Introduction to Resurgence of Quartic Scalar Theory

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Declaration of authorship

Hereby I certify that all content of this work was written and produced by myself and the work presented is my own unless explicitly stated otherwise in the text.

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Last thanks goes to the nature, for being interesting and complex enough to provide endless opportunities for exploration.

Abstract

A treatment of quantum mechanics as a quantum field theory is introduced. Instanton theory and large N limit are reviewed. A brief introduction to Borel resummation is given, and a basic example of resurgence is demonstrated. $\lambda\phi^4$ theory is studied in 0, 1, and 4 dimensions, with particular focus on the case $\lambda < 0$. The connection between asymptotics of perturbation series and non-perturbative effects is explained. The large N limit is taken and studied in the perspective of resurgence. The connection of planar diagrams to tree graphs is shown.

Contents

1 Introduction

Quantum field theory (abbreviated to QFT for the rest of the text), is one of the most successful theories we have. It is able to describe interactions of the elementary particles, moreover, it can be used to study condensed matter systems as well. The success of QFT lies in its robustness, since the starting point is a harmonic oscillator, and all we do after is merely a perturbation around it. If the coupling constant g, around which we are perturbing, is small, then we expect to find an expression for an observable O in the form of power series $O = \sum_n a_n g^n$. While this looks simple, difficulties with the theory were soon discovered. These were related to the appearance of divergences of loop integrals. These problems were solved using regularization, but new infinities appeared. These were taken care of with the renormalization procedure. At this point, many physicists called it a day, and all there was to do, was to "Shut up and calculate." The times were good. The theory of quantum electrodynamics(QED) was being established, we were able to calculate observables, and experiments were measuring them, confirming predictions.

However, as it turned out, if one is persistent enough, and calculates a lot of orders of the perturbation series, one finds that the terms in the series grow as $n!$. The problem of the series $\sum_n n! g^n$ is that it converges only for $g = 0$. Naive way of thinking would tell us that the more terms of the series we calculate, the better precision we can achieve. But, if the series is not convergent, it is difficult to trust the perturbation series. The experiments were confirming predictions, but, mathematically, it seemed something was very wrong. It turns out that for any reasonable QFT, the observables come in form of a divergent series. Since divergent series even nowadays are not completely understood, it seemed unappealing to define a theory just in form of perturbations. In 1952, Freeman Dyson wrote an argument [1] where he reasoned why the observables in QED should assume a form of divergent series. The argument goes like this: "Let us say that we compute an observable as a perturbation series $F(e) = a_0 + a_1 e^2 + a_2 e^4 + \dots$. Now, suppose that this series converges for some positive e^2. From this follows that the function $F(e)$ is analytic. Now, imagine a theory where $e^2 < 0$, in a theory like that electron and positrons repel each other. We know that an electron-positron pair can be created out of the vacuum. In case of $e^2 < 0$, these two particles will fly away from each other, and our universe would have a bunch of electrons on one side, and a bunch of positrons on the other. A system like that cannot be described with analytic functions, thus F(e) has to be non-analytic on the negative real axis." As it will be demonstrated later in this work on a simpler theory, Dyson was not far from the truth.

Soon, it was discovered that the solutions of the equations of motion do not come only in form of perturbation series. There are solutions which cannot be written as a perturbation series. It is analogous to the case where we try to expand a function in Taylor series. Take for example $e^{-1/x}$. To expand a function in a Taylor series, around let us say 0, we have to calculate the derivatives of said function at that point. But, all the derivatives of $e^{-1/x}$ at $x = 0$ are 0. This means that the Taylor expansion would be $e^{-1/x} = 0 + 0x + 0x^2 + \dots$ The problem is that the Taylor series approach fails for this function. Indeed, it was noticed that some observables in QFT and in quantum mechanics include terms like these. For example, the splitting of ground state energy of a double-well potential exhibits such behavior. The first two energy states have the form of $E_{\pm} = E_0 \pm e^{-S/g}$, where S is some constant, g a coupling constant, and E_0 a ground state energy of harmonic oscillator. Such effects are called non-perturbative effects. The idea was to do the perturbation, as before, but then add non-perturbative contributions to the observable by hand. This might be aesthetically unappealing, but it was working, and was continued to be used.

In 1969, Carl Bender and Tai Tsun Wu [2] have shown where is the source of divergence of perturbation series. They have considered a quantum mechanical quartic oscillator. This can be thought of as a one dimensional scalar QFT with quartic interaction, $H = \frac{p^2}{2} + \frac{1}{2}q^2 + \frac{g}{4}q^4$. They have found that the terms in perturbative expression diverge as a factorial, and they pointed out the reason for the divergence - the analytical cut on the negative real axis in terms of the coupling constant g. This is incredibly similar to Dyson's argument. The case where the theory is unphysical, described by $g < 0$, is causing the divergence of the physical theory ($g > 0$). We call the theory for $g < 0$ unphysical, because the potential, $V(x) = \frac{1}{2}x^2 + gx^4$, is unbounded from below. The candidate for the ground state of such theory is at $x = 0$. Due to the tunneling, this ground state is meta-stable, and it will tunnel in some finite time. The cause for the tunneling is explained through instanton solutions. These are stationary solutions of equations of motion in the inverted potential, and can be shown to lead to the contributions to the ground state energy of the form $e^{-S/g}$. As mentioned before, this is a typical non-perturbative contribution. Of course, to treat tunneling in quantum mechanics, the instantons are not needed. But, they offer a nice generalization for the QFT. The conclusion is that non-perturbative contributions can create the cuts in the complex plane, which cause the perturbative series to diverge. This seems very curious, since we were treating perturbative and non-perturbative contributions as separate, but now we realized that they are actually connected. This is a real paradigm shift. Bender and Wu kept extending the understanding of this connection by studying the problem in generalized theories, but staying in one-dimensional QFT that is quantum mechanics.

In 1977 Lipatov [3] extended the work of Bender and Wu to the actual QFT for scalar theory. This article gave rise to a whole new field, represented by Parisi [4], Zinn-Justin, and many others. Quantum theory can, in some cases bring some new issues. First, one has to deal with renormalization, and even after renormalization, there can appear factorially growing series, which cannot be assigned to instantons, namely renormalons. In 1980, Ecalle [5] has mathematically defined the field of resurgence. The name comes from the fact that functions can seem very "normal" on the positive real axis, but when we continue them into the complex plane, a rich set of non-analytic singularities can be found. At this point, the field of resurgence became slightly inactive. In 1990s, it has resurged with work of Dunne(QFT) [6], Marino and Schiappa (string theory). It has been conjectured that from the asymptotics of the perturbation theory, all of the non-perturbative contributions can be extracted. There were proposed also resolutions of renormalons, but it still remains an open problem. One of the uses of resurgence in physics is in topological string theory, where we can only treat the theory perturbatively, while the non-perturbative contributions are unknown. The hope is that we might extract some information about the latter from the former.

In 2011, Dvali, Gomez and Mukhanov [7] have published an article, where they explore classical $\lambda\phi^4$ theory with $\lambda < 0$. From the pure classical treatment, they show that the theory is asymptotically free, and it shows signs of confinement. Both of these are properties of QCD. The theory also has an instanton, and, for large N, it will also have renormalons. This means that it is a good starting point for exploring resurgence.

In this work, we start with describing the perturbative treatment of quantum mechanics as a quantum field theory. A short review of instantons, relevant for our model, will be presented. This will be followed by a revision of large N limit, which will be split in two parts, first one being more pictorial and the second one being more formal. In the main part, the $\lambda\phi^4$ theory will be analyzed in 0, 1 and 4 dimensions, focusing particularly on $\lambda < 0$ case. For each case, we will perform an analysis of the theory from perturbative side, and try to extract the non-perturbative part by using resurgence.

2 Perturbation theory in Quantum Mechanics

In this text, we will assume that the reader has already been exposed to the perturbation techniques in quantum field theory and quantum mechanics. Since the latter is treated in a much different way than the former in the standard curriculum, we will present how one can treat quantum mechanics as a 1-dimensional quantum field theory.

Take a theory with a Hamiltonian

$$H = \frac{1}{2}p^2 + W(q), \tag{2.1}$$

where $W(q)$ is the potential. The mass and $\hbar$ have been set to 1. This theory is just quantum mechanics disguised as a one-dimensional quantum field theory. To obtain a ground state, we need first a partition function

$$Z(\beta) = \operatorname{tr} e^{-\beta H} \tag{2.2}$$

Suppose that H has a discrete spectrum, then the partition function can be written as

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n}, \tag{2.3}$$

where E_n, is n-th eigenenergy. From here it follows that ground state energy, or E_0 is

$$E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \log Z(\beta). \tag{2.4}$$

A partition function can be also viewed in path integral formalism. Let us perform a Wick rotation to imaginary time:

$$t \to -it \tag{2.5}$$

We have to consider only periodic trajectories in imaginary time, since $e^{ix} = e^{i(x+2\pi)}$:

$$q(-\beta/2) = q(\beta/2), \tag{2.6}$$

where β is the period of the motion. After Wick rotation, Euclidean action $S(q)$ appears, defined as:

$$S(q) = \int_{-\beta/2}^{\beta/2} \mathrm{d}\, t \left[\frac{1}{2} \left(\dot{q}(t)\right)^2 + W(q(t))\right]. \tag{2.7}$$

Partition function is now given by

$$Z(\beta) = \int [\mathrm{d}\, q] e^{-S(q)}, \tag{2.8}$$

where the integration goes over periodic trajectories. We see that the Euclidean action can be seen as a Lagrangian action

$$S(q) = \int_{-\beta/2}^{\beta/2} \mathrm{d}\, t \left[\frac{1}{2} \left(\dot{q}(t)\right)^2 - V(q(t))\right], \tag{2.9}$$

where the potential is turned upside down: $V(q) = -W(q)$.

We can now compute the ground state energy by using perturbation theory. Let us assume that the shape of the potential is

$$W(q) = \frac{m}{2} q^2 + \lambda q^4 \tag{2.10}$$

We compute Z by using Feynman perturbation theory by expanding W_{int}. We will work in a limit where $\beta \to \infty$, to make calculations simpler. We are looking for $\log Z(\beta)$, so we need to consider just connected bubble diagrams. To do this, we need Feynman rules.

1. Expression for the propagator is $(E^2 - m^2)^{-1}$,
2. Every vertex carries a factor 24λ,
3. for a loop integration $(2\pi) \int_{-\infty}^{\infty} \mathrm{d}\, E$.

We see the similarity with Feynman rules for a higher dimensional case. The only difference is that our only dimension is time, so 4-momentum is simply energy.

The diagrams that we have to compute are the ones on the Figures 2.1-2.3.

Each diagram also has to be multiplied by its symmetry factor, in to not overcount. If we compute these diagrams we find for the first few terms [2]

$$E = \frac{1}{2} + \frac{3}{4}\lambda - \frac{21}{8}\lambda^2 + \frac{333}{16}\lambda^3 \pm \dots \tag{2.11}$$

It is interesting that the denominators of the terms are powers of 2, and enumerators are integers. The asymptotic behavior of this series is very interesting, namely, the series diverges. The reasons for this will be discussed later on. If we chose $\lambda < 0$, we would

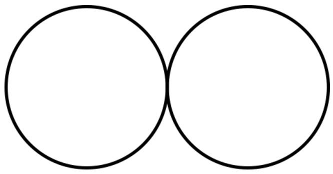

Figure 2.1: The only diagram of the order λ.

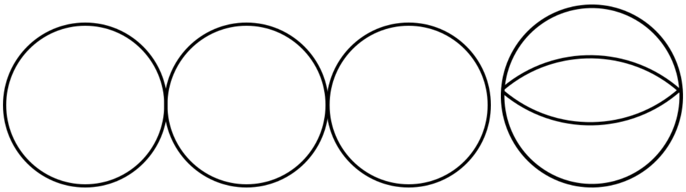

Figure 2.2: Diagrams of the order λ^2

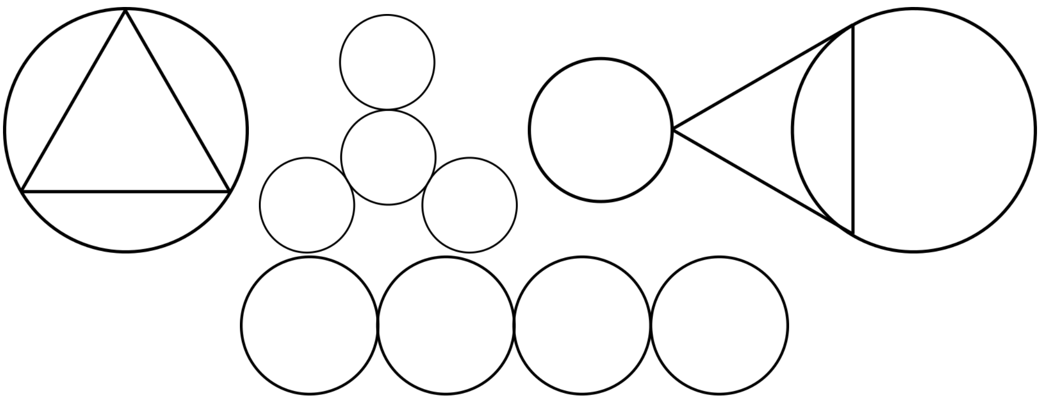

Figure 2.3: Diagrams of the order λ^3

expect the ground state energy to be of the form $E_0 = \mathrm{Re}\, E_0 + i\, \mathrm{Im}\, E_0$, since there is a finite probability for tunneling. But, the series we are getting is real, so there is no chance for finding an imaginary part of the ground state energy from perturbation theory. Or is there? The answer to this will also be given later.

3 Instantons

3.1 In Quantum Mechanics

Physicists have realized that very basic effects known from quantum mechanics cannot be seen in perturbative quantum field theory, for example tunneling. We will be performing a semi-classical treatment of quantum mechanics, so let us keep $\hbar$ in our expressions. Again, we are treating the Hamiltonian

$$H = \frac{1}{2}p^2 + W(q), \tag{3.1}$$

Let's calculate a following Feynman's sum

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = N \int [\mathrm{d}\, x] e^{-S/\hbar} \tag{3.2}$$

On the right hand side we recognize a partition function with the action

$$S = \int_{-T/2}^{T/2} \mathrm{d}\, t \left[\frac{1}{2} \left(\frac{\mathrm{d}\, x}{\mathrm{d}\, t} \right)^2 + V \right], \tag{3.3}$$

and N being a normalization coefficient, which is needed because $x_i \neq x_f$. The measure $[\mathrm{d}\, x]$ denotes the integration over all the functions $x(t)$ that satisfy the boundary conditions $x(-T/2) = x_i$ and $x(T/2) = x_f$. In this case, $|x_f\rangle$ and $|x_i\rangle$ in equation (3.2) are position eigenstates, and T is a positive number. We can expand the left-hand side of said equation in a complete set of energy eigenstates:

$$H|n\rangle = E_n |n\rangle. \tag{3.4}$$

Then we find

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = \langle x_f | \left(\sum e^{-E_n T/\hbar} |n\rangle\langle n| \right) |x_i\rangle = \sum_n e^{-E_n T/\hbar} \langle x_f | n \rangle \langle n | x_i \rangle. \tag{3.5}$$

From this expression, it is simple to extract the value of a ground state energy. In order to get it, one just has to take the limit $T \to \infty$ and look at the leading term.

We will expand the action (3.3) around the critical points.

$$S[x(t)] = S[\bar{x}(t)] + \int \mathrm{d}\,t' \, \frac{\delta S[x(t)]}{\delta x(t')}|_{x(t)=\bar{x}(t)} \delta x(t') + \frac{1}{2} \int \mathrm{d}\,t' \, \mathrm{d}\,t'' \, \frac{\delta^2 S[x(t)]}{\delta x(t') \delta x(t'')}|_{x(t)=\bar{x}(t)} \delta x(t') \delta x(t'') \tag{3.6}$$

We will be of course expanding around the critical point $\bar{x}(t)$,

$$\frac{\delta S[x(t)]}{x(t')}|_{x(t)=\bar{x}(t)} = -\ddot{\bar{x}}(t') + V'(\bar{x}(t')) = 0, \tag{3.7}$$

so the linear term of the expansion vanishes

$$S[x(t)] = S[\bar{x}(t)] + \frac{1}{2} \int \mathrm{d}\,t' \, \mathrm{d}\,t'' \, \frac{\delta^2 S[x(t)]}{\delta x(t') \delta x(t'')}|_{x(t)=\bar{x}(t)} \delta x(t') \delta x(t''). \tag{3.8}$$

The equation (3.7) describes a motion of the particle in the potential $-V(x)$ Let's take a look at what form our functional measure has. If we take a function $\bar{x}$ that obeys the boundary conditions specified above, then a general function obeying said conditions can be expressed as

$$x(t) = \bar{x}(t) + \sum_n c_n x_n(t), \tag{3.9}$$

where x_ns build an orthonormal basis of functions vanishing at the boundaries

$$\int_{-T/2}^{T/2} \mathrm{d}\, t x_n(t) x_m(t) = \delta_{nm}, \quad x_n(\pm T/2) = 0. \tag{3.10}$$

So the measure $[dx]$ is then defined as

$$[\mathrm{d}\, x] = \prod_n (2\pi\hbar)^{-1/2} \, \mathrm{d}\, c_n \tag{3.11}$$

We can see the the right-hand side of the equation (3.2) can be evaluated in the semi-classical limit. In this limit, the dominating parts will be the stationary points of the action S. Let us assume for now that the action has only one such stationary point, and denote it by $\bar{x}$:

$$\frac{\delta S}{\delta \bar{x}} = -\frac{d^2 \bar{x}}{dt^2} + V'(\bar{x}) = 0 \tag{3.12}$$

Let us now choose the set of orthogonal functions x_n being eigenfunctions of the second variational derivative of S at $\bar{x}$

$$-\frac{d^2 x_n}{dt^2} + V''(\bar{x}) x_n = \lambda_n x_n \tag{3.13}$$

In the semi-classical limit we get

$$\begin{aligned}
\langle x_f|e^{-HT/\hbar}|x_i\rangle &= Ne^{-S(\bar{x})/\hbar}\prod_n \lambda_n^{-1/2}\left[1+O(\hbar)\right] && (3.14)\\
&= Ne^{-S(\bar{x})/\hbar}\left[\det(-\partial_t^2+V''(\bar{x}))\right]^{-1/2}\left[1+O(\hbar)\right] && (3.15)
\end{aligned}$$

If we look at the stationary point equation (3.7), we notice, it is an equation of motion of a particle in a potential $-V$. So we find a constant of motion

$$E=\frac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2-V(\bar{x}) \tag{3.16}$$

If we take a potential of the form $V(x)=\frac{1}{2}\omega^2x^2$, with $x_i=x_f=0$, we find that there is only one stationary point of the action, $\bar{x}=0$, for which $S=0$. So, we get from (3.14)

$$\langle 0|e^{-HT/\hbar}|0\rangle = N\left[\det(-\partial_t^2+\omega^2)\right]^{-1/2}\left[1+O(\hbar)\right] \tag{3.17}$$

It can be shown that for large T

$$N\left[\det(-\partial_t^2+\omega^2)\right]^{-1/2}=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega T/2} \tag{3.18}$$

So, the ground state is

$$E_0=\frac{1}{2}\omega\hbar\left[1+O(\hbar)\right] \tag{3.19}$$

The same result will follow for any potential with one minimum at the origin and $V''(0)=\omega^2$, since in the first order of semi-classical approximation we obtain the harmonic-potential result.

The physics becomes much more exciting when we analyze a potential with more stationary points. Below, we will later on consider a potential of the form $\frac{1}{2}x^2+\frac{g}{4}x^4$. Of course, the analysis from above would not be very different for $g>0$, since there would not be any new stationary points. This changes when we take $g<0$. What we expect in this case when we compute ground-state energy is that we will obtain an imaginary part, $E\to E+i\Gamma/2$ which physically means tunneling, where Γ tells us the decay rate. We could find this solution through WKB, but since instantons can be applied also to Quantum Field Theory, we will consider it as an easier example of analogous calculation in QFT. So we go back to the equation (3.7), and we see that in our this case this equation has one more family of solutions. Thus we have $\bar{x}=0$, and $\bar{x}$ where the particle starts on top of the hill with minimum velocity at $t=-T/2$, slides down the potential, reaches the height it was at the origin at $t=0$, and slides back to the top at $t=T/2$. This kind of instantonic solution, represented in Figure 3.1, is called a bounce, so we will denote it by $\bar{z}_{\text{bounce}}(t)$. We do not need the exact expression for this solution. It is enough to see that the solution is very close to 0 for large and small times, and at some particular time it has a size of $1/\omega$, which is, as before $\omega = V''(0)$. To see this consider equation (3.16) with $E = 0$(needed when $T \to \infty$). we get

$$\frac{\mathrm{d}\, x}{\mathrm{d}\, t} = \sqrt{2V} \tag{3.20}$$

For large t we can approximate this with

$$\frac{\mathrm{d}\, x}{\mathrm{d}\, t} = -\omega x \Rightarrow x \sim e^{-\omega t} \tag{3.21}$$

So our solution has the typical size of $1/\omega$. Let's denote the action of the bounce for $T \to \infty$ by S_0. Because of the time translation invariance, we have whole family of solutions, where the bounce happens at any time $t_0 \in [-T/2, T/2]$, and the action of those is exponentially close to the S_0. We have even further configurations of n bounces, occurring at separated times (separation has to be much larger than $1/\omega$), with the action approximately nS_0. Of course we have to sum over all of these configurations.

In equation (3.14) we assumed that all the eigenvalues of z_ns are positive. In case that a certain eigenvalue is equal to zero, the whole expression. This is a sign that we have to rethink our calculation. Especially, since our system has such zero-mode. Due to the time translation invariance we have [8]

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + V''(\bar{z})\right)\dot{\bar{z}} = \frac{\mathrm{d}}{\mathrm{d}\, t}\left(-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2}\bar{z} + V'(\bar{z})\right) = 0 \tag{3.22}$$

So in our matrix element computation, we will have to compute an integral

$$\int \frac{\mathrm{d}\, c_0}{\sqrt{2\pi\hbar}} e^{-\lambda c_0^2} = \int \frac{\mathrm{d}\, c_0}{\sqrt{2\pi\hbar}} 1 = \infty \tag{3.23}$$

To deal with this infinity, we perform a substitution in the integral above. Take $\bar{z}$ to be an instanton solution. By differentiating we see that

$$\mathrm{d}\, \bar{z} = \dot{\bar{z}}\, \mathrm{d}\, t \tag{3.24}$$

But also

$$\mathrm{d}\, \bar{z} = z_0\, \mathrm{d}\, c_0, \tag{3.25}$$

Where z_0 is the normalized zero mode

$$z_0 = \frac{1}{\sqrt{S_0}} \dot{\bar{z}} \tag{3.26}$$

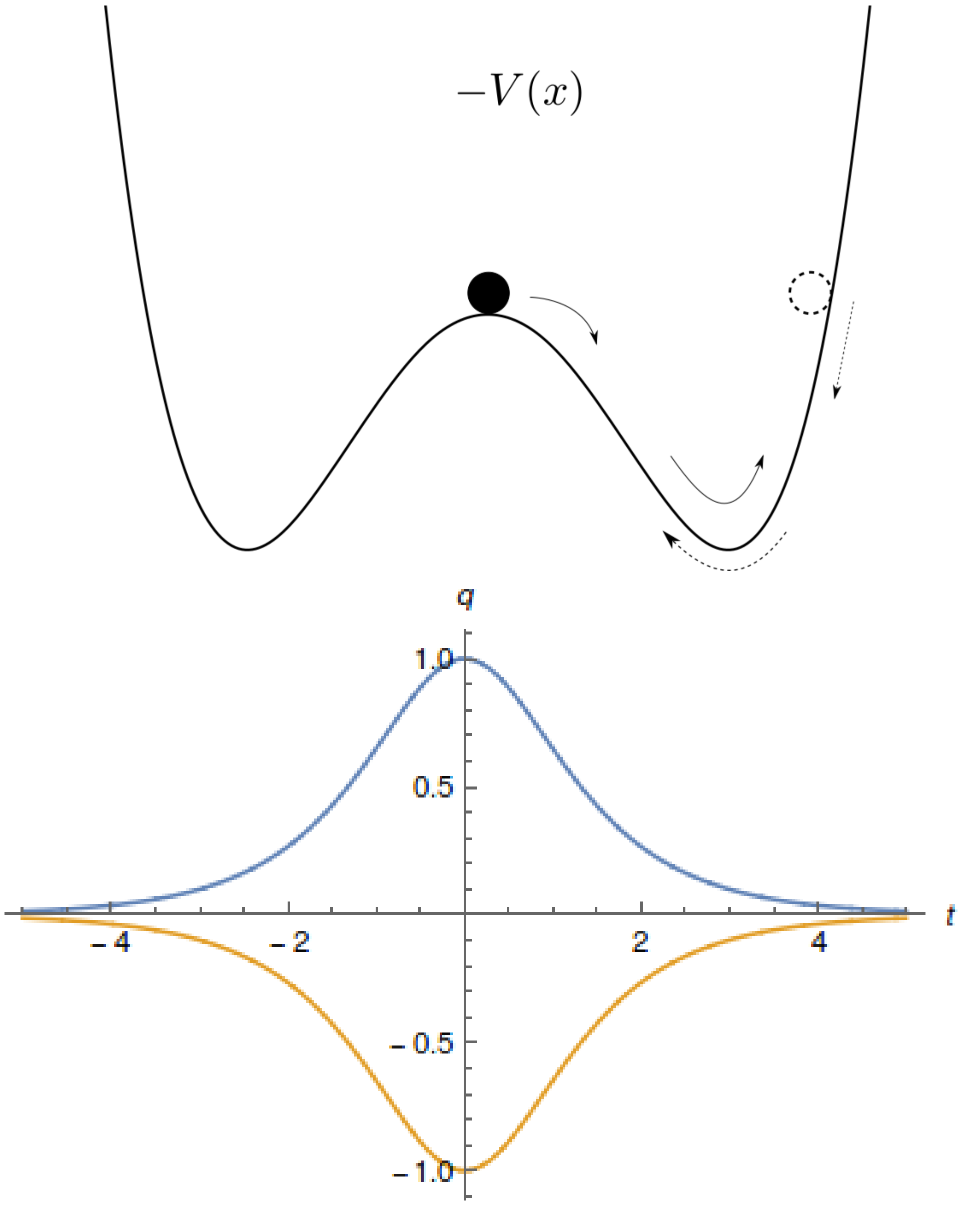

Figure 3.1: The picture above shows the motion which bounce describes. The particle starts at the local maximum of the inverted potential, rolls down the hill, passes the local minimum, rolls up, acquiring the starting height, and goes back. The plot below displays the said movement

So we see that

$$\frac{\mathrm{d}\, c_o}{\sqrt{2\pi\hbar}} = \sqrt{\frac{S_0}{2\pi\hbar}}\,\mathrm{d}\, t \tag{3.27}$$

when we substitute this into the Gaussian integral we get, unsurprisingly

$$\int \frac{\mathrm{d}\, c_0}{\sqrt{2\pi\hbar}} 1 = \int \mathrm{d}\, t \sqrt{\frac{S_0}{2\pi\hbar}} = \sqrt{\frac{S_0}{2\pi\hbar}} T, \tag{3.28}$$

which is still infinite in the end, but now we have at least some control over it. So we extract the infinite term out of the determinant and get

$$e^{-S_0/\hbar} \sqrt{\frac{S_0}{2\pi\hbar}} T N \left(\det{}' \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + V''(\bar{z}) \right] \right)^{-1/2}, \tag{3.29}$$

where $\det'$ means a determinant over non-zero eigenvalues

$$\left(\det{}' \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + V''(\bar{z}) \right] \right)^{-1/2} = \prod_{n, \lambda_n \neq 0} \lambda_n^{-1/2} \tag{3.30}$$

We also have to take into account all the possible multi-instanton configurations. Consider a sequence of n instantons, each occurring at particular time t_n. This yields an n-tuple integral

$$\int_{-T/2}^{T/2} \mathrm{d}\, t_1 \int_{-T/2}^{t_1} \mathrm{d}\, t_2 \cdots \int_{-T/2}^{t_{n-1}} \mathrm{d}\, t_n = T^n / n! \tag{3.31}$$

It can be shown through the tedious and long calculation that

$$\det{}' \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + V''(\bar{z}) \right] = \det \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + \omega^2 \right] K, \tag{3.32}$$

where K is independent of $\hbar$, and

$$K = \frac{\det' \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + V''(\bar{z}_n) \right]}{\det \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + \omega^2 \right]} \tag{3.33}$$

All things considered, we expect the form of the matrix element to be

$$\langle 0 | e^{-HT/\hbar} | 0 \rangle = N \left(\det \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + \omega^2 \right] \right)^{-1/2} \sum_{n=0}^{\infty} \left(\left(\frac{S_0}{2\pi\hbar} \right)^{1/2} T \right)^n \frac{e^{-nS_0/\hbar}}{n!} \cdot \tag{3.34}$$

$$\cdot \left(\frac{\det' \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + V''(\bar{z}_n) \right]}{\det \left[-\frac{\mathrm{d}^2}{\mathrm{d}\, t^2} + \omega^2 \right]} \right)^{-1/2}. \tag{3.35}$$

To find the imaginary part of the ground state energy we expand $\langle 0|e^{-HT/\hbar}|0\rangle$ only to find that

$$\operatorname{Im} E_0 = -\frac{1}{T} e^{T/2} \operatorname{Im}\langle 0|e^{-HT/\hbar}|0\rangle = \frac{S_0}{2\sqrt{2\pi}} \left[\frac{\det'\left[-\frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + V''(\bar{z}_n)\right]}{\det\left[-\frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + \omega^2\right]} \right] e^{-4/3\lambda} \tag{3.36}$$

We have extracted an imaginary part of the ground state energy. This is the result from the fact that we have one negative eigenvalue, which under the square root will give an imaginary contribution. The Gaussian integration for this eigenvalue doesn't even exist, so we have to modify our treatment. What saves us is that we are trying to compute the ratio of the determinants. For the computation, which is quite tedious, we refer the reader to [8]. In the end we find

$$\frac{\det'\left[-\frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + V''(\bar{z}_n)\right]}{\det\left[-\frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + \omega^2\right]} = -\frac{1}{12}. \tag{3.37}$$

This result is expected, since we know that one of the eigenvalues is negative. In the end we find that

$$\operatorname{Im} E_0 = \frac{4}{\sqrt{2\pi\lambda}} e^{-4/3\lambda} \tag{3.38}$$

To see where this negative eigenvalue comes from, we go back to our zero-mode. The zero mode has a node at the origin. So we know from basics of quantum mechanics that there exists a mode with lower energy. To define an analytic continuation properly, we shall write our potential with a free parameter, which will glue together different regimes, from the one that can be solved without analytic continuation, to the one we are currently treating.

3.2 In Quantum Field Theory

Let us consider now the quantum field theory version of our system. In general we have theory of a massive scalar field ϕ in d dimensions with the action

$$S(\phi) = \int \mathrm{d}^d\, x \left(\frac{1}{2} \left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} g m^{4-d} \phi^4 \right), \tag{3.39}$$

where m is mass, and g a dimensionless coupling. The complete n-point correlation function has the form

$$Z^{(n)}(x_1, \dots, x_n) = \int [\mathrm{d}\,\phi]\phi(x_1)\dots\phi(x_n)e^{-S(\phi)} \tag{3.40}$$

We expect analogously to the quantum mechanical case that for $g < 0$, the n-point functions will obtain an imaginary part due to the instanton contribution. As before we will look for the non-trivial saddle points. The procedure starts as before, through variation of the action, we get an equation of motion

$$\left(-\nabla^2 + m^2\right)\phi(x) + \frac{1}{6}gm^{4-d}\phi(x) = 0, \tag{3.41}$$

where ∇^2 is a 4-dimensional Laplace operator. We can use the scaling properties of the action to write the solution ansatz in the following form

$$\phi(x) = (-6/g)^{1/2}m^{d/2-1}f(mx), \tag{3.42}$$

Note that $g < 0$, so the square root is real. If we plug this ansatz into our action we find

$$S(f) = -\frac{6}{g}\int \mathrm{d}^d\,x\left[\frac{1}{2}\left(\partial_\mu f\right)^2 + \frac{1}{2}f^2 - \frac{1}{4}f^4\right]. \tag{3.43}$$

The equation of motion is then

$$\left(-\nabla^2 + 1\right)f(x) - f^3(x) = 0. \tag{3.44}$$

Using Sobolev inequality, it can be shown that the solutions minimizing the action are spherically symmetric. So we can rewrite the equation in spherically symmetric form with $r = |x - x_0|$:

$$\left[-\left(\frac{\mathrm{d}}{\mathrm{d}\,r}\right)^2 - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}\,r} + 1\right]f(r) - f^3(r) = 0. \tag{3.45}$$

If we interpret r as time, the equation is describing motion of a particle in the potential $-V(f)$, where $V(f) = \frac{1}{2}f^2 - \frac{1}{4}f^4$, and additionally being exposed to the viscous damping force. We are looking for the solution, where action is finite, so we have to impose the following boundary condition

$$f(r) \to 0, \text{ for } r \to \infty \tag{3.46}$$

If $f(r)$ goes to zero at infinity it will go exponentially, as seen from the equation (3.45). Because of the symmetry of the equation, the solutions will be even in r. As intuitively expected, for a generic initial condition $f(r)$, the solution will for $r \to \infty$ tend to one of the minima at $f = \pm 1$. The desired condition (3.46) will be satisfied only for a discrete set of initial values $f(0)$. One can show that the solution with the minimal action is the one for which $|f(0)|$ is minimal, and vanishes only at infinity.

Suppose we find the desired solution f_c that minimizes the action. Since the coupling g is dimensionless, the action will be of the form

$$S(\phi_c) = S(f_c) = -\frac{A}{g}, \quad A \in \mathbb{R}. \tag{3.47}$$

Through the Virial theorem it can be shown that for $d < 4$ holds

$$A = \frac{6}{d} \int [\partial_\mu f(x)]^2 \, \mathrm{d}^d x = \frac{3}{2} \int f^4(x) \, \mathrm{d}^d x = \frac{6}{4-d} \int f^2(x) \, \mathrm{d}^d x \tag{3.48}$$

For the dimension $d = 4$, last relations do not hold, but this case will be studied closer below. Supposing we have found the instanton solution, we can now perform an analogous procedure we did for the quantum mechanical case. Since the calculation gets very technical, we again refer the reader to the Zinn-Justin's book [8]. The result is analogous to the quantum mechanical case. There, the ground state energy acquired an imaginary part. In quantum field theory, the $n-$point functions acquire imaginary part proportional to $e^{A/g}$. The distinctive feature of the quantum field theory is that the operator M$=\frac{\delta^2 S}{\delta\phi_c \delta\phi_c}$ will appear as UV divergent, so it will require renormalization.

Let us now study the case in $d = 4$. Interesting property of this theory is that only the massless case possesses the instanton solutions. Intuitively this is surprising, since in the quantum mechanical case, there are no solutions for the massless case, only for massive. In the quantum field theory, the situation is opposite. Of course this holds solely for the ϕ^4 model considered here.

The action that we will be considering is

$$S(\phi) = \int \mathrm{d}^4 x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{4} g \phi^4 \right], \tag{3.49}$$

the corresponding equation of motion is

$$-\nabla^2 \phi(x) + g\phi^3(x) = 0 \tag{3.50}$$

Not that we have redefined the coupling constant g from equation (3.39). To restore the original g one just has to substitute $g \to g/6$. We follow the steps of the calculation in generic dimension. We know that the solution corresponding to the minimal action is spherically symmetric:

$$\phi(x) = \frac{1}{\sqrt{-g}} f(r), \quad r = |x - x_0| \tag{3.51}$$

With this we reduce the partial differential equation of motion to the ordinary differential equation

$$\left[-\left(\frac{\mathrm{d}}{\mathrm{d}\,r}\right)^2-\frac{3}{r}\frac{\mathrm{d}}{\mathrm{d}\,r}\right]f(r)-f^3(r)=0. \tag{3.52}$$

A significant property of ϕ^4 classical theory in $d=4$ is scalar invariance (even conformal). This symmetry is broken on the quantum level, but since we are considering classical solutions we can use it. If $\phi(x)$ is a solution of the equation of motion, then, by scale invariance, also $\psi(x)$ is a solution, where

$$\phi(x)=\lambda\psi(\lambda x). \tag{3.53}$$

This motivates the substitution

$$f(r)=e^{-t}h(t), \quad r=e^t. \tag{3.54}$$

The differential equation gets simplified down to

$$\ddot{h}(t)-h(t)+h^3(t), \tag{3.55}$$

which is same as the equation for an instanton in the quantum mechanical case. we can now write down the solution

$$h_c=\pm\frac{\sqrt{2}}{\cosh(t-t0)}. \tag{3.56}$$

So the solution for ϕ is

$$\phi_c(x)=\pm\frac{1}{\sqrt{-g}}\frac{2\sqrt{2}\lambda}{1+\lambda^2(x-x_0)^2}, \text{ where } \lambda=e^{-t_0}. \tag{3.57}$$

Because of scale invariance, also the solution is scale invariant. Consequently we will have 5 eigenvectors of operator M with eigenvalue zero, 4 for due to translational invariance, and one for scalar. In quantum mechanical case we had just translational invariance in time. The corresponding classical action of ϕ_c is

$$S(\phi_c)=-A/g, \quad A=8\pi^2/3. \tag{3.58}$$

For the standard normalization of g we find $A=16\pi^2$. The strategy is now very similar to the quantum mechanics case. We expand our action around the instanton solution. At the second order this is

$$M(x,x')=\frac{\delta^2 S}{\delta\phi_c(x)\delta\phi_c(x')}=\left[-\nabla^2-\frac{24\lambda^2}{(1+\lambda^2x^2)^2}\right]\delta^{(4)}(x-x'). \tag{3.59}$$

As said above, operator M has five eigenvectors, $\partial_\mu\phi_c(x)$ and $(\frac{d}{d\lambda})\phi_c(x)$. But the last eigenvector happens to be not normalizable with with the natural measure,

$$\int \left[\frac{\mathrm{d}}{\mathrm{d}\,\lambda}\phi_c(x)\right]^2 \mathrm{d}^4\,x = \infty \tag{3.60}$$

This IR problem arises because the theory is massless. We encounter another problem. We have to add the mass counterterm

$$\frac{1}{2}\delta m_0^2 \int \mathrm{d}^4\,x\,\phi_c(x)^2 = \infty. \tag{3.61}$$

So this integral is IR divergent, and it is expected to cancel the divergence from (3.60). So some kind of IR regularization is needed.

The resolution of these problems is quite technical. For details, we point reader to [8]. After this, also the coupling constant has to be renormalized. When all this is done, we find that 1PI $n-$point function will obtain an imaginary term, analogous to the one from ground state energy expression for 1-dimensional case.

$$\Gamma^{(n)}(p_1,\ldots,p_n) = \frac{C}{(-g)^{(n+5)/2}}\int_0^\infty \frac{\mathrm{d}\,\lambda}{\lambda}\lambda^{4-n}e^{8\pi^2/3g(\lambda)}\prod_{i=1}^{n}(p_i^2/\lambda^2)u(p_i/\lambda), \tag{3.62}$$

where

$$u(p) = 2\sqrt{2}\int e^{ipx}\frac{\mathrm{d}^4\,x}{1+x^2} \tag{3.63}$$

and $g(\lambda)$ is the effective coupling at the scale λ

$$\frac{8\pi^2}{3g} - 3\log\frac{\lambda}{\mu} = \frac{8\pi^2}{3g(\lambda)} \tag{3.64}$$

Since one component ϕ^4 theory can be tedious for calculating observables, the main text will focus on large N limit, where the calculations simplify.

4 Borel Resummation and Resurgence

The perturbative series of the observables in quantum field theory are usually asymptotic. The generic form the large terms have is

$$E = \sum a_n \lambda^n, \quad a_n \sim n! C^n \tag{4.1}$$

Series of such form are asymptotic. This means that the convergence radius is 0. The series will converge only for $\lambda = 0$. Which means that the whole expression will be zero. Because $\lambda \ll 1$, the terms will initially become smaller and smaller, but at typically $n \approx 1/(C\lambda)$ the terms start to become larger and larger. Typically we take first few terms of the perturbation series, sum them, and call this a result, while throwing higher terms away, with justification that we cannot trust the higher terms. There is however a procedure how to tend the wild behavior of asymptotic series. the Borel transformation of a power series $E(\lambda) = \sum a_n \lambda^{n+1}$ is defined as

$$BE(g) = \sum \frac{a_n}{n!} g^n \tag{4.2}$$

By doing this, we simply cancel out the problematic factorial growth. In case of the power series with the terms from equation (16), Borel transform has the form $BE(t) = \sum (Ct)^n$, in which we recognize geometric series, which can be resummed, namely $BE(t) = \frac{1}{1-Ct}$. This function has a pole at the $t = 1/C$. We also know that our resummation for $Ct \leq 1$ is not valid anymore. This is also the point where terms of power series start growing. An inverse transformation of the Borel one is the Laplace transformation. This can be easily verified by Taylor expanding the exponential function. Let us naively Laplace transform our Borel transform.

$$LBE(\lambda) = \int_0^\infty \mathrm{d}\, t \, e^{-t/\lambda} BE(t). \tag{4.3}$$

We see that we have to integrate from 0 to infinity, but somewhere at $t = C$ a pole awaits us. We are encountering an ambiguity of integration. We can continue our function to the complex plane, but we have two choices of integration paths, above or under the pole, as shown in Figure 4. We can define a lateral Laplace transformation:

$$S_{\theta^\pm} BE(\lambda) = \int_0^{e^{i\theta}(\infty \pm i\varepsilon)} \mathrm{d}\, t \, e^{-t/\lambda} BE(t) \tag{4.4}$$

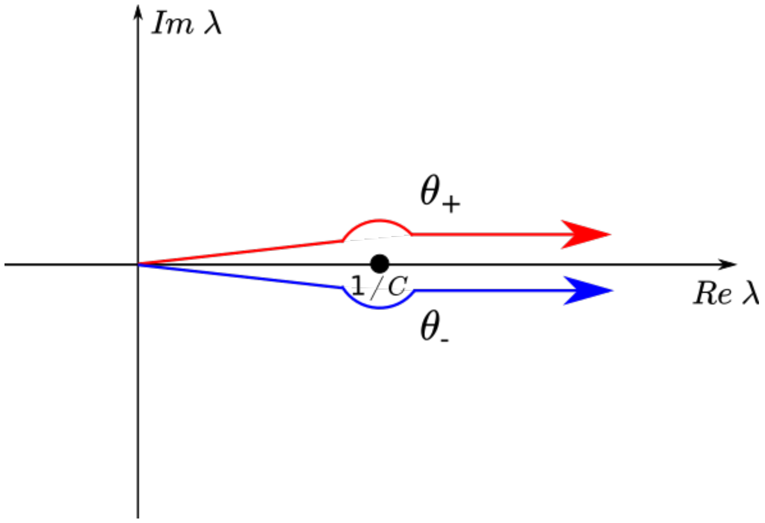

Figure 4.1: The Borel pole at $\lambda = 1/C$ is causing an ambiguity of Laplace transform. The picture shows the two lateral transforms; θ_+ and θ_-

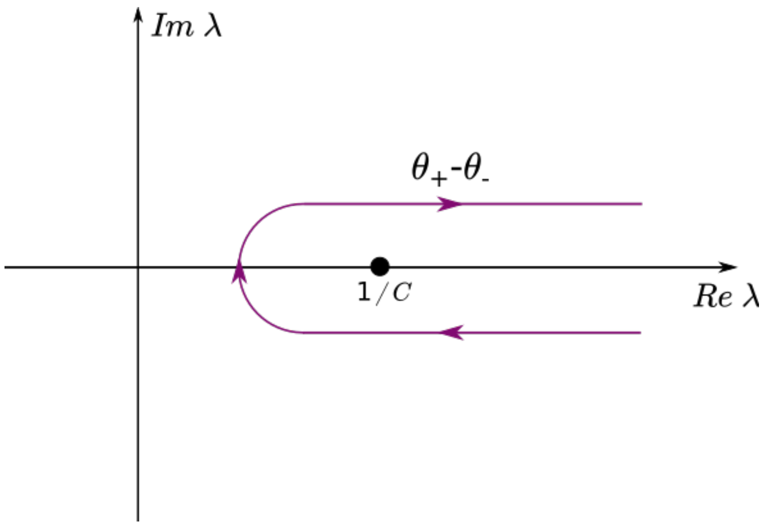

Figure 4.2: If we take a difference of the two lateral transforms, we obtain this path. If the integrand vanishes at infinity, the path can be closed around the origin at large distance from it. In this case, we can use Cauchy theorem to compute the difference of lateral resummations

What we find is that the integral will acquire an imaginary part, and the real part will simply be a function which is asymptotic to our perturbation expansion. If we simply subtract the upper lateral Laplace transformation from the lower one, as in Figure 4.2 , the real parts will cancel out, and we are left with the imaginary part, which we can extract using the residue theorem:

$$(S_{\theta^+} - S_{\theta^-}) = 2\pi i \operatorname{Res}|_{t=1/C} \left[\frac{e^{-t/\lambda}}{1 - Ct} \right] = 2\pi i \exp\left[\frac{1}{C\lambda} \right]. \tag{4.5}$$

This shows that our perturbation series was hiding something. Because of the ansatz we were using, a solution of non-analytic form has eluded us. But, nevertheless we were able to extract it.

For a more mathematically inclined reader, we recommend [9]

5 $1/N$ expansion

5.1 Diagramatic introduction à la Coleman

Consider a scalar theory with N scalar fields, governed by the Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}(\phi^2)^2, \text{ where } \phi = (\phi_1, \ldots, \phi_N) \tag{5.1}$$

Our goal is to explore our theory in the limit of a large N. We notice that every loop in a Feynman diagram gives a contribution of N, since there are N possible fields running in the loop. The contribution of the diagram in Figure 5.1 has a factor of $\lambda^3 N^2$, where λ is small, but N is large. What we also notice is that the diagram depicted in the figure is the only leading order diagram of a 4-point function with the factor of λ^3. For example, there is another possible diagram, as in Figure 5.2, which will be of the order $\lambda^3 N$, thus, subleading. Thus, we cannot expand our perturbation series in a small coupling constant λ anymore. It is useful to construct a new coupling $g = \lambda N$ We will then look at the limit where $\lambda \to 0$, $N \to \infty$, but g being kept constant. This can be achieved by introducing an auxilliary field σ, which we introduce as a Lagrangian multiplier:

$$L' = L + \left(\frac{N}{g}\right)\left(\sigma - \frac{1}{2}\left(\frac{g}{N}\right)\phi^2\right)^2 \tag{5.2}$$

If we vary this Lagrangian with σ, we get just a definition for the field σ:

$$\sigma = \left(\frac{g}{N}\right)\phi^2 \tag{5.3}$$

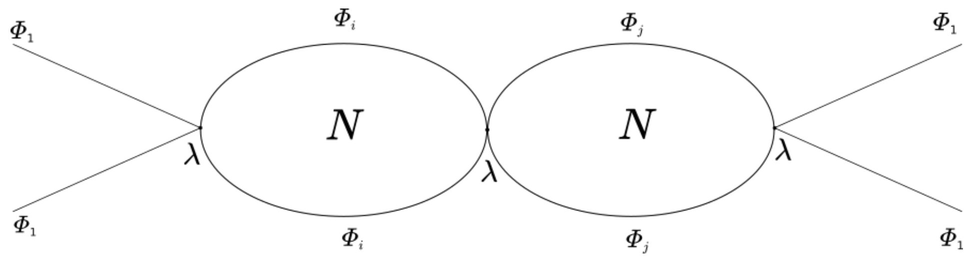

Figure 5.1: This is the leading order diagram of the 4-point function, it is of the order $\lambda^3 N^2$

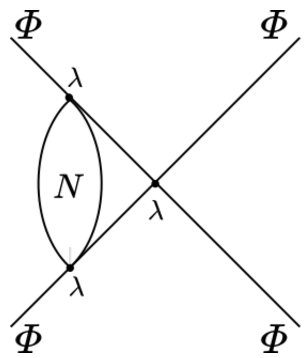

Figure 5.2: This diagram is subleading, since it is of the order $\lambda^3 N$. Even though it has the same number of vertices is this diagram subleading in comparison to the first one.

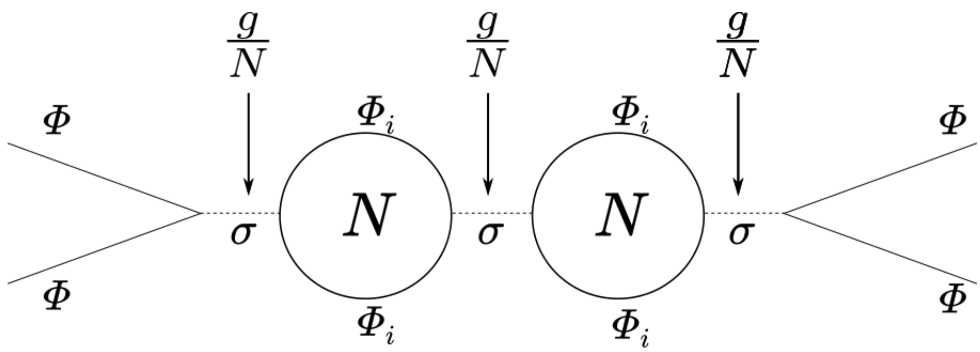

Figure 5.3: The diagram from figure 5.1 redrawn in the large N limit. It is of the order $g^3 N$

Our Lagrangian takes the form

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{N}{g}\sigma^2 - \sigma\phi^2 \tag{5.4}$$

We notice that the Feynman rules have just become much simpler. The only interacting term is the cubic vertex of σ and ϕ^2. The only term coming with the coupling constant is the propagator of σ. We see that the only scalar loops allowed, are the ones where all the scalar fields come with the same index. Because there is N possibilities for the index, every scalar loop comes with the contribution N. We can now redraw the diagram from the first picture, as in Figure 5.3. The diagram is of the order $\frac{g^3}{N}$, while the other is only of the order $\frac{g^3}{N^2}$

We have thus seen that the leading contributions of Feynman diagrams in the large N limit will come from the planar diagrams, because of the constraint that every loop consists of only one flavour of the scalar field. To summarize, the leading diagram is the one with the most loops. Below, we will treat the large N theory with the functional integral approach, but the intuition behind is still the same.

5.2 General scalar field formalism à la Zinn-Justin

Take a genera $O(N)$ symmetric Euclidean action of the N-component scalar field ϕ

$$S(\phi) = \int \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + NU(\phi^2/N)\right] \mathrm{d}^d\, x. \tag{5.5}$$

Here, $U(\rho)$ is a general polynomial, and the N dependence is chosen such as to lead to the large N limit. The partition function is

$$Z = \int [\mathrm{d}\,\phi] \exp[-S(\phi)] \tag{5.6}$$

In large N limit, the quantities like ϕ^2 will self-average, and thus the fluctuations will be small, as predicted by central limit theorem. Concretely

$$\langle\phi^2(x)\phi^2(y)\rangle \sim_{N\to\infty} \langle\phi^2(x)\rangle\langle\phi^2(y)\rangle \tag{5.7}$$

This fact invites us to use ϕ^2 as a dynamical variable instead of just ϕ. As above we will define two additional field λ and ρ and impose the constraint $\rho = \phi^2/N$. Note that ρ is taking now the role of σin the previous chapter. To include this constraint into the partition function, we use the identity

$$1 = N\int \mathrm{d}\,\rho\delta(\phi^2 - N\rho) = \frac{1}{4\pi i}\int \mathrm{d}\,\rho\,\mathrm{d}\,\lambda e^{\lambda(\phi^2-N\rho)/2}. \tag{5.8}$$

Our partition function has now the following form

$$Z = \int [\mathrm{d}\,\phi][\mathrm{d}\,\rho][\mathrm{d}\,\lambda] \exp[-S(phi, \rho, \lambda)], \tag{5.9}$$

the action now being

$$S(\phi, \rho, \lambda) = \int \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + NU(\rho) + \frac{1}{2}\lambda\left(\phi^2 - N\rho\right)\right] \mathrm{d}^d\, x \tag{5.10}$$

It is useful to separate the components of ϕ such that $\phi_1 = \sigma$ and the other $N-1$ components π and integrate over the πs. Finally we get the following action

$$S_N(\sigma,\rho,\lambda) = \int \left[\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + NU(\rho) + \frac{1}{2}\lambda(\sigma^2 - N\rho)\right] \mathrm{d}^d\, x + \frac{1}{2}(N-1)\,\mathrm{tr}\ln\left[-\nabla^2+\lambda\right] \tag{5.11}$$

We can now use the steepest descent method to find the ground state. We take $\sigma(x) = \sigma$, $\rho(x) = \rho$ and $\lambda(x) = m^2$, since otherwise we will find instantons and solitons. We now vary the action with respect to the σ,ρ, and m^2.Before we do so, we also integrate out the kinetic part of the σ field. We obtain the following system of equations

$$m^2\sigma = 0, \tag{5.12}$$

$$\frac{1}{2}m^2 = U'(\rho), \tag{5.13}$$

$$\sigma^2/n - \rho + \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d\, k}{k^2+m^2} = 0. \tag{5.14}$$

The last integral is cut off with a cut-off Λ for regularization.

Now, we want to consider ϕ^4 theory, namely a theory with action

$$S(\phi) = \int \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}r\phi^2 + \frac{1}{4!}\frac{u}{N}(\phi^2)^2\right] \tag{5.15}$$

The corresponding potential in the equation (5.5) is

$$U(\rho) = \frac{1}{2}r\rho + \frac{u}{4!}\rho^2 \tag{5.16}$$

We plug this in the equation (**??**) and find that the action for this case is

$$S_N(\sigma,\rho,\lambda) = \int \left[\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + N\left(\frac{1}{2}r\rho + \frac{u}{4!}\rho^2\right) + \frac{1}{2}\lambda(\sigma^2 - N\rho)\right] \mathrm{d}^d\, x + \dots \tag{5.17}$$

We can now integrate ρ out, since the integral is Gaussian, so effectively we just have to replace ρ with the solution of

$$\frac{1}{6}u\rho(x) + r = \lambda(x) \tag{5.18}$$

The obtained action is

$$S_N(\sigma,\rho,\lambda) = \int \left[\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \lambda\sigma^2 - \frac{3N}{u}\lambda^2 + \frac{6Nr}{u}\lambda\right] \mathrm{d}^d\, x + \frac{1}{2}(N-1)\,\mathrm{tr}\ln\left[-\nabla^2+\lambda\right] \tag{5.19}$$

What we got is the interaction term between σ and λ with coupling equal to 1, and propagator of λ is the only source of the perturbation coupling $\frac{u}{N}$. So the leading perturbative contributions come from the diagrams without loops of λ propagators. This will enable us to do calculations in the large N limit much more comfortable.

6 $\lambda\phi^4$ Theory in 0 dimensions

6.1 Scalar Case

Take a ϕ^4 theory in d dimensions. The action is the following

$$S = \int \mathrm{d}^d\, x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \lambda\phi^4\right], \quad \mu = 0, 1, \ldots, d-1 \tag{6.1}$$

The partition function in this case is

$$Z(0) = \int [\mathrm{d}\,\phi] e^{-iS} \tag{6.2}$$

We can transform this into Euclidean theory by Wick rotation $x_0 \to i\tau$ and by this we recover

$$Z(0) = \int [\mathrm{d}\,\phi] e^{-S_E}, \tag{6.3}$$

where

$$S_E = \int \mathrm{d}^d\, x \left[\frac{1}{2}\partial_i^2\phi + \frac{1}{2}m^2\phi^2 + \lambda\phi^4\right], \quad i = 0, 1, \ldots, d-1 \tag{6.4}$$

If we now take $d = 0$, the kinetic terms will disappear, and we are left with

$$Z(\lambda) = \int \mathrm{d}\, x e^{-\frac{1}{2}x^2 - \lambda x^4}, \tag{6.5}$$

Where we substituted x for ϕ to make the fact that this is just a simple integral more obvious, and set $m = 1$. For $|\lambda| << 1$ it seems legitimate to treat this integral perturbatively. We can expand

$$e^{-\lambda x^4} = \sum_{n=0}^{\infty} \frac{1}{n!}(-\lambda x^4)^n. \tag{6.6}$$

So our partition function will be

$$Z(\lambda) = \int \mathrm{d}\, x e^{-\frac{1}{2}x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\lambda x^4)^n = \sum_{n=0}^{\infty}(-1)^n \frac{\lambda^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}\, x\, x^{4n} e^{-\frac{1}{2}x^2} \tag{6.7}$$

We know that

$$\int_{-\infty}^{\infty} \mathrm{d}\, x\, x^n e^{-ax^2} = \frac{1}{2}(1+(-1)^n)a^{-\frac{1+n}{2}}\Gamma(\frac{1+n}{2}) \tag{6.8}$$

so in our partition function will be of the form

$$Z(\lambda) = \sum_n a_n \lambda^n, \quad a_n = (-1)^n \frac{1}{n!}\left(\frac{1}{2}\right)^{-(2n+\frac{1}{2})}\Gamma(2n+\frac{1}{2}), \tag{6.9}$$

or nicer

$$Z(\lambda) = \sqrt{2}\,\frac{\Gamma(2n+\frac{1}{2})}{\Gamma(n+1)}(-4\lambda)^n \tag{6.10}$$

For $\lambda < 0$, the series will cease to be alternating. By using Stirling's approximation $n! \approx n^n e^{-n}$, we arrive to the estimate $\lambda^n a_n \approx \left(\frac{-\lambda n}{e}\right)^n$. This is worrisome, since we know that for large enough n, the factor n^n will be greater than $(1/g)^n$. We know that the integral should be convergent for $\lambda > 0$, but our series is obviously asymptotic. This kind of series are very often encountered in quantum field theory. So, is our theory ill-defined? The usual treatment is such that we sum only first few terms, ideally to the term which still improves the approximation, an throw the rest away. This procedure yields such incredible accuracy that one of the most accurate measured quantity was predicted in this way. Let's estimate the error of our series. We want to estimate $|Z(\lambda) - \sum_{n=0}^{n_{\max}} \lambda^n a_n|$. We use that $\left|e^{-\lambda x^4} - \sum_{n=0}^{n_{\max}} \frac{1}{n!}(-\lambda x^4)^n\right| \leq \frac{(\lambda x^4)^{n_{\max}+1}}{(n_{\max}+1)!}$. We arrive to the following estimation

$$\left|Z(\lambda) - \sum_{n=0}^{n_{\max}} \lambda^n a_n\right| \leq \lambda^{n_{\max}+1}\left|a_{n_{\max}+1}\right| \sim \left(\frac{\lambda n_{\max}}{e}\right)^{n_{\max}} \tag{6.11}$$

We can now find a minimum of the error, and get $n_{\max} = \lambda^{-1}$. At this value error scales like $e^{-1/\lambda}$. Let us now take the example from QED, where the coupling $\alpha = \frac{1}{137}$. So the error in this case would be of the order e^{-137}, which is incredibly precise, as also shown in Figures 6.1-6.4.

Now consider the same integral, but now we're going to analyze it globally. So we are considering again

$$Z(\lambda) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\, x\, e^{-\frac{1}{2}x^2 - \lambda x^4}. \tag{6.12}$$

This integral is dominated by the saddle point at $x = 0$. So we can estimate $Z(\lambda) = 1 + O(\lambda)$. Now let's try to continue λ to the negative values. We know that the integral with negative λ will diverge, but the story has a small twist. There is an analytic cut on the negative real axis. This means we can continue λ in two different directions. Now choose to continue it into the upper imaginary plane $\lambda \to |\lambda| e^{i\phi}$. We can then rewrite this

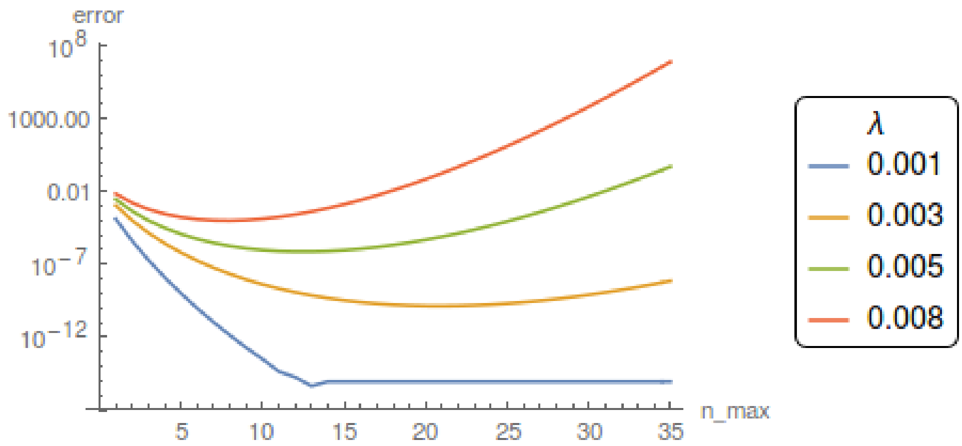

Figure 6.1: Here we see the error of the sum, depending on how far do we sum. We see that for small enough λ, the error is incredibly small. Also, it is enough to sum less than 10 terms to achieve a very small error. As expected, the lower the λ, the lower the minimal error.

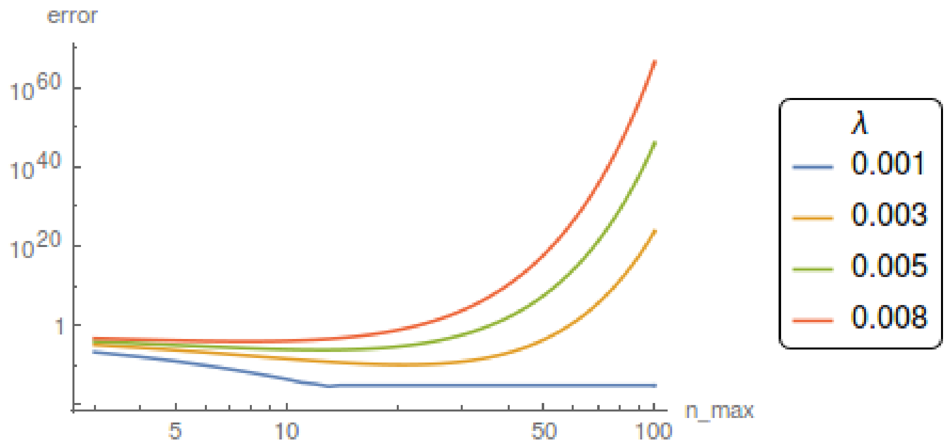

Figure 6.2: If we sum farther, the divergence is faster than exponential

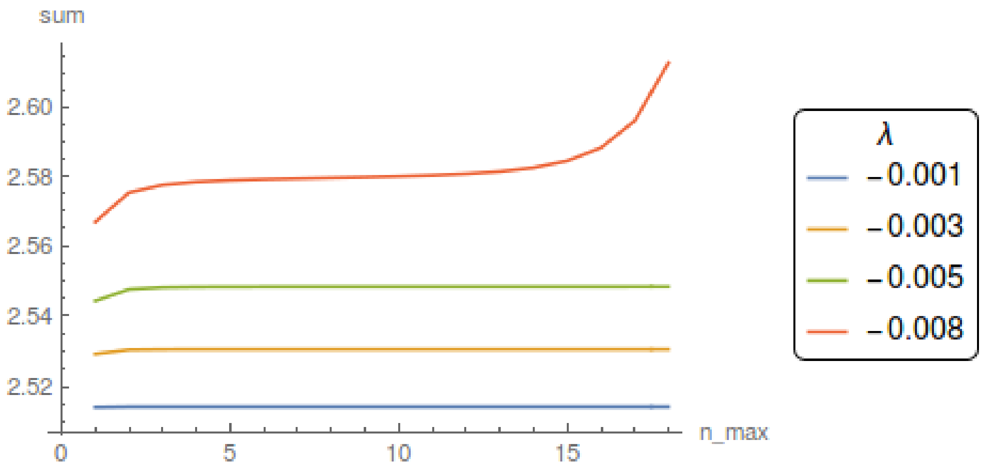

Figure 6.3: In case of $\lambda < 0$ we see that the value becomes stationary for some terms, and only then diverge wildly.

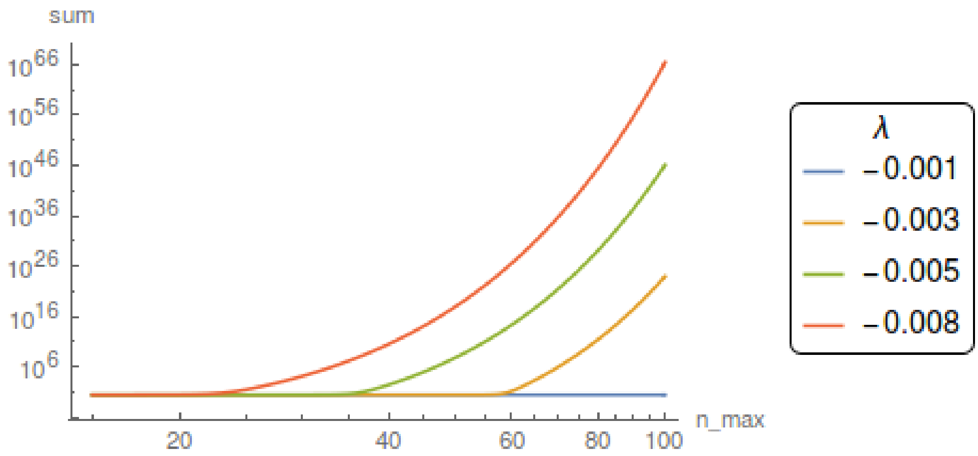

Figure 6.4: We see that the divergence is faster than exponential, qualitatively not very different from the divergence in case $\lambda > 0$

integral

$$\int_{-\infty}^{\infty} \mathrm{d}\, x\, e^{-x^2-(|\lambda|e^{i\phi})x^4} = \int_{C_+} \mathrm{d}\, x\, e^{-x^2-(|\lambda|e^{i\phi})(|x|e^{-i\phi})^4} = \int_{C_+} \mathrm{d}\, x\, e^{-x^2-\lambda x^4}, \tag{6.13}$$

where C_+ is denoting a real axis rotated for $\phi/4$. We do the similar procedure for continuing to the lower half-plane, and obtain similar integral, only now getting the path rotated for $\phi/4$. For $\phi = \pi$ we get the two paths going along the diagonals of the complex plane.

For $g \to 0_-$ the two integrals are still dominated by the saddle point at the origin. The other saddle points are at $x + 4\lambda x^3 = 0$. It is useful to define $g = 4\lambda$. In this case the saddle points are at $x^2 = \pm 1/g$. If we want to compute the discontinuity

$$Z(g + i0) - Z(g - i0) = 2i \,\mathrm{Im}\, I(g) = \int_{C_+ - C_-} \mathrm{d}\, x\, e^{-x^2/2+gx^4/4}, \tag{6.14}$$

we can deform the integration path so that it goes through the non-trivial saddle points, as in Figure 6.5. The real parts of the integrals will cancel and we are left with an imaginary part which is then

$$\mathrm{Im}\, Z(g) = e^{1/(4g)} = e^{1/\lambda}. \tag{6.15}$$

So we see that for $\lambda < 0$, the real part of the integral is given by perturbation theory, but the imaginary part is given by the contribution of non-trivial saddle points.

The fact that we are seeing the factor of $e^{1/\lambda}$ seems mysterious, so let us try to demystify

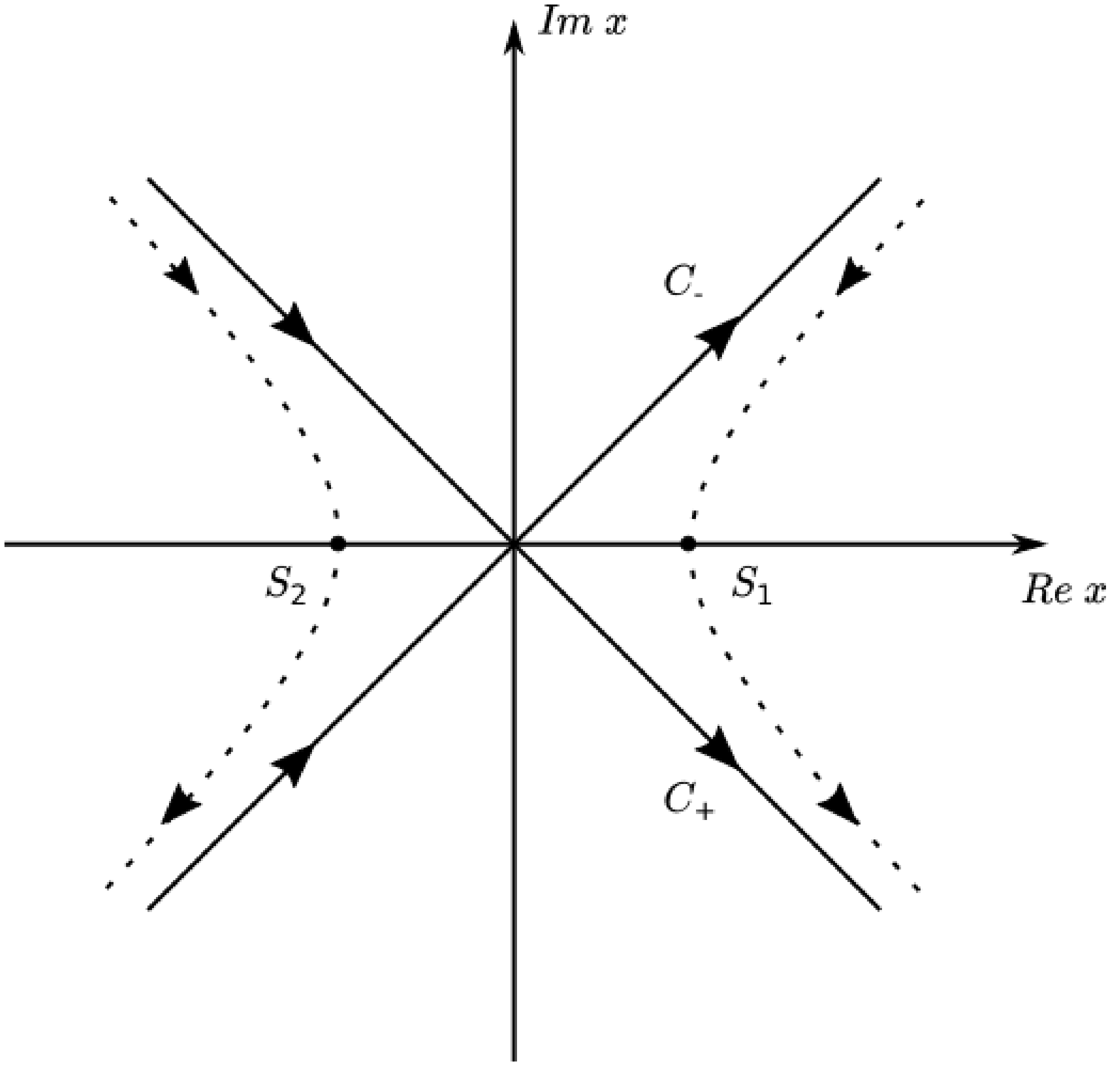

Figure 6.5: The difference of contributions of paths C_- and C_+ can be deformed into the path passing through the saddle points S_1 and S_2.

it. We return to the integral

$$Z(\lambda) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\, x\, e^{-\frac{1}{2}x^2 - \lambda x^4}. \tag{6.16}$$

This integral has a branch cut on the negative real axis. Its behavior at infinity is

$$Z(\lambda) \sim \lambda^{-1/4}, \tag{6.17}$$

which can be seen by substituting $x = \lambda^{-1/4} u$. We find

$$Z(\lambda) = \frac{\lambda^{-1/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\, u\, \exp\left[-\frac{\lambda^{-1/2} u^2}{2} - u^4\right] \tag{6.18}$$

$$\underset{\lambda\to\infty}{=} \frac{\lambda^{-1/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\, u\, \exp\left[-u^4\right] \tag{6.19}$$

$$Z(\lambda) = \frac{\Gamma(\frac{1}{4})}{\sqrt{8\pi}} \lambda^{-1/4}. \tag{6.20}$$

After Cauchy's theorem, we can select a point $x \in \mathbb{C}$, choose a contour C_x which stays away from the branch cut, and represent Z as

$$Z(z) = \frac{1}{2\pi i} \oint_C \mathrm{d}\, x\, \frac{Z(x)}{x - z} \tag{6.21}$$

But now, we can deform our contour, as in Figure 6.6, so that it goes along the branch cut, and at infinity makes a circle around the origin. Because the integral vanishes at the origin and infinity, the only remaining part is the contribution from the integral along the branch cut. So we have

$$Z(z) = \frac{1}{2\pi i} \int_{-\infty}^{0} \mathrm{d}\, x\, \frac{Z(x)}{x - z}, \tag{6.22}$$

where $D(x)$ is the discontinuity at the branch cut

$$D(x) = \lim_{\varepsilon\to 0} \left(Z(x + i\varepsilon) - Z(x - i\varepsilon)\right) = 2i\, \mathrm{Im}\, Z(x). \tag{6.23}$$

We find

$$Z(z) = \frac{1}{\pi} \int_{-\infty}^{0} \mathrm{d}\, x\, \frac{\mathrm{Im}\, Z(x)}{x - z} \tag{6.24}$$

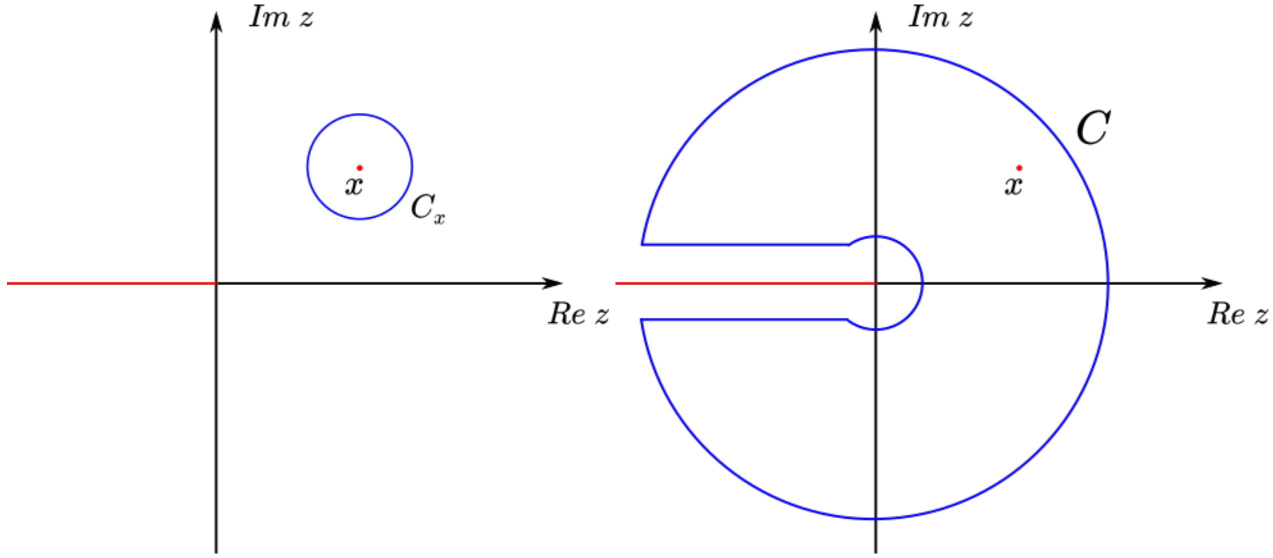

Figure 6.6: The path C_x around the point x, can be deformed into the path C where only the integration along the negative real axis contributes.

We have found the $\operatorname{Im} Z(\lambda) = e^{1/\lambda}$ earlier, so we can now find the series expansion of $Z(z)$.

$$
\begin{aligned}
Z(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}\, x\, e^{1/x}\frac{1}{x-\lambda} && (6.25)\\
&= \frac{1}{\pi}\pi\int_{-\infty}^{0} \mathrm{d}\, x\, e^{1/x}\frac{1}{x}\frac{1}{1-\frac{\lambda}{x}} && (6.26)\\
&= \frac{1}{\pi}\pi\int_{-\infty}^{0} \mathrm{d}\, x\, e^{1/x}\frac{1}{x}\sum_{k=0}^{\infty}\left(\frac{\lambda}{x}\right)^{k}. && (6.27)
\end{aligned}
$$

We can write $Z(\lambda) = \sum_k a_k \lambda^k$, with

$$
a_k = \frac{(-1)^k}{\pi}\int_0^{\infty} \mathrm{d}\, x\, e^{-1/x}\frac{1}{x^{k+1}} = \Gamma(k) \tag{6.28}
$$

In this way we see that the asymptotics of the series representation of this integral, is dictated by the discontinuities in the complex plane, in this case, branch cut on the negative real axis.

6.2 Large N limit

Now we take an example where the field ϕ has N components. We will perform a large N limit on it. Our partition function is

$$Z(0) = \int_{-\infty}^{\infty} \mathrm{d}\,\phi_1 \dots d\phi_{N+1} e^{-\frac{1}{2}m^2\left(\phi_1^2+\dots+\phi_{N+1}^2\right)-\lambda/4\left(\phi_1^2+\dots+\phi_{N+1}^2\right)^2} \tag{6.29}$$

Because the dimension is 0, this is just a casual one-dimensional integral.

To perform a large N limit, we define $N\rho^2 = \sum_{i=1}^{N+1}$. Integral now reads

$$Z(0) = \int \mathrm{d}\,\Omega \int \mathrm{d}\,\rho \exp\left[-N\left(\frac{1}{2}m^2\rho^2 + \frac{g}{4}\rho^4 - \log\rho\right)\right], \quad \int \mathrm{d}\,\Omega = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})} \tag{6.30}$$

We obtain similar series in powers of λ, but the coefficients will now be different [10]

$$a_n = (-1)^n \frac{1}{2n!}\left(\frac{m^2}{2}\right)^{-(\frac{1+4n+N}{2})} \Gamma\left(\frac{1+4n+N}{2}\right) \tag{6.31}$$

7 $\lambda\phi^4$ Theory in 1 Dimension

7.1 Scalar case

Take a 1-dimensional anharmonic oscillator, with the theory

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 - \lambda\phi^4. \tag{7.1}$$

This can be rewritten in Hamiltonian form

$$H = \frac{p^2}{2} + \frac{1}{2}m^2x^2 + \lambda x^4, \tag{7.2}$$

where we have replaced ϕ with x. This way, the equivalency with quantum mechanics is obvious. The ground state energy can be calculated with perturbation theory, and is merely a sum of all vacuum bubbles. We obtain an expansion in coupling λ

$$E_0(\lambda) = \frac{1}{2}m + \sum_{n=1}^{\infty} mA_n \left(\frac{\lambda}{m^3}\right)^n, \tag{7.3}$$

where the first term is a known solution for harmonic oscillator. The asymptotics of A_n is known to be [2]

$$A_n \sim (-1)^{n+1}\left(\frac{6}{\pi^3}\right)^{1/2}\Gamma(n+\frac{1}{2})3^n. \tag{7.4}$$

For simplicity we take $m = 1$, so that our series admits the form

$$E_0(\lambda) = \frac{1}{2} + \sum_{n=1}^{\infty} A_n\lambda^n \tag{7.5}$$

We want to inspect the properties for negative coupling, so we write $\lambda = -\alpha$, where $\alpha > 0$. The form of the series is

$$E_0(\lambda) = \frac{1}{2} + \sum_{n=1}^{\infty} \tilde{A}_n\alpha^n, \tag{7.6}$$

where the asymptotics for coefficients will now be $\tilde{A}_n = -|A_n|$. The series is non-alternating. It's not convergent, it's merely asymptotic. From the introduction(to be written) we know, we can extract additional, non-perturbative properties out of it. To do that, we have to Borel transform it. To do that, we rewrite our series in more useful form

$$E_0(g) = \frac{1}{2} + C\sum_{n=0}^{\infty} c_n g^{n+1}, \tag{7.7}$$

Where $C = -\left(\frac{6}{\pi^3}\right)^{1/2}$,$c_n = \Gamma(n+\frac{3}{2})$ and $g = 3\alpha = 3|\lambda|$. The Borel transform of $E_0(g)$ is

$$BE_0(t) = C\sum_{n=0}^{\infty} \frac{c_n}{n!}t^n = C\sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)}{\Gamma(n+1)}t^n \tag{7.8}$$

The power series can be written as a function of t. With the help of Mathematica, we find

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)}{\Gamma(n+1)}t^n = \frac{\sqrt{\pi}}{2(1-t)^{\frac{3}{2}}}. \tag{7.9}$$

This function has a branch cut on the interval $(1,\infty)$. To transform back, we have to apply the Laplace transform, to the Borel transformed function:

$$LBE_0(g) = \int_0^{\infty} \mathrm{d}\,t\, e^{-t/g} BE_0(t) \tag{7.10}$$

When performing this integral, we have to avoid the branch cut. So we perform a lateral Borel transformation

$$S_{\theta^{\pm}} BE_0(g) = \int_0^{e^{i\theta(\infty \pm i\varepsilon)}} \mathrm{d}\,t\, e^{-t/g} BE_0(t) \tag{7.11}$$

So we have an ambiguity in θ. Because of the branch cut, the lateral summations will differ. We find that

$$S_{\theta^{\pm}}\left[\frac{1}{(1-t)^{\frac{3}{2}}}\right](g) = -2 + 2e^{-1/g}\sqrt{\frac{\pi}{g}}\left(\pm i + \operatorname{erfi}\left(\sqrt{1/g}\right)\right), \tag{7.12}$$

where erfi is an imaginary error function, defined by $\operatorname{erfi}(z) = -i\operatorname{erfi}(iz)$. If we take the difference of these two sums, we get:

$$S_{\theta^{+}}\left[\frac{1}{(1-t)^{\frac{3}{2}}}\right](g) - S_{\theta^{-}}\left[\frac{1}{(1-t)^{\frac{3}{2}}}\right](g) = 4ie^{-1/g}\sqrt{\pi/g} \tag{7.13}$$

So we see that

$$S_{\theta^+}[BE_0](g) - S_{\theta^-}[BE_0](g) = 2i\sqrt{\frac{6}{\pi}}e^{-1/g}\sqrt{1/g}. \tag{7.14}$$

Now we can plug back our parameter $\lambda = -g/3$. We get

$$S_{\theta^+}[BE_0](g) - S_{\theta^-}[BE_0](g) = 2i\sqrt{\frac{2}{\pi}}e^{1/3\lambda}\sqrt{-1/\lambda}. \tag{7.15}$$

We have found that after the transformation from the Borel plane, we get an imaginary part of a ground state energy which was not there before. The existence of the imaginary part is telling us that the ground state is not a real one, it's an unstable vacuum, and it will tunnel with the characteristic time $\tau \sim \frac{1}{\operatorname{Im} E_0}$. To sum up, we have started with an asymptotic power series perturbative expansion of a ground state. With the use of resurgence, we have discovered an imaginary part of the ground state energy, which is a non-perturbative (instantonic) contribution. So the information about the vacuum not being a real one, was hidden all along in the perturbation expansion, and was just waiting to be discovered.

This again seems a bit miraculous. Let us do the same procedure as before, for the 0-dimensional case. We want to describe the ground state energy of our system, using the perturbative approach

$$E(g) = \sum a_k g^k, \quad a_0 = \frac{1}{2}. \tag{7.16}$$

It is useful to introduce a function

$$f(z) = \frac{1}{z}\left(E(z) - a_0\right) = \sum_{k=0}^{\infty} f_k z^k, \quad f_k = a_{k+1}. \tag{7.17}$$

Again, since our system is unstable for $g < 0$ (tunneling), the series is going to be asymptotic again. As before, $f(z)$ has a branch cut on negative real axis and vanishes at the origin. At the large g, the scaling is $f(z) \sim z^{-2/3}$. To see this, we notice as before that for large g

$$H \approx \frac{1}{2}p^2 + \frac{g}{4}x^4 \tag{7.18}$$

Under rescaling $x \to g^{-1/6}x$, we get

$$H \to g^{1/3}\left(\frac{1}{2}p^2 + \frac{1}{4}x^4\right) \tag{7.19}$$

So, the ground state energy will scale as

$$E(g) \sim Cg^{1/3}, \tag{7.20}$$

where C is the ground state energy of the anharmonic oscillator with $m = 0$ and $g = 1$.

Let us now argue as in the last chapter. Let us take the same contour deformation as before, so that we find again

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}\,x \frac{\operatorname{Im} f(x)}{x-z} \tag{7.21}$$

In terms of the ground state energy, this translates to

$$E(g) = a_0 + \frac{g}{\pi}\int_{-\infty}^{0} \mathrm{d}\,g' \frac{\operatorname{Im} E(g')}{g'(g'-g)} \tag{7.22}$$

After similar treatment as in the previous chapter, we find the expression for perturbation terms

$$a_k = \frac{(-1)^{k+1}}{\pi}\int_{0}^{\infty} \mathrm{d}\,z \frac{\operatorname{Im} E(-z)}{z^{k+1}} \tag{7.23}$$

We have obtained the $\operatorname{Im} E(g)$ in the introduction of this work through instanton calculus, so at this point we could straight plug it in, and obtain

$$a_k \sim (-1)^{k+1}\frac{\sqrt{6}}{\pi^{3/2}}\left(\frac{3}{4}\right)^k \Gamma\left(k+\frac{1}{2}\right). \tag{7.24}$$

7.2 Large N limit

Now we take a Hamiltonian

$$H = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 - \lambda(\phi^2)^2, \tag{7.25}$$

Where $\phi = (\phi_1, \dots, \phi_N)$. To treat this in large N limit, since we know that one dimensional QFT is equivalent to Quantum mechanics. We could do the perturbation in the manner of QFT, but in this case it is better to proceed with Schrödinger equation

$$\left[-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}\,r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}\,r}\right) + \frac{1}{2}r^2 + \frac{g}{N}r^4 - E\right]\psi = 0. \tag{7.26}$$

Now, we are able to calculate the asymptotics of the perturbation series for the ground state energy [11]. We get

$$E = \sum_{k=0}^{\infty} a_k \left(\frac{\lambda}{N}\right)^k, \quad a_k \sim -\frac{1}{\pi}\frac{6^{N/2}}{\Gamma(N/2)}\Gamma(k+\frac{N}{2})3^k. \tag{7.27}$$

Again, we perform a Borel transform. When we take the difference of the lateral sums we obtain:

$$\mathrm{Im}(E) = \frac{1}{\pi} 2^{1+N/2} 3^{N/2} e^{-x} \pi \frac{x^{N/2}}{(1 - N/2)!} \tag{7.28}$$

8 Large N $\lambda\phi^4$ in 4 dimensions

Here we continue the treatment we have done in the introduction for a $1/N$ expansion for any potential of scalar field theory. At the end we studied the case with

$$U(\rho) = \frac{1}{2}r\rho + \frac{u}{4!}\rho^2, \tag{8.1}$$

Which corresponds to the action

$$S(\phi) = \int \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}r\phi^2 + \frac{1}{4!}\frac{u}{N}\left(\phi^2\right)^2\right] \mathrm{d}^d\, x, \tag{8.2}$$

under the definition, $\rho = \frac{\phi^2}{N}$. It can be shown that for a coupling constant $g = u/N$, this theory's β function is

$$\beta(g) = \frac{N+8}{48\pi^2}g^2 \tag{8.3}$$

We rewrote $\phi_1 = \sigma$, integrated out all the other components, and replaced ρ by the solution of $\frac{1}{6}u\rho + r = \lambda$. We found the action

$$S = \frac{1}{2}^d x \left[(\partial_\mu\sigma)^2 + \lambda\sigma^2 - \frac{3N}{u}\lambda^2 + \frac{6Nr}{u}\lambda\right] + \frac{(N-1)}{2}\operatorname{tr}\log[-\nabla^2 + \lambda]. \tag{8.4}$$

Let us rewrite the equations (5.12) with the new potential

$$m^2\sigma = 0, \tag{8.5}$$

$$\frac{1}{2}m^2 = \frac{1}{2}r + \frac{u}{6}\rho, \tag{8.6}$$

$$\sigma^2/N - \rho + \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d\, k}{k^2+m^2} = 0. \tag{8.7}$$

As mentioned before, this theory is particularly simplified, since we have only the kinetic term for the field σ, and the coupling between σ and λ is equal to 1. The leading contribution diagrams in the perturbative treatment will be of the form (Insert the picture of the fish diagrams) To study the theory we need the propagators. Let us compute the propagator of the fieldλ, Δ_λ. We need to differentiate the action twice with respect to λ,

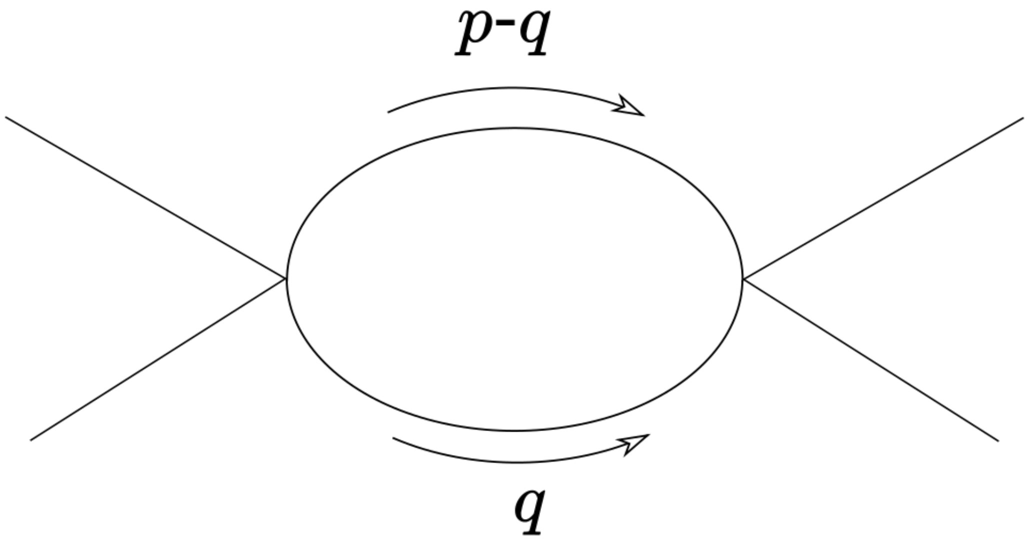

Figure 8.1: A diagram corresponding to $B_\Lambda(p,m)$

and we get

$$\Delta_\lambda(p) = -\frac{2}{N}\left[\frac{6}{u} + B_\Lambda(p,m)\right]^{-1}, \tag{8.8}$$

where $B_\Lambda(p,m)$, depicted in Figure 8.1, is defined as

$$B_\Lambda(p,m) = \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d\, q}{(q^2+m^2)((p-q)^2+m^2)} \tag{8.9}$$

Using the relation between λ, ϕ and ρ, we get

$$\langle \phi^2\phi^2\rangle = N^2\langle\rho\rho\rangle = -\frac{12N/u}{1+(u/6)B_\Lambda(p,m)}. \tag{8.10}$$

In order to proceed, we renormalize first the coupling constant g, which is the value of the vertex $\langle\sigma\sigma\sigma\sigma\rangle$ for momenta $p \ll \Lambda$. For the sake of simplicity, we will study the massless theory. We find

$$g_r = \frac{g}{1+\frac{1}{6}NgB_\Lambda(\mu,0)} \tag{8.11}$$

We will later be evaluating the n-point functions at low momenta, it is thus of interest to know the behavior of $B_\Lambda(p,0)$ at $p \ll \Lambda$. Also, we choose to work in 4 dimensions

$$\begin{aligned}
B_\Lambda(p,0) &= \frac{1}{(2\pi)^4}\int^\Lambda \frac{\mathrm{d}^4 q}{q^2(p-q)^2} && (8.12)\\
&= \frac{(2\pi)^2}{\Gamma(2)(2\pi)^4}\int^\Lambda \frac{q^3\,\mathrm{d}\,q}{q^2(p-q)^2} && (8.13)\\
&= \frac{1}{8\pi^2}\int^\Lambda \frac{\mathrm{d}\,q}{q(1-p/q)^2}, \quad \text{where } p \ll q && (8.14)\\
&= \frac{1}{8\pi^2}\left[\int_0^p \frac{\mathrm{d}\,q}{q(1-p/q)^2} + \int_p^\Lambda \frac{\mathrm{d}\,q}{q}\left(1+2\frac{p}{q}\right)\right] && (8.15)\\
&\sim \frac{1}{8\pi^2}\log\left(\frac{\Lambda}{p}\right) + \text{const} && (8.16)
\end{aligned}$$

So for the ratio $\mu/\Lambda \to 0$, we find that $B_\Lambda(\mu,0) \to \infty$, from where it follows that $g_r \to 0$. So if we take a limit of our cut-off to infinity, we will find a theory where the coupling is 0, so this theory is going to be free. This property is called triviality. It seems impossible to construct a ϕ^4 theory in 4 dimensions which has a non-vanishing coupling on all scales. Of course, if we look at this problem from the perspective of the effective field theory, there is no longer an issue; up to some scale we have a theory which has some non-vanishing coupling constant.

Let us for now work with the theory with $g > 0$. We are able to express the leading contribution to the 4-point function

$$\frac{g}{1+\frac{N}{48\pi^2}g\ln(\Lambda/p)} = \frac{g_r}{1+\frac{N}{48\pi^2}g\ln(\mu/p)}. \tag{8.17}$$

The last expression has a pole at $p = \mu e^{48\pi^2/(Ng_r)}$, to find this, we just have to solve equation for denominator equal to 0. This pole is called a Landau pole and is unphysical. It appears because our theory is only an effective theory. It will cause UV divergences of loop integrals even after the renormalization. For example, terms of the perturbative expansion of two-point function, corresponding to the diagram shown in Figure 8.2, in terms of g_r will be of the form

$$a_k = \int^\infty \frac{\mathrm{d}^4 q}{q^6}\left(-\frac{Ng_r}{48\pi^2}\ln(\mu/q)\right)^k \sim_{k\to\infty} \left(\frac{Ng_r}{96\pi^2}\right)^k k!, \tag{8.18}$$

We know from before that such series are asymptotic. For the positive g, the Borel pole will be generated on the positive real axis, and cause an ambiguity. In this case, we know that the asymptoticity of the perturbation series is a consequence of unphysical theory, so we may conclude that also the ambiguity is unphysical. We see that in the case when

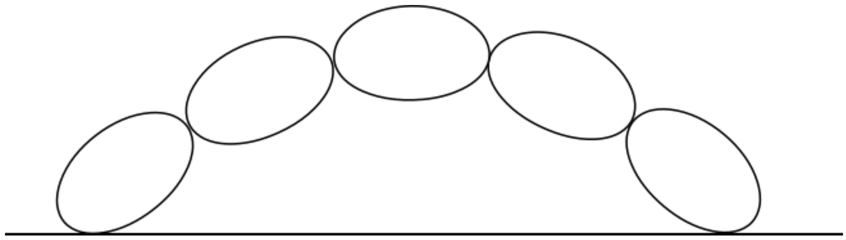

Figure 8.2: A diagram contributing to the 2-point function.

$g < 0$, the series will be alternating, which leads to the Borel pole on the negative real axis, and will lead to the Borel resummable series. Such factorial growth of a subclass of the Feynman diagrams is called a renormalon, analogously to instanton.

The problem that will be more interesting for us, is the case of IR renormalons. Those will appear only in the massless theory. Now we calculate the contribution of the small momentum range to the mass renormalization at the fixed Λ.(picture of the contributing diagrams)

We find that the diagrams will be proportional to

$$\int^{\Lambda} \frac{\mathrm{d}^4 q}{q^2(1+\frac{1}{6}NgB_\Lambda(q))} \sim \int \frac{\mathrm{d}^4 q}{q^2(1+\frac{N}{48\pi^2 g \ln(\Lambda/q)})}, \tag{8.19}$$

where we substituted for B_Λ result from (8.16). We find the expansion in powers of g, and for k large, we find

$$\sum_k a_k g^k,\ a_k \sim (-1)^k \left(\frac{N}{96\pi^2}\right)^k k! \tag{8.20}$$

It can be shown in general for a finite N that $a_k \sim (-\beta_2/2)^k k!$, where β_2 is the second order in the beta-function expansion. For $g < 0$, our series will be asymptotically free. We see that for a generic asymptotically free theory, the perturbation expansion will be non-alternating, hence, there will be a Borel pole generated on the positive real axis, so there will be an ambiguity. Such renormalon ambiguities are conjectured to be related with mass gap generation.

We still need to check if the theory with $g < 0$ can even be satisfyingly constructed. It is possible to show that the massless theory is inconsistent. If we consider a massive theory, we know that there is a metastable state, which we would like define as the vacuum. Indeed, the tunneling barrier gets stabilized by the large N, and can be shown that the decay rate scales as e^{-N} and thus stabilizes the state. But, for the finite temperature case, which is out of the scope of this work, the thermal fluctuations will cause the destabilization of the state, thus making the theory inconsistent.

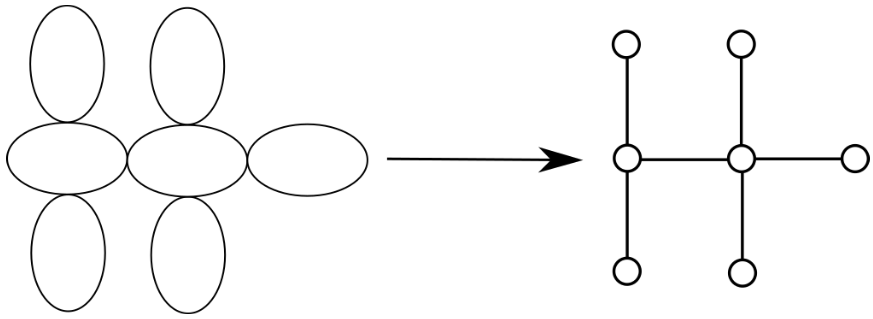

Figure 8.3: This picture demonstrates the connection between planar diagrams and tree graphs

8.1 The connection to the trees

Here, we try to calculate the ground state energy of our theory. We know that perturbatively, the result will be the sum of all the closed Feynman diagrams. Moreover, because of the large N limit, the only diagrams that will contribute at the leading order, will be diagrams like in Figure 8.3 . The leading order diagrams consist of single loops, and the diagrams should be planar. We see that in the perturbation series we will have

$$E = \sum_n a_n (uN)^n / N, \; a_n = I^n r(n). \tag{8.21}$$

Here, I is the loop integral of a single loop, and $r(n)$ is just a number of possible diagrams for a given number n of vertices. We can replace the loops in the diagrams with points, and vertices with connections between the points. What we get is a subclass of graphs, called trees. The number of possible trees has a known asymptotic behavior. But, the asymptotic form has been estimated:

$$r(n) \sim C\alpha^n n^{-5/2} \tag{8.22}$$

From this result follows that the perturbation series of E is not asymptotic. The Borel transformation of it will be merely a function akin to the exponential one that is, there will be no Borel poles. From this we can conclude that there are no instantons, as stated in above. It would be possible to approach the problem from the other direction. Suppose we find, through the instanton calculus the contribution to the $\operatorname{Im} E$. Using the result from complex analysis, assuming that we have such an expansion $E(\lambda) = \sum a_n \lambda^n$, we find for the large ns

$$a_n = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} E(\lambda) \, \mathrm{d}\, \lambda}{\lambda^{n+1}} \tag{8.23}$$

In this case we might be able to find the asymptotic behavior of our series that is one should find

$$a_n \sim_{n\to\infty} C^n n^b \tag{8.24}$$

Of course, this works only in the case when we have no other sources of Borel poles in our theory.

9 Conclusion and Outlook

We have seen that asymptotic behavior of perturbation series in quantum field theory is governed by non-perturbative contributions. Because of our naive ansatz, we are paying the price with asymptotic series. In 0 dimensions, the partition function was just an integral. We have seen that Taylor expansion yields a series that is asymptotic. We have shown that the factorial growth stems from the analytic cut in the negative real axis. Of course, for our parameter $\lambda < 0$, the integral is expected to diverge, but surprising is the fact that the divergence of integral for negative parameter can cause factorial divergence of the perturbative (Taylor) series for the positive parameter, even though we know that the integral should converge. In fact, this factorially diverging series “converges” much faster than a generic convergent series. If we stop summing the series before it starts diverging, we can achieve precision of the order $e^{-1/\lambda}$, so for small enough parameter, we can find a very satisfying precision very quickly. If we want to improve the result, we can improve convergence with Padé representation, which is out of the scope of this work.

In one dimension (quantum mechanics), we know that the theory for $\lambda < 0$ has an instanton, in other words, the ground state is meta-stable; over some time, it will tunnel. We have used the result of Bender and Wu, who computed the asymptotics of perturbation series for the ground state energy for $\lambda > 0$. Using resurgence, we were able to recover imaginary instanton contribution for the case $\lambda < 0$. Again, we are able to show that the divergence of the perturbation series comes from the analytic cut. One might use Dyson's argument that the theory for $\lambda < 0$ is not physical, since the potential is not bounded from below, hence, the perturbation series has to be divergent. Curiously, this might not be the case. Recent work of Carl Bender and others [12] show that a non-hermitian operator can have a real spectrum, bounded from below, yielding a physical theory.

In the large N limit, we have found that the asymptotics has very similar expression as in one component case, with some corrections. We notice that the Borel poles in this case start running towards infinity. This confirms the generic e^{-N} suppression of instantons in the large N limit. Issues with the large N limit in 0- and 1- dimensional cases have been reported recently [13] . Authors propose solution by treating the theory in a PT symmetric limit.

In the higher dimensional case, the situation becomes complicated rapidly. The difficulty of calculating higher order terms of perturbation theory increases to the level of impossibility. For this reason, we have turned to the large N limit, in which the theory gets simplified. We have shown that another effect causing the Borel poles arises, renormalons. It is interesting that in the large N limit instantons become suppressed, so renormalons may be the only singularities in the Borel plane. Moreover, the IR renormalons, which lie on the positive real axis, disappear when we add the mass term. At this point, it seems that our theory is well behaved, but even though instantons get suppressed, at finite temperature, which is out of the scope of this work, the theory would remain unstable. We have also shown the suppression of instantons as a growth of diagrams, which in the large N limit grow only exponentially, and not factorially.

So, what did we learn about resurgence? In quantum mechanics, it seems to work pretty well. A lot of different systems have been explored, and it has been confirmed that, indeed, one can recover the non-perturbative physics from only perturbative one. Even when the goal of the recent articles is to explore a quantum field theory, it usually ends up with obtaining an effective quantum mechanical theory for which we know it works. We have seen that in quantum field theory not everything is clear, since we have different sources of Borel poles. This makes it much harder to confidently say, which Borel poles belong to instantons and which to renormalons. At the moment it seems that the theory of resurgence brought an interesting point of view on quantum field theory, but it is not at the point where we are confident about it. Another difficulty is that we are not able to compute asymptotics of perturbation series very easily, so any progress on that matter might be very useful. Resurgence is a very dynamical field, which has already had an impact on how we understand quantum field theory [14], and is being applied to more and more models [15]. It has also found use in hydrodynamics [16]. We hope that this work will serve to provoke an interest for the field, and demonstrate the use of resurgence.

Bibliography

[1] F. J. Dyson. Divergence of perturbation theory in quantum electrodynamics. *Physical Review*, 85(4):631–632, 1952.

[2] C.M. Bender and T.T. Wu. Anharmonic Oscillator. *Physical Review*, 184(5):1231–1260, 1969.

[3] L N Lipatov. Calculation of the Gell-Mann-Low function in scalar theory with strong nonlinearity. (8):1055–1062, 1977.

[4] G. Parisi. Asymptotic estimates of Feynman diagrams. *Physics Letters B*, 68(4):361–364, jun 1977.

[5] Jean Écalle. Les fonctions résurgentes. *Publ. math. d'Orsay/Univ. de Paris, Dep. de math.*, 1981.

[6] Gerald V. Dunne and Mithat Unsal. Resurgence and Dynamics of O(N) and Grassmannian Sigma Models. (3), 2015.

[7] Gia Dvali, Cesar Gomez, and Slava Mukhanov. Classical dimensional transmutation and confinement. *Journal of High Energy Physics*, 2011(12), 2011.

[8] Jean Zinn-Justin. Quantum field theory and critical phenomena. 2002.

[9] Daniele Dorigoni. An Introduction to Resurgence, Trans-Series and Alien Calculus. page 48, 2014.

[10] Carl M.. Bender. Anharmonic Oscillator. II. A Study of Perturbation Theory in Large Order, 1973.

[11] S Hikami and E Brezin. Large-order behaviour of the 1 / N expansion in zero and one dimensions. 759.

[12] Carl M. Bender. Introduction to PT-symmetric quantum theory. *Contemporary Physics*, 46(4):277–292, 2005.

[13] Carl M Bender and Sarben Sarkar. Double-scaling limit of the o(n)-symmetric anhar-

monic oscillator. *Journal of Physics A: Mathematical and Theoretical*, 46(44):442001, 2013.

[14] Gerald V Dunne. What is QFT? Resurgent trans-series, Lefschetz thimbles, and new exact saddles. (July 2015).

[15] Sergei Gukov, Marcos Marino, and Pavel Putrov. Resurgence in complex Chern-Simons theory. (2):56, 2016.

[16] Gökçe Baar and Gerald V. Dunne. Hydrodynamics, resurgence, and transasymptotics. *Physical Review D - Particles, Fields, Gravitation and Cosmology*, 92(12):1–29, 2015.