left(s_{10}r\right) rdr \int_0^1 J_{l^*+1/2}^2\left(s_{1l^*}r\right) rdr} \\ &\equiv R^{-3}C_1^* \end{aligned} \tag{5.53}$$

Where C_1^* is defined as C^* for $R = 1$. The same can be argued about the integral C^0, which can analogously be expressed as $C^0 = R^{-3}C_1^0$. This way we can eliminate the dependence of the critical gN with the radius. Nevertheless, it still depends on the quantum numbers of the function involved in C_1^*. Therefore, the scaling of the critical effective coupling for this modes does not scale as the inverse energy cut-off, as it would be expected. On the contrary, C_1^0 is of course independent of the cut-off and equal to:

$$C_1^0 = \frac{\int_0^1 dr\left(\frac{1}{\pi}\sqrt{\frac{2}{r}}\right)^4 sin^4(\pi r)}{\left(\int_0^1 drr\left(\frac{1}{\pi}\sqrt{\frac{2}{r}}\right)^2 sin^2(\pi r)\right)^2} = 2\pi\left(2Si(2\pi) - Si(4\pi)\right) \simeq 8.44549 \tag{5.54}$$

The integral in the numerator was computed with help of Mathematica. There $Si(x)$ is the sine integral function, defined by:

$$Si(x) = \int_0^x \frac{sin(t)}{t}dt \tag{5.55}$$

However, the value of C_1^0 is irrelevant in the large cut-off limit.

As it can be seen in figure 5.3, C_1^* scales as some inverse power of l^*. The non-linear fitting curve that best describes its behavior is $C_1^* = 5.54l^{*-1.33}$. In the large l^* limit, the n-th zero of the $(l^* + 1/2)$-th Bessel function scales linearly in l^* [10]:

$$s_{1l} \simeq \left(l + \frac{1}{2}\right) + 1.8557571 \cdot \left(l + \frac{1}{2}\right)^{1/3} + \mathcal{O}\left(\left(l + \frac{1}{2}\right)^{-1/3}\right) \tag{5.56}$$

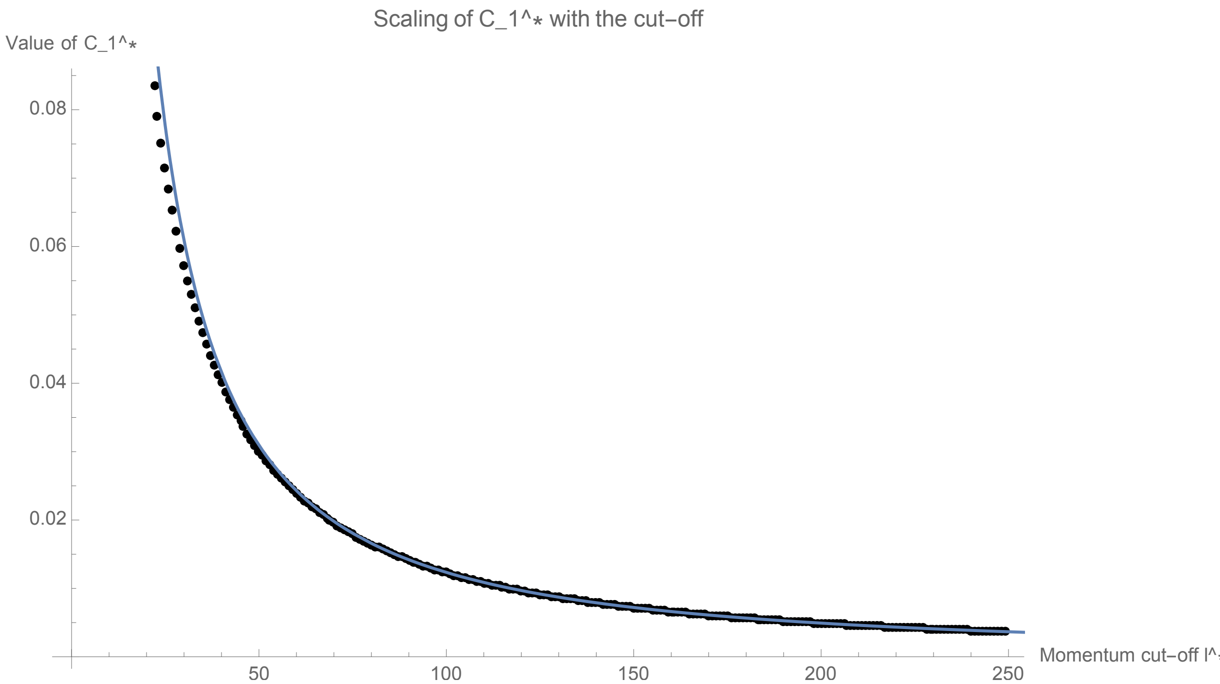


Figure 5.1: Scaling of C_1^* with the cut-off l^*

Then the energy cut-off scales quadratically in l^* and for this modes one can conclude that the effective coupling gN scales approximately as $l^{*-0.67}$, which is not the expected scaling. However, we should not be worried: it is still necessary to check the rest of the modes, those that are only numerically diagonalizable.

Most importantly, we will see that these are not the modes that become critical for the smallest value of gN given a certain cut-off E^*. Nevertheless, their analytic study provides some insights that are interesting regarding how the Bogoliubov transformation works in this model and how gapless modes emerge. Furthermore, it shows clearly a connection with the 3-sphere model in the limit $E_{10} \to 0$. In this particular limit the Hamiltonian becomes diagonal without needing any Bogoliubov transformation and furthermore $C^* \to 1$. Therefore, the particular properties of the model on the 2-ball differ of those on the 3-sphere due to this non-vanishing lowest eigenenergy, which is a consequence of the non-periodic boundary conditions of Dirichlet kind. However, the limit $E_{10} \to 0$ is not realizable unless $R \to \infty$ but then all eigenenergies vanish as well. Besides, its not equivalent to the limit $E^* \to \infty$, as one can see from the behavior of C_1^*. However, as we will see after the numerical diagonalization of the other modes, this latter limit does trigger the emergence of degenerate gap-less modes with an entropy that follows an area-law.

We finish the discussion of the analytic Bogoliubov transformation by finding the full expression for the particle depletion, which is given by:

$$n = |v|^2 = sinh^2\left(\frac{1}{2}tanh^{-1}\left(\frac{2gNR^3C^*E_{10}E^*}{E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2C^0E_{10}^2\right)}\right)\right) \tag{5.57}$$

The function $sinh^2\left(\frac{1}{2}tanh^{-1}(\alpha)\right)$ can be rewritten in terms of algebraic functions as follows by noting, respectively, that $tahn^{-1}(x) = \frac{1}{2}log\left(\frac{1+x}{1-x}\right)$ and $sinh(x) = \frac{e^x - e^{-x}}{2}$:

$$\begin{aligned} sinh^2\left(\frac{1}{2}tanh^{-1}(\alpha)\right) &= \frac{1}{4}\left(e^{\frac{1}{2}log\left(\frac{1+\alpha}{1-\alpha}\right)} + e^{-\frac{1}{2}log\left(\frac{1+\alpha}{1-\alpha}\right)} - 2\right) = \frac{1}{4}\left(\sqrt{\frac{1+\alpha}{1-\alpha}} + \sqrt{\frac{1-\alpha}{1+\alpha}} - 2\right) \\ &= \frac{1}{2}\left((1-\alpha^2)^{-1/2} - 1\right) \end{aligned} \tag{5.58}$$

Now identify:

$$\alpha = \frac{2gNR^3C^*E_{10}E^*}{E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2C^0E_{10}^2\right)} \tag{5.59}$$

Note the disappearance of the $(-1)^m$ factor. This is simply due to the symmetry properties $tanh^{-1}(-x) = -tanh^{-1}(x)$ and $sinh^2(-x) = sinh^2(x)$.

Therefore, in terms of algebraic functions the density of depleted particles is:

$$n = v^2 \ = 1/2\left(\left(1 - \frac{2gNR^3C^*E_{10}E^*}{E^* - gNR^3\left(C^*(E_{10}+E^*)^2 - 2C^0E_{10}^2\right)}\right)^{-1/2} - 1\right) \tag{5.60}$$

5.4 The effect of linear terms in the Hamiltonian

As a small detour here we show that, once the bilinear terms of the Hamiltonian have been conveniently diagonalized by means of a Bogoliubov transformation, the linear terms have no effect on the energy levels. It should be noted that the Bogoliubov transformation changes the linear terms as well, but they will stay linear.

Consider therefore a Hamiltonian of the form:

$$H = \sum_i \alpha_i a_i^\dagger a_i + \gamma_i(a_i^\dagger + a_i) \tag{5.61}$$

Where γ_i is assumed to be real for simplicity since the Hamiltonian is hermitian. If it is not the case, one can achieve it by phase-shifting the ladder operators. Now, we wish to achieve a Hamiltonian of the form:

$$H = \sum_i \beta_i b_i^\dagger b_i \tag{5.62}$$

And in order to do so we introduce the transformation:

$$a_i = b_i + c_i \tag{5.63}$$

Where c_i is some c-number times the Identity operator. Notice that such a transformation trivially respects the canonical commutation relation:

$$[b_i, b_j^\dagger] = [a_i - c_i, a_j^\dagger - c_j] = [a_i, a_j^\dagger] \tag{5.64}$$

We plug it in the Hamiltonian:

$$\begin{aligned} H &= \sum_i \alpha_i(b_i^\dagger + c_i^*)(b_i + c_i) + \gamma_i(b_i^\dagger + c_i^* + b_i + c_i) \\ &= \sum_i \alpha_i b_i^\dagger b_i + b_i^\dagger(\alpha_i c_i + \gamma_i) + b_i(\alpha_i c_i^* + \gamma_i) + |c_i|^2 + \gamma_i(c_i + c_i^*) \end{aligned} \tag{5.65}$$

This Hamiltonian is bilinear and diagonal for $c_i = -\frac{\gamma_i}{\alpha_i}$ and $c_i = c_i^*$. Furthermore, $\beta_i = \alpha_i$, so the eigenenergies are not affected by the linear terms.

It is interesting now to check how the particle states transform. For simplicity, let us reduce the model to one only mode. The conclusions can be immediately extended to the case with several modes due to the factorization of the Hilbert space. Consider first of all the original vacuum $|0\rangle_a$. The new destruction operator b acts like this:

$$b\,|0\rangle_a = (a - c)\,|0\rangle_a = -c\,|0\rangle_a \tag{5.66}$$

Therefore the original vacuum $|0\rangle_a$ has become a coherent state with respect to the new set of ladder operators and we can relabel it as $|-c\rangle_b$. From this we can get the new vacuum by means of the displacement operator:

$$|0\rangle_b = D(c)\,|-c\rangle_b = e^{ca^\dagger - c^* a}\,|-c\rangle_b \tag{5.67}$$

And the remaining particle states can simply be constructed by repeatedly applying the creation operator $b^\dagger$ to the vacuum state.

5.5 Numerical Bogoliubov transformation

One might be worried that the quantum levels for which the Hamiltonian can be analytically diagonalized are actually not the first ones to become critical, i.e. not the ones that become critical for a smaller effective coupling gN. This is in fact true, since the mixing between the n levels for $l < l^*$ causes a decrease in the energy due to the interaction. Nevertheless, we will see that a dependence of the effective coupling with the energy cut-off E^* arises as in the case with periodic boundary conditions. We can understand the basics of this mechanism by the following toy model. Consider a 2x2 diagonal matrix:

$$M_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad det(M_1) = ab \tag{5.68}$$

Since M_1 is diagonal its eigenvalues are simply a and b. Now consider another 2x2 matrix which is however not diagonal, but is still orthogonal:

$$M_2 = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \quad det(M_2) = ab - c^2 \tag{5.69}$$

Assume that $a, b \geq 0$ and furthermore $a > b$. Then the smallest eigenvalue λ is given by:

$$\lambda = \frac{a + b - \sqrt{(a+b)^2 - 4(ab - c^2)}}{2} \tag{5.70}$$

Which, in a Taylor expansion for small c becomes:

$$\lambda \simeq b - \frac{c^2}{a-b} + \mathcal{O}\left(c^3\right) \tag{5.71}$$

Therefore, the addition of an 'interacting' term c effectively decreases the lower eigenvalue of the matrix. In a similar way, the mixing between the eigenstates of the free Hamiltonian leads to a decrease in the lowest Bogoliubov energy.

As we reviewed in chapter 3, in general one cannot simply take the Hamiltonian in matrix form and diagonalize it algebraically, because these diagonalization need not respect the canonical commutation relations. In order to be able to apply the procedure described there, we need the Hamiltonian of the system to be positive-definite. Otherwise it is not guaranteed that it will be Bogoliubov-diagonalizable. It is easy to see that the free Hamiltonian, i.e. for $gN = 0$ is positive definite. Then it follows that it will also be for small enough gN, since the determinant is a polynomial and therefore a continuous and differentiable function of the matrix elements. This 'small enough' ceases to be valid at the point where one of the Bogoliubov energies becomes 0. Then the critical effective coupling has been achieved. Beyond this critical point, the first Bogoliubov energy becomes negative with respect to the background condensate and therefore unstable. This makes Bogoliubov diagonalization meaningless, since the assumption that there is background condensate at all is violated and the Bogoliubov replacement ceases to be applicable.

As long as we start studying the Bogoliubov energies from $gN = 0$ and for increasing gN, the situation should be kept under control. We can then proceed with the numerical diagonalization of the matrix $I\_\mathcal{H}$. It would be convenient not to carry around the mass m and the radius R, since they are free parameters and therefore we do not wish to fix them. The same goes for the Planck constant $\hbar$, whose value is known but we do not wish to insert it either. We could just use natural units and set $\hbar = 1$ but here it is more convenient to express all energies in units of $\frac{\hbar^2}{2mR^2}$. This includes the Hamiltonian itself and the energy levels, which become dimensionless. Correspondingly, since g has dimensions of inverse energy, it will carry units of $\left(\frac{\hbar^2}{2mR^2}\right)^{-1}$. The resulting dimension-less Hamiltonian is radius-independent: the term R^3 accompanying $gN\Omega$ is compensated by the scaling of the radial integrals with R, as we saw in the previous section. The level mixing does not change this. Al radial integrals will be assumed therefore to be computed for the case

$R = 1$ without loss of generality. This applies to both the integration interval and the argument of the Bessel functions.

Let us clarify the notation once more. The hamiltonian H (now dimensionless), in its non-diagonal form, can be written like this:

$$H = \vec{a}^{\dagger}\mathcal{H}\vec{a} \tag{5.72}$$

Where the vector notation introduced in section 3.6 was used. The only difference is that now the components span to all l, n and m. The Hamiltonian being diagonalizable means that it can be written like this:

$$H = \vec{b}^{\dagger}\mathcal{H}_{\mathcal{D}}\vec{b} \tag{5.73}$$

Where $\mathcal{H}_{\mathcal{D}}$ is a diagonal matrix with elements $\frac{1}{2}\epsilon_{nlm}$, the Bogoliubov energies. The factor $\frac{1}{2}$ is simply due to the redundancy in the vector notation. Furthermore, new ladder operators b_{nlm} and $b^{\dagger}_{nlm}$ are introduced. Recalling again section 3.6, in order to obtain the Bogoliubov energies it is simply necessary to algebraically diagonalize the matrix $I_{-}\mathcal{H}$ and take the absolute value of its eigenvalues.

The numerical Bogoliubov transformation was performed with the software Mathematica. The corresponding codes for all the computations shown in this section can be found in the appendix. The codes regarding plotting and fitting are not showed there, but rather understood to be trivial. As it was done in the analytic Bogoliubov transformation, we pick for simplicity the cut-off E^* to be equal to the first energy of some l-mode labeled by l^*, which the cut-off l-mode. Therefore, we have that $E^* = E_{1l^*}$.

An instance of the collapse of the first Bogoliubov energy can be seen in figure 5.2. For $gN = 0$ it corresponds to the lowest energy for the $l = 0$ mode of the free Hamiltonian (after the Bogoliubov replacement) i.e. E_{20}, because the Hamiltonian is automatically diagonal. With increasing gN, the Bogoliubov level ϵ_{20} decreases and reaches eventually 0, when its energy becomes degenerate with respect to the background condensate. However, it is not always the case that the lowest eigenvalue of the free Hamiltonian is the one that collapses first. Due to the mixing and the numerical algebraic diagonalization, it is difficult to keep track of which n-mode is collapsing, because the n-mode is only defined by the correspondence with a free n-mode in the $gN \to 0$ limit. However, from the knowledge obtained in the case with periodic boundary conditions it is expected that the first collapsing mode is the highest n-mode, at least in the large cut-off limit. The reason is that it has an eigenergy closer to the cut-off and therefore the factor E^2_{nl} multiplying the coupling gN becomes very relevant and it also overcomes the factor proportional to gNE^2_{10}.

After diagonalization one finds out that that the more n-levels are present under the cut-off, the sooner the criticality appears. This can be seen in figure 5.3. For a cut-off $l^* = 10$, the l-modes appear in four groups according approximately to their value for the critical coupling gN. Among each group, the highest l-mode has a lower value of the critical coupling, as one would expect from the fact the the corresponding highest eigenenergy $E_{n_{max}l} \lesssim E^*$ increases with l.

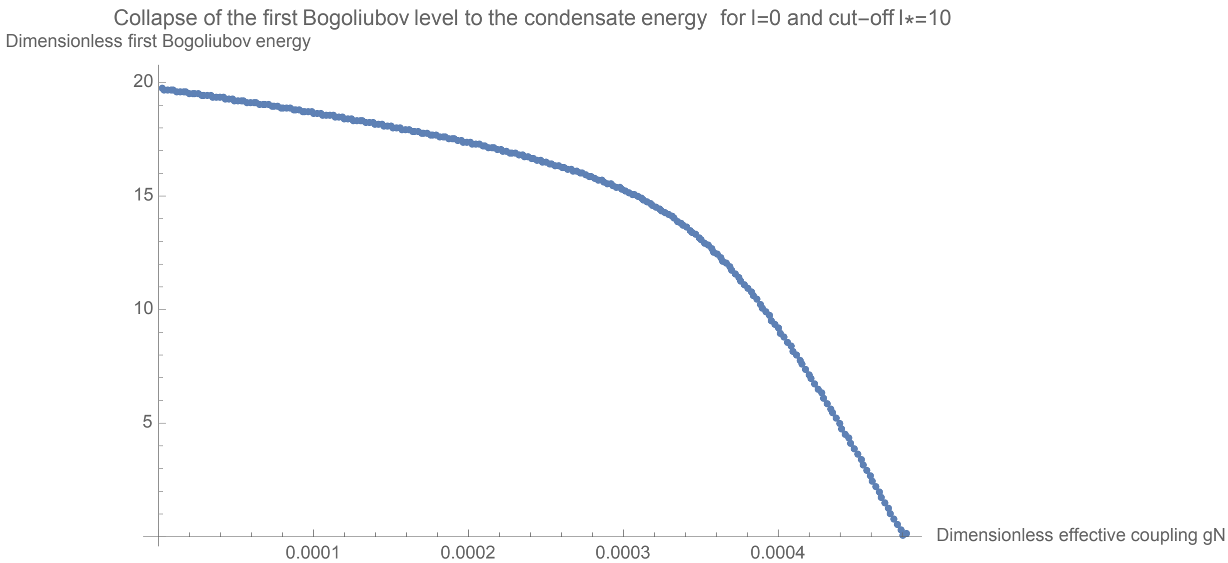


Figure 5.2: Collapse of the first (dimensionless) Bogoliubov energy for $l = 0$ and $l^* = 10$.

The difference between the groups arises from the fact that each of them has a different n_{max}, i.e. a different number of eigenenergies below the cut-off E^*. This number becomes very relevant for the diagonalization process, for the mixing between the energy levels decreases the lowest Bogoliubov level and therefore lowers the critical gN as well. In this particular example, we have that for $l = 0, 1$ then $n_{max} = 4$, for $l = 2, 3$ then $n_{max} = 3$, for $l = 4, 5, 6$ then $n_{max} = 2$ and for $l = 7, 8, 9, 10$ then $n_{max} = 1$.

A similar analysis can be done for any other cut-off, as can be seen in the figures 5.5 and 5.6.

Actually it should be noted that the quantum level with eigenenergy E_{10} is removed by the Bogoliubov replacement because it is the quantum state that forms the Bose-Einstein condensate. Nevertheless, even if $l = 0$ has one less n-mode than $l = 1$, this effect is compensated by a much higher first eigenenergy E_{20} as compared to E_{11}, so $l = 0$ and $l = 1$ are usually the most relevant modes in order to find the critical point. For them we have computed the critical coupling for several cut-offs ranging from $l^* = 5$ to $l^* = 100$ with spacing of 5 units. Their plot and fit can be seen in 5.4.

From dimensional analysis, for consistency with the problem without boundary conditions and from the results of the analytically diagonalizable energy levels, one would expect a scaling $gN \sim 1/E^* \sim l^{*2}$, at least for a sufficiently high cut-off energy. For large l, the first zero of the $l + 1/2$-th Bessel function is approximately located at:

$$s_{1l} \simeq l \tag{5.74}$$

And therefore we can safely identify $E_{1l} \simeq l^2$ in the large l limit.

The scaling $\sim l^{*2}$ is true, but there are corrections of order of higher inverse powers of l^*. The fits correspond to the functions:

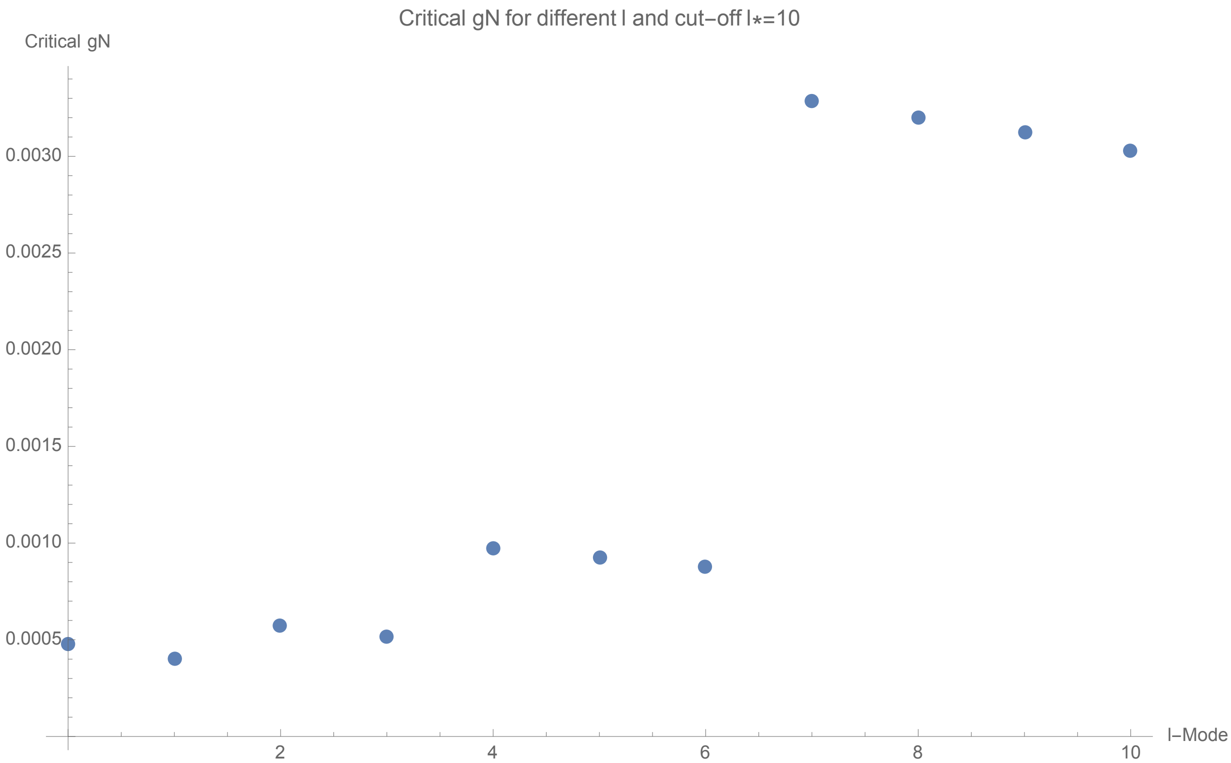


Figure 5.3: Critical coupling for different l with cut-off l*=10

$$f_0(l^*) = \frac{a_0}{l^{*2}} + \frac{b_0}{l^{*4}} + \frac{c_0}{l^{*6}} \quad \text{and} \quad f_1(l^*) = \frac{a_1}{l^{*2}} + \frac{b_1}{l^{*4}} + \frac{c_1}{l^{*6}} \tag{5.75}$$

With coefficients:

$$\begin{aligned} a_0 &= 0.0496122 \quad b_0 = -0.216974 \quad c_0 = 5.1981 \\ a_1 &= 0.0508407 \quad b_1 = -1.35978 \quad c_1 = 20.9693 \end{aligned} \tag{5.76}$$

5.6 Mode degeneracy and the area-law

In the case of periodic boundary conditions the emergence of the area law (Bekenstein-like entropy) was due to both quantum criticality and spherical symmetry. We have already established that for non-periodic boundary conditions some modes on top of the condensate can become critical as well. The role of the spherical symmetry is however less transparent here. Indeed we could expect only a "radius-law" thanks to the symmetry of B_2, but even this would work only if the $l = l^*$ mode was to become critical first, which is not the case. How can this be overcome? One could expect that, in the large cut-off limit (where the non-vanishing ground-state energy becomes almost negligible), spherical symmetry would be approximately restored. We argue in the following that this is true, at least in what concerns mode degeneracy.

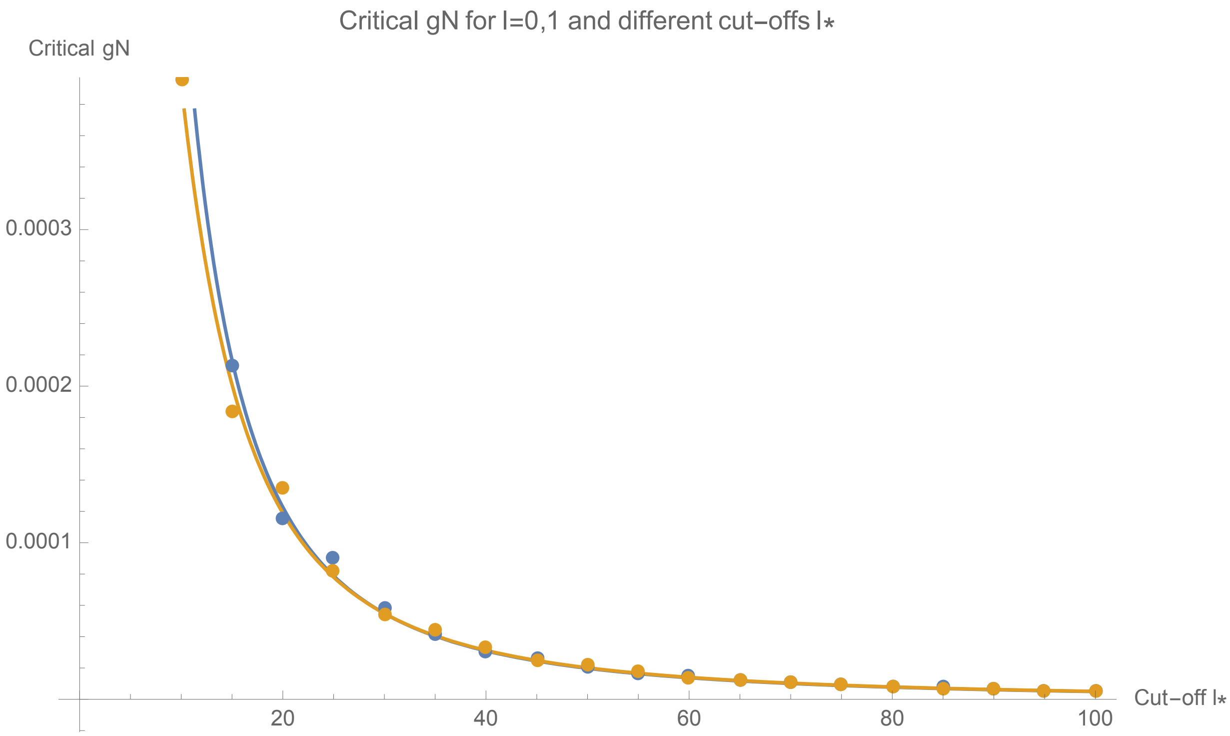


Figure 5.4: Critical gN for $l = 0$ (blue) and $l = 1$ (yellow) for different cut-offs l^*. Their corresponding fits are $f_0(l^*)$ and $f_1(l^*)$.

Take a look at figure 5.3 and its equivalent for cut-offs $l^* = 20, 30$ in figures 5.5 and 5.6. We see that, for the different cut-offs, roughly half of the l-modes achieve criticality for approximate the same small effective coupling gN, whereas the other half does it only for larger values. In order to compute this fraction explicitly, one should pick a maximal allowed deviation from the true lowest value of the critical coupling $\Delta gN/gN$. Let us assume that a reasonable estimate gives a fraction of $1/a$ of all l-modes. Then the total number of approximately degenerate modes can be obtained by summing all the l-modes and their corresponding m-modes:

$$\mathcal{N} = \sum_{l=0}^{l^*/a}(2l+1) = \frac{(a+l^*)^2}{a^2} \tag{5.77}$$

Therefore, the number of approximate degenerate modes grows as $\sim l^{*2}$ and, upon restoring dimensions and following the same arguments as for the case with periodic boundary conditions, we get that:

$$S = log(N_{states}) \sim log(n_{hol}+1)\left(\frac{R}{L_*}\right)^2 \tag{5.78}$$

Where L^* is defined as before: $L_* = \frac{\hbar}{\sqrt{2mE^*}}$.

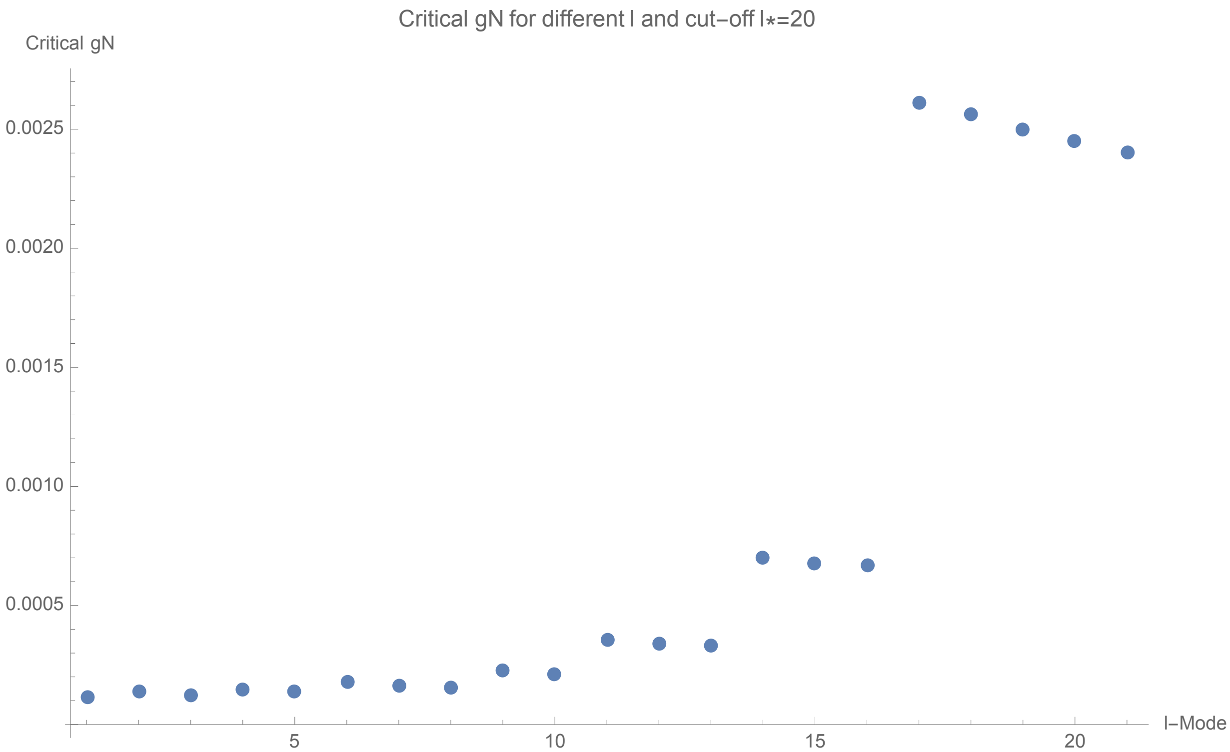


Figure 5.5: Critical coupling for different l with cut-off l*=20

We can see how spherical symmetry is approximately restored by studying the Bessel functions themselves and their integrals. Analytically this is a hard problem since involves the study of integrals of Bessel functions whose exact value is unknown. On the other hand, a full numerical analysis for high cut-offs and a large number of angular momenta is too expensive computationally for the extent of this work. However, we may pursue an approach based on the asymptotic (numerical) behavior of the Bessel functions, and therefore the elements of the Hamiltonian coefficient matrix.

If the Bessel functions have a reasonable asymptotic behavior, then it is expected that degeneracy due to the third dimension of the 3-sphere, which was broken by the non-periodic boundary conditions, would be restored, at least approximately. To elaborate a bit more: in this case the elements of the Hamiltonian coefficient matrix depend solely on the energy levels, which do become degenerate as the large zeros of the Bessel functions become exactly degenerate asymptotically [10]:

$$s_{nl} \simeq \left(n + \frac{l}{2}\right)\pi + \mathcal{O}\left(1/n\right) \tag{5.79}$$

And therefore:

$$s_{nl} \simeq s_{n-1,l+2} \simeq s_{n+1,l-2} \simeq \ldots \tag{5.80}$$

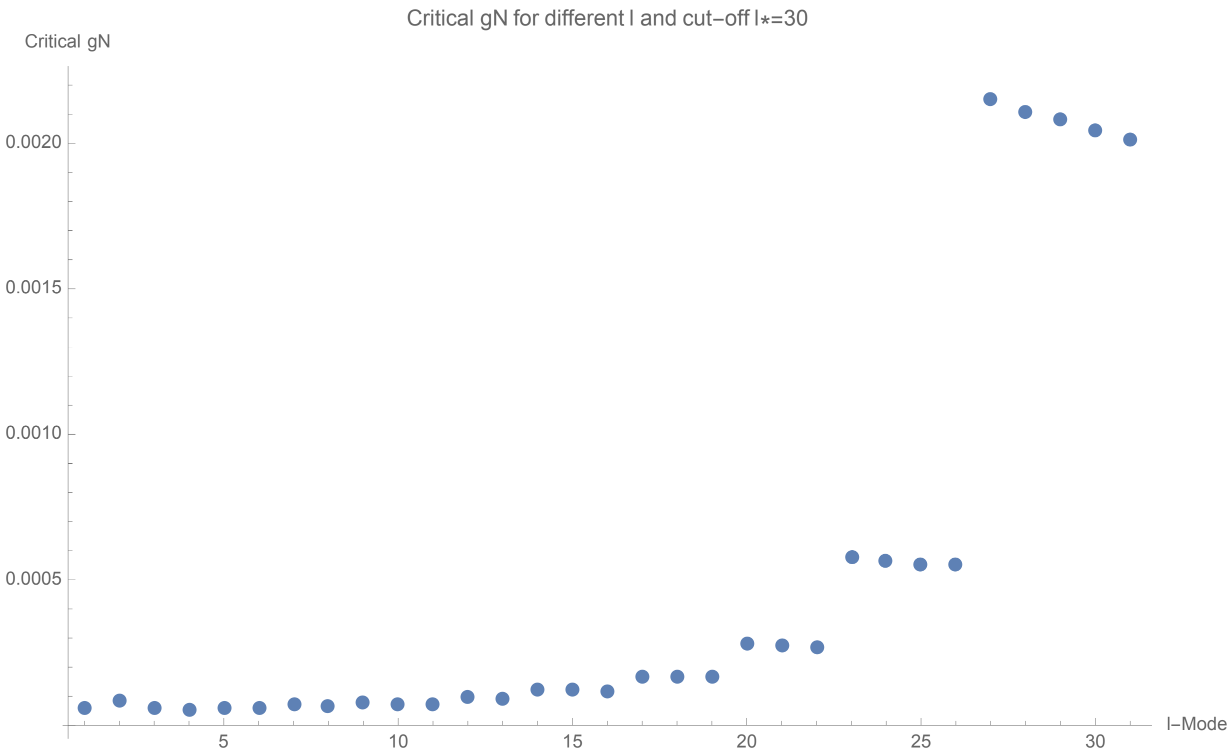


Figure 5.6: Critical coupling for different l with cut-off l*=30

By means of a numerical analysis we find the following asymptotic behavior for the Bessel functions:

1. At high energies the mixing between the high energy level and the ground state becomes some non-vanishing constant that is therefore independent of the cut-off. This explains why the critical point depends on the cut-off in a clean way. See figure 5.9. The considered integral is of the form:

$$\frac{\int_0^1 dr J^2_{1/2}(s_{10}r) J^2_{1+1/2}(s_{n1})}{\int_0^1 drr J^2_{1/2}(s_{10}r) \int_0^1 drr J^2_{1+1/2}(s_{n1}r)} \tag{5.81}$$

2. At high energies mixing between two different n-modes and the ground state also become constant but not zero (so the Hamiltonian does not become automatically diagonal). See figure 5.10. The considered integral is of the form:

$$\frac{\int_0^1 dr J^2_{1/2}(s_{10}r) J_{1+1/2}(s_{n-2,1}) J_{1+1/2}(s_{n,1})}{\int_0^1 drr J^2_{1/2}(s_{10}r) \sqrt{\int_0^1 drr J^2_{1+1/2}(s_{n1}r)} \sqrt{\int_0^1 drr J^2_{1+1/2}(s_{n-2,1}r)}} \tag{5.82}$$

3. The mixing between different n-modes for a same l-mode decreases very rapidly. See figure 5.11. The considered integral is of the form:

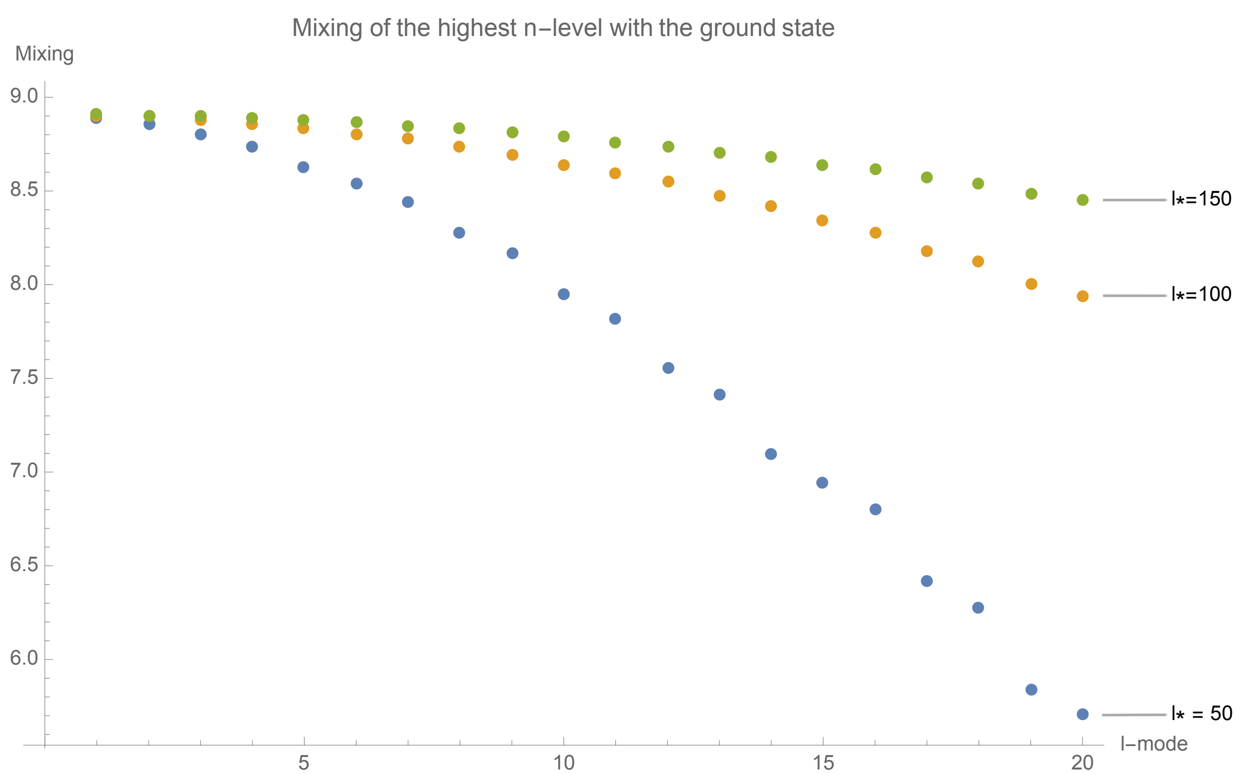


Figure 5.7: Mixing of the highest n-level with the ground state

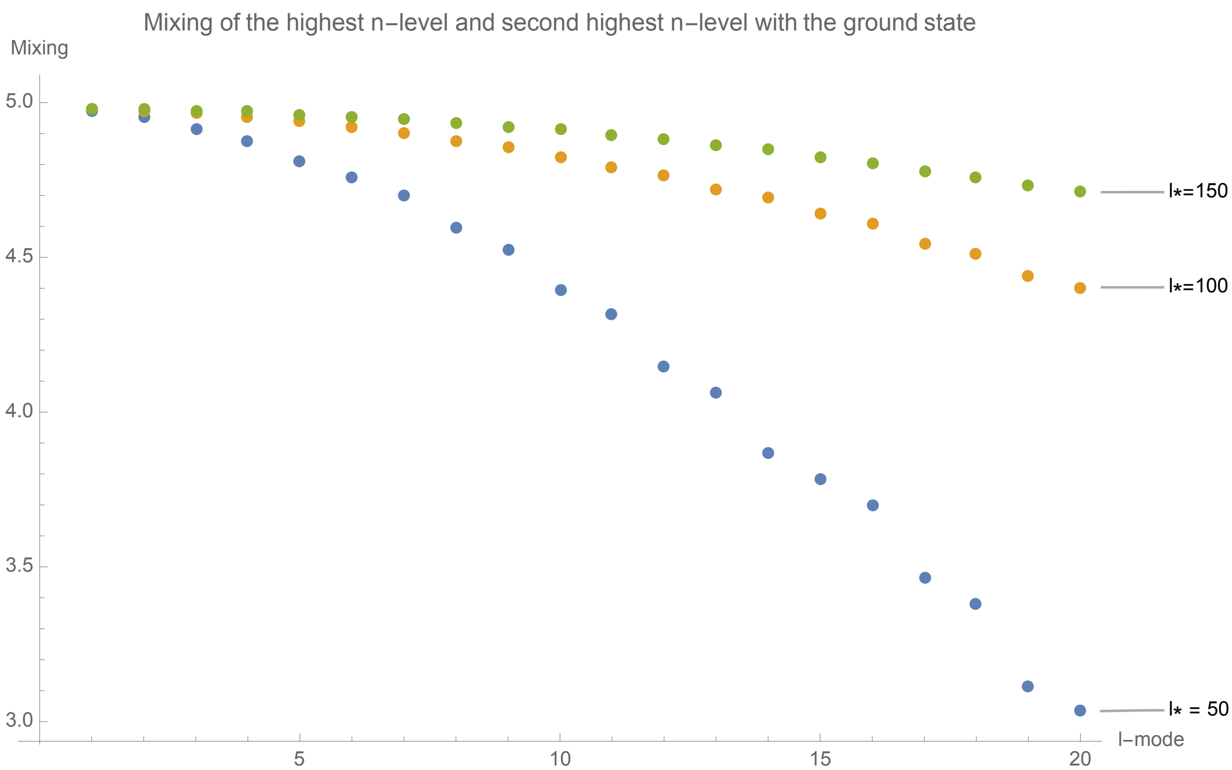


Figure 5.8: Mixing of the highest and second highest n-level with the ground state

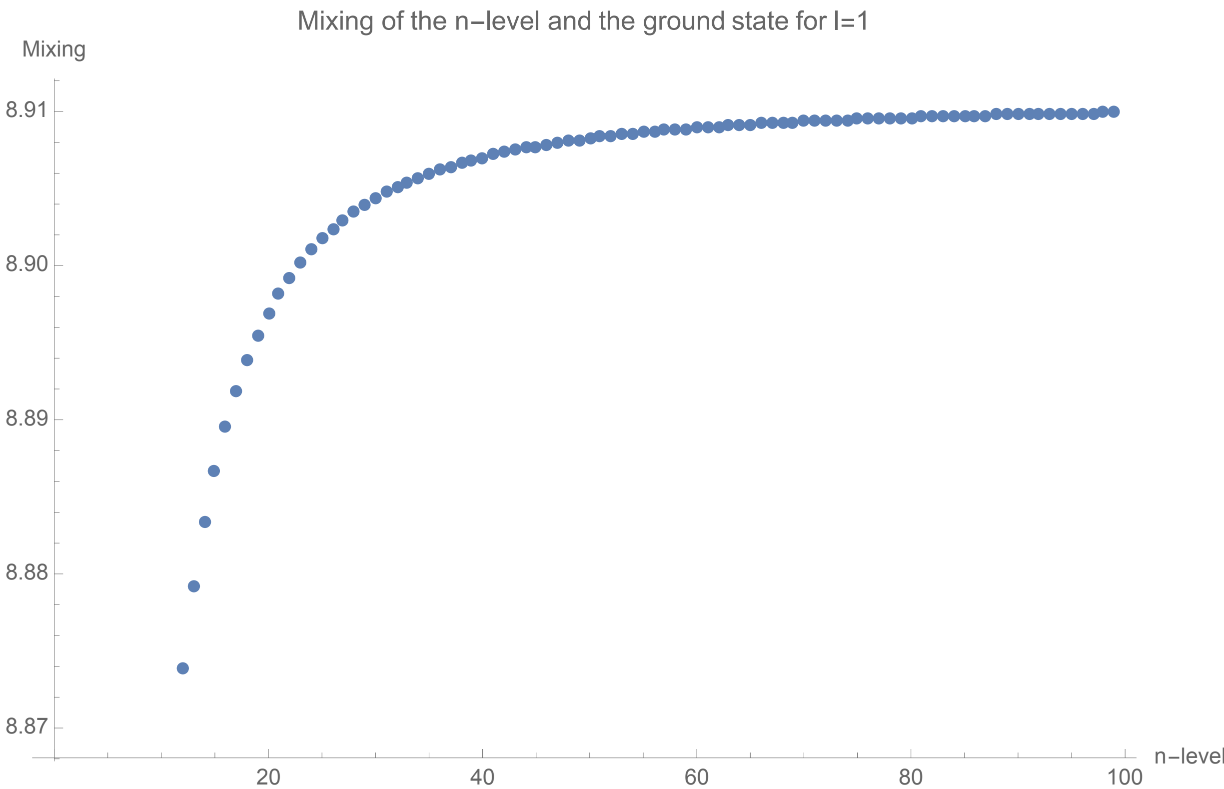


Figure 5.9: Mixing of the n-level with the ground state for l=1

$$\frac{\int_0^1 dr J_{1/2}^2(s_{10}r) J_{1+1/2}(s_{1,1}) J_{1+1/2}(s_{n,1})}{\int_0^1 drr J_{1/2}^2(s_{10}r)\sqrt{\int_0^1 drr J_{1+1/2}^2(s_{n1}r)}\sqrt{\int_0^1 drr J_{1+1/2}^2(s_{1,1}r)}} \tag{5.83}$$

As seen in the previous section, the higher cut-off, the higher the relevant n-modes are. Therefore, we can expect that the relevant elements of the Hamiltonian coefficient matrix will depend only on the energies E_{nl} and not the values of some integrals. On the one hand, the integrals corresponding to mixing with neighboring modes tend to a constant which depends on the ”distance” between the levels. On the other hand, the integrals corresponding to mixing with very low n-modes tend to vanish and therefore become irrelevant in the diagonalization procedure, even if they do not become constant.

It can be seen in figures 5.7 and 5.8 how this asymptotic behavior is nice: the higher the cut-off energy is, the more l-modes will have almost the same high energy coefficients (we can guess around the first 10% l-modes by checking the figures). This levels are the most important ones in determining the critical point. Furthermore, for a sufficiently high energy cut-off, the l-modes have a similar number of n-modes that mix with the high energy modes, being there only a difference in the very low-energy sector, which has almost no mixing for sufficiently high cut-offs.

To sum up, the basis of the argument for the emergence of Bekenstein-Entropy is:

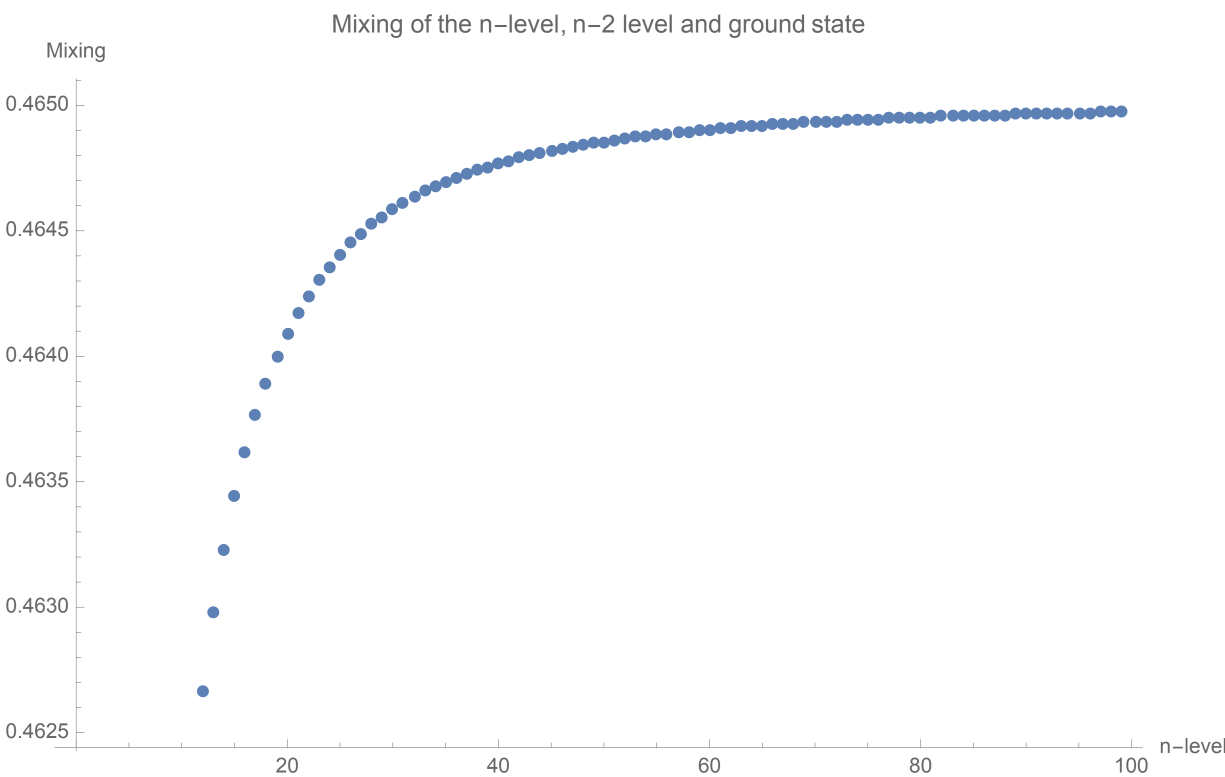


Figure 5.10: Mixing of the n- and (n-2)- levels with the ground state for l=1

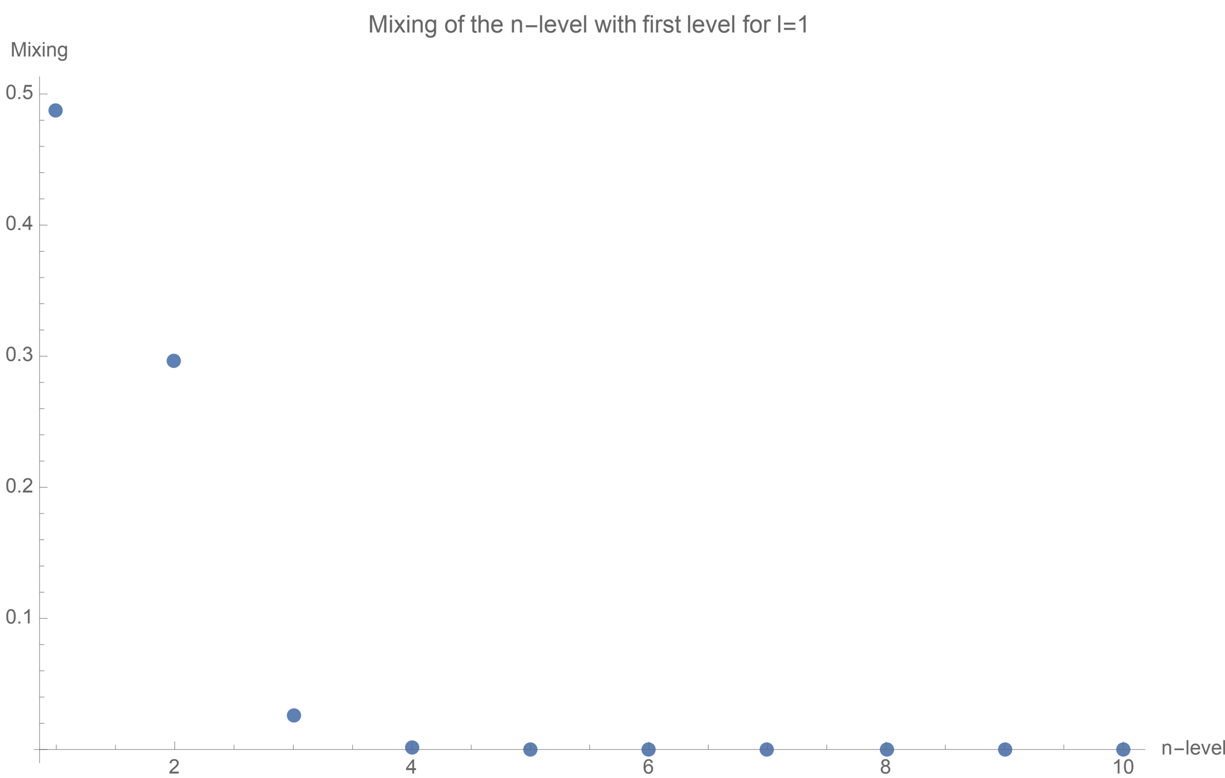


Figure 5.11: Mixing of the n- and 1- levels with the ground state for l=1

1. For a given cut-off l^*, the modes that will collapse and become degenerate with the background condensate are those corresponding to high energies E_{nl} but low l (and correspondingly, high n). The reasons for this are that: i) for highly energetic modes the derivative coupling pushes down the energy of the corresponding Bogoliubov modes and ii) for low l there are more available n-modes that mix with each other and bring non-diagonal elements of the Hamiltonian coefficient matrix that push down the corresponding Bogolibov modes.

2. For a large cut-off l^* the energy levels close to the cut-off become degenerate due to the degeneracy of the zeros of the Bessel functions for large n. This happens therefore not only to levels close to the cut-off (candidates to become critical), but also for their neighboring modes.

3. For large n-modes, the integral mixing with neighboring modes tends to constants depending on the "distance" between the modes. On the other hand, the mixing with low n-modes tends to vanish. This means that for different l-modes with degenerate high energy levels the presence of a few more low n-modes should not affect the Bogoliubov levels associated with the large n-modes.

For all these reasons, it is expected that the area-law emerges in the high energy-cut-off limit, i.e. this system would exhibit Bekenstein-like entropy.

Chapter 6

Conclusions and outlook

This master thesis has been devoted to Bekenstein entropy and its emergence in black holes and symmetric Bose-Einstein condensates at the critical point of phase transition. We have reviewed the theoretical arguments that historically lead to the introduction of black hole thermodynamics, being Bekenstein entropy and Hawking temperature two key concepts thereof. These arise rigorously from General Relativity and Quantum Field Theory in curved space-time. We have seen how the transition from the laws of black hole mechanics to the laws of black hole thermodynamics brings the need of a quantum description for black holes, most likely a many-body description as well. This motivates the introduction of the Quantum N-portrait for black holes.

The Quantum N-portrait has opened a new and very rich connection between condensed matter physics and black holes. We have reviewed its main features and results and we have explored how this connection could be useful to understand the emergence of Bekenstein entropy in black holes. In particular, we have reviewed the case of the model of a bosonic gas living on a d-sphere with periodic boundary conditions. Due to the spherical symmetry, Bekenstein-like entropy emerges when the Bose-Einstein condensate reaches the critical point of phase transition, where some of the Bogoliubov modes collapse and become degenerate with the background condensate. In other words, at the critical point gapless modes appear and their number scales the volume of the (d-1)-sphere.

Next, we have extended the study of this model to the case with non-periodic boundary conditions by putting the bosonic gas in a 2-ball. The topology of the 2-ball is of special relevance, since it corresponds to the event horizon and interior of a black hole. We have showed numerically that in this case some of the modes collapse and become degenerate with the background condensate as well. Again, these gapless modes are responsible for the emergence of Bekenstein entropy. Even though spherical symmetry is partially broken by the non-periodic boundary conditions, numerical computations strongly suggest an area scaling of the entropy. This can be explained by means of the asymptotic behaviour of the Bessel functions, which brings a degeneracy of the energy levels that can be understood as an approximate restoring of spherical symmetry.

It would be interesting to extend further the numerical analysis to larger cut-offs, as well as to numerically compute the matrices responsible for the Bogoliubov transformation in order to check the dependence with N of the number density of depleted particles for the whole Hamiltonian and not only for the analytically diagonalizable modes. Another relevant extension of the model could be the inclusion of higher derivative terms that create naturally a minimum in the potential, instead of imposing a cut-off by hand.

In light of the Bekenstein bound it is also very interesting that other physical systems can exhibit an area-scaling entropy. It suggests that these critical condensates could store much larger amounts of information than any other available system. If they were realizable experimentally, they could be of great interest in the field of quantum information and computation. In fact, the model discussed in section 3.4 has been indeed tested with a quasi one-dimensional bosonic gas living in torus of inner radius much smaller than its outer radius.

Being many-body physics the natural field to study the entropy arising from micro-state counting, we think that the approach of the Quantum N-Portrait to black holes will continue to provide results that will help us to improve our understanding of black holes from the point of view of a quantum theory and, therefore, from the point of view of fundamental physics.

Appendix A

Mathematica codes

For completeness we include here the Mathematica codes used for the numerical analysis described in chapter 5.

A.1 Preliminar definitions

- Function that computes the norm of the l+1/2-th order spherical Bessel function with boundary condition given by the n-th zero.

```
Norm[l_, n_] :=
  Sqrt[NIntegrate[
    r/N[BesselJZero[l + 1/2, n], 32]^2*(BesselJ[l + 1/2, r])^2, {r, 0,
      N[BesselJZero[l + 1/2, n], 32]}, WorkingPrecision -> 30,
    Method -> "DoubleExponential"]];
```

- Function that computes the mixing of two ground-state spherical Bessel functions with two l+1/2-th order spherical Bessel function with boundary conditions given by the n1-th and the n2-th zero.

```
TetraIntegral[n1_, n2_, l_] :=
 If[n1 > n2,
  NIntegrate[
   1/N[BesselJZero[l + 1/2, n1], 102]*
    BesselJ[1/2,
      N[BesselJZero[1/2, 1], 102]/N[BesselJZero[l + 1/2, n1], 102]*
       r]^2*BesselJ[l + 1/2, r]*
    BesselJ[l + 1/2,
     N[BesselJZero[l + 1/2, n2], 102]/
       N[BesselJZero[l + 1/2, n1], 102]*r], {r, 0,
    N[BesselJZero[l + 1/2, n1], 102]}, WorkingPrecision -> 100,
```

```
   MaxRecursion -> 100, Method -> "DoubleExponential"],
  NIntegrate[
   1/N[BesselJZero[l + 1/2, n2], 102]*
    BesselJ[1/2,
      N[BesselJZero[1/2, 1], 102]/N[BesselJZero[l + 1/2, n2], 102]*
       r]^2*BesselJ[l + 1/2, r]*
    BesselJ[l + 1/2,
     N[BesselJZero[l + 1/2, n1], 102]/
       N[BesselJZero[l + 1/2, n2], 102]*r], {r, 0,
    N[BesselJZero[l + 1/2, n2], 102]}, WorkingPrecision -> 100,
   MaxRecursion -> 100, Method -> "DoubleExponential"]]
```

- Set desired l to study for a given l^*. This code also computes the number of rows needed for the Hamiltonian coefficient matrix.

```
lmax = 50; (*Write here the desired cut-off*)
l = 0; (*Write here the desired l to study*)

cutoff = BesselJZero[lmax + 1/2, 1];
n = 0;
For[i = 1, BesselJZero[l + 1/2, i] <= cutoff, i++, n = i]
Print["n = ", n]
rows = n*(2 l + 1)
```

A.2 Numerical Bogoliubov transformation

- Loop that computes the Hamiltonian coefficient matrix elements for the case $l = 0$ and stores them in the array Hamiltonian[l,lmax]

```
Array[H1, {rows, rows}];

For[i = 1, i < rows + 1, i++,
  For[j = 1, j < rows + 1, j++,
   H1[i, j] = 0]];

Clear[gN]; For[i = 2, i < n + 1, i++,
 For[j = 2, j < n + 1, j++,
  temp = N[BesselJZero[l + 1/2, i]]^2*KroneckerDelta[i, j] -
     gN*(N[BesselJZero[l + 1/2, i]]^2 +
        N[BesselJZero[1/2, 1]]^2)*(N[BesselJZero[l + 1/2, j]]^2 +
        N[BesselJZero[1/2, 1]]^2)*TetraIntegral[i, j, l]*
      Norm[l, i]^(-1)*Norm[l, j]^(-1)*Norm[0, 1]^(-2))
```

```
      +2*gN*N[BesselJZero[1/2, 1]]^4*TetraIntegral[1, 1, 0]*
 Norm[0, 1]^(-4)*KroneckerDelta[i, j];
  H1[i, j] = temp]];

Array[H, {2*(rows - 1), 2*(rows - 1)}];

For[i = 1, i < rows, i++,
  For[j = 1, j < rows, j++,
   H[i, j] = 0.5*H1[i + 1, j + 1];
   H[rows - 1 + i, j] = 0;
   H[i, rows - 1 + j] = 0;
   H[rows - 1 + i, rows - 1 + j] = -0.5*H1[i + 1, j + 1]]];

For[k1 = 1, k1 < 2*rows, k1++,
 For[k2 = 1, k2 < 2*rows, k2++,
  Hamiltonian[l, lmax][k1, k2] = H[k1, k2]]];
```

- Loop that computes the Hamiltonian coefficient matrix elements for the case $l \neq 0$ and stores them in the array Hamiltonian[l,lmax]

```
Array[H1, {rows, rows}];

For[i = 1, i < rows + 1, i++,
 For[j = 1, j < rows + 1, j++,
  H1[i, j] = 0]];

Clear[gN]; For[i = 1, i < n + 1, i++,
 For[j = 1, j < n + 1, j++,
  temp = (N[BesselJZero[l + 1/2, i]]^2*KroneckerDelta[i, j] -
     gN*(N[BesselJZero[l + 1/2, i]]^2 +
        N[BesselJZero[1/2, 1]]^2)*(N[BesselJZero[l + 1/2, j]]^2 +
        N[BesselJZero[1/2, 1]]^2)*TetraIntegral[i, j, l]*
      Norm[l, i]^(-1)*Norm[l, j]^(-1)*Norm[0, 1]^(-2))
      +2*gN*N[BesselJZero[1/2, 1]]^4*TetraIntegral[1, 1, 0]*
 Norm[0, 1]^(-4)*KroneckerDelta[i, j];
  For[m1 = -l, m1 < l + 1, m1++,
   For[m2 = -l, m2 < l + 1, m2++,
    H1[(i - 1)*(2 l + 1) + 1 + (m1 + l), (j - 1)*(2 l + 1) +
       1 + (m2 + l)] = temp*KroneckerDelta[m1, m2]]]]];

Array[H2, {rows, rows}];
```

```
For[i = 1, i < rows + 1, i++,
 For[j = 1, j < rows + 1, j++,
  H2[i, j] = 0]];

For[i = 1, i < n + 1, i++,
 For[j = 1, j < n + 1, j++,
  temp = (-gN*N[BesselJZero[1/2, 1]]^2*
      N[BesselJZero[l + 1/2, 1]]^2)*TetraIntegral[i, j, l]*
    Norma[l, i]^(-1)*Norm[l, j]^(-1)*Norm[0, 1]^(-2);
  For[m1 = -l, m1 < l + 1, m1++,
   For[m2 = -l, m2 < l + 1, m2++,
    H2[(i - 1)*(2 l + 1) + 1 + (m1 + l), (j - 1)*(2 l + 1) +
       1 + (m2 + l)] = KroneckerDelta[m1, -m2]*(-1)^(m1)*temp]]]];

Array[H, {2*rows, 2*rows}];

For[i = 1, i < rows + 1, i++,
 For[j = 1, j < rows + 1, j++,
  H[i, j] = 0.5*H1[i, j];
  H[rows + i, j] = H2[i, j];
  H[i, rows + j] = -H2[i, j];
  H[rows + i, rows + j] = -0.5*H1[i, j]]];
For[k1 = 1, k1 < 2*rows + 1, k1++,
 For[k2 = 1, k2 < 2*rows + 1, k2++,
  Hamiltonian[l, lmax][k1, k2] = H[k1, k2]]]
```

A.3 Search for the critical point

- This code looks for the approximate critical point. Multiple runs with changed accepted values for gN allow for a refinement in the precision of the computation. It first computes the lowest energy level for the given values of gN, then picks the value of gN for which the lowest energy level is closest to 0.

```
gNmax = 0.005; (*It should be higher than the expected critical point*)
points = 1000; (*Precision*)

For[gN = N[gNmax/points], gN < gNmax, gN += N[gNmax/points],
 temp = Min[
   Abs[Eigenvalues[
     Table[N[Alles[l, lmax][i, j]], {i, 2*rows}, {j, 2*rows}]]]];
 LowestLevel[l, lmax][Round[gN*points/gNmax]] = {gN, temp}];
```

```
Print["Searching critical point"];

For[i = 1; j = LowestLevel[l, lmax][1][[2]],
 LowestLevel[l, lmax][i + 1][[2]] < j, i++,
 j = LowestLevel[l, lmax][i][[2]];
 Critical[l, lmax] = LowestLevel[l, lmax][i][[1]]];
gNmax = Critical[l, lmax];
Print["Critical point is at gN = ", Critical[l, lmax]]

];
```

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Statement of authorship

I hereby state that this master thesis has been written by me and is based on my own work. Every used source has been acknowledged and listed in the bibliography.

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