Master's Thesis

On the Quantization of the Hall Conductance for Many-Body Interactions on a Torus

Ludwig Fürst
01.08.2017

Advisor: Prof. Dr. Sven Bachmann

Theoretical Mathematical Physics
Ludwig-Maximilians-Universität
München

Contents

1 Introduction

The quantum Hall effect describes the fact that at low temperatures the Hall conductance of a given quantum system is quantized to integer multiples of e^2/h with the electron charge e and Planck's constant h. This effect was originally predicted in 1975 in [3] based on approximate calculations. Klitzling discovered in 1980 that this is an exact quantization, i.e. up to remarkable precision [12]. For this he was awarded the 1985 Nobel Prize in Physics. The first intuitive understanding of this phenomenon was given by Laughlin in his 1981 work [13]. But up to date there has been no fully rigorous mathematical proof for the reason behind this. There are many approaches under simplified conditions of free particles, which rely on non-commutative geometry [7] or under assumption of averaged or uniform conductances [4, 5, 15], which are based on Chern numbers as topological invariants. Other works base upon perturbations of simple models to add complexity ([1, 2, 11]) but only work as long as the perturbations stay small. In his 2004 paper [8] Hastings introduced a technique which he called "quasi-adiabatic continuation". This in combination with Lieb-Robinson bounds for quantum lattice dynamics allowed him to prove the Lieb-Schultz-Mattis theorem for higher dimensions. Afterwards that technique was also applied by Hastings as well as others for new applications [9, 16], mostly for systems with uniformly gapped Hamiltonians. Then in October 2013 Hastings and Michalakis submitted a paper [10] in which they proved Hall quantization for finite k-body interactions on a torus, provided a unique groundstate and conserved local charge. They managed to avoid any averaging assumptions and gave explicit bounds. This paper was finally accepted in June 2014 and published, it was however much disputed and critized for lack of readability and consistent notations. This can also be seen in the quite long time between submission and publishing. In this work based on that paper, we aim to provide a refined version of their paper, which should be easier to understand and resolve some questions and dubious parts.

In the next chapter we will begin with general systems of Hamiltonians on a torus, define the quasi-adiabatic continuation and derived operators and present properties shown in [6]. Further we prove some bounds which we will use later in the proof, but which can be formulated more generally than later used. In chapter 3 we will introduce the system for which we want to show quantization of the Hall conductance, state the main theorem and give an high outline of the proof. We construct families of so called twisted Hamiltonians on a flux-torus and derive first properties. Chapter 4 and 5 are dedicated to prove two estimates needed for the main theorem. The first one is based on partial traces which will lead to energy bounds for systems without a uniform gap. The second shows a certain uniformity of phase for the quasi-adiabatic evolution which follows amongst others from Lieb-Robinson bounds introduced in [14, 6]. Those two chapters are the most technical ones and need the most notations. The last chapter contains remarks on the work and possible extensions.

Where possible, all occurring bounds were given explicitly in the parameters of the system, such as interaction strength and range. In chapter 5 however there will be bounds with implicit dependency which will persist in the final result.

2 General Observations

2.1 Quasi-Adiabatic Continuation

In this section we consider differentiable one- and two-parameter families $H(s,t)$, $s,t \in [0,1]$ of Hamiltonians on the torus T. We define certain super-operators and the unitaries defined by them and give properties of these objects. Afterwards we define loop operators for 2-parameter families of Hamiltonians and compute a certain phase for uniformly gapped Hamiltonians.

In [6] the authors defined for $\Delta > 0$ a bounded, odd weight function $W_\Delta \in L^1(\mathbb{R})$ with the following properties:

i. $|W_\Delta(t)|$ *is continuous and monotone decreasing for* $t \geq 0$ *s.t.* $\|W_\Delta\|_\infty = W_\Delta(0) = 1/2$;

ii. $|W_\Delta(t)| \leq f(\Delta|t|)$ *for a subexponential function* f *of the form* $f(x) = cx^4 exp(-\frac{2}{7}\frac{x}{ln^2 x})$

iii. $\|W_\Delta\|_1 \leq \frac{K}{\Delta}$ *for some constant* $K > 0$.

A function $f : [0,\infty[\to [0,\infty[$ is said to be subexponential small if $f(x) = \mathcal{O}(exp(-x^c))$ for all $c \in [0,1[$. This set of functions has the property that it is closed under scaling and multiplication with polynomials. The function given above is a typical example.

Let H, A be operators on T. Then we define:

$$S_\Delta(H,A) = \int_{-\infty}^{\infty} dt\, W_\Delta(t) \cdot e^{itH} A e^{-itH} \tag{2.1}$$

Now consider a smooth one-parameter familiy of Hamiltonians $H(s)$ with $s \in [0,1]$. We define

$$D_\Delta(s) = S_\Delta(H(s), H'(s)) = \int_{-\infty}^{\infty} dt\, W_\Delta(t) \cdot e^{itH(s)} H'(s) e^{-itH(s)} \tag{2.2}$$

Then this defines a unitary $U_\Delta(s)$ by the differential equation

$$\frac{d}{ds} U_\Delta(s) = iD_\Delta(s) U_\Delta(s), \qquad U_\Delta(0) = \mathbb{I}, \tag{2.3}$$

which is called in [10] the quasi-adiabatic evolution corresponding to $H(s)$ with threshold Δ. The reason for this lies in the following lemma.

Lemma 2.1. *Assume that* $\{H(s)\}_{s\in[0,1]}$ *is a smooth path of gapped Hamiltonians with spectral gap* $\Delta(s) \geq \Delta > 0$ *and (up to phase) unique groundstate* $|\Psi_0(s)\rangle$*. Then* $U_\Delta(s)|\Psi_0(0)\rangle$ *is the unique groundstate of* $H(s)$ *fulfilling the parallel transport condition* $\langle\Psi(s)|\Psi'(s)\rangle = 0$.

This means that for smooth paths of Hamiltonians with a spectral gap uniformly bounded below by Δ, the quasi-adiabatic evolution with threshold Δ simulates the true adiabatic evolution of $H(s)$.

Proof. Set $|\Psi_0\rangle := |\Psi_0(0)\rangle$. Then corollary 2.8 in [6] states

$$|\Psi_0(s)\rangle\langle\Psi_0(s)| = U_\Delta(s)\,|\Psi_0\rangle\langle\Psi_0|\,U^*_\Delta(s) \tag{2.4}$$

and we only have to check that $U_\Delta(s)\,|\Psi_0\rangle$ fulfills the parallel transport condition. We have

$$\frac{d}{ds}(U_\Delta(s)\,|\Psi_0\rangle) = iD_\Delta(s)U_\Delta(s)\,|\Psi_0\rangle \tag{2.5}$$

and therefore by inserting $\langle\Psi_0|\Psi_0\rangle = 1$ twice we get with (2.4)

$$\begin{aligned}\langle\Psi_0|\,U^*_\Delta(s)\frac{d}{ds}U_\Delta(s)\,|\Psi_0\rangle &= i\,\langle\Psi_0|\Psi_0\rangle\langle\Psi_0|\,U^*_\Delta(s)D_\Delta(s)U_\Delta(s)\,|\Psi_0\rangle\langle\Psi_0|\Psi_0\rangle = \\ &= i\,\langle\Psi_0|\,U^*_\Delta(s)\,|\Psi_0(s)\rangle\langle\Psi_0(s)|\,D_\Delta(s)\,|\Psi_0(s)\rangle\langle\Psi_0(s)|\,U_\Delta(s)\,|\Psi_0\rangle = 0\end{aligned} \tag{2.6}$$

since

$$\begin{aligned}\langle\Psi_0(s)|\,D_\Delta(s)\,|\Psi_0(s)\rangle &= \int_{-\infty}^{\infty} dt\,W_\Delta(t)\,\langle\Psi_0(s)|\,e^{itH(s)}H'(s)e^{-itH(s)}\,|\Psi_0(s)\rangle = \\ &= \int_{-\infty}^{\infty} dt\,W_\Delta(t)\,\langle\Psi_0(s)|\,H'(s)\,|\Psi_0(s)\rangle = 0\end{aligned} \tag{2.7}$$

where the second equality holds since the $|\Psi_0(s)\rangle$ are groundstates for $H(s)$ and the last equality comes from the fact that W_Δ is an odd function. □

2.2 Loop Operators

Now we consider a two-parameter family of Hamiltonians $H(s,t)$, $s,t \in [0,2\pi]$ with groundstates $|\Psi_0(s,t)\rangle$ and introduce loop operators which describe the quasi-adiabatic evolution around a small (quadratic) loop in flux-space (s,t). Therefore we define the generators

$$D_{\Delta,x}(s,t) = S_\Delta(H(s,t), \partial_1 H(s,t)) \tag{2.8}$$

$$D_{\Delta,y}(s,t) = S_\Delta(H(s,t), \partial_2 H(s,t)) \tag{2.9}$$

and the corresponding unitaries for fixed (s,t) through the differential equations

$$\partial_r U_{\Delta,x}(s,t,r) = iD_{\Delta,x}(s+r,t)U_{\Delta,x}(s,t,r), \qquad U_{\Delta,x}(s,t,0) = \mathbb{I} \tag{2.10}$$

$$\partial_r U_{\Delta,y}(s,t,r) = iD_{\Delta,y}(s,t+r)U_{\Delta,y}(s,t,r), \qquad U_{\Delta,y}(s,t,0) = \mathbb{I} \tag{2.11}$$

so that e.g. $U_{\Delta,x}(s,t,r)$ is the unitary corresponding to the quasi-adiabatic evolution from (s,t) to $(s+r,t)$ in flux-space.
By differentiating both sides with respect to r' and using the uniqueness of solutions to the ODEs, one gets the following composition formulas:

$$U_{\Delta,x}(s+r,t,r')U_{\Delta,x}(s,t,r) = U_{\Delta,x}(s,t,r+r') \tag{2.12}$$

$$U_{\Delta,y}(s,t+r,r')U_{\Delta,y}(s,t,r) = U_{\Delta,y}(s,t,r+r') \tag{2.13}$$

The loop operator corresponding to a loop of side r, starting at (s,t) is then defined as

$$V_{\Delta,\circlearrowleft}(s,t,r) = U^*_{\Delta,y}(s,t,r)U^*_{\Delta,x}(s,t+r,r)U_{\Delta,y}(s+r,t,r)U_{\Delta,x}(s,t,r) \tag{2.14}$$

and is given by quasi-adiabatic evolution along the path $\Lambda(s,t,r)$, which runs along the sides of a square with length r in counter-clockwise direction, starting at (s,t).
From the previous lemma we then get the following:

Lemma 2.2. *Assume that* $\{H(s,t)\}_{(s,t)\in\Lambda(s,t,r)}$ *is a smooth two-parameter family of Hamiltonians which are gapped uniformly along* $\Lambda(s,t,r)$ *with lower bound* $\Delta > 0$. *Denote* $|\Psi_0(a,b)\rangle$ *the groundstate of* $H(a,b)$. *Then we have:*

$$\langle\Psi_0(s,t)|\, V_{\Delta,\circlearrowleft}(s,t,r)\, |\Psi_0(s,t)\rangle = e^{i\Phi(s,t,r)} \tag{2.15}$$

where

$$\Phi(s,t,r) = 2\int_s^{s+r} dx \int_t^{t+r} dy\, Im\,\langle\partial_2\Psi_0(x,y)|\partial_1\Psi_0(x,y)\rangle \tag{2.16}$$

and $Im(\cdot)$ *denotes the imaginary part.*

Proof. Since we have uniformly gapped Hamiltonians, lemma 2.1 tells us that every unitary of $V_{\Delta,\circlearrowleft}(s,t,r)$ gives us the groundstate fulfilling the parallel transport condition. So in the four sides of the loop we acquire the following four phases (since those exactly ensure the parallel transport condition):

$$i\Phi_1 = -\int_s^{s+r} dx\ \langle\Psi_0(x,t)|\partial_1\Psi_0(x,t)\rangle \tag{2.17}$$

$$i\Phi_2 = -\int_t^{t+r} dx\ \langle\Psi_0(s+r,x)|\partial_2\Psi_0(s+r,x)\rangle \tag{2.18}$$

$$i\Phi_3 = \int_s^{s+r} dx\ \langle\Psi_0(x,t+r)|\partial_1\Psi_0(x,t+r)\rangle \tag{2.19}$$

$$i\Phi_4 = \int_t^{t+r} dx\ \langle\Psi_0(s,x)|\partial_1\Psi_0(s,x)\rangle \tag{2.20}$$

Putting those together we get the claim from $\Phi(s,t,r) = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$ and the Theorem of Stokes. □

2.3 Useful Bounds

Remark 2.3. *Let* $g(x,y)$ *be a smooth function of two variables. Then*

$$\varphi(r) := \int_0^r dx \int_0^r dy\, g(x,y) \tag{2.21}$$

has the following Taylor expansion around 0*:*

$$\varphi(r) = g(0,0)r^2 + \int_0^r ds\, \frac{(r-s)^2}{2}\left(3\partial_1 g(s,s) + 3\partial_2 g(s,s) + \int_0^s dt\,(\partial_1^2 g(s,t) + \partial_2^2(t,s))\right) \tag{2.22}$$

In particular, we get the following bound:

$$\left|\frac{\varphi(r)}{r^2} - g(0,0)\right| \le r\cdot\left(\frac{1}{2}\sup_{0\le s\le r}(|\partial_1 g(s,s)| + |\partial_2 g(s,s)|) + \frac{r}{24}\sup_{0\le s,t\le r}(|\partial_1^2 g(s,t)| + |\partial_2^2 g(s,t)|)\right) \tag{2.23}$$

Later we will relate $g(0,0)$ for the function $\Phi(0,0,r)$ from Lemma 2.2 with the Hall conductance.

To get explicit bounds in (2.23) we need to bound the norm of derivates of groundstates. We show how this is principally done in the following

Remark 2.4. *Let* $\{H(s)\}_{s\in[0,1]}$ *be a smooth family of Hamiltonians with spectral gap* $\Delta(s) > 0$ *and* $|\Psi_0(s)\rangle$ *the corresponding groundstates with energies* $E_0(s)$ *fulfilling the parallel transport condition. So we have the relations*

$$(H(s) - E_0(s))\,|\Psi_0(s)\rangle = 0 \tag{2.24}$$

$$\langle\Psi_0(s)|\Psi_0'(s)\rangle = 0 \tag{2.25}$$

$$H(s) - E_0(s) \geq \Delta(s)(\mathbb{I} - |\Psi_0(s)\rangle\langle\Psi_0(s)|) \tag{2.26}$$

Differentiating (2.24), *multiplying with* $Q_0(s) := \mathbb{I} - |\Psi_0(s)\rangle\langle\Psi_0(s)|$ *and inserting* (2.25) *yields*

$$|\Psi_0'(s)\rangle = \frac{Q_0(s)}{H(s) - E_0(s)} H'(s)\,|\Psi(s)\rangle \tag{2.27}$$

so that with (2.26) *we get the bound*

$$\|\,|\Psi_0'(s)\rangle\,\| \leq \frac{\|H'(s)\|}{\Delta(s)} \tag{2.28}$$

From differentiating (2.24) *twice and* (2.25) *once we get the next bound*

$$\|\,|\Psi_0''(s)\rangle\,\| \leq 3\frac{\|H'(s)\|^2}{\Delta(s)^2} + \frac{\|H''(s)\|}{\Delta(s)} \tag{2.29}$$

By repeating this procedure we see that we can bound the norm of derivatives of $|\Psi_0(s)\rangle$ *as polynomials in the norm of derivatives of* $H(s)$ *and* $\Delta(s)$.

In the situation of the remark above, we have the following bound on $\Delta(s)$:

Lemma 2.5. *For* $r \in [0,1]$ *it is*

$$\Delta(r) \geq \Delta(0) - 2r \sup_{s\in[0,r]} \|H'(s)\| \tag{2.30}$$

Proof. When we look at the spectrum of $H(r)$, a triangle inequality gives us

$$\Delta(r) \geq \Delta(0) - 2\|H(r) - H(0)\| \tag{2.31}$$

and the claim readily follows. □

So we see that later we will only need bounds for the derivatives of the Hamiltonians.

In chapter 5 we will also use the following

Proposition 2.6. *For* $a \in \{1,2\}$ *let* U_a *be the unitary generated by* D_a, *i.e.*

$$\frac{d}{ds}U_a(s) = iD_a(s)U_a(s), \qquad U_a(0) = \mathbb{I} \tag{2.32}$$

Then for $r \geq 0$ *we have*

$$\|U_1(r) - U_2(r)\| \leq r \cdot \sup_{0\leq s\leq r} \|D_1(s) - D_2(r)\| \tag{2.33}$$

Proof. We have

$$\frac{d}{dr}(U_1^*(r)U_2(r)) = -iU_1^*(r)(D_1(r) - D_2(r))U_2(r) \tag{2.34}$$

and therefore the claim follows simply by the fundamental theorem of calculus and the unitary invariance of the norm. □

2.4 Localization

At the end of this chapter we state another important property of the generator for the quasi-adiabatic evolution, mainly Lemma 4.7 in [6].
For a subset $X \subset T$ and $n \in \mathbb{N}$ denote $X(n) := \{s \in T | d(X, s) \leq n\}$ the broadening of X by n. Let $H(s) = \sum_{Z \subseteq T} \Phi(Z, s)$ be the sum of local interactions and denote by

$$\tau_t^{H(s)}(A) = e^{itH(s)} A e^{-itH(s)} \tag{2.35}$$

the Heisenberg dynamics of $H(s)$. We assume a uniform, exponential Lieb-Robinson bound for $\tau_t^{H(s)}$. Then we have:

Lemma 2.7. *Let A be an operator with $X := supp(A) \subseteq T$ and let $n \in \mathbb{N}$ Then there is an operator $S_\Delta^{(n)}(H(s), A)$ supported on $X(n)$ such that the following hold:*

i. $\|S_\Delta^{(n)}(H(s), A)\| \leq \|A\| \cdot K/\Delta$ *for some constant $K > 0$.*

ii. $\|S_\Delta(H(s), A) - S_\Delta^{(n)}(H(s), A)\| \leq \|A\| g_\Delta(n)$ *for a subexponential function g_Δ*

iii. If we define $H_\Lambda(s) := \sum_{Z \subseteq \Lambda} \Phi(Z, s)$ for a subset $\Lambda \subseteq T$, then $S_\Delta^{(n)}(H, A) = S_\Delta^{(n)}(H_{X(n)}, A)$.

Remark 2.8. *The subexponential function g_Δ fulfills the bound $g_\Delta(t) \leq C \cdot t^{22} exp(-c\frac{t}{ln^2 t})$ for some constants C, c depending on Δ and the Lieb-Robinson bound.*

3 Introducing our System

With the preliminary work done, we will now formally introduce the setting in which we are interested as it is defined in [10]. We consider the Torus T of length L as lattice $[0, L] \times [0, L] \subset \mathbb{Z}^2$, where we identify the sides in the usual way. On T we have a metric $d(\cdot, \cdot)$ defined by $d(s_1, s_2) = |x(s_1) - x(s_2)| + |y(s_1) - y(s_2)|$ where $|\cdot|$ is the absolutely smallest residue mod L, i.e. $|\cdot| \leq L/2$. Then we have the known algebra $\mathcal{A}_X$ of local operators for a subset $X \subseteq T$. At each side $s \in T$ we define the charge operator $q_s \in \mathcal{A}_{\{s\}}$ with eigenvalues $0, 1, ..., q_{max}$. We are interested in properties of $H_0 = \sum_{Z \subseteq T} \Phi(Z)$ which satisfies the following conditions:

1. The interaction terms $\Phi(Z)$ are local k_{max}-body interactions with finite strength J and finite range R, i.e.
   i. $\Phi(Z) = \Phi(Z)^* \in \mathcal{A}_Z$,
   ii. $sup_{s \in T} \sum_{Z \ni s} \|\Phi(Z)\| \leq J$,
   iii. For all $Z \subseteq T$, if $diam(Z) > R$ or $|Z| > k_{max}$, then $\Phi(Z) = 0$.
2. The Hamiltonian H_0 has a unique groundstate $|\Psi_0\rangle$ and a spectral gap of $\gamma > 0$. We denote $P_0 = |\Psi_0\rangle\langle\Psi_0|$ the projector onto the groundstate.
3. The total charge $Q = \sum_{s \in T} q_s$ is conserved, i.e. $[Q, H_0] = 0$.

By replacing $\Phi(Z)$ with

$$\Phi'(Z) := \frac{1}{2\pi} \int_0^{2\pi} dt\, e^{itQ} \Phi(z) e^{-itQ} \tag{3.1}$$

we can assume $[Q, \Phi(Z)] = 0$, since $\Phi'(Z) \in \mathcal{A}_Z$, $\|\Phi'(Z)\| \leq \|\Phi(Z)\|$, $[Q, \Phi'(Z)] = 0$ (since $exp(2\pi i Q) = 1$ because Q is sum of commuting charges with integer spectrum) and $\sum_{Z \subseteq T} \Phi'(Z) = H_0$ because of $[Q, H_0] = 0$.

In this work we want to prove the following Theorem:

Theorem 3.1. *Let H_0 be a Hamiltonian satisfying the above properties for fixed, L-independent R, J, k_{max}, q_{max} and γ. Then for large enough L the difference between the Hall conductance σ and the nearest integer multiple of e^2/h is subexponential small in the linear size L.*

Here, e and h are the known fundamental physical constants and the Hall conductance σ is given by Kubo's formula in (3.13).
This theorem says that the Hall conductance is quantified up to an error subexponential small in the size of the system. It does not, however, say that this quantization is independent of the size L since the integer given by the theorem may depend upon L.

We will now give a short sketch of how the proof will be done. In the next section we construct a family of twisted Hamiltonians $H(s, t)$ by introducing charge fluxes of strength (s, t) at the sides and center of the torus. Since H_0 is gapped, this family will be uniformly gapped as long as we stay close to H_0. Kubo's formula then relates σ with the function given by Lemma 2.2, which is the phase picked up by a small loop around the origin in flux space. In the next step we will show that a big loop of size 2π picks up a phase which is trivial up to subexponential errors. We will then decompose such a big loop into paths in flux-space of the form $(0, 0) \to (0, t) \to (s, t)$, followed by a small loop around (s, t) and back to the origin. By going such a path one picks

up a phase $\phi(s,t)$. In the crucial step we show that this phase is independent of s,t up to subexponential errors and thus can be correlated to the phase $\phi(0,0)$ which is linked to the Hall conductance.
We have to stress that the phase $\phi(s,t)$ can only be related to the Hall conductance of $H(s,t)$ by Lemma 2.2 if this Hamiltonian is gapped. Since this needs only be true at the origin in flux space, we in particular do not prove that the Hall conductance is uniform in s,t. This uniformity is a consequence of the quasi-adiabatic evolution used for the paths and is not necessarily true for the true adiabatic evolution.
To make this clearer, we look into more details of the sketch given above. From now on, we will fix the threshold $\Delta = \gamma/2$, with the γ given by H_0. If we take a loop in flux space small enough around $(0,0)$, then the Hamiltonians $H(s,t)$ along this loop will have a gap which is lower bounded uniformly by Δ. So in this region, the quasi-adiabatic evolution with threshold Δ coincides with the true adiabatic evolution and therefore $\phi(0,0)$ can be related to the Hall conductance σ. If we leave the gapped area, we cannot control the adiabatic evolution anymore, but we can still control the quasi-adiabatic one via Δ. In [6] the authors show properties of the quasi-adiabatic evolution and the spectral flow derived from it. We will use those properties to show the uniformity of $\phi(s,t)$. The size of the small loops is left as a parameter in the beginning. In the end we will set it to be subexponential in L.

3.1 Twisted Hamiltonians

Following the notation in [10] we construct the following family of Hamiltonians:

$$H(\theta_x, \phi_x, \theta_y, \phi_y) = \sum_{Z \subseteq T} \Phi(Z; \theta_x, \phi_x, \theta_y, \phi_y) \tag{3.2}$$

which corresponds to a twisted Hamiltonian with fluxes at the lines $x = 0$ with strength θ_x, $x = L/2$ with strength ϕ_x and accordingly for y. For this we introduce for a subset $\Lambda \subseteq T$ the unitary equivalence

$$R_\Lambda(t, A) = e^{itQ_\Lambda} A e^{-itQ_\Lambda}, \qquad Q_\Lambda = \sum_{s \in \Lambda} q_s. \tag{3.3}$$

We note that R_Λ is 2π-periodic in t and for $Z := supp(A)$ we have $R_\Lambda(t, A) = R_{\Lambda \cap Z}(t, A) \in \mathcal{A}_Z$ and $\frac{d}{dt} R_\Lambda(t, A) = i[Q_{\Lambda \cap Z}, R_\Lambda(t, A)]$. We also see that $R_{\Lambda_1}(s, \cdot)$ and $R_{\Lambda_2}(t, \cdot)$ commute for arbitrary $\Lambda_1, \Lambda_2 \subseteq T$.
In particular we are interested in R_X and R_Y corresponding to the sets $X = \{s \in T \mid x(s) \in [0, L/2]\}$ and $Y = \{s \in T \mid y(s) \in [0, L/2]\}$.

The interactions $\Phi(Z; \theta_x, \phi_x, \theta_y, \phi_y)$ are then defined by the following rules:
X-1: If $\exists s \in Z : |x(s)| < R$, *then* $\Phi(Z; \theta_x, \phi_x, \theta_y, \phi_y) = R_X(\theta_x, \Phi(Z; 0, 0, \theta_y, \phi_y))$.
X-2: If $\exists s \in Z : |x(s) - L/2| < R$, *then* $\Phi(Z; \theta_x, \phi_x, \theta_y, \phi_y) = R_X(\phi_x, \Phi(Z; 0, 0, \theta_y, \phi_y))$.
X-3: Otherwise $\Phi(Z; \theta_x, \phi_x, \theta_y, \phi_y) = \Phi(Z; 0, 0, \theta_y, \phi_y)$.

Further, we define $\Phi(Z; 0, 0, \theta_y, \phi_y)$ by:
Y-1: If $\exists s \in Z : |y(s)| < R$, *then* $\Phi(Z; 0, 0, \theta_y, \phi_y) = R_Y(\theta_y, \Phi(Z))$.
Y-2: If $\exists s \in Z : |y(s) - L/2| < R$, *then* $\Phi(Z; 0, 0, \theta_y, \phi_y) = R_Y(\phi_y, \Phi(Z))$.
Y-3: Otherwise $\Phi(Z; 0, 0, \theta_y, \phi_y) = \Phi(Z)$.

Since those definitions are quite formal, we will discuss them for a bit.
First, we note that this is well-defined, i.e. X-1 and X-2 cannot hold simultaneously, as long as $L > 6R$ since the interactions Φ have finite range R and $\Phi(Z;\theta_x,\phi_x,\theta_y,\phi_y) \in \mathcal{A}_Z$.
Since $0 = [Q,\Phi(Z)] = [Q_Z,\Phi(Z)]$, we get $R_X(t,\Phi(Z)) = \Phi(z)$ as long as $Z \cap X = \emptyset$ or $Z \subseteq X$ and similarly for R_Y. So R_X only acts non-trivially on $\Phi(Z)$ if Z crosses the lines $x = 0$ or $x = L/2$. Therefore, $\Phi(Z)$ only feels the twist at $x = 0$ if Z lies within a strip of width $2R$, centered at $x = 0$. Since we will use that way of saying quite often, we denote by $B_x(a,d)$ the strip (or band) centered at $x = a$ of width $2d$ and similarly $B_y(a,d)$. In particular, if Z is such that X-3 applies, R_X acts trivially on $\Phi(Z)$ so that we get the formulas

$$R_Y(t,(R_X(s,\Phi(Z))) = \Phi(Z;s,s,t,t) \quad for\, all\, Z \subseteq T \quad and \tag{3.4}$$
$$R_Y(t,(R_X(s,H_0)) = H(s,s,t,t) \tag{3.5}$$

which means that H_0 and $H(s,s,t,t)$ are unitarily equivalent. Later we will only consider the two-parameter families of the forms $H(s,0,t,0)$ and $H(s,s,t,t)$. The former is the one that leads to the Hall conductance but is in general not unitarily equivalent to H_0. Therefore it will be useful to introduce the second family.
The probably most recent and general Lieb-Robinson bounds from [14] apply to our families so that we can use the approximations introduced in (2.7) for those two families.

3.2 Bounding Derivatives

As we have seen in the last chapter, we need bounds on the Hamiltonians and their derivatives. We will derive them now. We have $\|\Phi(Z;\theta_x,\phi_x,\theta_y,\phi_y)\| = \|\Phi(Z)\|$ and therefore the trivial bound

$$\|H(\theta_x,\phi_x,\theta_y,\phi_y)\| \leq L^2 J \tag{3.6}$$

Next we bound $\frac{d}{dt}H(t,\phi_x,\theta_y,\phi_y)$. We note that the only contributing entries come from Z which lie in $B_x(0,R)$. For such a Z we have

$$\begin{aligned}\frac{d}{dt}\Phi(Z;t,\phi_x,\theta_y,\phi_y) &= \frac{d}{dt}R_X(t,\Phi(Z;0,0,\theta_y,\phi_y) = \\ &= i[Q_{X\cap Z},\Phi(Z;t,0,\theta_y,\phi_y)] = -i[Q_{X^c\cap Z},\Phi(Z;t,0,\theta_y,\phi_y)]\end{aligned} \tag{3.7}$$

where X^c is the complement of X in T. This gives a norm bound of

$$\|\frac{d}{dt}\Phi(Z;t,\phi_x,\theta_y,\phi_y)\| \leq min\{|X\cap Z|,|X^c\cap Z|\}q_{max}\|\Phi(Z)\| \leq \frac{Q_{max}}{2R}\|\Phi(Z)\| \tag{3.8}$$

where we set $Q_{max} := Rk_{max}q_{max}$ and used $\|[A,B]\| \leq \|A\|\|B\|$ if $A \geq 0$. Since the strip only contains $2RL$ sites, we get the norm bounds

$$\|\partial_j H(\theta_x,\phi_x,\theta_y,\phi_y)\| \leq Q_{max}JL \quad for\ j \in \{1,2,3,4\} \tag{3.9}$$

To bound the higher derivatives, we note the following: $\frac{d}{dt}\Phi(Z;t,\phi_x,\theta_y,\phi_y)$ is independent of ϕ_x, therefore higher derivatives of $H(\theta_x,\phi_x,\theta_y,\phi_y)$ are zero whenever we include ∂_1 and ∂_2 or ∂_3 and ∂_4. Since the charges commute with each other, if we take another derivative in (3.7), we simply get another commutator with a charge and the following bound holds, where ∂^k stands for a combination of k derivatives:

$$\|\partial^k \Phi(Z;\theta_x,\phi_x,\theta_y,\phi_y)\| \leq \left(\frac{Q_{max}}{2R}\right)^k \|\Phi(Z)\| \tag{3.10}$$

If all derivatives apply to a flux R_x, then the Z above are supported in some strip of width $2R$, but if there are derivatives that apply onto R_X and some that apply onto R_Y, then the only Z which contribute have to be supported in some square of length $2R$, so that we get for example the following bounds for k-th derivatives:

$$\|\partial_1^k H(\theta_x,\phi_x,\theta_y,\phi_y)\| \leq \left(\frac{Q_{max}}{2R}\right)^k 2RJL \tag{3.11}$$

$$\|\partial_1^l \partial_3^{k-l} H(\theta_x,\phi_x,\theta_y,\phi_y)\| \leq \left(\frac{Q_{max}}{2R}\right)^k 4R^2 J \quad for\ l \in \{1,...,k-1\} \tag{3.12}$$

In particular are all bounds at most $\mathcal{O}(L)$.

3.3 Introducing the Hall conductance

We want to show that a certain value t is close to an integer. For this it is enough to show that $e^{2\pi it}$ is close to 1, which can be seen as follows:
Let n be the integer closest to t. Then $|e^{2\pi it}-1| \leq 1$ implies $|t-n| \leq \pi/3$ and we can conclude from some simple geometrics that $|t-n| \leq \sqrt{2}/(2\pi)|e^{2\pi it}-1|$.

As stated in [10], to compute the Hall conductance we use Kubo's formula from linear response theory applied to the setting of a torus pierced by two solenoids carrying magnetic fluxes θ_x and θ_y in the x and y directions. See also [17].
Therefore we look at the two-parameter family $H(\theta_x,0,\theta_y,0)$ of Hamiltonians, whose groundstates we denote by $|\Psi_0(\theta_x,\theta_y)\rangle$. Then the Hall conductance for H_0 is given by

$$\sigma = 2\,Im\,\langle\partial_2\Psi_0(\theta_x,\theta_y)|\partial_1\Psi_0(\theta_x,\theta_y)\rangle\,|_{\theta_x=\theta_y=0}\cdot\left(2\pi\frac{e^2}{h}\right) \tag{3.13}$$

Here we already see the familiarity with Lemma 2.2. We can now formulate the first estimate: Consider $V_{\Delta,\circlearrowleft}$ the loop operator corresponding to the quasi-adiabatic evolution of the family $\{H(s,0,t,0)\}_{s,t\in[0,2\pi]}$ with threshold $\Delta = \gamma/2$. Then we have

Lemma 3.2. *For $0 < r < (8Q_{max}JL/\gamma)^{-1}$ we have for some numerical constant C:*

$$\left|\langle\Psi_0|\,V_{\Delta,\circlearrowleft}(0,0,r)\,|\Psi_0\rangle^{(\frac{2\pi}{r})^2} - e^{2\pi i\sigma(e^2/h)^{-1}}\right| \leq C\left(Q_{max}\frac{J}{\gamma}L\right)^3\cdot r \tag{3.14}$$

Proof. According to Lemma 2.5 and the assumption on r, our family of Hamiltonians is uniformly gapped by Δ on $\Lambda(0,0,r)$, since we have the norm estimate (3.9) and we need at most two paths of length r to reach every point on $\Lambda(0,0,r)$ from the origin. Therefore we can use Lemma 2.2 to see

$$\langle\Psi_0|\,V_{\Delta,\circlearrowleft}(0,0,r)\,|\Psi_0\rangle^{(\frac{2\pi}{r})^2} = e^{i\Phi(0,0,r)(\frac{2\pi}{r})^2}. \tag{3.15}$$

If we use the notation of Remark 2.3, then we see $g(0,0) = \sigma \left(2\pi \frac{e^2}{h}\right)^{-1}$, so we get

$$\left| e^{i\Phi(0,0,r)(\frac{2\pi}{r})^2} - e^{4\pi^2 i g(0,0)} \right| \leq 4\pi^2 \left| \frac{\Phi(0,0,r)}{r^2} - g(0,0) \right| . \tag{3.16}$$

Therefore we only need to bound the rhs of (2.23). Using Remark 2.4 and the bounds given in (3.11),we see that we get bounds of the form $|\partial^k g(s,t)| = \mathcal{O}((Q_{max}JL/\gamma)^{k+2})$. So we have an upper bound of the form

$$r \cdot \left(\mathcal{O}((Q_{max}JL/\gamma)^3) + r\, \mathcal{O}((Q_{max}JL/\gamma)^4) \right) \tag{3.17}$$

and the claim follows since $r = \mathcal{O}(Q_{max}JL/\gamma)^{-1})$. □

4 The Big Loop

In this chapter we will show the second estimate, namely that the phase around a big loop is trivial up to an subexponential error. We will continue with the notation from before.

Lemma 4.1. *For some numerical constant C the following holds for L large enough:*

$$|\langle\Psi_0| V_{\Delta,\circlearrowleft}(0,0,2\pi)|\Psi_0\rangle - 1| \leq CQ_{max}\frac{J^2}{\gamma}L^3 \cdot g_\Delta(L/4-2R) \tag{4.1}$$

Here g_Δ is the function from Lemma 2.7.

Remark 4.2. *If we assume, that the family $\{H(s,0,t,0)\}_{s,t\in[0,2\pi]}$ is uniformly gapped with lower bound Δ, then the lemma above holds with the trivial bound 0 since we can use Lemma 2.2 and $\Phi(0,0,2\pi)$ is given by a Chern number, in particular it is an integer. This assumption leads to the usual proofs of quantization of an averaged Hall conductance.*
Since we don't assume this, we have to put some more work into this.

4.1 Partial Traces

If we split up the path $\Lambda(0,0,2\pi)$ into the four sides, then each side alone defines a closed loop in flux space since R_X and R_Y and therefore $H(s,0,t,0)$ is 2π-periodical. So we only need to consider one-dimensional paths in this case.
With the notations from chapter 2 in mind we define the unitaries $U_{\Delta,x}(r)$ and $U_{\Delta,y}(r)$ to be the quasi-adiabatic evolutions with threshold Δ corresponding to the Hamiltonians $H(s,0,0,0)$ and $H(0,0,s,0)$. Then we define the quasi-adiabatic evolved states

$$|\Psi_x(r)\rangle = U_{\Delta,x}(r)|\Psi_0\rangle, \quad |\Psi_y(r)\rangle = U_{\Delta,y}(r)|\Psi_0\rangle \tag{4.2}$$

and the corresponding density matrices

$$\rho_x(r) = |\Psi_x(r)\rangle\langle\Psi_x(r)|, \quad \rho_y(r) = |\Psi_y(r)\rangle\langle\Psi_y(r)|. \tag{4.3}$$

It is important to note that $|\Psi_x(r)\rangle$ will in general not be a groundstate for $H(r,0,0,0)$ since without a gap quasi-adiabatic evolution fails to simulate the true one. But later we will give an energy estimate how the energy of $|\Psi_x(r)\rangle$ relates to that of $|\Psi_0\rangle$.
Since $H(s,s,0,0)$ is unitarily equivalent to H_0 via $R_X(s,\cdot)$, and therefore uniformly gapped by γ, we know on the other hand from Lemma 2.1 that the density matrix for the ground state of $H(s,s,0,0)$, namely $R_X(s,P_0)$, is generated by the quasi-adiabatic evolution with threshold Δ.

The idea behind the proof is that $H(s,0,0,0)$ looks away from $x=0$ like H_0, and away from $x=L/2$ like $H(s,s,0,0)$ which is unitarily equivalent to H_0. So we want to restrict us to those regions where we have good approximations. In particular, we want to be able to distinguish between the fluxes at $x=0$ and $x=L/2$. For this we denote the strips $\Omega_x := B_x(0,L/4)$ and

$\Omega_y := B_y(0, L/4)$ which separate the fluxes.
For some $Z \subseteq T$ we define $\overline{Z} := Z^c(R)$, in particular for strips we have $\overline{B_x(a,d)} = B_x(a + L/2, L/2 - d + R)$. The reason behind this definition is that if an interaction is not supported in Z, it has to be supported in $\overline{Z}$ and therefore, we have for any $Z \subseteq T$ that H_Z is supported in Z and $H_0 - H_Z$ is supported in $\overline{Z}$.
With this notation we now define the following restricted Hamiltonians:

$$H_{\Omega_x}(r) = \sum_{Z \subseteq \Omega_x} \Phi(Z; r, 0, 0, 0), \quad H_{\overline{\Omega_x}}(r) = \sum_{Z \not\subseteq \Omega_x} \Phi(Z; 0, r, 0, 0) \tag{4.4}$$

$$H_{\Omega_y}(r) = \sum_{Z \subseteq \Omega_y} \Phi(Z; 0, 0, r, 0), \quad H_{\overline{\Omega_y}}(r) = \sum_{Z \not\subseteq \Omega_y} \Phi(Z; 0, 0, 0, r) \tag{4.5}$$

Then the considerations above show that any of the H_A above is supported in A. As interpretation one could say that H_{Ω_x} measures how strong the flux is at $x = 0$ and $H_{\overline{\Omega_x}}$ measures the strength of the flux at $x = L/2$. We have for example the following decompositions:

$$H(r, 0, 0, 0) = H_{\Omega_x}(r) + H_{\overline{\Omega_x}}(0), \quad H(r, r, 0, 0) = H_{\Omega_x}(r) + H_{\overline{\Omega_x}}(r) \tag{4.6}$$

$$H(0, 0, r, 0) = H_{\Omega_y}(r) + H_{\overline{\Omega_y}}(0), \quad H(0, 0, r, r) = H_{\Omega_y}(r) + H_{\overline{\Omega_y}}(r) \tag{4.7}$$

From the definition we can also see that $H_A(r) = R_{X/Y}(r, H_A(0))$ for any of the H_A above since we include into H_A only those interactions that actually see the corresponding flux twist. So from the decompositions in (4.6) we can see again that $H(r, r, 0, 0)$ is unitarily equivalent to H_0 but furthermore we can see how $H(r, 0, 0, 0)$ fails to be so.
From the remarks above one could now suppose that $\rho_x(r)$ should look away from $x = 0$ similar to P_0 and away from $x = L/2$ similar to $R_X(r, P_0)$. In which sense this is in fact true, states the following

Lemma 4.3. *With the definitions above we have the following partial-trace norm bounds:*

$$\| Tr_{\overline{\Omega_x}^c}(\rho_x(r) - P_0) \| \leq 2r\, Q_{max} JL \cdot g_\Delta(L/4 - 2R) \tag{4.8}$$

$$\| Tr_{\Omega_x^c}(\rho_x(r) - R_X(r, P_0)) \| \leq 6r\, Q_{max} JL \cdot g_\Delta(L/4 - 2R) \tag{4.9}$$

The same bounds hold for Y instead of X.

To prove this we bound the partial-trace norm of the derivatives of ρ, i.e. commutators with the generator of the quasi-adiabatic evolution. We use Lemma 2.7 to switch to the localized versions of those because we know their support.

Proof. To shorten notation we set $H^{(1)}(s) = H(s, 0, 0, 0)$ and $H^{(2)}(s) = H(s, s, 0, 0)$. Then from (4.6) and the remark before the Lemma, we have the following:

$$\partial_s \rho_x(s) = i[S_\Delta\left(H^{(1)}(s), \partial_s H_{\Omega_x}(s)\right), \rho_x(s)] \tag{4.10}$$

$$\begin{aligned} \partial_s R_X(s, P_0) &= i[S_\Delta\left(H^{(2)}(s), \partial_s H^{(2)}(s)\right), R_X(s, P_0)] = \\ &= i[S_\Delta\left(H^{(2)}(s), \partial_s H_{\Omega_x}(s)\right), R_X(s, P_0)] + i[S_\Delta\left(H^{(2)}(s), \partial_s H_{\overline{\Omega_x}}(s)\right), R_X(s, P_0)] \end{aligned} \tag{4.11}$$

Now we recall that $supp(\partial_s H_{\Omega_x}(s)) \subseteq B_x(0, R)$ and we get from Lemma 2.7 that

$$S_\Delta^{(n)}(H^{(1)}(s), \partial_s H_{\Omega_x}(s)) = S_\Delta^{(n)}(H^{(2)}(s), \partial_s H_{\Omega_x}(s)) \in \mathcal{A}_{\overline{\Omega_x}^c} \tag{4.12}$$

as long as $H^{(1)}_{B_x(0,R+n)}(s) = H^{(2)}_{B_x(0,R+n)}(s)$ and $B_x(0, R+n) \subseteq \overline{\Omega_x}^c = B_x(0, L/4-R)$, i.e. as long as $R+n \leq L/2-R$ and $R+n \leq L/4-R$. In particular this holds for $n := L/4-2R$.
From Lemma 2.7 and (3.9) we conclude that $S(j,A)(s) := S_\Delta(H^{(j)}(s), \partial_s H_A(s))$ and their localized version differ in norm by at most $Q_{max}JL \cdot g_\Delta(n)$, for $j \in \{1,2\}$ and $A \in \{\Omega_x, \overline{\Omega_x}\}$. Since $\partial_s H_{\overline{\Omega_x}}(s)$ has support on $B_x(L/2, R)$, we note that the support of $S^{(n)}_\Delta(H^{(2)}(s), \partial_s H_{\overline{\Omega_x}}(s))$ lies in $B_x(L/2, L/4-R) \subset \Omega_x^c$.
With this we get:

$$\begin{aligned}
\|Tr_{\overline{\Omega_x}^c}[S(1,\Omega_x)(s), \rho_x(s)]\| &= \sup_{A \in \mathcal{A}_{\overline{\Omega_x}}, \|A\|=1} |Tr(A[S(1,\Omega_x)(s), \rho_x(s)])| = \\
&= \sup_{A \in \mathcal{A}_{\overline{\Omega_x}}, \|A\|=1} |Tr([A, S(1,\Omega_x)(s)]\rho_x(s))| \leq \\
&\leq \sup_{A \in \mathcal{A}_{\overline{\Omega_x}}, \|A\|=1} \|[A, S(1,\Omega_x)(s)]\| = \\
&= \sup_{A \in \mathcal{A}_{\overline{\Omega_x}}, \|A\|=1} \|[A, S(1,\Omega_x)(s) - S^{(n)}(1,\Omega_x)(s)]\| \leq \\
&\leq 2Q_{max}JL \cdot g_\Delta(n),
\end{aligned} \tag{4.13}$$

where the last equality holds because the supports of A and $S^{(n)}(1,\Omega_x)(s)$ are disjoint. The first claim of the lemma follows simply by integrating this relation.

The second bound is more tedious. The term with $S(2,\overline{\Omega_x})$ in (4.10) can be dealt with as in the first bound. But for the ones with Ω_x we can't insert the localizations via the commutator. So we have to find another way. Therefore let $U(s)$ be the unitary generated by $S^{(n)}(1,\Omega_x) = S^{(n)}(2,\Omega_x)$. Then U is supported in $\overline{\Omega_x}^c$ and since on $\overline{\Omega_x}$ $H^{(1)}(s)$ and $H^{(2)}(s)$ coincide, we have

$$\rho_x(s) - R_X(s,P_0) = U^*(s)(\rho_x(s) - R_X(s,P_0))U(s) \tag{4.14}$$

and differentiating and rearranging this gives us the three summands

$$iU^*(s)[S(1,\Omega_x)(s) - S^{(n)}(1,\Omega_x)(s), \rho_x(s)]U(s) \tag{4.15}$$
$$-iU^*(s)[S(2,\Omega_x)(s) - S^{(n)}(2,\Omega_x)(s), R_X(s,P_0)]U(s) \tag{4.16}$$
$$-iU^*(s)[S(2,\overline{\Omega_x})(s), R_X(s,P_0)]U(s) \tag{4.17}$$

From there on we can for each of the three summands conclude as in the first bound to get the claim.
For Y instead of X one argues in the exact same way. □

4.2 Energy Estimates

The Lemma 4.3 will be used again in the next chapter. Here we need it to prove the aforementioned energy estimates.

Lemma 4.4. *Denote E_0 the groundstate energy of H_0. Then with the notations from above we have the following bound:*

$$|\langle\Psi_x(r)|\, H(r,0,0,0)\, |\Psi_x(r)\rangle - E_0| \leq 8r\, Q_{max}J^2L^3 \cdot g_\Delta(L/4-2R) \tag{4.18}$$

The same bound holds for y instead of x.

As mentioned before, we cannot say that $|\Psi_x(r)\rangle$ is still a groundstate for $H(r,0,0,0)$ but we can at least estimate its energy. One should also note that E_0 need not be the groundstate energy for $H(r,0,0,0)$ since we have no unitary equivalence with H_0.
What this Lemma says is that quasi-adiabatically evolving the groundstate of H_0 gives us a state which energy is subexponentially close to the groundstate energy E_0.

Proof. From the unitary equivalence, the decompositions in (4.6) and $H_{\overline{\Omega_x}}(r) = R_X(r, H_{\overline{\Omega_x}}(0))$ we get:

$$\langle\Psi_x(r)|\, H(r,0,0,0)\,|\Psi_x(r)\rangle = Tr(H_{\Omega_x}(r)\rho_x(r)) + Tr(H_{\overline{\Omega_x}}(0)\rho_x(r)) \tag{4.19}$$

$$E_0 = Tr(H(r,r,0,0)R_X(r,P_0)) = Tr(H_{\Omega_x}(r)R_X(r,P_0)) + Tr(H_{\overline{\Omega_x}}(0)P_0) \tag{4.20}$$

Therefore, we can bound the difference by

$$\begin{aligned} &|Tr(H_{\Omega_x}(r)(\rho_x(r) - R_X(r,P_0))| + Tr(H_{\overline{\Omega_x}}(0)(\rho_x(r) - P_0)| \leq \\ &\leq \|H_{\Omega_x}(r)\|\|Tr_{\Omega_x^c}(\rho_x(r) - R_X(r,P_0))\| + \|H_{\overline{\Omega_x}}(0)\|\|Tr_{\overline{\Omega_x}^c}(\rho_x(r) - P_0)\| \end{aligned} \tag{4.21}$$

and the claim follows with (3.6) and Lemma 4.3. For y one argues in the same way. □

We can now proof the second estimate, given at the beginning of this chapter.

Proof of Lemma 4.1. As said before, we split $\Lambda(0,0,2\pi)$ into the four components. Since each one gives a closed loop in flux space, we can see with the notation of this chapter:

$$V_{\Delta,\circlearrowleft}(0,0,2\pi) = U^*_{\Delta,y}(2\pi)U^*_{\Delta,x}(2\pi)U_{\Delta,y}(2\pi)U_{\Delta,x}(2\pi) \tag{4.22}$$

For the rest of this proof we will suppress the Δ and 2π in the notation, since this is the only point where the unitaries are evaluated. Denoting $\mathbb{I} = P_0 + Q_0$, we define further

$$\begin{aligned} |\Psi_x\rangle &= U_x\,|\Psi_0\rangle\,, \quad |\delta_x\rangle = Q_0\,|\Psi_x\rangle \\ |\Psi'_x\rangle &= U^*_x\,|\Psi_0\rangle\,, \quad |\delta'_x\rangle = Q_0\,|\Psi'_x\rangle \end{aligned} \tag{4.23}$$

and similarly for Y. Since we expect $|\Psi_x\rangle$ to be a state close to $|\Psi_0\rangle$, we expect $|\delta_x\rangle$ to be small. And in fact, we get the following bound from the gap $Q_0 \leq (H_0 - E_0)/\gamma$:

$$\langle\delta_x|\delta_x\rangle = \langle\Psi_x|\,Q_0\,|\Psi_x\rangle \leq \gamma^{-1}\,\langle\Psi_x|\,(H_0 - E_0)\,|\Psi_x\rangle \tag{4.24}$$

which is bounded from above by $\delta := 16\pi Q_{max}J^2L^3\gamma^{-1}\cdot g_\Delta(L/4 - 2R)$ by 2π-periodicity of $H(s,0,0,0)$ and Lemma 4.4. The same bound holds for $|\delta'_x\rangle$ and x replaced by y.
Furthermore, we have the relation $|\langle\Psi_x|\Psi_0\rangle|^2 = 1 - \langle\delta_x|\delta_x\rangle$ which follows from $P_0 = \mathbb{I} - Q_0$.
Then we can expand by inserting $P_0 + Q_0$ three times:

$$\begin{aligned} &\langle\Psi_0|\,U^*_yU^*_xU_yU_x\,|\Psi\rangle = \langle\Psi_y|\,U^*_xU_y\,|\Psi_x\rangle = \\ &= \langle\Psi_y|\Psi_0\rangle\langle\Psi_x|\Psi_y\rangle\langle\Psi_0|\Psi_x\rangle + \langle\Psi_y|\Psi_0\rangle\langle\Psi_x|\,U_y\,|\delta_x\rangle + \langle\delta_y|\,U^*_x\,|\Psi_y\rangle\langle\Psi_0|\Psi_x\rangle + \langle\delta_y|\,U^*_xU_y\,|\delta_x\rangle = \\ &= |\langle\Psi_y|\Psi_0\rangle|^2|\langle\Psi_x|\Psi_0\rangle|^2 + \langle\Psi_y|\Psi_0\rangle\,\langle\delta_x|\delta_y\rangle\,\langle\Psi_0|\Psi_x\rangle + \langle\delta_y|\,U^*_xU_y\,|\delta_x\rangle + \\ &\quad + \langle\Psi_y|\Psi_0\rangle\langle\Psi_x|\,U_y\,|\delta_x\rangle + \langle\delta_y|\,U^*_x\,|\Psi_y\rangle\langle\Psi_0|\Psi_x\rangle \end{aligned} \tag{4.25}$$

Inserting again $P_0 + Q_0$ and using $Q_0 = Q_0^2$ we see

$$\langle \Psi_x | \, U_y \, | \delta_x \rangle = \langle \Psi_x | \Psi_0 \rangle \langle \Psi'_y | \delta_x \rangle + \langle \delta_x | \, U_y \, | \delta_y \rangle = \langle \Psi_x | \Psi_0 \rangle \langle \delta'_y | \delta_x \rangle + \langle \delta_x | \, U_y \, | \delta_y \rangle \tag{4.26}$$

$$\langle \delta_y | \, U_x^* \, | \Psi_y \rangle = \langle \delta_y | \Psi'_x \rangle \langle \Psi_0 | \Psi_y \rangle + \langle \delta_y | \, U_x^* \, | \delta_y \rangle = \langle \delta_y | \delta'_x \rangle \langle \Psi_0 | \Psi_y \rangle + \langle \delta_y | \, U_x^* \, | \delta_y \rangle \tag{4.27}$$

If we now extract the 1 which sits inside the first summand of (5.26), then we readily get the bound

$$| \, \langle \Psi_0 | \, U_y^* U_x^* U_y U_x \, | \Psi \rangle - 1 | \leq 8\delta, \tag{4.28}$$

from which the claim follows. □

5 Uniformity

The last and most difficult part of the proof is to show that the phase picked up by small quasi-adiabatic loops around (s,t) in flux space is independent of s,t up to some subexponential error. From here on we will only use the two-parameter family $H(s,t) = H(s,0,t,0)$ of twisted Hamiltonians. And all unitaries of quasi-adiabatic evolutions will be generated by those. We needed to introduce the Hamiltonians with twist in the middle only to show the bounds in the last chapter, although we will be using those bounds in this chapter again.

5.1 Splitting the Loop

Let us first show how we decompose the big loop into small ones. To this we define the unitary

$$V_{\Delta}(s,t) := U_{\Delta,x}(0,t,s)U_{\Delta,y}(0,0,t) \tag{5.1}$$

which evolves quasi-adiabatically from $(0,0)$ to (s,t) by going first into the y-direction. Let now $r>0$ be the size of a small loop and $N = 2\pi/r$. Then we will decompose the big loop into N^2 small loops in the following way: Define

$$U_{nN+N-m} = V_{\Delta}^{*}(mr,nr)V_{\Delta,\circlearrowleft}(mr,nr,r)V_{\Delta}(mr,nr) \quad for\ m,n \in \{0,...,N-1\}. \tag{5.2}$$

Those all define closed paths starting at the origin which go to some point, make there a small loop and then return their way back to the origin. The index j of U_j is chosen in such a way that $\lfloor j/N \rfloor$ gives the row of the loop and j $(mod\ N)$ gives the column of the loop. One sees that the indexing is defined s.t. we start counting at the lower right and when the index increases, we go to the left end of the row and then start the row above again from the right.
By doing this in this order we build succesively bigger loops, e.g. gives $U_N \cdots U_2U_1$ the loop $(0,0) \to (0,2\pi) \to (r,2\pi) \to (r,0) \to (0,0)$. Therfore we have the identity

$$V_{\Delta,\circlearrowleft}(0,0,2\pi) = U_{N^2}U_{N^2-1}\cdots U_2U_1. \tag{5.3}$$

We will now prove that it is enough to show uniformity of the phase picked up by the small loops to show the third and last estimate needed for the Theorem.

Lemma 5.1. *Let r be as in Lemma 3.2. Given the definitions above let*

$$\delta := max_{j=1,\dots,N^2} |\langle\Psi_0| U_j |\Psi_0\rangle - \langle\Psi_0| V_{\Delta,\circlearrowleft}(0,0,r) |\Psi_0\rangle|. \tag{5.4}$$

Then we have the bound

$$\left|\langle\Psi_0| V_{\Delta,\circlearrowleft}(0,0,2\pi) |\Psi_0\rangle - \langle\Psi_0| V_{\Delta,\circlearrowleft}(0,0,r) |\Psi_0\rangle^{(\frac{2\pi}{r})^2}\right| \le 4\pi^2 \left(\sqrt{2\delta r^{-4}} + e^{4\pi^2\delta r^{-2}}\delta r^{-2}\right) \tag{5.5}$$

From this form we already see what kind of bound we will have to show for δ. For the assumption on r in 3.2 to hold, we know that $r \to 0$ for $L \to \infty$. Then the given bound will be subexponential small in L if and only if δr^{-4} is subexponential in L. Therefore any term in a bound for δ will need to contain either some subexponential function in L by itself, or at least a fifth power of r, so that we can choose that to be small enough. This explains why we will later have to consider Taylor expansions up to the fifth order.

Proof. To shorten notation we introduce the following notations for $a \leq b \in \{1, ..., N^2\}$:

$$p_{[a,b]} = \langle \Psi_0 | U_b \cdots U_{a+1} U_a | \Psi_0 \rangle \quad \textit{and} \quad q_{[a,b]} = \langle \Psi_0 | U_b \cdots U_{a+1} Q_0 U_a | \Psi_0 \rangle \tag{5.6}$$

Further, we set $p_a = p_{[a,a]}$. Then by definition $\delta = max_{j=1,...,N^2} |p_j - p_N|$ and we want to bound

$$|p_{[1,N^2]} - (p_N)^{N^2}| \leq |p_{[1,N^2]} - p_1 p_2 \cdots p_{N^2}| + |p_1 p_2 \cdots p_{N^2} - (p_N)^{N^2}|. \tag{5.7}$$

Using $P_0 + Q_0 = \mathbb{I}$ one sees that $p_{[j,N^2]} = p_j p_{[j+1,N^2]} + q_{[j,N^2]}$ so that we get by iteration the formula

$$p_{[1,N^2]} - \prod_{j=1}^{N^2} p_j = \sum_{j=1}^{N^2-1} p_1 \cdots p_{j-1} \, q_{[j,N^2]} \tag{5.8}$$

We know $|p_j| \leq 1$ and $|p_N| = 1$ by Lemma 2.2 since by assumption on r near the origin we are uniformly gapped by Δ. Therefore we have

$$|q_{[j,N^2]}|^2 \leq \| Q_0 U_j |\Psi_0\rangle \|^2 = 1 - |p_j|^2 \leq 2(|p_j| - |p_N|) \leq 2\delta \tag{5.9}$$

For the other term we have

$$\left| \prod_{j=1}^{N^2} p_j - (p_N)^{N^2} \right| = \left| \prod_{j=1}^{N^2} (p_N + (p_j - p_N)) - (p_N)^{N^2} \right| = \left| \sum_{k=1}^{N^2} (p_N)^{N^2-k} \sum_{\alpha_1 < ... < \alpha_k} \prod_{j=1}^{k} (p_{\alpha_j} - p_{\alpha_N}) \right| \leq$$

$$\leq \sum_{k=1}^{N^2} \sum_{\alpha_1 < ... < \alpha_k} \delta^k = (1+\delta)^{N^2} - 1 \leq e^{\delta N^2} - 1 \leq e^{\delta N^2} \delta N^2. \tag{5.10}$$

Putting those two together and recalling $N = 2\pi/r$ we get the claim. □

5.2 Lieb-Robinson

In this section we will use Lieb-Robinson bounds which were shown in [6]. Therefore we will give a short presentation of facts and properties which we will use from that paper. Also it should be noted that until now we were able in all our given bounds to make the dependency of J, Δ, Q_{max} and R explicit. In this chapter we won't any longer be always capable of that. Therefore the constants appearing here will always depend on those parameters but not on the size L of the system. The reason for this is that those constants involve Lieb-Robinson velocities of the quasi-adiabatic evolution for our twisted Hamiltonians derived in [14].
The main ingredient from [6] we want to use is Theorem 4.5 therein. For this theorem one assumes a family of Hamiltonians $H_\Lambda(s)$ on $\Lambda \subseteq T$ uniformly gapped by γ_0 which fulfill a uniform exponential Lieb-Robinson bound. Then the unitaries $U_\Lambda(s)$ corresponding to the adiabatic evolution of $H(s)$ define an operator called by the authors the spectral flow, namely

$$\alpha_s^\Lambda(A) = U_\Lambda(s)^* A U_\Lambda(s) \quad for \ A \in \mathcal{A}_\Lambda. \tag{5.11}$$

This operator is shown to fulfill again a Lieb-Robinson bound, namely let $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. Then we have

$$\|[\alpha_s^\Lambda(A), B]\| \leq \|A\|\|B\| g(s) \sum_{x\in X, y\in Y} F(d(x,y)) \tag{5.12}$$

Here the function $g(s)$ is independent of the size L, but depends (exponentially) on the Lieb-Robinson velocity of $H(s)$. F is monotonically decreasing and can be chosen in such a way that $F(r) \leq C(1+r)^{-3} exp(-c\frac{r}{ln^2 r})$ for constants C, c again depending only on properties of H, in particular also the spectral gap.

We want to apply this Lieb-Robinson bound in our situation. The problem is that we don't assume uniformly gapped Hamiltonians. But if one looks at the proof of (5.12) in [6], then one sees that the authors use the existence of the gap to conclude as in Lemma 2.1 that the unitaries defining the spectral flow are actually generated by the quasi-adiabatic evolution with threshold γ_0. Then they proceed to show the Lieb-Robinson bound for this evolution with fixed threshold. Since we are already only looking at the quasi-adiabatic evolution with threshold Δ, we see that we can indeed apply Theorem 4.5 of [6].

We also want to bound the sum given in (5.12). This can be easily done if we assume that $d := dist(X,Y) > 0$. Then we get the following bound.

Proposition 5.2. *Given A, B and d as above. Then*

$$\sum_{x\in X, y\in Y} F(d(x,y)) \leq min\{|X|,|Y|\} \cdot h(d) \tag{5.13}$$

for some subexponential function h fulfilling $h(r) \leq C' \, exp(-c'\frac{r}{ln^2 r})$, where C', c' are constants derived from the constants C, c guaranteed by the properties of F.

Proof. In $\mathbb{Z}^2$ with the usual metric we have for fixed $x \in \mathbb{Z}^2$ and $r > 0$:

$$|\{y \in \mathbb{Z}^2 \,|\, d(x,y) = r\}| = 4r, \tag{5.14}$$

where it would even suffice to have simply a linear bound in r. This bound then clearly also holds on the Torus.
Since $dist(X,Y) = d$ and F is monotonously decreasing, we can write:

$$\sum_{x\in X, y\in Y} F(d(x,y)) = \sum_{x\in X}\sum_{r\geq d}\sum_{\substack{y\in Y\\ d(x,y)=r}} F(r) \leq \sum_{x\in X}\sum_{r\geq d} 4rC(1+r)^{-3}e^{-c\frac{r}{ln^2 r}} \leq$$
$$\leq 4C|X|d^{-2}\sum_{r\geq d} e^{-c\frac{r}{ln^2 r}} \leq C'|X|e^{-c\frac{d}{ln^2 d}}. \tag{5.15}$$

In the last step we used Lemma 2.5 from [6], which proves certain properties of this subexponential function. The claim follows from the symmetry of the Proposition in X and Y. ☐

5.3 Localizing Operators

In the following Lemmas, f always denotes a subexponential function and $\widetilde{C}$ a constant depending on J,R,Δ and Q_{max}, but not on L.
f can be bounded by $\widetilde{C}exp(-\widetilde{c}\frac{\cdot}{ln^2(\cdot)})$ for such $\widetilde{C}, \widetilde{c}$.

To use (5.12) we see that we need to be able to bound the support of our operators in question. To reach this, for the rest of the chapter we will make extensive use of localized unitaries which are generated by localizations given by Lemma 2.7. To keep track of supports we call $rect_{(x,y)}(s,t) \subseteq T$ the rectangle centered at (x,y) in real-space with half-sides s,t in x- and y-direction. In this notation, our strips can for example be expressed as $B_x(0,R) = rect_{(0,0)}(R, L/2)$.
We want to reduce our operators such that they are localized near the origin, where they feel the flux-twists, since if they are supported away from the mid-lines, we have unitary equivalences. To measure this we define the set $\Omega_0 := rect_{(0,0)}(L/8 - R, L/8 - R) \subseteq T$, a square of half-side $L/8 - R$ around the origin.

From here on, our fixed threshold Δ for the quasi-adiabatic evolutions will be kept implicit in the unitaries. We recall the definition of $U_x(s,t,r)$ as evolving from (s,t) to $(s+r,t)$ in flux-space. We want to replace this with a unitary which is supported away from $y = L/2$. In more detail, we set $\Omega_1 := B_y(0, \frac{5}{24}L - R)$ and for fixed s,t we let $U_{\Omega_1}(s,t,r)$ be the unitary generated by

$$S_\Delta^{(L/24)}(H(s+r,t), \partial_1 H_{\Omega_1}(s+r,t)). \tag{5.16}$$

We note that since H_{Ω_1} is supported on Ω_1, $\partial_1 H_{\Omega_1}(s+r,t)$ is supported on $\Omega_1 \cap B_x(0,R) = rect_{(0,0)}(R, (5/24)L - R)$. In total we see by Lemma 2.7 that $U_{\Omega_1}(s,t,r)$ is supported on $rect_{(0,0)}(R, (5/24)L - R)(L/24) \subseteq rect_{(0,0)}(R + L/24, L/4 - R) \subset \Omega_x \cap \Omega_y$.
So we see that it is supported away from $y = L/2$ and therefore we have the unitary equivalence

$$U_{\Omega_1}(s,t,r) = R_Y(t, U_{\Omega_1}(s,0,r)). \tag{5.17}$$

Evolving from $(0,t)$ over (r,t) to (s,t), we conclude from the composition rule (2.12) the following relation:

$$U^*_{\Omega_1}(r,t,s-r) = U_{\Omega_1}(0,t,r)U^*_{\Omega_1}(0,t,s), \tag{5.18}$$

so that we can see with (5.16) that for fixed s,t, $U^*_{\Omega_1}(r,t,s-r)$ is generated by

$$S_\Delta^{(L/24)}(H(r,t), \partial_1 H_{\Omega_1}(r,t)). \tag{5.19}$$

As done before we make the decomposition $H(s,t) = H_{\Omega_1}(s,t) + H_{\overline{\Omega_1}}(s,t)$.

With these preliminaries done, we can now state the following approximation Lemma:

Lemma 5.3. *For any $s,t \in [0,2\pi]$ and any $A \in \mathcal{A}_{\Omega_0}$ we have the following bound:*

$$\|U^*_x(0,t,s)AU_x(0,t,s) - U^*_{\Omega_1}(0,t,s)AU_{\Omega_1}(0,t,s)\| \leq \widetilde{C}s\|A\|f(L) \tag{5.20}$$

Proof. We call $\delta_U(s,t) := U^*_x(0,t,s)AU_x(0,t,s) - U^*_{\Omega_1}(0,t,s)AU_{\Omega_1}(0,t,s)$ what we want to bound. Then we can compute with (5.16) and (5.19):

$$\begin{aligned}\delta_U(s,t) &= \int_0^s dr\, \partial_r (U^*_x(0,t,r)U^*_{\Omega_1}(r,t,s-r)AU_{\Omega_1}(r,t,s-r)U_x(0,t,r)) = \\ &= i\int_0^s dr\, U^*_x(0,t,r)\left[U^*_{\Omega_1}(r,t,s-r)AU_{\Omega_1}(r,t,s-r), \delta_S(r,t)\right]U_x(0,t,r)\end{aligned} \tag{5.21}$$

where we have defined

$$\delta_S(r,t) = S_\Delta(H(r,t), \partial_1 H(r,t)) - S_\Delta^{(L/24)}(H(r,t), \partial_1 H_{\Omega_1}(r,t)). \tag{5.22}$$

If we denote with $\alpha_r^{\Omega_1}$ the spectral flow corresponding to $U_{\Omega_1}(r,t,s-r)$, we can bound

$$\|\delta_U(s,t)\| \leq s \sup_{r\in[0,s]} \|[\alpha_r^{\Omega_1}(A), \delta_S(r,t)]\|. \tag{5.23}$$

Here we want to use the Lieb-Robinson bound from (5.12). Therefore we need to separate the supports of the operators. $\delta_S(r,t)$ has arbitrary support, so let us look instead at the localization $S_\Delta^{(L/24)}(H(r,t), \partial_1 H_{\overline{\Omega_1}}(r,t))$. $\partial_1 H_{\overline{\Omega_1}}$ has support within

$$\begin{aligned} B_x(0,R) \cap \overline{\Omega_1} &= B_x(0,R) \cap \overline{B_y(0,(5/24)L-R)} = \\ &= B_x(0,R) \cap B_y(L/2,(7/24)L+2R) = rect_{(0,L/2)}(R,(7/24)L+2R) \end{aligned} \tag{5.24}$$

Therefore the support of $S_\Delta^{(L/24)}(H(r,t), \partial_1 H_{\overline{\Omega_1}}(r,t))$ lies within $rect_{(0,L/2)}(R+L/24, L/3+2R)$ and this is separated from $\Omega_0 = rect_{(0,0)}(L/8-R, L/8-R)$ by

$$L/2 - (L/3+2R) - (L/8-R) = L/24 - R. \tag{5.25}$$

So for this we know a good Lieb-Robinson bound. Furthermore, we see

$$\delta_S(r,t) = \delta_{\Omega_1}(r,t) + \delta_{\overline{\Omega_1}}(r,t) + S_\Delta^{(L/24)}(H(r,t), \partial_1 H_{\overline{\Omega_1}}(r,t)) \text{ , } where \tag{5.26}$$

$$\delta_{\Omega_1}(r,t) = S_\Delta(H(r,t), \partial_1 H_{\Omega_1}(r,t)) - S_\Delta^{(L/24)}(H(r,t), \partial_1 H_{\Omega_1}(r,t)) \text{ , } and \tag{5.27}$$

$$\delta_{\overline{\Omega_1}}(r,t) = S_\Delta(H(r,t), \partial_1 H_{\overline{\Omega_1}}(r,t)) - S_\Delta^{(L/24)}(H(r,t), \partial_1 H_{\overline{\Omega_1}}(r,t)) \tag{5.28}$$

and the first two summands in (5.26) we can bound by Lemma 2.7. In total we get

$$\|\delta_U(s,t)\| \leq 4s\|A\|(Q_{max}JL)\cdot g_\Delta(L/24) + s\|A\|(Q_{max}JL)|\Omega_0| \cdot h(L/24-R) \tag{5.29}$$

with the subexponential function h from Proposition 5.2. From this we readily get the claim. □

In the next Lemma we show what happens when we evolve and twist an operator that's supported near the origin to an arbitrary place by the unitaries $V(s,t)$ defined in (5.1) (suppressing the Δ).

Lemma 5.4. *Let $A \in \mathcal{A}_{\Omega_0}$. Then for all $s,t \in [0,2\pi]$ we have the following bound:*

$$|\langle\Psi_0| V^*(s,t) R_Y(t, R_X(s,A)) V(s,t) |\Psi_0\rangle - \langle\Psi_0| A |\Psi_0\rangle| \leq \widetilde{C}\|A\|(s+t)f(L) \tag{5.30}$$

Proof. Recalling the notation from section 4.1 we recognize the expectation values above as the following traces:

$$Tr(\rho_y(t) U_x^*(0,t,s) R_Y(t, R_X(s,A)) U_x(0,t,s)) - Tr(P_0 A) \tag{5.31}$$

This we can expand into the following four summands:

$$\begin{aligned} &Tr\left(\rho_y(t)\left[U_x^*(0,t,s)R_Y(t,R_X(s,A))U_x(0,t,s) - U_{\Omega_1}^*(0,t,s)R_Y(t,R_X(s,A))U_{\Omega_1}(0,t,s)\right]\right) \\ &Tr\left(\left[\rho_y(t) - R_Y(t,P_0)\right]U_{\Omega_1}^*(0,t,s)R_Y(t,R_X(s,A))U_{\Omega_1}(0,t,s)\right) \\ &Tr\left(P_0\left[U_{\Omega_1}^*(0,0,s)R_X(s,A)U_{\Omega_1}(0,0,s) - U_x^*(0,0,s)R_X(s,A)U_x(0,0,s)\right]\right) \\ &Tr\left(\left[\rho_x(s) - R_X(s,P_0)\right]R_X(s,A)\right) \end{aligned}$$

where we used (5.17) and $Tr(R_Y(t,B)) = Tr(B)$ for any operator B by the cyclicity of the trace for the third summand and $Tr(P_0 A) = Tr(R_X(s, P_0 A))$ for the last one.

Since $R_X(s, A)$ and $R_Y(t, R_X(s, A))$ are also supported within $\mathcal{A}_{\Omega_0}$, we can use Lemma 5.3 to bound the first and the third summand. Since U_{Ω_1} is supported within $\Omega_x \cap \Omega_y$ we can use partial traces for the second and last summand to get the following bound in norm:

$$2\widetilde{C}s\|A\|f(L) + \|A\|\|Tr_{\Omega_y^c}(\rho_y(t) - R_Y(t, P_0))\| + \|A\|\|Tr_{\Omega_x^c}(\rho_x(s) - R_x(s, P_0))\| \tag{5.32}$$

From Lemma 4.3 we get the claim with a new subexponential function f. □

The following Lemma will provide us with the remaining tool we need to prove the last estimate for the main theorem. In this Lemma the aforementioned Talor expansion will be done. We will show that $V_{\circlearrowleft}(s, t, r)$ is up to a small error a twisted version of $V_{\circlearrowleft}(0, 0, r)$, which can further be approximated by an operator supported in Ω_0. More precisely, we have the following:

Lemma 5.5. *For any $s, t \in [0, 2\pi]$ we have the following bound for a numerical constant C and sufficiently large L:*

$$\|V_{\circlearrowleft}(s, t, r) - R_Y(t, R_X(s, V_{\circlearrowleft}(0, 0, r)))\| \leq C\left(Q_{max}JLr \cdot g_\Delta(L/48) + (Q_{max}\frac{J}{\Delta}Lr)^5\right) \tag{5.33}$$

Further, there is an operator $V_0(r)$ supported in $\mathcal{A}_{\Omega_0}$ such that for $V_{\circlearrowleft}(0, 0, r) - V_0(r)$ the same bound holds and we have

$$\|V_0(r) - \mathbb{I}\| \leq CQ_{max}\frac{J}{\Delta}Lr \tag{5.34}$$

Proof. If we denote for the unitaries U_x, U_y their localized versions defined by $S_\Delta^{(n)}$ as $U_{x,n}$ resp. $U_{y,n}$, then from Proposition 2.6, Lemma 2.7 and (3.9) we get the following bounds for all s, t, r:

$$\|U_x(s, t, r) - U_{x,n}(s, t, r)\| \leq r(Q_{max}JL) \cdot g_\Delta(n) \tag{5.35}$$

$$\|U_y(s, t, r) - U_{y,n}(s, t, r)\| \leq r(Q_{max}JL) \cdot g_\Delta(n) \tag{5.36}$$

So if we denote with $V_{\circlearrowleft,n}(s, t, r)$ the corresponding product of localized unitaries, we get the bound

$$\|V_{\circlearrowleft}(s, t, r) - V_{\circlearrowleft,n}(s, t, r)\| \leq 4r(Q_{max}JL) \cdot g_\Delta(n) \tag{5.37}$$

So going to the localizations gives us the first summand of the bound in the lemma. For the second summand we consider the Taylor expansion of $V_{\circlearrowleft,n}(s, t, r)$ in r. At $r = 0$ we have the identity. Since differentiating can be executed by computing commutators, each time we take a derivative we increase the support of the operator in terms of n and R. As long as we stay away from the mid-lines, the resulting operator at s, t is simply the twisted version of the operator at $s = t = 0$. So it is clear that as long as we choose n small enough, the fourth order Taylor approximation $V^{(4)}(s, t, r)$ of $V_{\circlearrowleft,n}(s, t, r)$ fulfills

$$V^{(4)}(s, t, r) = R_Y(t, R_X(s, V^{(4)}(0, 0, r))) \quad \text{and} \tag{5.38}$$

$$\|V_{\circlearrowleft,n}(s, t, r) - V^{(4)}(s, t, r)\| \leq C(Q_{max}\frac{J}{\Delta}L)^5 \cdot r^5 \tag{5.39}$$

where the prefactor $Q_{max}\frac{J}{\Delta}L$ comes from the norm-bounds in (3.9). With this we get the first part of the Lemma for n small enough and setting $V_0(r) := V^{(4)}(0, 0, r)$ gives the second part since $V_0(0) = \mathbb{I}$ and for n small enough $V_0(r)$ is still supported in $\mathcal{A}_{\Omega_0}$.

It is shown in Appendix D of [10] that choosing $n = L/48$ suffices to get the claim. If one wants $V_0(r)$ to be supported in an even smaller region around the origin, or wants higher polynomial bounds in r, one simply has to take some smaller n. □

Now we can finally bound the δ defined in Lemma 5.1. For that let

$$|\Psi_{\circlearrowleft}(s,t,r)\rangle := V^*(s,t)V_{\circlearrowleft}(s,t,r)V(s,t)\,|\Psi_0\rangle\,. \tag{5.40}$$

Then the following holds:

Theorem 5.6. *For sufficiently large L, we have for all $s,t \in [0,2\pi]$*

$$|\,\langle\Psi_0|\Psi_{\circlearrowleft}(s,t,r)\rangle - \langle\Psi_0|\Psi_{\circlearrowleft}(0,0,r)\rangle\,| \leq \widetilde{C}\cdot\Big(L^5r^5 + rLf(L)\Big)\,. \tag{5.41}$$

Proof. We expand into the following summands:

$$\langle\Psi_0|\,V^*(s,t)\,[V_{\circlearrowleft}(s,t,r) - R_Y(t,R_X(s,V\circlearrowleft(0,0,r)))]\,V(s,t)\,|\Psi_0\rangle \tag{5.42}$$
$$\langle\Psi_0|\,V^*(s,t)R_Y(t,R_X(s,[V\circlearrowleft(0,0,r) - V_0(r)]))V(s,t)\,|\Psi_0\rangle \tag{5.43}$$
$$\langle\Psi_0|\,V^*(s,t)R_Y(t,R_X(s,V_0(r)))V(s,t)\,|\Psi_0\rangle - \langle\Psi_0|\,V_0(r)\,|\Psi_0\rangle \tag{5.44}$$
$$\langle\Psi_0|\,[V_0(r) - V_{\circlearrowleft}(0,0,r)]\,|\Psi_0\rangle \tag{5.45}$$

For the first, second and last summand we get a bound from Lemma 5.5. In the third summand we subtract the Identity from both terms and then get a bound from Lemma 5.4 using $A = V_0(r) - \mathbb{I} \in \mathcal{A}_{\Omega_0}$ and again Lemma 5.5.
Putting this together we get the bound

$$\begin{aligned}&|\,\langle\Psi_0|\Psi_{\circlearrowleft}(s,t,r)\rangle - \langle\Psi_0|\Psi_{\circlearrowleft}(0,0,r)\rangle\,| \leq\\ &\leq C\left(Q_{max}JLr\cdot g_\Delta(L/48) + (Q_{max}\frac{J}{\Delta}Lr)^5\right) + \widetilde{C}Q_{max}\frac{J}{\Delta}Lr\cdot f(L)\end{aligned} \tag{5.46}$$

From here the claim readily follows. □

Now that we have proven uniformity of the phase we can go back to Lemma 5.1 to get the last estimate:

Lemma 5.7. *Let f be the subexponential decreasing function from Lemma 5.6. Then for*

$$r := 2\pi\left(\left\lfloor 2\pi\left(\frac{L^4}{3f(L)}\right)^{1/4}\right\rfloor\right)^{-1} \tag{5.47}$$

we have:

$$\left|\langle\Psi_0|\,V_{\circlearrowleft}(0,0,2\pi)\,|\Psi_0\rangle - \langle\Psi_0|\,V_{\circlearrowleft}(0,0,r)\,|\Psi_0\rangle^{(\frac{2\pi}{r})^2}\right| \leq \widetilde{C}L^2f(L)^{1/8} \tag{5.48}$$

The r is chosen in such a way that $\frac{2\pi}{r}$ is sure to be a integer. The bound given here is again subexponential in L.

Proof. We note that this r is indeed subexponential in L. Therefore we can apply Lemma 5.1 and therein the first term dominates, as stated in the remark after that Lemma. So we get with the bound on δ from Lemma 5.6 the following:

$$\left|\langle\Psi_0|\,V_{\circlearrowleft}(0,0,2\pi)\,|\Psi_0\rangle - \langle\Psi_0|\,V_{\circlearrowleft}(0,0,r)\,|\Psi_0\rangle^{(\frac{2\pi}{r})^2}\right| \leq \widetilde{C}\sqrt{L^5r + r^{-3}Lf(L)} \tag{5.49}$$

If we minimize the term under the square root we get the r stated above. Inserting this yields the claim. □

Therefore, we can now conclude the main Theorem:

Proof of Theorem 3.1. Taking the r from Lemma 5.7, this r is subexponential in L, so for L large enough, the assumption of Lemma 3.2 holds, so 3.2, 4.1 and 5.7 together give us a subexponential bound for

$$\left| e^{2\pi i \sigma (e^2/h)^{-1}} - 1 \right| \tag{5.50}$$

and the claim follows by the remark at the beginning of section 3.3 □

6 Conclusion

We have now proven the quantization of the Hall conductance for our given system. This system applies for example to a system of finitely many electrons on a torus under the assumption of finite-range interactions. In reality this is not exactly true but due to screening effects of the Coulomb force this is a valid approximation.
Hopefully the motivation and especially the technical details of this proof can be seen more clearly than in the original work by M. Hastings and S. Michalakis. It was also a big concern to keep the notations more consistent and less redundant. In particular the last chapter about uniformity of the phase has considerably altered bounds since the original work for example still had explicit dependencies on J, R, Δ and Q_{max} although the same bound contained some constants that were only implicitly dependent. To get a more explicit bound in terms of these parameters, one needs to get explicit bounds on the Lieb-Robinson velocity from [14]. If we have such bounds, then one could chase the constants appearing in [6] and [14] throughout the proof to get explicit descriptions for the dependency of the bound in the main theorem. For our purposes, however, it is enough to show existence of a bound subexponential in the size of the system.
The authors themselves mention at the end of their paper also possible extensions and generalizations. On the one hand, one could consider the fractional quantum Hall effect, where there is not a unique, but a degenerated groundstate. The authors say that under some assumption of topological order, one could give a proof along the same line but with more work. This was not elaborated upon in this work for lack of time, therefore I won't make any statement about the correctness of this claim. On the other hand they mention systems not with a spectral gap but a mobility gap or on an annulus instead of the torus. Since for those cases even the assumptions are still not defined satisfactory, this will be left for future work.
Another obvious question would be how the integer closest to the Hall conductance depends on the size of the system. It would be especially interesting to know whether the limit for $L \to \infty$ exists. But it is not clear if such a statement is possible to show with the techniques appearing in this proof.

Acknowledgements

I would like to thank Prof. Bachmann for his support, insightful discussions and the material he provided. I thank Prof. Müller that he agreed to be the second referee for my defense.

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Statement of Authorship

I hereby declare that I am the sole author of this master's thesis and that I have not used any sources other than those listed in the bibliography and identified as references. I further declare that I have not submitted this thesis at any other institution in order to obtain a degree.

Munich, the 01.08.2017

Ludwig Fürst