Graviton Quantum Tree Graphs

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Title: Graviton Quantum Tree Graphs

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Abstract: The Schwarzschild metric is a well known solution to the Einstein field equations, when considering a spherically symmetric mass, with no charge or angular momentum.
In this thesis we consider two quantum field theory approaches for computing the vacuum expectation value of the graviton, in the presence of the source of the Schwarzschild model.
The first one is the straightforward calculation of the S-matrix element, which requires computing the 3-vertex function.
The second one avoids this inconvenient calculation, by going directly to the equations of motion and solving them iteratively. Each iteration corresponds to considering a new diagram contribution to the vacuum expectation value.

Contents

1 Introduction

Since the time of Newton, scientists have always thought of gravity as a force acting between two bodies. The acceleration of a mass was seen as the result of the action of a gravitational field on it and, on the other hand, the gravitational field was expressed as a consequence of the mass configuration present in the universe.

This has changed dramatically with the development of Einstein's theory of General Relativity, which treats gravity as the curvature of spacetime in the following sense: matter curves spacetime and a curved spacetime tells the matter how to move. This is similar to Newton's theory, in the sense that we have the double role of matter, that is both a source of gravity and is acted upon by it, but it is also fundamentally different, because gravity is no longer treated as a force and it becomes part of the very fabric of our universe.
Although Einstein's field equations look very simple, they are indeed extremely non-linear. The solution of Einstein's field equations is known only for some particular mass configurations, one of which will be discussed in this thesis.

In order to apply General Relativity to the physical world, it is useful to look at one of the simplest examples that can approximate a physical system, namely that of a perfectly static star. To make the model as simple as possible, the star is assumed to be a perfect fluid, i.e. without any shear stresses. The reasonable prediction is that the metric tensor should then be independent of time, that it should be spherically symmetric and that it should be asymptotically flat, at large distances from the star. The first part of this thesis computes precisely this solution, introducing at the same time the notation and the conventions that will be used throughout the whole work.

Also at the beginning of the 20th century, another groundbreaking theory was developed: Quantum Mechanics.
What started as a theory of a fixed amount of particles described by wave equations was then developed into a theory of fields, where particles are described by the modes of the fields and can be created or annihilated.
This is of course the Quantum Field Theory, which unites Quantum Mechanics and Special Relativity. From Quantum Mechanics it takes the quantization procedure and the whole formalism of observables, while it inherits causality from Relativity.

Quantum Field Theory has obtained many successes, the biggest of them being probably Quantum Electrodynamics, which is the best fundamental physical theory that we have at the moment. It also paved the way for the Electroweak Theory and for Quantumchromodynamics.
But gravity still can't be included in these conceptual schemes.
The focus of the second part of my work is to show, following [1], that it is indeed possible to reconstruct the Schwarzschild solution perturbatively, at least up to order G^2, using the Feynman-Dyson expansion known from Quantum Field Theory.
The usual gravitational potential of the Schwarzschild problem is recovered as the vacuum expectation value of the graviton in the presence of an appropriate external source.

The computations involved using this method are quite lengthy: among other things, the 3-point-vertex function of the graviton, which consists of 171 terms, needs to be determined.
A more compact method, which is also more easily extendable to higher orders, is the core of

the third part of this thesis. In it the Einstein field equations are derived in such a way that they can be solved iteratively to get higher orders of perturbation. This yields once more the well known result.

2 The Schwarzschild solution and mass renormalization

The first part of the thesis focuses on a detailed derivation of the Schwarzschild solution, following the usual approach of general relativity.
We introduce the most generic spherically symmetric metric tensor in both spherical and cartesian coordinates. This ansatz contains unknown functions that can be determined by considering the 00-component of the Einstein field equations.
The method of curvature 2-forms proves to be the quickest to compute the 00-component of the Einstein tensor.

Then we can make a particular choice for the energy-momentum tensor, corresponding to a spherically symmetric mass density of radius $r = \varepsilon$.
At this point we can define two different radii that will turn out to be relevant for the final solution: one is obtained from the length of a circumference on the equatorial plane and is denoted by ε_c and the other is the proper invariant radial distance, which will be called ε_r.

As the concept of mass is not well defined in general relativity, we need to be careful with it. The bare mass m_0 is defined as the integral along the proper radius of the mass density, while the renormalized mass m is just the mass density, that we are assuming to be constant, multiplied by the volume of a sphere of radius ε_c.
The problem with the bare mass m_0 is that then the gravitational potential expressed in these terms is divergent for $\varepsilon \to 0$. Introducing the renormalized mass m, which is indeed what an observer would measure using Kepler's third law, cures the divergency.
The result obtained in this way does not depend on whether the ansatz for the energy-momentum tensor has vanishing pressure or not.

In order to determine the last unfixed parts of the metric, we need to choose a gauge. By imposing the de Donder gauge condition we finally arrive at the Schwarzschild solution and can expand it in orders of Newton's constant G for later comparison.

We will use the $\begin{pmatrix} - & + & + & + \end{pmatrix}$ convention for the Minkowski metric and the following index labels for the metric tensor in spherical coordinates: $0 = t$, $1 = r$, $2 = \theta$ and $3 = \varphi$.

2.1 Generic classical solution

2.1.1 The generic spherically symmetric metric tensor

The goal of this first section is to find a generic classical solution to Einstein's equations

$$G_\mu{}^\nu = -\frac{1}{2}\kappa^2 T_\mu{}^\nu, \tag{2.1}$$

where $\kappa^2 = 16\pi G$.

We can make the following ansatz for a generic spherically symmetric line element

$$ds^2 = F^2 \mathrm{d}r^2 + H^2 \mathrm{d}\Omega^2 - N^2 \mathrm{d}t^2, \tag{2.2}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$.

We can rewrite it in rectangular coordinates using the following relations:

$$\mathrm{d}r^2 = \left(x^2+y^2+z^2\right)\left(x\mathrm{d}x+y\mathrm{d}y+z\mathrm{d}z\right)^2 \tag{2.3}$$

$$\begin{aligned} \mathrm{d}\theta^2 \;=\; & \frac{1}{r^4}\frac{1}{x^2+y^2}\Big[z^2x^2\mathrm{d}x^2+z^2y^2\mathrm{d}y^2+\left(x^2+y^2\right)^2\mathrm{d}z^2 \\ & +z^2xydxdy+z^2xydydx-zx\left(x^2+y^2\right)dxdz \\ & -zx\left(x^2+y^2\right)\mathrm{d}z\mathrm{d}x-zy\left(x^2+y^2\right)\mathrm{d}y\mathrm{d}z-zy\left(x^2+y^2\right)\mathrm{d}z\mathrm{d}y\Big] \end{aligned} \tag{2.4}$$

$$\sin^2\theta\mathrm{d}\varphi^2 = \frac{1}{\left(x^2+y^2\right)r^2}\left(y^2\mathrm{d}x^2+x^2\mathrm{d}y^2-xy\mathrm{d}x\mathrm{d}y-xy\mathrm{d}y\mathrm{d}x\right). \tag{2.5}$$

This yields an expression for the line element ds^2 with respect to the cartesian coordinates x, y and z.
Then the spatial components of the metric can be immediately read off:

$$g_{xx} = \frac{H^2}{r^2}\eta_{xx}+\left(F^2-\frac{H^2}{r^2}\right)\frac{x^2}{r^2} \tag{2.6}$$

$$g_{yy} = \frac{H^2}{r^2}\eta_{yy}+\left(F^2-\frac{H^2}{r^2}\right)\frac{y^2}{r^2} \tag{2.7}$$

$$g_{zz} = \frac{H^2}{r^2}\eta_{zz}+\left(F^2-\frac{H^2}{r^2}\right)\frac{z^2}{r^2} \tag{2.8}$$

$$g_{xy}=g_{yx} = \frac{xy}{r^2}\left(F^2-\frac{H^2}{r^2}\right) \tag{2.9}$$

$$g_{xz}=g_{zx} = \frac{xz}{r^2}\left(F^2-\frac{H^2}{r^2}\right) \tag{2.10}$$

$$g_{yz}=g_{zy} = \frac{zy}{r^2}\left(F^2-\frac{H^2}{r^2}\right) \tag{2.11}$$

These can be written in the more compact way

$$g_{ij} = \frac{H^2}{r^2}\eta_{ij}+\left(F^2-\frac{H^2}{r^2}\right)\frac{x_ix_j}{r^2}, \tag{2.12}$$

where i and j indicate the cartesian coordinates x, y and z in this case.

The 00-component of the metric can be read off immediately: $g_{00}=-N^2$.

2.1.2 The Einstein tensor

Our goal is now to use this generic form of the metric to compute the 00-component of the Einstein tensor.
This allows us to write down the 00-component of the Einstein field equations and then solve it to obtain $g^{\mu\nu}$.
To this purpose we use the method of curvature 2-forms.
Let us briefly outline the idea of this method, before starting the explicit computations.

Given a linear connection ∇, its corresponding curvature tensor is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{2.13}$$

for some vector fields $X, Y, Z \in \Gamma(TM)$.
Let $\{e_\mu\}$ be a coordinate frame. We usually represent the curvature tensor with the curvature matrix $R^\mu{}_\nu$, which satisfies

$$R(X,Y)e_\mu = R^\nu{}_\mu e_\nu. \tag{2.14}$$

The curvature matrix is the usual Ricci tensor in which we are interested.

After defining $\omega^\mu{}_\nu$ as the matrix of one-forms satisfying $\nabla_X e_\nu = \omega^\mu{}_\nu(X)e_\mu$, Cartan's equation tells us that

$$R^\mu{}_\nu = \mathrm{d}\omega^\mu{}_\nu + \omega^\mu{}_\gamma \wedge \omega^\gamma{}_\nu. \tag{2.15}$$

Thus all we need to do, in order to compute the Ricci tensor, is determine $\omega^\mu{}_\nu$.
This can be done in the following way.
We consider the $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$-tensor $\mathrm{d}\mathcal{P} = e_\mu\omega^\mu$, where $\{\omega^\mu\}$ is the basis of one-forms, corresponding to $\{e_\mu\}$. Since we know that $\mathrm{d}^2\mathcal{P} = 0$, we immediately have

$$\begin{aligned} 0 &= \mathrm{d}e_\mu \wedge \omega^\mu + e_\mu \mathrm{d}\omega^\mu \\ &= e_\mu\left(\omega^\mu{}_\nu \wedge \omega^\nu + \mathrm{d}\omega^\mu\right), \end{aligned}$$

where we have used $\mathrm{d}e_\nu = \omega^\mu{}_\nu e_\nu$.
This means that the condition needed to determine $\omega^\mu{}_\nu$ is

$$\omega^\mu{}_\nu \wedge \omega^\nu + \mathrm{d}\omega^\mu = 0. \tag{2.16}$$

At this point an appropriate choice of the one-forms ω^μ is necessary, to make the computations as easy as possible.

The line element in spherical coordinates can be explicitly written as

$$ds^2 = F^2\mathrm{d}r^2 + H^2\mathrm{d}\theta^2 + H^2\sin^2(\theta)\,\mathrm{d}\varphi^2 - N^2\mathrm{d}t^2. \tag{2.17}$$

We can then define the 1-forms:

$$\begin{aligned} \omega^t &= N\mathrm{d}t \\ \omega^r &= F\mathrm{d}r \\ \omega^\theta &= H\mathrm{d}\theta \\ \omega^\varphi &= H\sin\theta\mathrm{d}\varphi \end{aligned} \tag{2.18}$$

With this notation:

$$ds^2 = -\left(\omega^t\right)^2 + \left(\omega^r\right)^2 + \left(\omega^\theta\right)^2 + \left(\omega^\varphi\right)^2 \tag{2.19}$$

Recalling that the exterior derivative can be computed by

$$\mathrm{d}\left(\sum_J{}'\omega_J \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k}\right) = \sum_J{}'\sum_i \frac{\partial \omega_J}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^{j_1} \wedge \ \ldots \ \wedge \mathrm{d}x^{j_k}, \tag{2.20}$$

we can write down the following two-forms:

$$\begin{aligned}
\mathrm{d}\omega^t &= \tfrac{N'}{NF}\omega^r \wedge \omega^t \\
\mathrm{d}\omega^r &= \tfrac{\dot{F}}{NR}\omega^t \wedge \omega^r \\
\mathrm{d}\omega^\theta &= \tfrac{\dot{H}}{HN}\omega^t \wedge \omega^\theta + \tfrac{H'}{HF}\omega^r \wedge \omega^\theta \\
\mathrm{d}\omega^\varphi &= \tfrac{\dot{H}}{HN}\omega^t \wedge \omega^\varphi + \tfrac{H'}{HF}\omega^r \wedge \omega^\varphi + \tfrac{1}{H}\cot\theta\omega^\theta \wedge \omega^\varphi.
\end{aligned} \tag{2.21}$$

We can then find the connection forms $\omega_{\mu\nu}$ satisfying (2.16) by choosing

$$\omega_{\mu\nu} = \frac{1}{2}\left(c_{\mu\nu\alpha} + c_{\mu\alpha\nu} - c_{\nu\alpha\mu}\right)\omega^\alpha. \tag{2.22}$$

${c_{\mu\nu}}^\alpha$ are the commutation coefficients of the basis and are defined by $[e_\alpha, e_\beta] = {c_{\alpha\beta}}^\gamma e_\gamma$.
Note that this implies that ${c_{\alpha\beta}}^\gamma = -{c_{\beta\alpha}}^\gamma$.
$c_{\alpha\beta\gamma}$ and ${c_{\alpha\beta}}^\gamma$ are related by: $c_{\alpha\beta\gamma} = g_{\gamma\mu}{c_{\alpha\beta}}^\mu$.

Using the definition of the commutation relation coefficients and $\langle \mathrm{d}\alpha, u \wedge v\rangle = \partial_u \langle \alpha, u\rangle - \partial_v \langle \alpha, u\rangle - \langle \alpha, [u, v]\rangle$ for a one-form α and two vectors u and v, one can show that

$$d\omega^\alpha = -{c_{|\mu\ \nu|}}^\alpha \omega^\mu \wedge \omega^\nu. \tag{2.23}$$

From this we see that $\omega_{\mu\nu}$ defined as in (2.22) satisfies $\mathrm{d}\omega^\mu + {\omega^\mu}_\nu \wedge \omega^\nu = 0$.

We can use (2.23) to read the non-vanishing ${c_{\mu\nu}}^\alpha$'s directly off of (2.21):

$${c_{tr}}^t = \tfrac{N'}{NF} \qquad {c_{tr}}^r = -\tfrac{\dot{F}}{FN} \qquad {c_{t\theta}}^\theta = -\tfrac{\dot{H}}{HN} \qquad {c_{r\theta}}^\theta = -\tfrac{H'}{HN}$$

$${c_{r\varphi}}^\varphi = -\tfrac{H'}{HF} \qquad {c_{\theta\varphi}}^\varphi = -\tfrac{1}{H}\cot(\theta)$$

With these we can calculate $\omega_{\mu\nu}$:

$$\begin{aligned}
\omega_{tr} &= -\tfrac{N'}{NF}\omega^t - \tfrac{\dot{F}}{FN}\omega^r \\
\omega_{t\theta} &= -\tfrac{\dot{H}}{HN}\omega^\theta \\
\omega_{t\varphi} &= -\tfrac{\dot{H}}{HN}\omega^\varphi \\
\omega_{r\theta} &= -\tfrac{H'}{HF}\omega^\theta \\
\omega_{r\varphi} &= -\tfrac{H'}{HF}\omega^\varphi \\
\omega_{\theta\varphi} &= -\tfrac{1}{H}\cot(\theta)\,\omega^\varphi
\end{aligned} \tag{2.24}$$

From the definition of the Einstein tensor via the double dual of the Riemann tensor $G^{\beta}{}_{\delta} = \mathfrak{R}^{\mu\beta}{}_{\mu\delta} = \frac{1}{2}\epsilon^{\mu\beta\mu_1\nu_1} R_{\mu_1\nu_1}{}^{\mu_2\nu_2} \frac{1}{2}\epsilon_{\mu_2\nu_2\mu\delta}$ it is easy to see that $G_0{}^0 = -\left(R^{12}{}_{12} + R^{23}{}_{23} + R^{31}{}_{31}\right)$.

In order to proceed, we must compute these components of the Riemann tensor. They are obtained from $R^{\mu\nu} = R^{\mu\nu}{}_{|\alpha\beta|}\omega^{\alpha} \wedge \omega^{\beta}$ and $R^{\mu}{}_{\nu} = \mathrm{d}\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\alpha} \wedge \omega^{\alpha}{}_{\nu}$.
The result is

$$R^{r\theta}{}_{r\theta} = \frac{1}{FH}\left(\frac{\dot{F}\dot{H}}{N^2} - \frac{H''}{F} + \frac{H'F'}{F^2}\right) \tag{2.25}$$

$$R^{\theta\varphi}{}_{\theta\varphi} = \frac{1}{H^2} + \frac{\dot{H}^2}{H^2N^2} - \frac{H'^2}{H^2F^2} \tag{2.26}$$

$$R^{r\varphi}{}_{r\varphi} = -\frac{1}{HF}\left[\frac{H''}{F} - \frac{H'F'}{F^2} - \frac{\dot{F}\dot{H}}{N^2}\right], \tag{2.27}$$

which means that for $G_0{}^0$ we can write

$$\begin{aligned} G_0{}^0 &= -\left(\frac{1}{FH}\left(\frac{\dot{F}\dot{H}}{N^2} - \frac{H''}{F} + \frac{H'F'}{F^2}\right) + \frac{1}{H^2}\right. \\ &\quad \left. + \frac{\dot{H}^2}{H^2N^2} - \frac{H'^2}{H^2F^2} - \frac{1}{FH}\left(\frac{H''}{F} - \frac{H'F'}{F^2} - \frac{\dot{F}\dot{H}}{N^2}\right)\right). \end{aligned} \tag{2.28}$$

We are interested in the solution corresponding to a spherically symmetric mass density. This solution wil be in general time dependent, since the mass might be expanding and then recollapsing. At the turning point, however, we expect the metric to be locally static, i.e. $\dot{F} = \dot{H} = 0$. This will become true at all times once we assume a static mass distribution by allowing a non-vanishing pressure.
Under these conditions the final result for the 00-component of the Einstein tensor is:

$$G_0{}^0 = -\left(\frac{1}{H^2} - \frac{H'^2}{H^2F^2} - 2\frac{H''}{HF^2} + 2\frac{H'F'}{F^3H}\right) \tag{2.29}$$

2.1.3 Energy-momentum tensor and mass renormalization

Now we can pick an explicit energy-momentum tensor. We know: $T^{\mu}{}_{\nu} = \mu u^{\mu} u_{\nu}$.
At the turning point we have: $u^{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$.
Thus from $g_{\mu\nu}u^{\mu}u^{\nu} = -1$ follows: $u^0 u_0 = -1$ and

$$-T^0_{\ 0} = \mu(r) = \rho\theta(\varepsilon - r), \tag{2.30}$$

where ρ is the density of the cloud of dust and θ is as usual $\theta(\varepsilon - r) = \begin{cases} 1 & \varepsilon - r \geq 0 \\ 0 & \varepsilon - r < 0 \end{cases}$.

Inserting this in (2.1) with the help of (2.29), we get

$$\frac{2}{H^2H'}\left[H\left(1 - \frac{H'^2}{F^2}\right)\right]' = 16\pi G\rho\theta(\varepsilon - r). \tag{2.31}$$

After defining $K = \frac{H'}{F}$, we can rewrite the equation as

$$\left[H\left(1-K^2\right)\right]' = \frac{1}{2}H^2H'16\pi G\rho\theta\left(\varepsilon - r\right). \tag{2.32}$$

We can now integrate over r and get two solutions K_- and K_+, inside and outside of the dust cloud respectively.

Inside of the dust cloud, i.e. for $x < \varepsilon$, we get

$$H\left(x\right)\left(1-K_-^2\left(x\right)\right) - H\left(0\right)\left(1-K_-^2\left(0\right)\right) = \int_0^x H^2H'8\pi G\rho dr, \tag{2.33}$$

which means, since $H\left(0\right)=0$,

$$K_-^2\left(r\right) = 1 - \frac{8\pi G\rho}{3}H^2\left(r\right) = 1 - \frac{H^2}{R^2}\left(r\right) \tag{2.34}$$

with $R^2 = \frac{3}{8\pi G\rho}$.

Outside of the dust cloud, i.e. for $x > \varepsilon$,

$$H\left(x\right)\left(1-K_+^2\left(x\right)\right) = \int_0^\varepsilon H^2H'8\pi G\rho dr = \frac{8\pi G\rho}{3}H^3\left(\varepsilon\right), \tag{2.35}$$

which means

$$K_+^2\left(r\right) = 1 - \frac{8\pi G\rho}{3}H^3\left(\varepsilon\right)\frac{1}{H\left(r\right)} = 1 - \frac{2Gm}{H\left(r\right)} \tag{2.36}$$

with $m = \frac{4\pi\rho}{3}H^3\left(\varepsilon\right)$.

We can now define two different radii:

1. The first one is derived from the invariant circumference

$$\int_\gamma ds = \int_0^{2\pi}\sqrt{g_{\varphi\varphi}}d\varphi = \int_0^{2\pi}H\left(\varepsilon\right)d\varphi = 2\pi H\left(\varepsilon\right). \tag{2.37}$$

   This is the path length of a maximal cicumference, i.e. one with $\theta = \frac{\pi}{2}$.
   We see that $H\left(\varepsilon\right)$ is the radius of this invariant circumference.
   We denote it by

$$\varepsilon_c = H\left(\varepsilon\right). \tag{2.38}$$

2. The second radius that we can define is the invariant one. It is defined as the distance from the origin to a point at $r=\varepsilon$. Since the path length is $\mathrm{d}s = \sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} = \sqrt{g_{rr}}\mathrm{d}r = F\left(r\right)\mathrm{d}r$ in this case, the invariant radius can be computed from

$$\varepsilon_r = \int_0^\varepsilon F\left(r\right)dr. \tag{2.39}$$

   Of course one should be careful and take the interior solution $F_-\left(r\right)$ for F, since we are integrating inside the dust cloud.

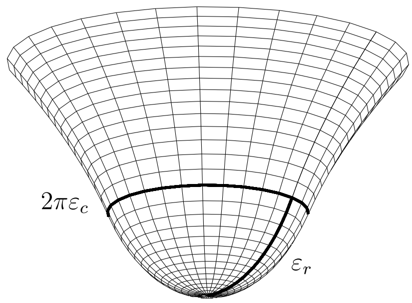

Figure 1: An embedding diagram of the equatorial plane of a spherically symmetric star with constant mass density.
Represented are also the invariant circumference of length $2\pi\varepsilon_c$ and the invariant radius of length ε_r.

These two definitions of radii are represented in Figure 1.

From $F_- = \frac{H'}{K_-}$ we have

$$F_- = \frac{RH'}{\left(R^2 - H^2\right)^{\frac{1}{2}}}. \tag{2.40}$$

The invariant radius is then

$$\begin{aligned} \varepsilon_r &= \int_0^{H(\varepsilon)} \frac{1}{\left(1 - \frac{H^2}{R^2}\right)^{\frac{1}{2}}} dH & (2.41) \\ &= R \arcsin\left(\frac{\varepsilon_c}{R}\right) & (2.42) \end{aligned}$$

with $\varepsilon_c = H\left(\varepsilon\right)$.

Let us now define ${}^3g$ to be the determinant of the spatial part of the metric. Then after long computations using (2.12) we get

$${}^3g = \det\left[\frac{H^2}{r^2}\eta_{ij} + \left(F^2 - \frac{H^2}{r^2}\right)\frac{x^i x^j}{r^2}\right] = \left(\frac{FH^2}{r^2}\right)^2. \tag{2.43}$$

Analogous to the two definitions of radii, we can consider two different masses as well.

On the one hand we can define the bare mass, which is simply the volume integral over the mass density

$$m_0 = \int_0^{\varepsilon} \left({}^3g\right)^{\frac{1}{2}} d^3r. \tag{2.44}$$

We define the renormalized mass as the integral over the radius obtained from the invariant circumferences, i.e. H, of the mass density:

$$m = \int_0^{H(\varepsilon)} \rho d^3 H. \tag{2.45}$$

There are various reasons why we call this the renormalized mass. This is the mass that we would measure if we were to use Kepler's third law. The adjective renormalized refers to the fact that it allows us to write the gravitational potential in a form that is no longer divergent for $\varepsilon \to 0$.

Let us now compute the bare mass explicitly:

$$\begin{aligned} m_0 &= \int \mu \frac{d^3 H}{K} \\ &= 4\pi \int_0^{\varepsilon_c} \mu H^2 \frac{dH}{K} \\ &= 4\pi\rho \int_0^{\varepsilon_c} \frac{H^2 dH}{\left(1 - \frac{H^2}{R^2}\right)^{\frac{1}{2}}} \\ &= 2\pi\rho R^3 \left[\arcsin\left(\frac{\varepsilon_c}{R}\right) - \frac{\varepsilon_c}{R}\left(1 - \frac{\varepsilon_C^2}{R^2}\right)^{\frac{1}{2}}\right], \end{aligned}$$

where we have used $\int \frac{x^2 dx}{(1-x^2)^{\frac{1}{2}}} = \frac{1}{2}\left(\arcsin(x) - x\sqrt{1-x^2}\right)$.

Using the Taylor expansions of $\arcsin\left(\frac{\varepsilon_c}{R}\right)$

$$\arcsin\left(\frac{\varepsilon_c}{R}\right) = \frac{\varepsilon_c}{R} + \frac{1}{6}\frac{\varepsilon_c^3}{R^3} + \frac{3}{40}\frac{\varepsilon_c^5}{R^5} + \mathcal{O}\left(\varepsilon_c^7\right) \tag{2.46}$$

and of $\left(1 - \frac{\varepsilon_c^2}{R^2}\right)^{\frac{1}{2}}$

$$\left(1 - \frac{\varepsilon_c^2}{R^2}\right)^{\frac{1}{2}} = 1 - \frac{\varepsilon_c^2}{2R^2} - \frac{\varepsilon_c^4}{8R^4} + \mathcal{O}\left(\varepsilon_c^6\right), \tag{2.47}$$

it is possible to write down a polynomial expression for m_0:

$$m_0 = \frac{4}{3}\pi\rho\varepsilon_c^3 \left[1 + \frac{3}{10}\frac{\varepsilon_c^2}{R^2} + \mathcal{O}\left(\varepsilon_c^2\right)\right]. \tag{2.48}$$

Using $m = \frac{4}{3}\pi\rho\varepsilon_c^3$ and $R^2 = \frac{3}{8\pi G\rho}$ we can finally find a relation between m and m_0:

$$m_0 = m + \frac{3}{5}G\frac{m^2}{\varepsilon_c} + \mathcal{O}\left(G^2\right). \tag{2.49}$$

Inverting this we get the following mass renormalization formula up to order G:

$$m = m_0 - \frac{3}{5}G\frac{m_0^2}{\varepsilon_c} + \mathcal{O}\left(G^2\right). \tag{2.50}$$

2.1.4 Mass renormalization for a generic energy-momentum tensor

It is possible to generalize this result for a spherically symmetric energy-momentum tensor of the form $T^{\nu}{}_{\mu} = (\mu + p)\, u^{\mu} u_{\nu} + p\delta^{\mu}_{\nu}$.

We are of course assuming a constant mass density, i.e. $\dot{\mu} = 0$.

The mass renormalization formula remains unchanged, beacuse

$$T^{0}{}_{0} = (\mu + p)\, u^{0} u_{0} + p\delta^{0}_{0} = -\mu - p + p = -\mu. \tag{2.51}$$

2.2 Solution in the de Donder gauge

2.2.1 De Donder gauge condition

We now pick a specific gauge, in order to solve the Einstein equations completely. The one that we choose is the de Donder gauge, which can be expressed by imposing the condition

$$\left[(-g)^{\frac{1}{2}}\, g^{\mu\nu}\right]_{,\nu} = 0. \tag{2.52}$$

In our case we have $g^{0i} = 0$ and a static metric, such that

$$\left[(-g)^{\frac{1}{2}}\, g^{\mu\nu}\right]_{,\nu} = \left[(-g)^{\frac{1}{2}}\, g^{ij}\right]_{,j} = 0. \tag{2.53}$$

Using the determinant computed in (2.43) we can easily calculate the determinant for the whole metric:

$$\det(g) = -N^2 \det\left({}^3 g\right) = -\left(\frac{NFH^2}{r^2}\right)^2, \tag{2.54}$$

which means

$$(-g)^{\frac{1}{2}} = \frac{NFH^2}{r^2}. \tag{2.55}$$

Furthermore we need to compute the inverse of g_{ij}.
After some calculations we find

$$g_{ij} = \frac{r^2}{H^2}\eta_{ij} + \left(\frac{1}{F^2} - \frac{r^2}{H^2}\right)\frac{x^i x^j}{r^2}. \tag{2.56}$$

This allows us to rewrite

$$(-g)^{\frac{1}{2}}\, g^{ij} = NF\eta^{ij} + \frac{NFH^2}{r^2}\left(\frac{1}{F^2} - \frac{r^2}{H^2}\right)\frac{x^i x^j}{r^2}. \tag{2.57}$$

Using $\frac{\partial r}{\partial x^j} = \frac{x^j}{r}$ and $\frac{\partial}{\partial x^j}\frac{1}{r^4} = -\frac{4x^j}{r^3}$ we can compute the derivative with respect to a spatial coordinate of the above expression.
The derivative of the first term with respect to x^j is

$$\frac{\partial}{\partial x^j}\left(NF\eta^{ij}\right) = \frac{\partial r}{\partial x^j}\left(NF\right)'\eta^{ij} = \frac{x^i}{r}\left(NF\right)'. \tag{2.58}$$

The derivative of the second term is

$$\frac{\partial}{\partial x^j}\left[\frac{x^i x^j}{r^4}\left(\frac{NH^2}{F}-r^2NF\right)\right] = \frac{x^i}{r^3}\left(\frac{NH^2}{F}\right)' - \frac{x^i}{r^3}\left(2rNF + r^2\left(NF\right)'\right). \tag{2.59}$$

Thus, the de Donder gauge condition can be rewritten as

$$\frac{x^i}{r}\left(NF\right)' + \frac{x^i}{r^3}\left(\frac{NH^2}{F}\right)' - \frac{2x^i}{r^2}NF - \frac{x^i}{r}\left(NF\right)' = 0, \tag{2.60}$$

which reduces to the compact expression

$$\left(\frac{NH^2}{F}\right)' = 2rNF. \tag{2.61}$$

2.2.2 Solution

A solution of this differential equation outside of the dust cloud is given by

$$H_+ = r + Gm \quad , \quad F_+^2 = \frac{r+Gm}{r-Gm} \quad , \quad N_+^2 = \frac{r-Gm}{r+Gm}. \tag{2.62}$$

Indeed, if we insert this in the right hand side of (2.61) we get

$$2rN_+F_+ = \pm 2r\sqrt{\frac{(r+Gm)(r-Gm)}{(r-Gm)(r+Gm)}} = \pm 2r \tag{2.63}$$

and on the left hand side of (2.61)

$$\begin{aligned}
\frac{N_+'H_+^2}{F_+} + \frac{2N_+H_+H_+'}{F_+} - \frac{N_+H_+^2F_+'}{F_+^2} &= \frac{(r+Gm)^2}{\sqrt{\frac{r+Gm}{r-Gm}}}\frac{d}{dr}\sqrt{\frac{r-Gm}{r+Gm}} \\
&\quad +2\sqrt{\frac{r-Gm}{r+Gm}}\sqrt{\frac{r-Gm}{r+Gm}}\left(r+Gm\right) \\
&\quad -\sqrt{\frac{r-Gm}{r+Gm}}\left(r+Gm\right)^2\left(\frac{r-Gm}{r+Gm}\right)\frac{d}{dr}\sqrt{\frac{r+Gm}{r-Gm}} \\
&= \frac{2Gm}{(r+Gm)^2}\frac{(r+Gm)^2}{2\sqrt{\frac{r-Gm}{r+Gm}}}\sqrt{\frac{r-Gm}{r+Gm}} + 2\left(r-Gm\right) \\
&\quad +\frac{2Gm}{(r-Gm)^2}\frac{1}{2\sqrt{\frac{r+Gm}{r-Gm}}}\left(r+Gm\right)^2\left(\frac{r-Gm}{r+Gm}\right) \\
&= Gm + 2r - 2Gm + Gm \\
&= 2r
\end{aligned} \tag{2.64}$$

With this solution the invariant size of the sphere becomes

$$\varepsilon_c = H_+\left(\varepsilon\right) = \varepsilon + Gm. \tag{2.65}$$

We can use this result to reexpress (2.50).
Indeed we obtain

$$m = m_0 - \frac{3}{5}\frac{Gm_0^2}{\varepsilon + Gm_0}. \tag{2.66}$$

The Taylor expansion of $\frac{Gm_0^2}{\varepsilon+Gm_0}$ is

$$\frac{Gm_0^2}{\varepsilon + Gm_0} = \frac{m_0^2}{\varepsilon}G + \mathcal{O}\left(G^2\right) \tag{2.67}$$

and thus

$$m = m_0 - \frac{3}{5}\frac{Gm_0^2}{\varepsilon} + \mathcal{O}\left(G^2\right). \tag{2.68}$$

Note that, since the final result is going to be proportional to positive powers of m, it would be divergent for $\varepsilon \to 0$ if we were to express it in terms of the bare mass m_0.

We can now look at the interior solutions.
They are to lowest order

$$H_- = r + \left(\frac{3}{2}\frac{r}{\varepsilon} - \frac{1}{2}\frac{r^3}{\varepsilon^3}\right) Gm \quad , \quad F_- = 1 + \left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right) Gm \tag{2.69}$$

$$N_- = \frac{3}{2}\left(1 - \frac{Gm}{\varepsilon}\right) - \frac{1}{2}\left(1 - \frac{Gmr^2}{\varepsilon^3}\right) \quad , \quad p\left(r\right) = \frac{1}{24}\kappa^2 \rho\left(\varepsilon^2 - r^2\right)\theta\left(\varepsilon - r\right). \tag{2.70}$$

2.2.3 Final result

We can finally insert these results in the metric to obtain the Schwarzschild solution up to order G^2.

For $r > \varepsilon$ the 00-component of the metric is

$$\begin{aligned} g^{00} &= -\frac{1}{N_+^2} && (2.71)\\ &= -\frac{r+Gm}{r-Gm} && (2.72)\\ &= -1 - \frac{2Gm}{r} - \frac{2m^2}{r^2}G^2 + \mathcal{O}\left(G^3\right). && (2.73) \end{aligned}$$

The ij-component is

$$g^{ij} = \frac{r^2}{H_+^2}\eta^{ij} + \left(\frac{1}{F_+^2} - \frac{r^2}{H_+^2}\right)\frac{x^i x^j}{r^2}. \tag{2.74}$$

We can calculate the following Taylor expansions, which will allow us to write g^{ij} as a polynomial in G:

$$\frac{1}{H_+^2} = \frac{1}{\left(r + Gm\right)^2} = \frac{1}{r^2} - \frac{2Gm}{r^3} + \frac{3m^2}{r^4}G^2 + \mathcal{O}\left(G^3\right) \tag{2.75}$$

$$\frac{1}{F_+^2} = \frac{r - Gm}{r + Gm} = 1 - \frac{2mG}{r} + \frac{2m^2G^2}{r^2} + \mathcal{O}\left(G^3\right) \tag{2.76}$$

Thus we obtain

$$g^{ij} = \left(1 - \frac{2Gm}{r} + \frac{3m^2G^2}{r^2}\right)\eta^{ij} - \frac{m^2G^2}{r^4}x^ix^j + \mathcal{O}\left(G^3\right). \tag{2.77}$$

For $r < \varepsilon$ the 00-component of the metric is

$$\begin{aligned} g^{00} &= -\frac{1}{N_-^2} && (2.78)\\ &= -\frac{1}{\left[\frac{3}{2}\left(\frac{Gm}{\varepsilon} - 1\right) + \frac{1}{2}\left(1 - \frac{Gmr^2}{\varepsilon^3}\right)\right]^2 + \mathcal{O}\left(G^2\right)}. && (2.79)\end{aligned}$$

After a Taylor expansion we obtain the final result

$$\begin{aligned} g^{00} &= -\frac{1}{\left[-1 + \frac{3}{2}\frac{Gm}{\varepsilon} - \frac{1}{2}\frac{Gmr^2}{\varepsilon^3}\right]^2}\\ &= -\frac{1}{1 + Gm\left(-\frac{3}{\varepsilon} + \frac{r^2}{\varepsilon^3} + G^2m^2\left(\frac{9}{4}\frac{1}{\varepsilon^2} + \frac{1}{4}\frac{r^4}{\varepsilon^6} - \frac{3}{2}\frac{r^2}{\varepsilon^4}\right)\right)}\\ &= -1 - \frac{3Gm}{\varepsilon} + \frac{Gmr^2}{\varepsilon^3} + \mathcal{O}\left(G^2\right). && (2.80)\end{aligned}$$

The ij-component is

$$g^{ij} = \frac{r^2}{H_-^2}\eta^{ij} + \left(\frac{1}{F_-^2} - \frac{r^2}{H_-^2}\right)\frac{x^ix^j}{r^2}. \tag{2.81}$$

Again computing the analogous Taylor expansions

$$\frac{1}{H_-^2} = \frac{1}{\left(r + \left(\frac{3}{2}\frac{r}{\varepsilon} - \frac{1}{2}\frac{r^3}{\varepsilon^3}\right)Gm\right)^2} = \frac{1}{r^2} - \frac{3Gm}{\varepsilon r^2} + \frac{Gm}{\varepsilon^3} + \mathcal{O}\left(G^2\right) \tag{2.82}$$

$$\frac{1}{F_-^2} = \frac{1}{\left[1 + \left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right)Gm\right]^2} = 1 - \frac{3mG}{\varepsilon} + \frac{mGr^2}{\varepsilon^3} + \mathcal{O}\left(G^2\right) = \frac{r^2}{H_-^2} + \mathcal{O}\left(G^2\right) \tag{2.83}$$

we obtain

$$g^{ij} = \frac{r^2}{H_-^2}\eta^{ij} = \eta^{ij} - \frac{3Gm}{\varepsilon}\eta^{ij} + \frac{Gmr^2}{\varepsilon^3}\eta^{ij} + \mathcal{O}\left(G^2\right). \tag{2.84}$$

3 Field theory approach to gravity

In this section we will treat gravity using the quantum field theory approach and try to reproduce the solution for the spacetime metric obtained in section 2.
In order to do this, we will consider the field given by the metric $g^{\mu\nu}$ and compute its vacuum expectation value in the presence of a source J, that represents the mass density of section 2.

As in any field theory, we will start with a Lagrangian density. This will consist of the usual Einstein-Hilbert term, another term that ensures that we are working in the harmonic gauge, and a source term, which describes the coupling of the gravitational field to the spherical and homogeneous mass density.

Once we have written the Lagrangian, we can define $g^{\mu\nu} = \eta^{\mu\nu} + \kappa\phi^{\mu\nu}$ for some symmetric $\phi^{\mu\nu}$. This will allow us to expand the Lagrangian in orders of perturbation from the flat spacetime. With this expedient we obtain the Lagrangian written as a kinetic part plus some interaction terms. The kinetic part gives the free propagator for the field $\phi^{\mu\nu}$. The first interaction term gives us the 3-point ineteraction represented in Figure 2.
Since the computations for the 3-point interaction are already quite involved, we will not consider interactions of higher order.

A big part of the computations involves calculating the 3-point vertex function, because it requires variating the action three times.
This will be done by introducing an auxiliary field $\tilde{\phi}^{\mu\nu} = \frac{1}{\kappa}\left(\sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}\right)$ and using [2] and [3] for the actual computations. This implies some technicalities that will be explained better in this section and in the appendix C.

Once we have obtained the 3-point vertex function, we can insert everything in the expression for the vacuum expectation value, including the explicit form of the external source. After solving some integrals and going back from $\phi^{\mu\nu}$ to $g^{\mu\nu}$, we will be able to compare the result with the one obtained in section 2.

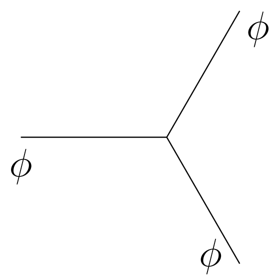

Figure 2: The three graviton interaction that will be considered for the vacuum expectation value of the gravitational field.

3.1 Lagrangian and gauge implementation

As anticipated in the introduction, we will be using the following action:

$$A = \int d^4x \left(\mathcal{L}_G + \mathcal{L}_\phi + \mathcal{L}_J\right) = A_G + A_\phi + A_J \tag{3.1}$$

$\mathcal{L}_G$ is the usual Einstein-Hilbert Lagrangian

$$\mathcal{L}_G = \frac{1}{\kappa^2} \left(-g\right)^{\frac{1}{2}} R. \tag{3.2}$$

$\mathcal{L}_\phi$ is the gauge fixing term and $\mathcal{L}_J$ describes the coupling to the external source.

In this section we will focus on expressing the Einstein-Hilbert action in a nicer way and on picking the correct gauge.
Since the source term is not important at this stage, we will temporarily leave it aside.

3.1.1 Einstein-Hilbert action

Let us first focus on the Einstein-Hilbert part of the Lagrangian.
We start by defining

$$\mathfrak{g}^{\mu\nu} = \left(-g\right)^{\frac{1}{2}} g^{\mu\nu}. \tag{3.3}$$

Then its inverse must satisfy $\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\nu\rho} = \delta^\mu{}_\rho$ and thus must be given by

$$\mathfrak{g}_{\mu\nu} = \left(-g\right)^{-\frac{1}{2}} g_{\mu\nu}. \tag{3.4}$$

Note that this implies the following relation between the determinant of $\mathfrak{g}_{\mu\nu}$ and the one of $g_{\mu\nu}$

$$\begin{aligned} \mathfrak{g} &= \det\left(\mathfrak{g}_{\alpha\beta}\right) \\ &= \det\left(\left(-g\right)^{-\frac{1}{2}} g_{\alpha\beta}\right) \\ &= \left(-g\right)^{-2} g \\ &= g^{-1}. \end{aligned} \tag{3.5}$$

We can now express the Einstein-Hilbert Lagrangian with respect to the newly defined $\mathfrak{g}^{\mu\nu}$.
To this purpose we use Goldberg's expression [4]

$$\begin{aligned} A_G &= \int d^nx \frac{1}{\kappa^2} \left(-g\right)^{\frac{1}{2}} g^{\mu\nu} R_{\mu\nu} \\ &= \frac{1}{4\kappa^2} \int d^nx \left[\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\lambda\alpha}\mathfrak{g}_{\kappa\tau}\mathfrak{g}^{\alpha\kappa}{}_{,\rho}\mathfrak{g}^{\lambda\tau}{}_{,\sigma} - \frac{1}{n-2}\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\alpha\kappa}\mathfrak{g}_{\lambda\tau}\mathfrak{g}^{\alpha\kappa}{}_{,\rho}\mathfrak{g}^{\lambda\tau}{}_{,\sigma} - 2\mathfrak{g}_{\alpha\tau}\mathfrak{g}^{\alpha\kappa}{}_{,\rho}\mathfrak{g}^{\rho\tau}{}_{,\kappa}\right] \end{aligned} \tag{3.6}$$

where n is the dimensionality of the spacetime.
More details on this relation can be found in the appendix A.

In our case, for $n = 4$, we get

$$\begin{aligned} \mathcal{L}_G &= \frac{1}{8\kappa^2}\left(2\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\lambda\alpha}\mathfrak{g}_{\kappa\tau}\mathfrak{g}^{\alpha\kappa}{}_{,\rho}\mathfrak{g}^{\lambda\tau}{}_{,\sigma} - \mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\alpha\kappa}\mathfrak{g}_{\lambda\tau}\mathfrak{g}^{\alpha\kappa}{}_{,\rho}\mathfrak{g}^{\lambda\tau}{}_{,\sigma} - 4\mathfrak{g}_{\alpha\tau}\mathfrak{g}^{\alpha\kappa}{}_{,\rho}\mathfrak{g}^{\rho\tau}{}_{,\kappa}\right) \\ &= \frac{1}{8\kappa^2}\left(2\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\lambda\iota}\mathfrak{g}_{\kappa\tau} - \mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\iota\kappa}\mathfrak{g}_{\lambda\tau} - 4\delta^\sigma{}_\kappa\delta^\rho{}_\lambda\mathfrak{g}_{\iota\tau}\right)\mathfrak{g}^{l\kappa}{}_{,\rho}\mathfrak{g}^{\lambda\tau}{}_{,\sigma}. \end{aligned} \tag{3.7}$$

We can now write $\mathfrak{g}^{\mu\nu}$ as the flat spacetime metric plus a perturbation from it,

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa\tilde{\phi}^{\mu\nu} \tag{3.8}$$

for some symmetric $\tilde{\phi}^{\mu\nu}$.

We would like to rewrite the Lagrangian through this new perturbation field $\tilde{\phi}^{\mu\nu}$.

First of all we need to express the inverse of $\mathfrak{g}^{\mu\nu}$ via $\tilde{\phi}^{\mu\nu}$ as well.
$\mathfrak{g}_{\mu\nu}$ must satisfy

$$\mathfrak{g}_{\alpha\beta}\mathfrak{g}^{\beta\gamma} = \delta_\alpha{}^\gamma. \tag{3.9}$$

We can make the ansatz $\mathfrak{g}_{\alpha\beta} = A^{(1)}_{\alpha\beta} + \kappa A^{(2)}_{\alpha\beta} + \kappa^2 A^{(3)}_{\alpha\beta} + \mathcal{O}\left(\kappa^3\right)$ and thus get the equation

$$\left(A^{(1)}_{\alpha\beta} + \kappa A^{(2)}_{\alpha\beta} + \kappa^2 A^{(3)}_{\alpha\beta} + \mathcal{O}\left(\kappa^3\right)\right)\left(\eta^{\beta\gamma} + \kappa\tilde{\phi}^{\beta\gamma}\right) = \delta_\alpha{}^\gamma. \tag{3.10}$$

To order κ^0 we obtain

$$A^{(1)}_{\alpha\beta}\eta^{\beta\gamma} = \delta_\alpha{}^\gamma, \tag{3.11}$$

i.e.

$$A^{(1)}_{\alpha\beta} = \eta_{\alpha\beta}. \tag{3.12}$$

To order κ we obtain

$$\eta_{\alpha\beta}\kappa\tilde{\phi}^{\beta\gamma} + \kappa A^{(2)}_{\alpha\beta}\eta^{\beta\gamma} = 0 \tag{3.13}$$

$$\Rightarrow \quad A^{(2)}_{\alpha\beta}\eta^{\beta\gamma} = -\eta_{\alpha\beta}\tilde{\phi}^{\beta\gamma}, \tag{3.14}$$

i.e.

$$A^{(2)}_{\alpha\lambda} = -\tilde{\phi}_{\alpha\lambda}. \tag{3.15}$$

Amd finally to order κ^2 we obtain

$$-\kappa^2\tilde{\phi}_{\alpha\beta} + \kappa^2 A^{(3)}_{\alpha\beta}\eta^{\beta\gamma} = 0 \tag{3.16}$$

$$\Rightarrow \qquad A^{(3)}_{\alpha\lambda} = \tilde{\phi}_{\alpha\beta}\tilde{\phi}^{\beta\gamma}\eta_{\gamma\lambda}. \tag{3.17}$$

Thus, up to order κ^2, $\mathfrak{g}_{\alpha\beta}$ is given by

$$\mathfrak{g}_{\alpha\beta} = \eta_{\alpha\beta} - \kappa\tilde{\phi}_{\alpha\beta} + \kappa^2\eta_{\gamma\beta}\tilde{\phi}_{\alpha\lambda}\tilde{\phi}^{\lambda\gamma} + \mathcal{O}\left(\kappa^3\right). \tag{3.18}$$

We furthermore note that $\mathfrak{g}^{\alpha\beta}{}_{,\gamma} = \kappa\tilde{\phi}^{\alpha\beta}{}_{,\gamma}$ and thus

$$\mathcal{L}_G = \frac{1}{8}\left(2\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\lambda\iota}\mathfrak{g}_{\kappa\tau} - \mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\iota\kappa}\mathfrak{g}_{\lambda\tau} - 4\delta^\sigma{}_\kappa\delta^\rho{}_\lambda\mathfrak{g}_{\iota\tau}\right)\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma}. \tag{3.19}$$

After eliminating $\mathfrak{g}^{\mu\nu}$ from this relation, we can write the Lagrangian as a sum of terms with increasing powers of κ

$$\mathcal{L}_G = \tilde{\mathcal{L}}^{(0)}_G + \kappa\tilde{\mathcal{L}}^{(1)}_G + \kappa^2\tilde{\mathcal{L}}^{(2)}_G + \ldots, \tag{3.20}$$

where the $\tilde{\mathcal{L}}^{(i)}_G$'s have no κ dependence.
Then we see that $\tilde{\mathcal{L}}^{(0)}_G$ is obtained by taking the 0-th order of both $\mathfrak{g}^{\alpha\beta}$ and $\mathfrak{g}_{\alpha\beta}$. Thus:

$$\tilde{\mathcal{L}}^{(0)}_G = \frac{1}{8}\left(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau} - \eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau} - 4\delta^\sigma{}_\kappa\delta^\rho{}_\lambda\eta_{\iota\tau}\right)\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma}. \tag{3.21}$$

In order to compute $\tilde{\mathcal{L}}_G^{(1)}$ we insert $\mathfrak{g}_{\alpha\beta} = \eta_{\alpha\beta} - \kappa\tilde{\phi}_{\alpha\beta}$ and $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \kappa\tilde{\phi}^{\alpha\beta}$ in $\mathcal{L}_G$ and then take all the terms of order κ.

$$\begin{aligned}\mathcal{L}_G &= \frac{1}{8}\Big(2\left(\eta^{\rho\sigma} + \kappa\tilde{\phi}^{\rho\sigma}\right)\left(\eta_{\lambda\iota} - \kappa\tilde{\phi}_{\lambda\iota}\right)\left(\eta_{\kappa\tau} - \kappa\tilde{\phi}_{\kappa\tau}\right) - \left(\eta^{\rho\sigma} + \kappa\tilde{\phi}^{\rho\sigma}\right) \\ &\quad \times \left(\eta_{\iota\kappa} - \kappa\tilde{\phi}_{\iota\kappa}\right)\left(\eta_{\lambda\tau} - \kappa\tilde{\phi}_{\lambda\tau}\right) - 4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\left(\eta_{\iota\tau} - \kappa\tilde{\phi}_{\iota\tau}\right)\Big)\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma}\end{aligned} \tag{3.22}$$

Then

$$\begin{aligned}\kappa\tilde{\mathcal{L}}_G^{(1)} &= \frac{1}{8}\Big(-2\eta^{\rho\sigma}\eta_{\lambda\iota}\kappa\tilde{\phi}_{\lambda\tau} - 2\eta^{\rho\sigma}\eta_{\kappa\tau}\kappa\tilde{\phi}_{\lambda\iota} + 2\kappa\tilde{\phi}^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau} + \eta^{\rho\sigma}\eta_{\iota\kappa}\kappa\tilde{\phi}_{\lambda\tau} + \eta^{\rho\sigma}\eta_{\lambda\tau}\kappa\tilde{\phi}_{\iota\kappa} \\ &\quad -\kappa\tilde{\phi}^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau} + 4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\kappa\tilde{\phi}_{\iota\tau}\Big)\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma} \\ &= \frac{\kappa}{8}\left(-2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\alpha}\eta_{\tau\beta} - 2\eta^{\rho\sigma}\eta_{\kappa\tau}\eta_{\lambda\alpha}\eta_{\iota\beta} + 2\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\eta_{\lambda\iota}\eta_{\kappa\tau} + \eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\alpha}\eta_{\beta\tau}\right. \\ &\quad \left.+\eta^{\rho\sigma}\eta_{\lambda\tau}\eta_{\iota\alpha}\eta_{\kappa\beta} - \delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\eta_{\iota\kappa}\eta_{\lambda\tau} + 4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\alpha\iota}\eta_{\tau\beta}\right)\tilde{\phi^{\alpha\beta}}\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma} \\ &= \frac{\kappa}{8}\left(-4\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\alpha}\eta_{\tau\beta} + 2\eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\alpha}\eta_{\beta\tau} + 2\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\eta_{\lambda\iota}\eta_{\kappa\tau} - \delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\eta_{\iota\kappa}\eta_{\lambda\tau}\right. \\ &\quad \left.+4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\alpha\iota}\eta_{\tau\beta}\right)\tilde{\phi}^{\alpha\beta}\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma}.\end{aligned} \tag{3.23}$$

Thus, to summarize, we can write the Lagrangian as $\mathcal{L}_G = \tilde{\mathcal{L}_G^{(0)}} + \kappa\tilde{\mathcal{L}}_G^{(1)} + \ldots$ with

$$\tilde{\mathcal{L}}_G^{(0)} = \frac{1}{8}\left(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau} - \eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau} - 4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\iota\tau}\right)\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma} \tag{3.24}$$

$$\begin{aligned}\tilde{\mathcal{L}}_G^{(1)} &= \frac{1}{8}\left(-4\eta^{\rho\sigma}\eta_{\lambda l}\eta_{\kappa\alpha}\eta_{\tau\beta} + 2\eta^{\rho\sigma}\eta_{l\kappa}\eta_{\lambda\alpha}\eta_{\tau\beta} + 2\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\eta_{\kappa\tau}\eta_{\lambda l} - \delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\eta_{l\kappa}\eta_{\lambda\tau}\right. \\ &\quad \left.+4\delta^{\rho}{}_{\lambda}\delta^{\sigma}{}_{\kappa}\eta_{l\alpha}\eta_{\tau\beta}\right)\tilde{\phi}^{\alpha\beta}\tilde{\phi}^{l\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma}.\end{aligned} \tag{3.25}$$

3.1.2 Gauge fixing

We want to work in the harmonic gauge, i.e.

$$\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0, \tag{3.26}$$

which in our case means

$$\tilde{\phi}^{\mu\nu}{}_{,\nu} = 0. \tag{3.27}$$

The equations of motion for the free field are

$$\begin{aligned}\partial_\alpha\frac{\partial\mathcal{L}_G^{(0)}}{\partial\left(\partial_\alpha\tilde{\phi}^{\beta\gamma}\right)} &= \frac{1}{8}\partial_\alpha\left\{\left(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau} - \eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau} - 4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\iota\tau}\right)\right. \\ &\quad \left.\times\left(\delta^{\alpha}{}_{\rho}I^{\iota\kappa}_{\beta\gamma}\tilde{\phi}^{\lambda\tau}{}_{\sigma} + \delta^{\alpha}{}_{\sigma}I^{\lambda\tau}_{\beta\gamma}\tilde{\phi}^{\iota\kappa}{}_{\rho}\right)\right\} \\ &= \frac{1}{4}\partial_\alpha\left\{\left(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau} - \eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau} - 4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\iota\tau}\right)\delta^{\alpha}{}_{\rho}I^{\iota\kappa}_{\beta\gamma}\tilde{\phi}^{\lambda\tau}{}_{\sigma}\right\} \\ &= \frac{1}{4}\left\{\left(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau} - \eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau} - 4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\iota\tau}\right)I^{\iota\kappa}_{\beta\gamma}\tilde{\phi}^{\lambda\tau}{}_{\sigma\rho}\right\} \\ &= \frac{1}{4}\left\{\eta_{\lambda\beta}\eta_{\gamma\tau}\Box\tilde{\phi}^{\lambda\tau} - \eta_{\beta\gamma}\eta_{\lambda\tau}\Box\tilde{\phi}^{\lambda\tau}\right\} - \eta_{\iota\tau}I^{\iota\kappa}_{\beta\gamma}\tilde{\phi}^{\lambda\tau}{}_{,\kappa\lambda} \\ &= \frac{1}{4}\left\{\eta_{\lambda\beta}\eta_{\gamma\tau}\Box\tilde{\phi}^{\lambda\tau} - \eta_{\beta\gamma}\eta_{\lambda\tau}\Box\tilde{\phi}^{\lambda\tau}\right\} - \frac{1}{2}\left(\eta_{\beta\tau}\tilde{\phi}^{\lambda\tau}{}_{,\gamma\lambda} + \eta_{\gamma\tau}\tilde{\phi}^{\lambda\tau}{}_{,\beta\lambda}\right).\end{aligned} \tag{3.28}$$

In order to get rid of the last two terms we then must introduce the following gauge fixing term

$$\mathcal{L}_\phi = \frac{1}{2\kappa^2}\eta_{\mu\nu}\mathfrak{g}^{\mu\alpha}{}_{,\alpha}\mathfrak{g}^{\nu\beta}{}_{,\beta} = \frac{1}{2}\eta_{\mu\nu}\tilde{\phi}^{\mu\alpha}{}_{,\alpha}\tilde{\phi}^{\nu\beta}{}_{,\beta}. \tag{3.29}$$

Indeed:

$$\begin{aligned}
\partial_\alpha\frac{\partial\mathcal{L}_\phi}{\partial\left(\partial_\alpha\tilde{\phi}^{\beta\gamma}\right)} &= \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\left[I^{\mu\alpha}_{\beta\gamma}\tilde{\phi}^{\nu\sigma}{}_{,\sigma} + I^{\nu\alpha}_{\beta\gamma}\tilde{\phi}^{\mu\rho}{}_{,\rho}\right] \\
&= \frac{1}{2}\eta_{\mu\nu}I^{\mu\alpha}_{\beta\gamma}\tilde{\phi}^{\nu\sigma}{}_{,\sigma\alpha} \\
&= \frac{1}{2}\left[\eta_{\beta\nu}\tilde{\phi}^{\nu\sigma}{}_{,\sigma\gamma} + \eta_{\gamma\nu}\tilde{\phi}^{\nu\sigma}{}_{,\sigma\beta}\right].
\end{aligned} \tag{3.30}$$

This cancels the unwanted terms in (3.28).

3.2 The free propagator

We now compute the free propagator of this theory. This can be obtained with the usual generating functional method, which will be outlined here for the case of gravity. Some more detailed computations can be found in the appendix.

The free propagator is usually defined as

$$\left\langle 0\left|T\left\{\tilde{\phi}^{\alpha_1\beta_1}(x)\,\tilde{\phi}^{\alpha_2\beta_2}(y)\right\}\right|0\right\rangle = \frac{1}{Z[J]}\left(-i\frac{\delta}{\delta J_{\alpha_1\beta_1}(x)}\right)\left(-i\frac{\delta}{\delta J_{\alpha_2\beta_2}(y)}\right)Z[J]\Big|_{J=0}, \tag{3.31}$$

where the generating functional is

$$Z[J] = \int\mathcal{D}\tilde{\phi}\exp\left\{iS_0\left[\tilde{\phi}\right] + iS_{src}\left[\tilde{\phi},J\right]\right\}. \tag{3.32}$$

J is the auxiliary source that will be then set to 0 in this prescription.

The action for the free theory is

$$S_0\left[\tilde{\phi}\right] = \int d^4x\frac{1}{8}\left(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau} - \eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau} - 4\delta^\sigma{}_\kappa\delta^\rho{}_\lambda\eta_{\iota\tau} + 4\eta_{\iota\lambda}\delta^\rho{}_\kappa\delta^\sigma{}_\tau\right)\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma} \tag{3.33}$$

Note how we are considering only the kinetic part of the Lagrangian and its gauge fixing term. On the other hand, the action for the coupling of $\tilde{\phi}^{\mu\nu}$ to the auxiliary source $J_{\mu\nu}$ is

$$S_{src}\left[\tilde{\phi},J\right] = \int d^4x J_{\mu\nu}(x)\,\tilde{\phi}^{\mu\nu}(x). \tag{3.34}$$

After following the usual procedure, we find the following expression for the generating functional

$$Z[J] = \exp\left\{-\frac{1}{2}\int d^4x d^4y J_{\mu\nu}\left(-iG^{\mu\nu\alpha\beta}(x-y)\right)J_{\alpha\beta}(x)\right\}Z[0], \tag{3.35}$$

where $-iG^{\mu\nu\alpha\beta}$ solves the equation

$$\frac{1}{8}\left(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau}-\eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau}-4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\iota\tau}+4\eta_{\iota\lambda}\delta^{\rho}{}_{\kappa}\delta^{\sigma}{}_{\tau}\right)$$
$$\partial_\rho\partial_\sigma\left(-iG^{\lambda\tau\mu\nu}\left(x-y\right)\right)=-\frac{i}{2}I^{\mu\nu}{}_{\iota\kappa}\delta\left(x-y\right) \tag{3.36}$$

and has the symmetries $G^{\alpha\beta\mu\nu}=G^{\mu\nu\alpha\beta}=G^{\beta\alpha\mu\nu}=G^{\alpha\beta\nu\mu}$.

A quick computation, which can be found in the appendix B.1, shows

$$\left\langle 0\left|T\left\{\tilde{\phi}^{\alpha_1\beta_1}\left(x\right)\tilde{\phi}^{\alpha_2\beta_2}\left(y\right)\right\}\right|0\right\rangle=-iG^{\alpha_2\beta_2\alpha_1\beta_1}\left(x-y\right), \tag{3.37}$$

i.e. $G^{\alpha_2\beta_2\alpha_1\beta_1}$ is indeed the propagator, for which we werte looking.
It is easy to check that the propagator defined by (3.36) in momentum space is given by

$$G^{\mu\nu\rho\sigma}\left(k^2\right)=\left(\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\rho\nu}\eta^{\mu\sigma}-\eta^{\mu\nu}\eta^{\rho\sigma}\right)\frac{1}{k^2}. \tag{3.38}$$

A proof of this can be found in appendix B.2.
For simplicity, let us define

$$d^{\mu\nu\rho\sigma}=\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}-\eta^{\mu\nu}\eta^{\rho\sigma} \tag{3.39}$$

and then write the propagator as

$$G^{\mu\nu\rho\sigma}\left(k^2\right)=d^{\mu\nu\rho\sigma}\frac{1}{k^2}. \tag{3.40}$$

3.3 3-graviton interaction

In order to compute the vacuum expectation value for the gravitational field, we will consider the interaction between 3 gravitons.

To this purpose, we compute in this section the 1-particle irreducible 3-point vertex, represented in Figure 3.

The 3-graviton vertex function can be computed from

$$\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(x^1,x^2,x^3\right)=\left.\frac{\delta^3 A}{\delta g^{\alpha_1\beta_1}\left(x^1\right)\delta g^{\alpha_2\beta_2}\left(x^2\right)\delta g^{\alpha_3\beta_3}\left(x^3\right)}\right|_{g^{\mu\nu}=\eta^{\mu\nu}}. \tag{3.41}$$

There are various possibilities to tackle the task of computing these three variations.
We will first follow the method suggested in [1]. Then we will try to do the variations directly on a computer. Finally we will consider the result for the vertex given in [5].

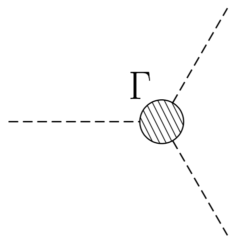

Figure 3: The 1-particle irreducible 3-point vertex

3.3.1 First method

We will start by computing an easier variation

$$\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(x^1, x^2, x^3\right) = \left.\frac{\delta^3 A}{\delta\mathfrak{g}^{\alpha_1\beta_1}\left(x^1\right)\delta\mathfrak{g}^{\alpha_2\beta_2}\left(x^2\right)\delta\mathfrak{g}^{\alpha_3\beta_3}\left(x^3\right)}\right|_{\mathfrak{g}^{\mu\nu}=\eta^{\mu\nu}}. \tag{3.42}$$

Since A_ϕ depends only quadratically on $\mathfrak{g}^{\mu\nu}$, we have $\frac{\delta^3 A_\phi}{\delta\mathfrak{g}^{\alpha_1\beta_1}(x^1)\delta\mathfrak{g}^{\alpha_2\beta_2}(x^2)\delta\mathfrak{g}^{\alpha_3\beta_3}(x^3)} = 0$ and thus

$$\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(x^1, x^2, x^3\right) = \left.\frac{\delta^3 A_G}{\delta\mathfrak{g}^{\alpha_1\beta_1}\left(x^1\right)\delta\mathfrak{g}^{\alpha_2\beta_2}\left(x^2\right)\delta\mathfrak{g}^{\alpha_3\beta_3}\left(x^3\right)}\right|_{\mathfrak{g}^{\mu\nu}=\eta^{\mu\nu}}. \tag{3.43}$$

Since doing the three variations by hand is quite demanding, we can do them on a computer to get the following result in momentum space:

$$\begin{aligned}\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(k_1, k_2, k_3\right) \;=\;& -\mathrm{sym}P_6\frac{\kappa}{8}\left(-4\eta_{\alpha_3\alpha_2}\eta_{\beta_2\alpha_1}\eta_{\beta_3\beta_1}k_2\cdot k_3 + 2\eta_{\alpha_2\beta_2}\eta_{\alpha_3\alpha_1}k_2\cdot k_3\right.\\ & -\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}k_{2\alpha_1}k_{3\beta_1} + 2\eta_{\alpha_3\alpha_2}\eta_{\beta_2\beta_3}k_{2\alpha_1}k_{3\beta_1}\\ & \left.+4\eta_{\alpha_2\alpha_1}\eta_{\beta_3\beta_1}k_{2\alpha_3}k_{3\beta_2}\right),\end{aligned} \tag{3.44}$$

where sym means that we have to symmetrize over $\alpha_1, \beta_1,\ \alpha_2, \beta_2$ and α_3, β_3 and P_6 means that we have to sum over all permutations (cyclic and anticyclic) of $\alpha_1, \beta_1, k_1,\ \alpha_2, \beta_2, k_2$ and α_3, β_3, k_3. There are then 78 terms in the explicit expression for $\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$.
A sketch of the idea behind the computation can be found in the appendix C.1.

Then, in order to get the complete 3-graviton vertex, we can use the following trick:

$$\begin{aligned}\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(x^1, x^2, x^3\right) &= \left.\frac{\delta^3\left(A_G + A_\phi\right)}{\delta g^{\alpha_1\beta_1}\left(x^1\right)\delta g^{\alpha_2\beta_2}\left(x^2\right)\delta g^{\alpha_3\beta_3}\left(x^3\right)}\right|_{g^{\mu\nu}=\eta^{\mu\nu}}\\ &= \left.\frac{\delta^3 A_G}{\delta g^{\alpha_1\beta_1}\left(x^1\right)\delta g^{\alpha_2\beta_2}\left(x^2\right)\delta g^{\alpha_3\beta_3}\left(x^3\right)}\right|_{g^{\mu\nu}=\eta^{\mu\nu}}\\ &\quad+ \left.\frac{\delta^3 A_\phi}{\delta g^{\alpha_1\beta_1}\left(x^1\right)\delta g^{\alpha_2\beta_2}\left(x^2\right)\delta g^{\alpha_3\beta_3}\left(x^3\right)}\right|_{g^{\mu\nu}=\eta^{\mu\nu}}.\end{aligned} \tag{3.45}$$

Now note that

$$\frac{\delta A_G}{\delta g^{\alpha_1\beta_1}(x^1)} = \int d^4y \frac{\delta A_G}{\delta \mathfrak{g}^{\mu_1\nu_1}(y^1)} \frac{\delta \mathfrak{g}^{\mu_1\nu_1}(y^1)}{\delta g^{\alpha_1\beta_1}(x^1)}. \tag{3.46}$$

We need to vary this two more times and use the Leibniz rule before inserting it in the expression for the vertex function. I.e. the expression to be computed is

$$\begin{aligned}
\frac{\delta^3 A_G}{\delta g^{\alpha_3\beta_3}(x_3)\,\delta g^{\alpha^2\beta^2}(x_2)\,\delta g^{\alpha_1\beta_1}(x_1)} &= \int d^4\tilde{x}_1 \frac{\delta^3 \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)}{\delta g^{\alpha_3\beta_3}(x_3)\,\delta g^{\alpha^2\beta^2}(x_2)\,\delta g^{\alpha_1\beta_1}(x_1)} \frac{\delta A_G}{\delta \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)} \\
&+ \int d^4\tilde{x}_1 d^4\tilde{x}_2 \frac{\delta^2 \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)}{\delta g^{\alpha_2\beta_2}(x_2)\,\delta g^{\alpha_1\beta_1}(x_1)} \frac{\delta \mathfrak{g}^{\mu_2\nu_2}(\tilde{x}_2)}{\delta g^{\alpha_3\beta_3}(x_3)} \\
&\times \frac{\delta^2 A_G}{\delta \mathfrak{g}^{\mu_2\nu_2}(\tilde{x}_2)\,\delta \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)} + \int d^4\tilde{x}_2 d^4\tilde{x}_1 \frac{\delta^2 \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)}{\delta g^{\alpha_3\beta_3}(x_3)\,\delta g^{\alpha_1\beta_1}(x_1)} \\
&\times \frac{\delta \mathfrak{g}^{\mu_2\nu_2}(\tilde{x}_2)}{\delta g^{\alpha_2\beta_2}} \frac{\delta^2 A_G}{\delta \mathfrak{g}^{\mu_2\nu_2}(\tilde{x}_2)\,\delta \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)} + \int d^4\tilde{x}_1 d^4\tilde{x}_2 \frac{\delta \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)}{\delta g^{\alpha_1\beta_1}(x_1)} \\
&\times \frac{\delta^2 \mathfrak{g}^{\mu_2\nu_2}(\tilde{x}_2)}{\delta g^{\alpha_3\beta_3}(x_3)\,\delta g^{\alpha_2\beta_2}(x_2)} \frac{\delta^2 A_G}{\delta \mathfrak{g}^{\mu_2\nu_2}(\tilde{x}_2)\,\delta \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)} \\
&+ \int d^4\tilde{x}_1 d^4\tilde{x}_2 d^4\tilde{x}_3 \frac{\delta \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)}{\delta g^{\alpha_1\beta_1}(x_1)} \frac{\delta \mathfrak{g}^{\mu_2\nu_2}(\tilde{x}_2)}{\delta g^{\alpha_2\beta_2}(x_2)} \frac{\delta \mathfrak{g}^{\mu_3\nu_3}(\tilde{x}_3)}{\delta g^{\alpha_3\beta_3}(x_3)} \\
&\times \frac{\delta^3 A_G}{\delta \mathfrak{g}^{\mu_3\nu_3}(\tilde{x}_3)\,\delta \mathfrak{g}^{\mu_2\nu_2}(\tilde{x}_2)\,\delta \mathfrak{g}^{\mu_1\nu_1}(\tilde{x}_1)}.
\end{aligned} \tag{3.47}$$

This is a quite lengthy expression, but it contains only variations of A_G with respect to $\mathfrak{g}$, which we have already computed.
We also need to determine $\frac{\delta \mathfrak{g}^{\mu\nu}(x')}{\delta g^{\alpha\beta}(x)}$. We'll do this by first finding $\frac{\delta g^{\alpha\beta}(x)}{\delta \mathfrak{g}^{\mu\nu}(x')}$ and then inverting it. We can write

$$\begin{aligned}
\frac{\delta g^{\alpha\beta}(x)}{\delta \mathfrak{g}^{\mu\nu}(x')} &= \delta(x,x') \frac{\delta g^{\alpha\beta}}{\delta \mathfrak{g}^{\mu\nu}} \\
&= \delta(x,x') \frac{\delta\left((-g)^{-\frac{1}{2}} g^{\alpha\beta}\right)}{\delta \mathfrak{g}^{\mu\nu}} \\
&= \delta(x,x') \left[(-g)^{-\frac{1}{2}} \frac{1}{2}\left(\delta^\alpha{}_\mu \delta^\beta{}_\nu + \delta^\alpha{}_\nu \delta^\beta{}_\mu\right) + \mathfrak{g}^{\alpha\beta} \frac{\delta(-g)^{-\frac{1}{2}}}{\delta \mathfrak{g}^{\mu\nu}}\right].
\end{aligned} \tag{3.48}$$

We have to write the last term in a better form.

Remembering that $\mathfrak{g} = g^{-1}$, we can write

$$\delta(-g)^{-\frac{1}{2}} = \delta(-\mathfrak{g})^{\frac{1}{2}}. \tag{3.49}$$

Using the following relation

$$\delta\sqrt{h} = -\frac{1}{2}\sqrt{h} h_{\alpha\beta}\delta h^{\alpha\beta}, \tag{3.50}$$

which is known for example from [6], we get

$$\delta(-g)^{-\frac{1}{2}} = \delta(-\mathfrak{g})^{\frac{1}{2}} = -\frac{1}{2}(-\mathfrak{g})^{\frac{1}{2}} \mathfrak{g}_{\mu\nu}\delta \mathfrak{g}^{\mu\nu}. \tag{3.51}$$

Thus

$$\frac{\delta\left(-g\right)^{-\frac{1}{2}}}{\delta\mathfrak{g}^{\mu\nu}} = -\frac{1}{2}\left(-g\right)^{-\frac{1}{2}}\mathfrak{g}_{\mu\nu}. \tag{3.52}$$

Inserting this in (3.48) we obtain the final result:

$$\frac{\delta g^{\alpha\beta}\left(x\right)}{\delta\mathfrak{g}^{\mu\nu}\left(x'\right)} = \frac{1}{2}\left(-g\right)^{-\frac{1}{2}}\left(\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} + \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu} - \mathfrak{g}_{\mu\nu}\mathfrak{g}^{\alpha\beta}\right)\delta\left(x,x'\right). \tag{3.53}$$

For $\frac{\delta\mathfrak{g}^{\mu\nu}(x')}{\delta g^{\alpha\beta}(x)}$ we only need to compute the inverse of (3.53):

$$\frac{\delta\mathfrak{g}^{\mu\nu}\left(x'\right)}{\delta g^{\alpha\beta}\left(x\right)} = \frac{1}{2}\left(-g\right)^{\frac{1}{2}}\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \delta^{\nu}{}_{\alpha}\delta^{\mu}{}_{\beta} - \mathfrak{g}_{\alpha\beta}\mathfrak{g}^{\mu\nu}\right)\delta\left(x,x'\right) \tag{3.54}$$

It is also useful to know that:

$$\begin{aligned}
\frac{\delta g_{\alpha_1\beta_1}}{\delta g^{\alpha_2\beta_2}} &= \frac{\delta\left(g_{\mu\alpha_1}g_{\nu\beta_1}g^{\mu\nu}\right)}{\delta g^{\alpha_2\beta_2}} \\
&= I^{\mu\nu}{}_{\alpha_2\beta_2}g_{\mu\alpha_1}g_{\nu\beta_1} + g^{\mu\nu}g_{\mu\alpha_1}\frac{\delta g_{\nu\beta_1}}{\delta g^{\alpha_2\beta_2}} + g^{\mu\nu}g_{\nu\beta_1}\frac{\delta g_{\mu\alpha_1}}{\delta g^{\alpha_2\beta_2}} \\
&= I^{\mu\nu}{}_{\alpha_2\beta_2}g_{\mu\alpha_1}g_{\nu\beta_1} + 2\frac{\delta g_{\alpha_1\beta_1}}{\delta g^{\alpha_2\beta_2}}
\end{aligned}$$

which implies:

$$\frac{\delta g_{\alpha_1\beta_1}}{\delta g^{\alpha_2\beta_2}} = -I^{\mu\nu}{}_{\alpha_2\beta_2}g_{\mu\alpha_1}g_{\nu\beta_1} = -\frac{1}{2}\left(g_{\alpha_2\alpha_1}g_{\beta_2\beta_1} + g_{\beta_2\alpha_1}g_{\alpha_2\beta_1}\right) \tag{3.55}$$

We can analogously compute the second and third variation ot $\mathfrak{g}$ with respect to g, but this is best done on a computer.
Unfortunately, after doing so, we obtain a result that differs from the one given in [1]. The latter is explicitly given by:

$$\begin{aligned}
\frac{1}{8}d^{\mu_1\nu_1\alpha_1\beta_1}d^{\mu_2\nu_2\alpha_2\beta_2}d^{\mu_3\nu_3\alpha_3\beta_3}\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} &= \tilde{\Gamma}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} - \frac{\kappa}{4}P_3\left(\delta^{\mu_1\nu_1\mu_2\nu_2}\eta^{\mu_3\nu_3} - \delta^{\mu_3\nu_3\mu_1\nu_1}\eta^{\mu_2\nu_2}\right. \\
&\quad \left. -\delta^{\mu_2\nu_2\mu_3\nu_3}\eta^{\mu_1\nu_1} + \frac{1}{2}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\right)k_3^2
\end{aligned} \tag{3.56}$$

where P_3 indicates a sum over all cyclic permutations of $\{\mu_1,\nu_1,k_1\}$, $\{\mu_2,\nu_2,k_2\}$ and $\{\mu_3,\nu_3,k_3\}$. Furthermore $\delta^{\mu_1\nu_1\mu_2\nu_2}$ is defined as:

$$\delta^{\mu_1\nu_1\mu_2\nu_2} = \frac{1}{2}\left(\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} + \eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2}\right) \tag{3.57}$$

3.3.2 Second method

We will now attempt to compute the three variations of $A_G^{(1)}$ and A_ϕ directly with respect to g using [2] and [3].
This is done in position space and then Fourier transformed, for later comparison with the

result given in section 3.3.1.

We start by defining the following variation rule: $\frac{\delta \mathfrak{g}^{\alpha_1\beta_1}(x')}{\delta g^{\mu_1\nu_1}(x)} = I^{\alpha_1\beta_1}{}_{\mu_1\nu_1}\,\delta\left(x'-x\right)$.

We also need to define the variation of the derivative as: $\frac{\delta \mathfrak{g}^{\alpha_1\beta_1}{}_{,\gamma}(x')}{\delta g^{\mu_1\nu_1}(x)} = I^{\alpha_1\beta_1}{}_{\mu_1\nu_1}\,\partial_\gamma\delta\left(x'-x\right)$.
We will later get rid of the derivative on the delta function via integration by parts.

After variating three times, we can perform the integration over d^4x included in the action to get rid of one of the delta functions. We choose to eliminate the one that is not acted upon by a partial derivative.

At this point we are ready to Fourier transform. This is done by substituting for example:

$$\partial_\alpha\left[\delta\left(x_1-x_2\right)\right]\partial_\beta\left[\delta\left(x_3-x_1\right)\right]\longrightarrow -\,k_{2\alpha}k_{3\beta} \tag{3.58}$$

Unfortunately, the final result does not coincide with the one found in [1].
The crucial points of the code used for the calculations can be found in the appendix C.2.

3.3.3 Result from the literature

Finally, we quote the result known from [5]:

$$\begin{aligned}\Gamma^{\mu\nu\sigma\tau\rho\lambda} \;=\; \mathrm{sym}\Big[&-\frac{1}{4}P_3\left(k_1\cdot k_2\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\right)-\frac{1}{4}P_6\left(k_1^\sigma p_1^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\right)+\frac{1}{4}P_3\left(k_1\cdot k_2\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\right)\\&+\frac{1}{2}P_6\left(k_1\cdot k_2\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\right)+P_3\left(k_1^\sigma k_1^\lambda\eta^{\mu\nu}\eta^{\tau\rho}\right)-\frac{1}{2}P_3\left(k_1^\tau k_2^\mu\eta^{\nu\sigma}\eta^{\rho\lambda}\right)+\frac{1}{2}P_3\left(k_1^\rho k_2^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\right)\\&+P_6\left(k_1^\sigma k_2^\lambda\eta^{\tau\mu}\eta^{\nu\rho}\right)+P_3\left(k_1^\sigma k_2^\mu\eta^{\tau\rho}\eta^{\lambda\nu}\right)-P_3\left(k_1\cdot k_2\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\right)\Big]\end{aligned} \tag{3.59}$$

It is important to note, that this expression for the vertex function is different from one given in [1], but this doesn't make any difference in the end, because the final results obtained for the vacuum expectation value coincide anyways.

3.4 Preliminary considerations on the vacuum expectation value

3.4.1 External source

Let us now consider the source term of the action.
We begin by defining:

$$J_{\mu\nu} = (-g)^{\frac{1}{2}}\,T_{\mu\nu} \tag{3.60}$$

Remember that we have in the rest frame:

$$T^{\mu\nu} = \begin{pmatrix} \mu & 0 & 0 & 0\\ 0 & p & 0 & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & p\end{pmatrix} \tag{3.61}$$

If we multiply the Einstein equations by $(-g)^{\frac{1}{2}}$ we get:

$$\frac{1}{\kappa^2}(-g)^{\frac{1}{2}}\,G_{\mu\nu}+\frac{J_{\mu\nu}}{2}=0 \tag{3.62}$$

Now, since $\frac{\delta A_G}{\delta g^{\mu\nu}} = \frac{1}{\kappa^2}(-g)^{\frac{1}{2}} G_{\mu\nu}$, we are left with:

$$\frac{\delta A_J}{\delta g^{\mu\nu}} = \frac{1}{2} J_{\mu\nu} \tag{3.63}$$

Taking

$$A_J = \frac{1}{2} \int d^4x g^{\mu\nu}(x) J_{\mu\nu}(x) \tag{3.64}$$

is then the correct choice, since then

$$\frac{\delta A_J}{\delta g^{\alpha\beta}} = \frac{1}{2} \delta^\mu{}_\alpha \delta^\nu{}_\beta J_{\mu\nu} = \frac{1}{2} J_{\alpha\beta} \tag{3.65}$$

3.4.2 Generic expression for the vacuum expectation value

We can then compute the S-matrix using the Feynman-Dyson expression:

$$S_J = T\left\{\exp\left(i \int d^4x \left[\mathcal{L}_{\text{int}} + \mathcal{L}_J(x)\right]\right)\right\} \tag{3.66}$$

Since we are only considering the 3 graviton interaction, the interaction part of the Lagrangian is in our case $\mathcal{L}_{int} = \kappa \tilde{\mathcal{L}}_G^{(1)}$.

The coupling with the external source J is described by (3.64).

We can expand up to order κ^2 the part of $\exp\left(i \int d^4x \mathcal{L}_J\right)$ that contains a $\phi^{\mu\nu}$ dependence in the following way:

$$\begin{aligned}\exp\left(\frac{i\kappa}{2} \int d^4x \phi^{\mu\nu}(x) J_{\mu\nu}(x)\right) &= 1 + \frac{i\kappa}{2} \int d^4x \phi^{\mu\nu}(x) J_{\mu\nu}(x) \\ &\quad - \frac{1}{8}\kappa^2 \int d^4x d^4y \phi^{\mu\nu}(x) \phi^{\alpha\beta}(y) J_{\mu\nu}(x) J_{\alpha\beta}(y)\end{aligned} \tag{3.67}$$

On the other hand, the expansion of $\exp\left(i \int d^4x \mathcal{L}_{int}\right)$ gives:

$$\exp\left(i \int d^4x \mathcal{L}_{int}\right) = 1 + i\kappa \int d^4x \overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \phi^{\alpha_1\beta_1}(x) \phi^{\alpha_2\beta_2}(x) \phi^{\alpha_3\beta_3}(x) \tag{3.68}$$

where we have defined $\overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} = \frac{1}{\kappa}\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$, so that the dependence on κ is clear and only in front of the integral.

The product of the two up to order κ^3 is then:

$$
\begin{aligned}
\exp\left(i\int d^4x\left[\mathcal{L}_{int}+\mathcal{L}_J\right]\right) = {} & 1+i\kappa\int d^4x\overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi^{\alpha_1\beta_1}(x)\,\phi^{\alpha_2\beta_2}(x)\,\phi^{\alpha_3\beta_3}(x) \\
& +\frac{i\kappa}{2}\int d^4x\phi^{\mu\nu}(x)\,J_{\mu\nu}(x) \\
& -\frac{\kappa^2}{2}\int d^4xd^4y\overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi^{\alpha_1\beta_1}(x)\,\phi^{\alpha_2\beta_2}(x)\,\phi^{\alpha_3\beta_3}(x)\,\phi^{\mu\nu}(y)\,J_{\mu\nu}(y) \\
& -\frac{1}{8}\kappa^2\int d^4xd^4y\phi^{\mu\nu}(x)\,\phi^{\alpha\beta}(y)\,J_{\mu\nu}(x)\,J_{\alpha\beta}(y) \\
& -\frac{i\kappa^3}{8}\int d^4xd^4yd^4z\overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi^{\alpha_1\beta_1}(x)\,\phi^{\alpha_2\beta_2}(x)\,\phi^{\alpha_3\beta_3}(x)\,\phi^{\mu\nu}(y) \\
& \times\phi^{\alpha\beta}(z)\,J_{\mu\nu}(y)\,J_{\alpha\beta}(z)
\end{aligned}
\tag{3.69}
$$

Mutiplying this by $\phi^{\mu\nu}(x)$ yields:

$$
\begin{aligned}
\phi^{\mu\nu}(x)\,S_J = {} & \phi^{\mu\nu}(x)+i\kappa\phi^{\mu\nu}(x)\int d^4y\overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi^{\alpha_1\beta_1}(y)\,\phi^{\alpha_2\beta_2}(y)\,\phi^{\alpha_3\beta_3}(y) \\
& +\frac{i\kappa}{2}\phi^{\mu\nu}(x)\int d^4y\phi^{\alpha\beta}(y)\,J_{\alpha\beta}(y) \\
& -\frac{\kappa^2}{2}\phi^{\mu\nu}(x)\int d^4yd^4z\overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi^{\alpha_1\beta_1}(y)\,\phi^{\alpha_2\beta_2}(y)\,\phi^{\alpha_3\beta_3}(y)\,\phi^{\alpha\beta}(z)\,J_{\alpha\beta}(z) \\
& -\frac{\kappa^2}{8}\phi^{\mu\nu}(x)\int d^4yd^4z\phi^{\alpha\beta}(y)\,\phi^{\rho\sigma}(z)\,J_{\alpha\beta}(y)\,J_{\rho\sigma}(z) \\
& -\frac{i\kappa^3}{8}\phi^{\mu\nu}(x)\int d^4yd^4zd^4w\overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi^{\alpha_1\beta_1}(y)\,\phi^{\alpha_2\beta_2}(y)\,\phi^{\alpha_3\beta_3}(y)\,\phi^{\alpha\beta}(z)\,\phi^{\rho\sigma}(w) \\
& \times J_{\alpha\beta}(z)\,J_{\rho\sigma}(w)
\end{aligned}
\tag{3.70}
$$

We can then use Wick's theorem to compute the expectation value of $T\left\{\phi^{\mu\nu}(x)\,S_J\right\}$.
The only non-vanishing terms are the ones containing an even number of ϕ's.
Furthermore, we want to consider only tree diagrams. This means that we must neglect any terms involving contractions between two fields evaluated at the same point, e.g. no contraction $\phi^{\alpha_1\beta_1}(y)\,\phi^{\alpha_2\beta_2}(y)$ is allowed in the second term of $\phi^{\mu\nu}(x)\,S_J$.

The only non-vanishing terms remaining are then:

$$
\begin{aligned}
\langle 0\left|T\left\{\phi^{\mu\nu}(x)\,S_J\right\}\right|0\rangle = {} & \frac{i\kappa}{2}\int d^4y\phi^{\mu\nu}(x)\,\phi^{\alpha\beta}(y)\,J_{\alpha\beta}(y) \\
& -\frac{i\kappa^3}{8}\int d^4yd^4zd^4w\phi^{\mu\nu}(x)\,\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\phi^{\alpha_1\beta_1}(y) \\
& \times\phi^{\alpha_2\beta_2}(y)\,\phi^{\alpha_3\beta_3}(y)\,\phi^{\alpha\beta}(z)\,\phi^{\rho\sigma}(w)\,J_{\alpha\beta}(z)\,J_{\rho\sigma}(w)
\end{aligned}
\tag{3.71}
$$

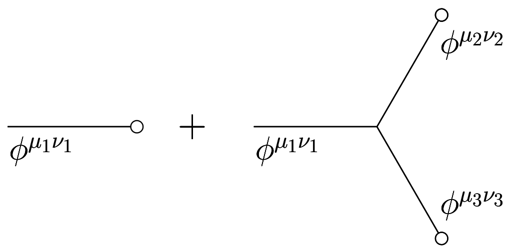

Figure 4: The expansion in tree-level diagrams of the vacuum expectation value.

Then the vacuum expectation value can be written in momentum space as:

$$\begin{aligned}\kappa\left\langle\phi^{\mu_1\nu_1}\left(k_1\right)\right\rangle_J &= \frac{1}{2}\kappa^2 G^{\mu_1\nu_1\alpha_1\beta_1}\left(k_1^2\right)J_{\alpha_1\beta_1}\left(k_1\right)-\frac{1}{8}\kappa^4\int đ^4k_2 đ^4k_3 G^{\mu_1\nu_1\alpha_1\beta_1}\left(k_1^2\right)\\ &\times G^{\mu_2\nu_2\alpha_2\beta_2}\left(k_2^2\right)G^{\mu_3\nu_3\alpha_3\beta_3}\left(k_3^2\right)\delta^4\left(k_1+k_2+k_3\right)\overline{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(k_1,k_2,k_3\right)\\ &\times J_{\mu_2\nu_2}\left(k_2\right)J_{\mu_3\nu_3}\left(k_3\right)\end{aligned} \tag{3.72}$$

where $đk=\frac{dk}{2\pi}$.
A diagrammtical representation of this can be found in Figure 4.

Remembering (3.38), we can write:

$$\begin{aligned}\kappa\left\langle\phi^{\mu_1\nu_1}\left(k_1\right)\right\rangle_J &= \frac{1}{2}\kappa^2 G^{\mu_1\nu_1\alpha_1\beta_1}\left(k_1^2\right)J_{\alpha_1\beta_1}\left(k_1\right)-\frac{1}{8}\kappa^4\int\frac{đ^4k_2 đ^4k_3}{k_1^2k_2^2k_3^2}d^{\mu_1\nu_1\alpha_1\beta_1}\\ &\times d^{\mu_2\nu_2\alpha_2\beta_2}d^{\mu_3\nu_3\alpha_3\beta_3}\delta^4\left(k_1+k_2+k_3\right)\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(k_1,k_2,k_3\right)\\ &\times J_{\mu_2\nu_2}\left(k_2\right)J_{\mu_3\nu_3}\left(k_3\right)\end{aligned} \tag{3.73}$$

The explicit form for the contraction of the vertex with $\frac{1}{8}d^{\mu_1\nu_1\alpha_1\beta_1}\times d^{\mu_2\nu_2\alpha_2\beta_2}\times d^{\mu_3\nu_3\alpha_3\beta_3}$ is known from (3.56).

3.4.3 Explicit form of the external gravitational potential

Let us now abandon briefly these computations for some considerations on the source.
We can define:

$$\lambda=\frac{4}{3}\pi\rho\varepsilon^3 \tag{3.74}$$

Remember the following relations from section 2:

$$m = \frac{4}{3}\pi\rho H^3(\varepsilon) = \frac{4}{3}\pi\rho\varepsilon^3 + 4\pi\rho\varepsilon^2 Gm + \mathcal{O}\left(G^2\right) \tag{3.75}$$

$$H(\varepsilon) = \varepsilon + Gm \tag{3.76}$$

$$m = m_0 - \frac{3}{5}\frac{Gm_0^2}{\varepsilon} + \mathcal{O}\left(G^2\right) \tag{3.77}$$

Thus:

$$\begin{aligned}
\lambda &= \frac{4}{3}\pi\rho\varepsilon^3 \\
&= m - \underbrace{4\pi\rho\varepsilon^2}_{=3\frac{m}{\varepsilon}+\mathcal{O}(G)} Gm + \mathcal{O}\left(G^2\right) \\
&= m - \frac{3Gm^2}{\varepsilon} + \mathcal{O}\left(G^2\right)
\end{aligned} \tag{3.78}$$

Using (3.77) we obtain:

$$\begin{aligned}
\lambda &= m_0 - \frac{3}{5}\frac{Gm_0^2}{\varepsilon} - \frac{3G}{\varepsilon}m_0^2 + \mathcal{O}\left(G^2\right) \\
&= m_0 - \frac{18}{5}\frac{Gm_0^2}{\varepsilon} + \mathcal{O}\left(G^2\right)
\end{aligned} \tag{3.79}$$

Let us now define:

$$V(x) = \frac{1}{4}\kappa^2 \int đ^4k \frac{e^{ikx}}{k^2}\mu(k) \tag{3.80}$$

where $\mu(k)$ is the Fourier transform of $\mu(x)$, i.e.

$$\begin{aligned}
\mu(k) &= \int d^4x e^{-ik\cdot x}\mu(\vec{x}) \\
&= \int dx^0 e^{-ik_0x^0}\int d^3x e^{-i\vec{k}\cdot\vec{x}}\mu(\vec{x}) \\
&= (2\pi)\,\delta\left(k^0\right)\int d^3x e^{-i\vec{k}\cdot\vec{x}}\mu(\vec{x}) \\
&= (2\pi)\,\delta\left(k^0\right)\mu\left(\vec{k}\right)
\end{aligned} \tag{3.81}$$

Thus we can rewrite $V(x)$ as:

$$\begin{aligned}
V(x) &= \frac{1}{4}\kappa^2 \int đ^3\vec{k}dk^0\delta\left(k^0\right)e^{-ik^0x^0}\frac{e^{i\vec{k}\vec{x}}}{-k^{0\,2}+\vec{k}^2}\mu\left(\vec{k}\right) \\
&= \frac{1}{4}\kappa^2 \int đ^3\vec{k}\frac{e^{i\vec{k}\vec{x}}}{\vec{k}^2}\mu\left(\vec{k}\right)
\end{aligned} \tag{3.82}$$

We can directly read off from this expression the Fourier transform of $V(x)$:

$$V(k) = \frac{1}{\kappa^2}\frac{\mu\left(\vec{k}\right)}{\vec{k}^2} \tag{3.83}$$

Furthermore we should remember that:

$$\mu(\vec{x}) = \mu(|\vec{x}|) = \mu(r) = \rho\theta(\varepsilon - r) \tag{3.84}$$

i.e. $\mu(\vec{x})$ depends only on the absolute value of $\vec{x}$.
This implies:

$$V(x) = V(|\vec{x}|) = \frac{1}{4}\kappa^2 \int đ^3\vec{k}\frac{e^{i\vec{k}\vec{x}}}{\vec{k}^2}\mu\left(\vec{k}\right) = \frac{1}{4}\kappa^2 \int đ^3\vec{k}\frac{e^{-i\vec{k}\vec{x}}}{\vec{k}^2}\mu\left(\vec{k}\right) \tag{3.85}$$

An explicit calculation of the integral, which can be found in the appendix E.1, yields:

$$V(x') = (-G\lambda)\left[\theta(\varepsilon - r')\left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r'^2}{\varepsilon^3}\right) + \theta(r' - \varepsilon)\frac{1}{r'}\right] \tag{3.86}$$

3.4.4 Some integrals

Let us now state three relations that will prove to be useful when computing the vacuum expectation value of $\phi^{\mu\nu}$.

$$-\int d^3x'\frac{p(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} = \frac{1}{\nabla^2}p(\vec{x}) \tag{3.87}$$

$$\frac{1}{16}\kappa^4 \int đ^3k_1 đ^3k_2 đ^3k_3 e^{i\vec{k}_1\cdot\vec{x}}\delta^3\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right)\frac{k_2^i k_2^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) = \frac{1}{\Delta}\left(V\partial^i\partial^j V\right) \tag{3.88}$$

$$\frac{1}{16}\kappa^4 \int đ^3k_1 đ^3k_2 đ^3k_3 e^{i\vec{k}_1\cdot\vec{x}}\delta^3\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right)\frac{k_2^i k_3^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) = \frac{1}{\Delta}\left(\partial^i V\partial^j V\right) \tag{3.89}$$

The first one is equivalent to

$$p(\vec{x}) = -\int d^3x'\Delta_{\vec{x}}\frac{p(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \tag{3.90}$$

and is clear, since we know that

$$-\Delta_{\vec{x}}\frac{1}{4\pi|\vec{x} - \vec{x}'|} = \delta(\vec{x} - \vec{x}')\,. \tag{3.91}$$

A proof of the other two relations can be found in the appendix E.2.

3.5 Explicit computation of the vacuum expectation value

With (3.87), (3.88) and (3.89) we are now in a position to compute the vacuum expectation value from (3.73).
We will rely again on a computer for some particularly long calculations.

3.5.1 00-component

Let us first consider the 00-component.
We have:

$$\begin{aligned}\kappa\left\langle\phi^{00}\left(k_1\right)\right\rangle_J &= \frac{1}{2}\kappa^2 G^{00\alpha_1\beta_1}\left(k_1^2\right)J_{\alpha_1\beta_1}\left(k_1\right)+\frac{1}{8}\kappa^4\int đ^4k_2đ^4k_3 G^{00\alpha_1\beta_1}\left(k_1^2\right)G^{\mu_2\nu_2\alpha_2\beta_2}\left(k_2^2\right)\\ &\quad\times G^{\mu_3\nu_3\alpha_3\beta_3}\left(k_3^2\right)\delta^4\left(k_1+k_2+k_3\right)\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(k_1,k_2,k_3\right)\\ &\quad\times J_{\mu_2\nu_2}\left(k_2\right)J_{\mu_3\nu_3}\left(k_3\right)\end{aligned} \tag{3.92}$$

We begin with the first term. Using (3.38) it can be written as:

$$\begin{aligned}\frac{1}{2}\kappa^2 G^{00\alpha_1\beta_1}\left(k_1^2\right)J_{\alpha_1\beta_1}\left(k_1\right) &= \frac{1}{2}\kappa^2\left(\eta^{0\alpha_1}\eta^{0\beta_1}+\eta^{0\beta_1}\eta^{0\alpha_1}-\eta^{00}\eta^{\alpha_1\beta_1}\right)\frac{1}{k_1^2}J_{\alpha_1\beta_1}\left(k_1\right)\\ &= \kappa^2\eta^{00}\eta^{00}\frac{J_{00}\left(k_1\right)}{k_1^2}-\frac{1}{2}\kappa^2\eta^{00}\frac{1}{k_1^2}\eta^{\alpha_1\beta_1}J_{\alpha_1\beta_1}\left(k_1\right)\\ &= \kappa^2\frac{\mu\left(k_1\right)}{k_1^2}+\frac{1}{2}\kappa^2\frac{1}{k_1^2}\eta^{00}J_{00}\left(k_1\right)+\frac{1}{2}\kappa^2\frac{1}{k_1^2}\eta^{ij}J_{ij}\left(k_1\right)\\ &= \kappa^2\frac{\mu\left(k_1\right)}{k_1^2}-\frac{1}{2}\kappa^2\frac{1}{k_1^2}\mu\left(k_1\right)+\frac{1}{2}\kappa^2\frac{1}{k_1^2}\eta^{ij}\eta_{ij}p\left(k_1\right)\\ &= \frac{1}{2}\kappa^2\frac{\mu\left(k_1\right)}{k_1^2}+\frac{3}{2}\kappa^2\frac{1}{k_1^2}p\left(k_1\right)\\ &= 2V-\frac{3}{2}\kappa^2\frac{1}{\Delta}p\end{aligned} \tag{3.93}$$

where we used that $V\left(k\right)=\frac{1}{4}\kappa^2\frac{\mu(k)}{k^2}$, $J_{00}\left(k\right)=\mu\left(k\right)$ and $J_{ij}\left(k\right)=\eta_{ij}p\left(k\right)$.

Next we note that the second term in (3.92) contains the expression that we computed explicitly in (3.56). Thus we can rewrite the second term as:

$$\begin{aligned}&\kappa^4\int đ^4k_2đ^4k_3\frac{1}{k_1^2k_2^2k_3^2}\delta^4\left(k_1+k_2+k_3\right)J_{\mu_2\nu_2}\left(k_2\right)J_{\mu_3\nu_3}\left(k_3\right)\tilde{\Gamma}^{00\mu_2\nu_2\mu_3\nu_3}\\ &-\frac{1}{4}\kappa^4\int đ^4k_2đ^4k_3\frac{1}{k_1^2k_2^2k_3^2}\delta^4\left(k_1+k_2+k_3\right)J_{\mu_2\nu_2}\left(k_2\right)J_{\mu_3\nu_3}P_3\left(\delta^{00\mu_2\nu_2}\eta^{\mu_3\nu_3}\right.\\ &\left.-\delta^{\mu_3\nu_3 00}\eta^{\mu_2\nu_2}-\delta^{\mu_2\nu_2\mu_3\nu_3}\eta^{00}+\frac{1}{2}\eta^{00}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\right)\end{aligned} \tag{3.94}$$

All of the terms involved can be calculated by using the relations found in section 3.4.4.
Some more details on how this was done can be found in the appendix D.
Here we can show just two examples explicitly that should give an idea of how the procedure works.

The first example that we consider is the following term from (3.94):

$$\frac{1}{4}\kappa^4\int đ^4k_2đ^4k_3\frac{1}{k_1^2k_2^2k_3^2}\delta^4\left(k_1+k_2+k_3\right)J_{\mu_2\nu_2}\left(k_2\right)J_{\mu_3\nu_3}\delta^{00\mu_2\nu_2}\eta^{\mu_3\nu_3}k_3^2 \tag{3.95}$$

Note that:

$$\delta^{00\mu_2\nu_2}=\frac{1}{2}\left(\eta^{0\mu_2}\eta^{0\nu_2}\eta^{0\nu_2}\eta^{0\mu_2}\right)=\eta^{0\mu_2}\eta^{0\nu_2} \tag{3.96}$$

Thus our example can be rewritten as:

$$\begin{aligned}
&\frac{1}{4}\kappa^4 \int đ^4k_2 đ^4k_3 \frac{1}{k_1^2 k_2^2}\delta^4\left(k_1+k_2+k_3\right) \\
&\times J_{\mu_2\nu_2}\left(k_2\right) J_{\mu_3\nu_3}\left(k_3\right) \eta^{0\mu_2}\eta^{0\nu_2}\eta^{\mu_3\nu_3} = \frac{1}{4}\kappa^4 \int đ^4k_2 đ^4k_3 \frac{1}{k_1^2 k_2^2}\delta^4\left(k_1+k_2+k_3\right)\mu\left(k_2\right)\eta^{\mu_3\nu_3}J_{\mu_3\nu_3}\left(k_3\right) \\
&= \frac{1}{4}\kappa^4 \int đ^4k_2 đ^4k_3 \frac{1}{k_1^2 k_2^2}\delta^4\left(k_1+k_2+k_3\right)\mu\left(k_2\right) \\
&\times\left[-J_{00}\left(k_3\right)+J_{ii}\left(k_3\right)\right]
\end{aligned} \tag{3.97}$$

We consider now only the part proportional to J_{00}, because we are interested in the dependence on V.
Using $J_{00}\left(k_3\right)=\mu\left(k_3\right)$ we then have:

$$\begin{aligned}
&-\frac{1}{4}\kappa^4 \int đ^4k_2 đ^4k_3 \frac{1}{k_1^2 k_2^2} \\
&\times\delta^4\left(k_1+k_2+k_3\right)\mu\left(k_2\right)\mu\left(k_3\right) = -\frac{1}{4}\kappa^4 \int đ^4k_2 đ^4k_3 \frac{1}{k_1^2 k_2^2}\delta^4\left(k_1+k_2+k_3\right)(2\pi)\,\delta\left(k_2^0\right)\mu\left(\vec{k}_2\right) \\
&\times(2\pi)\,\delta\left(k_3^0\right)\mu\left(\vec{k}_3\right) \\
&= -\frac{1}{4}\kappa^4 \int đ^3k_2 đ^3k_3 \frac{1}{\vec{k}_1^2\vec{k}_2^2}\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&= -\frac{1}{4}\kappa^4 \int đ^3k_2 đ^3k_3 \delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\frac{k_3^i k_{3\,i}}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&= -\frac{1}{4}\kappa^2\frac{16}{\kappa^4}\frac{1}{\Delta}\left(V\Delta V\right)
\end{aligned} \tag{3.98}$$

where we used (3.89) in the last step.

As a second example we would like to find a term that leads to a proportionality to $\frac{1}{\Delta}\partial^i V\partial^j V$.
In (3.94) the first term looks like:

$$\kappa^4 \int đ^4k_2 đ^4k_3 \tilde{\Gamma}^{00\mu_2\nu_2\mu_3\nu_3}\frac{J_{\mu_2\nu_2}J_{\mu_3\nu_3}}{k_1^2k_2^2k_3^2}\delta^4\left(k_1+k_2+k_3\right) \tag{3.99}$$

Let us consider as an example just one term in $\tilde{\Gamma}^{00\mu_2\nu_2\mu_3\nu_3}$, namely $-4\eta^{0\mu_3}\eta^{\nu_3\mu_2}\eta^{0\nu_2}k_3\cdot k_1$. Thus we get:

$$
\begin{aligned}
-4\kappa^4 \int & đ^4k_2 đ^4k_3 \eta^{0\mu_3}\eta^{\nu_3\mu_2}\eta^{0\nu_2}\frac{k_3\cdot k_1}{k_1^2k_2^2k_3^2} \\
\times J_{\mu_2\nu_2}J_{\mu_3\nu_3}\delta^4\left(k_1+k_2+k_3\right) &= -4\kappa^4\int đ^3k_2 đ^3k_3\eta^{00}\eta^{00}\eta^{00}\frac{\vec{k}_3\cdot\vec{k}_1}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&\quad \times\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right) \\
&= 2\kappa^4\int đ^3k_2 đ^3k_3\frac{\left(\vec{k}_3+\vec{k}_1\right)^2-\vec{k}_3^2-\left(\vec{k}_2+\vec{k}_3\right)^2}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right) \\
&\quad \times\mu\left(\vec{k}_3\right)\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right) \\
&= 2\kappa^4\int đ^3k_2 đ^3k_3\frac{\vec{k}_2^2-\vec{k}_3^2-\vec{k}_2^2-\vec{k}_3^2-2\vec{k}_2\cdot\vec{k}_3}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&\quad \times\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right) \\
&= -4\kappa^4\int đ^3k_2 đ^3k_3\frac{\vec{k}_3^2+\vec{k}_2\cdot\vec{k}_3}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&\quad \times\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right) \\
&= -4\left(\frac{16}{\Delta}V\Delta V+\eta_{kl}\frac{16}{\Delta}\partial^kV\partial^lV\right) \qquad (3.100)
\end{aligned}
$$

where we have used again (3.88) and (3.89).

After doing this repeatedly (see appendix D), we obtain the following final result:

$$\kappa\left\langle\phi^{00}\right\rangle_J = 2V+\frac{3\kappa^2}{2}\frac{1}{\Delta}p-\frac{4}{\Delta}\left(\eta_{kl}\partial^kV\partial^lV\right)-\frac{8}{\Delta}\left(V\Delta V\right) \tag{3.101}$$

Note that doing the same calculations with the vertex given in [5] yields the same expression. Doing this with the expression calculated explicitly as described in 3.3.2 gives instead:

$$\kappa\left\langle\phi^{00}\right\rangle_J = 2V+\frac{3\kappa^2}{2}\frac{1}{\Delta}p-\frac{8}{\Delta}\left(\eta_{kl}\partial^kV\partial^lV\right)-\frac{16}{\Delta}\left(V\Delta V\right) \quad , \tag{3.102}$$

i.e. there is a difference of a factor of 2.

3.5.2 *ij*-component

Let us now consider the ij-components of the vacuum expectation value:

$$
\begin{aligned}
\kappa\left\langle\phi^{ij}\left(k_1\right)\right\rangle_J &= \frac{1}{2}\kappa^2G^{ij\alpha_1\beta_1}\left(k_1^2\right)J_{\alpha_1\beta_1}\left(k_1\right)+\frac{1}{8}\kappa^4\int đ^4k_2 đ^4k_3G^{ij\alpha_1\beta_1}\left(k_1^2\right)G^{\mu_2\nu_2\alpha_2\beta_2}\left(k_2^2\right) \\
&\quad \times G^{\mu_3\nu_3\alpha_3\beta_3}\left(k_3^2\right)\delta^4\left(k_1+k_2+k_3\right)\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(k_1,k_2,k_3\right) \\
&\quad \times J_{\mu_2\nu_2}\left(k_2\right)J_{\mu_3\nu_3}\left(k_3\right) \qquad (3.103)
\end{aligned}
$$

We can again compute immediately the first term to get:

$$\begin{aligned}\frac{1}{2}\kappa^2 G^{ij\alpha_1\beta_1}\left(k_1^2\right)J_{\alpha_1\beta_1} &= \frac{1}{2}\kappa^2 d^{ij\alpha_1\beta_1}\frac{1}{k_1^2}J_{\alpha_1\beta_1}\left(k_1\right)\\ &= \frac{1}{2}\kappa^2\frac{1}{k_1^2}d^{ij00}J_{00}\left(k_1\right)+\frac{1}{2}\kappa^2\frac{1}{k_1^2}d^{ijab}J_{ab}\left(k_1\right)\end{aligned} \tag{3.104}$$

Now:

$$d^{ij00} = \eta^{i0}\eta^{j0}+\eta^{i0}\eta^{j0}-\eta^{ij}\underbrace{\eta^{00}}_{=-1} = \eta^{ij} \tag{3.105}$$

$$d^{ijab} = \eta^{ia}\eta^{jb}+\eta^{ib}\eta^{ja}-\eta^{ij}\eta^{00} = 2\eta^{ia}\eta^{jb}-\eta^{ij}\eta^{ab} \tag{3.106}$$

where we used the fact that J_{ab} is symmetric with respect to a and b.
Thus:

$$\begin{aligned}\frac{1}{2}\kappa^2 G^{ij\alpha_1\beta_1}\left(k_1^2\right)J_{\alpha_1\beta_1} &= \underbrace{\frac{1}{2}\kappa^2\frac{1}{k_1^2}\eta^{ij}\mu\left(k_1\right)}_{=2V\eta^{ij}}+\frac{1}{2}\kappa^2\frac{1}{k_1^2}\left(2\eta^{ia}\eta^{jb}-\eta^{ij}\eta^{ab}\right)\eta_{ab}p\left(k_1\right)\\ &= 2V\eta^{ij}+\frac{1}{2}\kappa^2\frac{1}{k_1^2}\left(2\eta^{ij}-3\eta^{ij}\right)p\left(k_1\right)\\ &= 2V\eta^{ij}-\frac{1}{2}\kappa^2\underbrace{\frac{1}{k_1^2}p\left(k_1\right)}_{=-\frac{1}{\Delta}p(k_1)}\eta^{ij}\\ &= \eta^{ij}\left(2V+\frac{1}{2}\kappa^2\frac{1}{\Delta}p\right)\end{aligned} \tag{3.107}$$

Next we will use again the relations from section 3.4.4 to simplify the second term in (3.103). We show here just two examples, but a more complete discussion can be found in the appendix D.

For the first example we note that $\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$ contains a term of the form: $-\frac{1}{8}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}k_{2\,\alpha_1}k_{3\,\beta_1}$. This means that $\kappa\left\langle\phi^{ij}\right\rangle$ includes a term of the form:

$$-\frac{1}{8}\kappa^4\int\frac{đ^4k_2đ^4k_3}{k_1^2k_2^2k_3^2}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}k_2^ik_3^jJ_{\mu_2\nu_2}\left(k_2\right)J_{\mu_3\nu_3}\left(k_3\right)\delta^4\left(k_1+k_2+k_3\right) \tag{3.108}$$

Again we are only interested in the V proportionalities, so we consider $\mu_2=\nu_2=\mu_3=\nu_3=0$. This leaves us with:

$$\begin{aligned}-\frac{1}{8}\kappa^4\int\frac{đ^4k_2đ^4k_3}{k_1^2k_2^2k_3^2}k_2^ik_3^j\mu\left(k_2\right)\mu\left(k_3\right)\delta^4\left(k_1+k_2+k_3\right) &= -\frac{1}{8}\kappa^4\int\frac{đ^3k_2đ^3k_3}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}k_2^ik_3^j\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right)\\ &\quad\times\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\\ &= -\frac{2}{\Delta}\left(\partial^iV\partial^jV\right)\end{aligned} \tag{3.109}$$

where we have used (3.89) in the last step.

As a second example we consider now the case $i = j$.
We know that $\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$ contains the term $-\frac{1}{4}\eta_{\alpha_2\beta_2\alpha_3\alpha_1}\eta_{\beta_3\beta_1}k_2 \cdot k_3$ and its permutations.
In particular it contains the term: $-\frac{1}{4}\eta_{\alpha_1\beta_1}\eta_{\alpha_3\alpha_2}\eta_{\beta_3\beta_2}k_1 \cdot k_3$.
So we focus on:

$$-\frac{\kappa^4}{4}\int \frac{đ^4 k_2 đ^4 k_3}{k_1^2 k_2^2 k_3^2} k_1 \cdot k_3 \eta^{ii}\eta^{\mu_3\mu_2}\eta^{\nu_3\nu_2} J_{\mu_2\nu_2}(k_2)\, J_{\mu_3\nu_3}(k_3)\, \delta^4(k_1 + k_2 + k_3) \tag{3.110}$$

After setting $\mu_2 = \nu_2 = \mu_3 = \nu_3 = 0$, this leads us to:

$$\begin{aligned}
-\frac{\kappa^4}{4}\int \frac{đ^4 k_2 đ^4 k_3}{k_1^2 k_2^2 k_3^2} k_1 \cdot k_3 (2\pi)^2 \delta\left(k_2^0\right)\delta\left(k_3^0\right) & \\
\times \mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right)\delta^3\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right) &= -\frac{\kappa^4}{4}\int \frac{đ^3 k_2 đ^3 k_3}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\vec{k}_1 \cdot \vec{k}_3 \mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&\quad \times \delta^3\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right) \\
&= \frac{\kappa^4}{4}\int \frac{đ^3 k_2 đ^3 k_3}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\left(\vec{k}_2 + \vec{k}_3\right)\cdot \vec{k}_3 \mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&\quad \times \delta^3\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right) \\
&= \frac{\kappa^4}{4}\int \frac{đ^3 k_2 đ^3 k_3}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\left(\vec{k}_2 \cdot \vec{k}_3 + \vec{k}_3^2\right)\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&\quad \times \delta^3\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right) \\
&= 4\left[\frac{1}{\Delta}(V\Delta V) + \frac{1}{\Delta}\eta_{ij}\partial^i V \partial^j V\right]
\end{aligned} \tag{3.111}$$

All these computations can be implemented on a computer and the final result that we get is:

$$\kappa\left\langle\phi^{ij}\right\rangle_J = \left(2V - \frac{\kappa^2}{2}\frac{1}{\Delta}p + \frac{4}{\Delta}\left(\eta_{kl}\partial^k V \partial^l\right)\right)\eta^{ij} + \frac{4}{\Delta}\left(\partial^i V \partial^j V\right) \tag{3.112}$$

Again the result obtained using the vertex from [5] coincides with this one, while the one calculated in 3.3.2 yields:

$$\kappa\left\langle\phi^{ij}\right\rangle_J = \left(2V - \frac{\kappa^2}{2}\frac{1}{\Delta}p - \frac{8}{\Delta}\left(\eta_{kl}\partial^k V \partial^l\right)\right)\eta^{ij} + \frac{4}{\Delta}\left(\partial^i V \partial^j V\right) \quad , \tag{3.113}$$

i.e. the only difference is in the third term, where there is a factor of -8 instead of a factor of 4.

3.5.3 Final reformulation of the result

In order to write our results in a better form we need to insert the expression for V that is known from (3.86).
We will make use of the following relations:

$$\frac{1}{\Delta}V\Delta V = \left(\frac{6}{5r\varepsilon}\right)G^2\lambda^2\theta(r-\varepsilon) + \left(\frac{15}{8\varepsilon^2} - \frac{3r^2}{4\varepsilon^4} + \frac{3}{40}\frac{r^4}{\varepsilon^6}\right)G^2\lambda^2\theta(\varepsilon - r) \tag{3.114}$$

$$\frac{1}{\Delta}\eta_{kl}\partial^k V\partial^l V = \left(\frac{1}{2r^2} - \frac{6}{5r\varepsilon}\right) G^2\lambda^2\theta\left(r-\varepsilon\right) + \left(-\frac{3}{4\varepsilon^2} + \frac{r^4}{20\varepsilon^6}\right) G^2\lambda^2\theta\left(\varepsilon - r\right) \tag{3.115}$$

$$\frac{\kappa^2}{4}\frac{1}{\Delta}p = \left(-\frac{1}{5r\varepsilon}\right) G^2\lambda^2\theta\left(r-\varepsilon\right) + \left(-\frac{3}{8\varepsilon^2} + \frac{r^2}{4\varepsilon^4} - \frac{3r^4}{40\varepsilon^6}\right) G^2\lambda^2\theta\left(\varepsilon - r\right) \tag{3.116}$$

A proof of these relations can be found in the appendix E.

Using these relations, we have for $r > \varepsilon$:

$$\begin{aligned}
\kappa\left\langle\phi^{00}\right\rangle_J &= 2V + \frac{3\kappa^2}{2}\frac{1}{\nabla^2}p - \frac{4}{\nabla^2}\left(\eta_{kl}\partial^k V\partial^l V\right) - \frac{8}{\nabla^2}\left(V\Delta V\right) \\
&= -\frac{2G\lambda}{r} + 6\left(-\frac{1}{5r\varepsilon}\right) G^2\lambda^2 + \left[-4\left(\frac{1}{2r^2} - \frac{6}{5r\varepsilon}\right) - 8\frac{6}{5r\varepsilon}\right] G^2\lambda^2 \\
&= -\frac{2G\lambda}{r} + \left(-\frac{2}{r^2} - \frac{6}{r\varepsilon}\right) G^2\lambda^2
\end{aligned} \tag{3.117}$$

and for $r < \varepsilon$:

$$\begin{aligned}
\kappa\left\langle\phi^{00}\right\rangle_J &= 2V + \frac{3\kappa^2}{2}\frac{1}{\nabla^2}p - \frac{4}{\nabla^2}\left(\eta_{kl}\partial^k V\partial^l V\right) - \frac{8}{\nabla^2}\left(V\Delta V\right) \\
&= G\lambda\left(-\frac{3}{\varepsilon} + \frac{r^2}{\varepsilon^3}\right) + 6\left(-\frac{3}{8\varepsilon^2} + \frac{r^2}{4\varepsilon^4} - \frac{3r^4}{40\varepsilon^6}\right) G^2\lambda^2 \\
&\quad + G^2\lambda^2\left(\frac{12}{4\varepsilon^2} - \frac{4r^4}{20\varepsilon^6} - \frac{15}{\varepsilon^2} + \frac{24r^2}{4\varepsilon^4} - \frac{3}{5}\frac{r^4}{\varepsilon^6}\right) \\
&= G\lambda\left(-\frac{3}{\varepsilon} + \frac{r^2}{\varepsilon^3}\right) + \left(-\frac{57}{4\varepsilon^2} + \frac{15}{2}\frac{r^2}{\varepsilon^4} - \frac{5}{4}\frac{r^4}{\varepsilon^6}\right) G^2\lambda^2
\end{aligned} \tag{3.118}$$

Now we can do the same for $\kappa\left\langle\phi^{ij}\right\rangle_J$, but we first need one more relation:

$$\partial^i V\partial^j V = \Delta\left[\left(\frac{1}{4r^2} - \frac{2}{5r\varepsilon}\right)\eta^{ij} - \frac{x^i x^j}{4r^4}\right] G^2\lambda^2 \qquad \text{for } r > \varepsilon \tag{3.119}$$

A proof of this relation can be found in the appendix E.6.

With this we obtain:

$$\begin{aligned}
\kappa\left\langle\phi^{ij}\right\rangle_J &= 2V\eta^{ij} - \frac{\kappa^2}{2}\frac{1}{\nabla^2}p + \frac{4}{\nabla^2}\left(\eta_{kl}\partial^k V\partial^l\right)\eta^{ij} + \frac{4}{\nabla^2}\left(\partial^i V\partial^j V\right) \\
&= -\frac{2G\lambda}{r}\eta^{ij} - 2\left(-\frac{1}{5r\varepsilon}G^2\lambda^2\right)\eta^{ij} + 4G^2\lambda^2\left(\frac{1}{2r^2} - \frac{6}{5r\varepsilon}\right)\eta^{ij} \\
&\quad + 4G^2\lambda^2\left[\left(\frac{1}{4r^2} - \frac{2}{5r\varepsilon}\right)\eta^{ij} - \frac{x^i x^j}{4r^4}\right] \\
&= \eta^{ij}\left(-\frac{2G\lambda}{r} + \frac{3}{r^2}G^2\lambda^2 - \frac{6G^2\lambda^2}{r\varepsilon}\right) - G^2\lambda^2\frac{x^i x^j}{r^4} + \mathcal{O}\left(G^3\right)
\end{aligned} \tag{3.120}$$

for $r > \varepsilon$.

Inserting now (3.78) in the expression for $\kappa \langle \phi^{00} \rangle_J$ for $r > \varepsilon$ we obtain finally:

$$\begin{aligned} g^{00} &= -1 + \kappa \phi^{00} \\ &= -1 - \frac{2G}{r}\left(m - \frac{3Gm^2}{\varepsilon} + \mathcal{O}\left(G^2\right)\right) + \left(-\frac{2}{r^2} - \frac{6}{r\varepsilon}\right)\left(m - \frac{3Gm^2}{\varepsilon} + \mathcal{O}\left(G^2\right)\right) G^2 \\ &= -1 - \frac{2Gm}{r} - \frac{2G^2m^2}{r^2} + \mathcal{O}\left(G^3\right) \qquad \text{for } r > \varepsilon \end{aligned} \tag{3.121}$$

And doing the same for $r < \varepsilon$ the result is:

$$\begin{aligned} g^{00} &= -1 + \left(-\frac{3}{\varepsilon} + \frac{r^2}{\varepsilon^3}\right) G\lambda + \left(-\frac{57}{4\varepsilon^2} + \frac{15r^2}{2\varepsilon^4} - \frac{5r^4}{4\varepsilon^6}\right) G^2\lambda^2 + \mathcal{O}\left(G^3\right) \\ &= -1 + \left(-\frac{3}{\varepsilon} + \frac{r^2}{\varepsilon^3}\right) G\left(m - \frac{3Gm^2}{\varepsilon} + \mathcal{O}\left(G^2\right)\right) + \left(-\frac{57}{4\varepsilon^2} + \frac{15r^2}{2\varepsilon^4} - \frac{5r^4}{4\varepsilon^6}\right) G^2m^2 + \mathcal{O}\left(G^2\right) \\ &= -1 - \frac{3Gm}{\varepsilon} + \frac{r^2Gm}{\varepsilon^3} + G^2m^2\left(-\frac{21}{4\varepsilon^2} + \frac{9r^2}{2\varepsilon^4} - \frac{5r^4}{4\varepsilon^6}\right) + \mathcal{O}\left(G^3\right) \end{aligned} \tag{3.122}$$

And finally the exterior solution for the spatial part of the metric is:

$$\begin{aligned} g^{ij} &= \eta^{ij}\left(1 - \frac{2G\lambda}{r} + \frac{3}{r^2}G^2\lambda^2 - \frac{6G^2\lambda^2}{r\varepsilon}\right) - G^2\lambda^2\frac{x^ix^j}{r^4} + \mathcal{O}\left(G^3\right) \\ &= \eta^{ij}\left(1 - \frac{2Gm}{r} + \frac{6G^2m^2}{r\varepsilon} + \frac{3G^2m^2}{r^2} - \frac{6G^2m^2}{r\varepsilon}\right) - \frac{G^2m^2x^ix^j}{r^4} + \mathcal{O}\left(G^3\right) \\ &= \eta^{ij}\left(1 - \frac{2Gm}{r} + \frac{3G^2m^2}{r^2}\right) - \frac{G^2m^2x^ix^j}{r^4} + \mathcal{O}\left(G^3\right) \end{aligned} \tag{3.123}$$

These results coincide with the ones obtained in section 2.

4 $\phi^{\mu\nu}$ as the iterative solution to the equations of motion

In this section, we consider a different approach to the same problem.
We start by rewriting the Einstein-Hilbert action a bit differently. We split it into a part relevant for the equations of motion S^{eom} and a boundary term S^{∂}, which vanishes upon variation. S^{eom} can then be written as a Taylor expansion around the perturbation from flat spacetime $\phi^{\mu\nu}$.
After carefully variating S^{eom}, we can define the coupling of the gravitational field to an external source. The external source will of course be chosen to be the same as the one described in sections 2 and 3.
We will then obtain a final equation of motion that can be solved iteratively. We will first find a solution of order κ. Then we will use this solution to find further contributions, this time of order κ^3.
The main difference with respect to section 3 is that we are solving directly the equations of motion and not using the S-matrix formalism.
This will allow us to find a final expression for $\phi^{\mu\nu}$ that can be compared to the ones already obtained.

4.1 Reformulation of the Einstein-Hilbert action

One of the crucial parts of this different approach is to rewrite the Einstein-Hilbert action in a more compact way.
To this purpose we first of all split it into a boundary part, that will not contribute to the equations of motion, and an inner part, that will be relevant for the further computations.

Indeed the Einstein-Hilbert action $S^{EH}[g]$ can be rewritten as:

$$S^{EH}[g] = S^{\text{eom}}[g] + S^{\partial}[g] \tag{4.1}$$

with

$$S^{\text{eom}}[g] = \frac{1}{\kappa^2}\int d^4x\,(-g)^{\frac{1}{2}}\,g^{\mu\rho}\left(\Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\nu}{}_{\alpha\nu} - \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\nu}{}_{\alpha\mu}\right) \tag{4.2}$$

and

$$S^{\partial}[g] = \frac{1}{\kappa^2}\int d^4x\partial_\lambda T^\lambda \tag{4.3}$$

We can see this with a straightforward computation:

$$\begin{aligned}
S^{EH}[g] &= \frac{1}{\kappa^2}\int d^4x\,(-g)^{\frac{1}{2}}\,R \\
&= \frac{1}{\kappa^2}\int d^4x\,(-g)^{\frac{1}{2}}\,g^{\mu\rho}\left\{\partial_\lambda\Gamma^{\lambda}{}_{\mu\rho} - \partial_\rho\Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\lambda\nu}\Gamma^{\nu}{}_{\mu\rho} - \Gamma^{\lambda}{}_{\rho\nu}\Gamma^{\nu}{}_{\mu\lambda}\right\} \\
&= \underbrace{\frac{1}{\kappa^2}\int d^4x\,(-g)^{\frac{1}{2}}\,g^{\mu\rho}\left(\Gamma^{\lambda}{}_{\lambda\nu}\Gamma^{\nu}{}_{\mu\rho} - \Gamma^{\lambda}{}_{\rho\nu}\Gamma^{\nu}{}_{\mu\lambda}\right)}_{=S^{\text{eom}}[g]} \\
&\quad + \underbrace{\frac{1}{\kappa^2}\int d^4x\,(-g)^{\frac{1}{2}}\,g^{\mu\rho}\left(\partial_\lambda\Gamma^{\lambda}{}_{\mu\rho} - \partial_\rho\Gamma^{\lambda}{}_{\mu\lambda}\right)}_{=S^{\partial}[g]}
\end{aligned} \tag{4.4}$$

We can rewrite T^λ in a more compact form as:

$$\begin{aligned}
T^\lambda &= (-g)^{\frac{1}{2}} \left(g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} - g^{\lambda\mu}\Gamma^\nu{}_{\mu\nu}\right) \\
&= (-g)^{\frac{1}{2}} \left(g^{\mu\nu}\frac{1}{2}g^{\lambda\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) - g^{\lambda\mu}\frac{1}{2}g^{\nu\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)\right) \\
&= (-g)^{\frac{1}{2}} \frac{1}{2}\left[2g^{\mu\nu}g^{\lambda\alpha}g_{\alpha\mu,\nu} - g^{\mu\nu}g^{\lambda\alpha}g_{\mu\nu,\alpha} - g^{\lambda\mu}g^{\nu\alpha}g_{\alpha\mu,\nu} - g^{\lambda\mu}g^{\nu\alpha}g_{\alpha\nu,\mu} + g^{\lambda\mu}g^{\nu\alpha}g_{\mu\nu,\alpha}\right] \\
&= (-g)^{\frac{1}{2}} \left[g^{\mu\nu}g^{\lambda\alpha} - g^{\mu\alpha}g^{\lambda\nu}\right] g_{\alpha\mu,\nu} \\
&= (-g)^{\frac{1}{2}} \mathcal{M}^{\alpha\lambda\mu\nu}(g)\,\partial_\nu g_{\mu\alpha} \qquad (4.5)
\end{aligned}$$

where we defined:

$$\mathcal{M}^{\alpha\beta\mu\nu}(g) = g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\beta\nu} \qquad (4.6)$$

A bit more computations are needed to write S^{eom} in a more compact form, but, using a computer, they are straightforward too.
The result is:

$$S^{\text{eom}} = \int d^4x \partial_\alpha g_{\beta\gamma} M^{\alpha\beta\gamma\mu\nu\rho}(g)\,\partial_\mu g_{\nu\rho} \qquad (4.7)$$

with

$$M^{\alpha\beta\gamma\mu\nu\rho}(g) = \frac{1}{8}\left[g^{\alpha\mu}\mathcal{M}^{\beta\nu\gamma\rho} + g^{\alpha\mu}\mathcal{M}^{\nu\beta\rho\gamma} + 2g^{\alpha\rho}\mathcal{M}^{\beta\gamma\nu\mu} + 2g^{\mu\gamma}\mathcal{M}^{\mu\rho\beta\alpha}\right](-g)^{\frac{1}{2}} \qquad (4.8)$$

Now we can expand g around the flat metric, i.e. set $g^{\mu\nu} = \eta^{\mu\nu} + \kappa\phi^{\mu\nu}$. As seen in section 3, for the inverse we have $g_{\mu\nu} = \eta_{\mu\nu} - \kappa\phi_{\mu\nu} + \mathcal{O}(\kappa^2)$.
Note that $\partial_\alpha g_{\beta\gamma} = -\kappa\partial_\alpha\phi_{\beta\gamma}$.
Furthermore we can introduce a directional functional derivative:

$$D_\phi[g] = -\kappa\int d^4x \phi_{\alpha\beta}(x)\frac{\delta}{\delta g_{\alpha\beta}} \qquad (4.9)$$

With this notation we can write down the Taylor expansion of $M(g)$ around ϕ:

$$\begin{aligned}
M(g) = M(\eta + \kappa\phi) &= \exp\{D_\phi[g]\}\,M(g)\big|_{g=\eta} \\
&= \sum_{n=0}^{\infty}\frac{(D_\phi[g])^n}{n!}M(g)\bigg|_{g=\eta} \\
&= M(\eta) - \kappa\int d^4x\phi_{\alpha\beta}(x)\frac{\delta}{\delta g_{\alpha\beta}(x)}M(g)\bigg|_{g=\eta} + \mathcal{O}\left(h^2\right) \qquad (4.10)
\end{aligned}$$

This allows us to obtain our final form for $S^{\text{eom}}[g]$:

$$\begin{aligned}
S^{\text{eom}}[g] &= \int d^4x\left(\partial_\alpha\phi_{\beta\gamma}\right)(x)\,M^{\alpha\beta\gamma\mu\nu\rho}(\eta + \kappa\phi)(x)\left(\partial_\mu\phi_{\nu\rho}\right)(x) \\
&= \int d^4x\left(\partial_\alpha\phi_{\beta\gamma}\right)(x)\left[\exp\{D_\phi[g]\}\,M^{\alpha\beta\gamma\mu\nu\rho}(g)\right]_{g=\eta}(x)\left(\partial_\mu\phi_{\nu\rho}\right)(x) \qquad (4.11)
\end{aligned}$$

4.2 Variation of S^D

Now that we have obtained this very compact form of writing S^{eom}, we can vary it with respect to the perturbation ϕ from flat spacetime. I.e. the goal of this section is to compute $\frac{\delta}{\delta\phi_{\mu\nu}} S^{\text{eom}}[\phi]$.

Using the product rule we can immediately obtain:

$$\begin{aligned}
\frac{\delta}{\delta\phi_{\mu\nu}(x)} S^{\text{eom}}[\phi] &= \frac{\delta}{\delta\phi_{\mu\nu}(x)} \left\{ \int d^4y \left(\partial_\alpha\phi_{\beta\gamma}\right)(y) \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\sigma\tau} (y) \left(\partial_\rho\phi_{\sigma\tau}\right)(y) \right\} \\
&= \int d^4y \frac{\delta\left(\partial_\alpha\phi_{\beta\gamma}\right)(y)}{\delta\phi_{\mu\nu}(x)} \left\{ \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\sigma\tau} (y) \left(\partial_\rho\phi_{\sigma\tau}\right)(y) \right\} \\
&\quad + \int d^4y \left\{ \left(\partial_\alpha\phi_{\beta\gamma}\right)(y) \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\sigma\tau} (y) \right\} \frac{\delta\left(\partial_\rho\phi_{\sigma\tau}\right)(y)}{\delta\phi_{\mu\nu}(x)} \\
&\quad + \int d^4y \left(\partial_\alpha\phi_{\beta\gamma}\right)(y) \left\{ \frac{\delta}{\delta\phi_{\mu\nu}(x)} \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\sigma\tau} (y) \right\} \left(\partial_\rho\phi_{\sigma\tau}\right)(y) \\
&= -\int d^4y \frac{\delta\phi_{\beta\gamma}(y)}{\delta\phi_{\mu\nu}(x)} \partial_\alpha \left\{ \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\sigma\tau} \left(\partial_\rho\phi_{\sigma\tau}\right) \right\} (y) \\
&\quad - \int d^4y \partial_\rho \left\{ \left(\partial_\alpha\phi_{\beta\gamma}\right) \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\sigma\tau} \right\} (y) \frac{\delta\phi_{\sigma\tau}(y)}{\delta\phi_{\mu\nu}(x)} \\
&\quad + \int d^4y \left(\partial_\alpha\phi_{\beta\gamma}\right)(y) \left\{ \frac{\delta}{\delta\phi_{\mu\nu}(x)} \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\sigma\tau} (y) \right\} \left(\partial_\rho\phi_{\sigma\tau}\right)(y) \\
&= -\partial_\alpha \left\{ \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\mu\nu\rho\sigma\tau} \left(\partial_\rho\phi_{\sigma\tau}\right) \right\} (x) \\
&\quad -\partial_\rho \left\{ \left(\partial_\alpha\phi_{\beta\gamma}\right) \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\mu\nu} \right\} (x) \\
&\quad + \int d^4y \left(\partial_\alpha\phi_{\beta\gamma}\right)(y) \left\{ \frac{\delta}{\delta\phi_{\mu\nu}(x)} \left[\exp\left\{D_\phi[g]\right\} M(g)\big|_{g=\eta} \right]^{\alpha\beta\gamma\rho\sigma\tau} (y) \right\} \\
&\quad \times \left(\partial_\rho\phi_{\sigma\tau}\right)(y)
\end{aligned} \tag{4.12}$$

where we also used integration by parts.

Let us now make a couple of considerations.
First of all we note that:

$$\begin{aligned}
\frac{\delta}{\delta\phi_{\mu\nu}(x)} D_\phi[g] &= \frac{\delta}{\delta\phi_{\mu\nu}(x)} \left[-\kappa \int d^4y \phi_{\alpha\beta}(y) \frac{\delta}{\delta g_{\alpha\beta}(y)} \right] \\
&= -\kappa \int d^4y \frac{\delta\phi_{\alpha\beta}(y)}{\delta\phi_{\mu\nu}(x)} \frac{\delta}{\delta g_{\alpha\beta}(y)} \\
&= -\kappa \int d^4y \frac{1}{2} \left(\delta^\mu{}_\alpha \delta^\nu{}_\beta + \delta^\mu{}_\beta \delta^\nu{}_\alpha \right) \delta(y-x) \frac{\delta}{\delta g_{\alpha\beta}(y)} \\
&= -\kappa \frac{\delta}{\delta g_{\mu\nu}(x)}
\end{aligned} \tag{4.13}$$

Secondly we can use this to vary $\exp\{D_\phi[g]\}$:

$$\begin{aligned}\frac{\delta}{\delta\phi_{\mu\nu}(x)}\exp\{D_\phi[g]\} &= \frac{\delta}{\delta\phi_{\mu\nu}(x)}\sum_{n=0}^{\infty}\frac{(D_\phi(g))^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{n(D_\phi[g])^{n-1}}{n!}\frac{\delta D_\phi[g]}{\delta\phi_{\mu\nu}(x)}\\ &= -\kappa\sum_{n=1}^{\infty}\frac{(D_h[g])^{n-1}}{(n-1)!}\frac{\delta}{\delta g_{\mu\nu}(x)}\\ &= -\kappa\exp\{D_\phi[g]\}\frac{\delta}{\delta g_{\mu\nu}(x)}\end{aligned}\tag{4.14}$$

We may now use this and $\frac{\delta}{\delta g_{\mu\nu}(x)}M(g)(y)=\frac{\partial M(g)}{\partial g_{\mu\nu}}\delta(y-x)$ to go on with the variation of S^{eom}:

$$\begin{aligned}\frac{\delta}{\delta\phi_{\mu\nu}(x)}S^{\text{eom}}[\phi] &= -\partial_\alpha\left\{\left[\exp\{D_\phi[g]\}M(g)|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}(\partial_\rho\phi_{\sigma\tau})\right\}(x)\\ &\quad -\partial_\rho\left\{(\partial_\alpha\phi_{\beta\gamma})\left[\exp\{D_\phi[g]\}M(g)|_{g=\eta}\right]^{\alpha\beta\gamma\rho\mu\nu}\right\}(x)\\ &\quad -\kappa\int d^4y\,(\partial_\alpha\phi_{\beta\gamma})(y)\left[\exp\{D_\phi[g]\}\frac{\partial M(g)}{\partial g_{\mu\nu}}\delta(y-x)\Big|_{g=\eta}\right]^{\alpha\beta\gamma\rho\sigma\tau}(y)(\partial_\rho\phi_{\sigma\tau})(y)\\ &= -\partial_\alpha\left\{\left[\exp\{D_\phi[g]\}M(g)|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}(\partial_\rho\phi_{\sigma\tau})\right\}(x)\\ &\quad -\partial_\rho\left\{(\partial_\alpha\phi_{\beta\gamma})\left[\exp\{D_\phi[g]\}M(g)|_{g=\eta}\right]^{\alpha\beta\gamma\rho\mu\nu}\right\}(x)\\ &\quad -\kappa(\partial_\alpha\phi_{\beta\gamma})(x)\left[\exp\{D_\phi[g]\}\frac{\partial M(g)}{\partial g_{\mu\nu}}\Big|_{g=\eta}\right]^{\alpha\beta\gamma\rho\sigma\tau}(x)(\partial_\rho\phi_{\sigma\tau})(x)\\ &= -2\left(\partial_\alpha\left[\exp\{D_\phi[g]\}M(g)|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}\right)(x)(\partial_\rho\phi_{\sigma\tau})(x)\\ &\quad -2\left[\exp\{D_\phi[g]\}M(g)|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}(x)(\partial_\alpha\partial_\rho\phi_{\sigma\tau})(x)\\ &\quad -\kappa(\partial_\alpha\phi_{\beta\gamma})(x)\left[\exp\{D_\phi[g]\}\frac{\partial M(g)}{\partial g_{\mu\nu}}\Big|_{g=\eta}\right]^{\alpha\beta\gamma\rho\sigma\tau}(x)(\partial_\rho\phi_{\sigma\tau})(x)\end{aligned}\tag{4.15}$$

where we have relabeled some indices and used the symmetry property $M^{\alpha\beta\gamma\mu\nu\rho}=M^{\mu\nu\rho\alpha\beta\gamma}$.

We are now going to expand this first variation in orders of ϕ. This corresponds to an expansion in powers of κ, since each ϕ comes with such a factor.
Note that every term in the expansion has at least one power of ϕ, and the last term in (4.15) has at least two.
There are of course no 0-th order terms, since we haven't coupled the field to an external source yet.
We can summarize the expansion as:

$$\frac{\delta S^{\text{eom}}[\phi]}{\delta\phi_{\mu\nu}}=\delta S^{\text{eom}}_{(1)}+\delta S^{\text{eom}}_{(2)}+\mathcal{O}\left(\phi^3\right)\tag{4.16}$$

Taking only the terms containing one power of ϕ, we get:

$$\begin{aligned}
\delta S^{\text{eom}}_{(1)} &= -2\underbrace{\left(\partial_\alpha M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\right)}_{=0}(x)\left(\partial_\rho\phi_{\sigma\tau}\right)(x) - 2M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\left(\partial_\alpha\partial_\rho\phi_{\sigma\tau}\right)(x) \\
&= -2M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\left(\partial_\alpha\partial_\rho\phi_{\sigma\tau}\right)(x) \qquad (4.17)
\end{aligned}$$

Analogously, but remembering this time to take into account the last term in (4.15), we can compute:

$$\begin{aligned}
\delta S^{\text{eom}}_{(2)} &= -2\left(\partial_\alpha\left[D_\phi[g]\,M(g)\big|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}\right)(x)\left(\partial_\rho\phi_{\sigma\tau}\right)(x) \\
&\quad -2\left[D_\phi[g]\,M(g)\big|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}(x)\left(\partial_\alpha\partial_\rho\phi_{\mu\nu}\right)(x) \\
&\quad -\kappa\left(\partial_\alpha\phi_{\beta\gamma}\right)(x)\left.\frac{\partial M^{\alpha\beta\gamma\rho\sigma\tau}(g)}{\partial g_{\mu\nu}}\right|_{g=\eta}\left(\partial_\rho\phi_{\sigma\tau}\right)(x) \qquad (4.18)
\end{aligned}$$

In order to simplify further this expression, we need to know that:

$$\begin{aligned}
D_\phi[g]\,M(g)(x) &= -\kappa\int d^4y\,\phi_{\alpha\beta}(y)\frac{\delta}{\delta g_{\alpha\beta}(y)}M(g)(x) \\
&= -\kappa\int d^4y\,\phi_{\alpha\beta}(y)\frac{\partial M}{\partial g_{\alpha\beta}}\delta(y-x) \\
&= -\kappa\phi_{\alpha\beta}(x)\frac{\partial M}{\partial g_{\alpha\beta}}(x) \qquad (4.19)
\end{aligned}$$

Then $\delta S^{\text{eom}}_{(2)}$ can be written as:

$$\begin{aligned}
\delta S^{\text{eom}}_{(2)} &= 2\kappa\left(\partial_\alpha\left[\phi_{\alpha_1\beta_1}\left.\frac{\partial M^{\alpha\mu\nu\rho\sigma\tau}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\right]\right)(x)\left(\partial_\rho\phi_{\sigma\tau}\right)(x) + 2\kappa\phi_{\alpha_1\beta_1}\left.\frac{\partial M^{\alpha\mu\nu\rho\sigma\tau}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}(x)\left(\partial_\alpha\partial_\rho\phi_{\sigma\tau}\right)(x) \\
&\quad -\kappa\left(\partial_\alpha\phi_{\beta\gamma}\right)(x)\left.\frac{\partial M^{\alpha\beta\gamma\rho\sigma\tau}}{\partial g_{\mu\nu}}\right|_{g=\eta}\left(\partial_\rho\phi_{\sigma\tau}\right)(x) \qquad (4.20)
\end{aligned}$$

4.3 Coupling to an external source and corresponding equations of motion

Let us now introduce the coupling to an external source:

$$S^{\text{source}}[\phi] = \kappa\int d^4x\,\phi_{\alpha\beta}(x)\,J^{\alpha\beta}(x) \qquad (4.21)$$

Note that since $J^{\alpha\beta}$ represents an external source, its variation with respect to ϕ vanishes. Thus the first variation of S^{source} is:

$$\frac{\delta S^{\text{source}}[\phi]}{\delta\phi_{\mu\nu}(x)} = \kappa J^{\mu\nu}(x) \qquad (4.22)$$

After imposing the coupling to an external source, the equations of motion coming from imposing $\delta\left(S^{\text{eom}}+S^{\text{source}}\right)=0$ are then:

$$\frac{\delta S^{\text{eom}}}{\delta\phi_{\mu\nu}(x)}=-\kappa J^{\mu\nu}(x) \tag{4.23}$$

We already know the expression for $\frac{\delta S^{\text{eom}}}{\delta\phi_{\mu\nu}(x)}$ from (4.15), so that we obtain:

$$\begin{aligned}
-2\left(\partial_\alpha\left[\exp\{D_\phi[g]\}M(g)|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}\right)(x)\left(\partial_\rho\phi_{\sigma\tau}\right)(x)&\\
-2\left[\exp\{D_\phi[g]\}M(g)|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}\left(\partial_\alpha\partial_\rho\phi_{\sigma\tau}\right)(x)&\\
-\kappa\left(\partial_\alpha\phi_{\beta\gamma}\right)(x)\left[\exp\{D_\phi[g]\}\left.\frac{\partial M(g)}{\partial g_{\mu\nu}}\right|_{g=\eta}\right]^{\alpha\beta\gamma\rho\sigma\tau}\left(\partial_\rho\phi_{\sigma\tau}\right)(x)&=-\kappa J^{\mu\nu}(x)
\end{aligned} \tag{4.24}$$

We can now write down the following expansion of $\exp\{D_\phi[g]\}M(g)|_{g=\eta}$:

$$\exp\{D_\phi[g]\}M(g)|_{g=\eta}=M(\eta)+D_\phi[g]M(g)|_{g=\eta}+\sum_{n=2}^{\infty}\left.\frac{(D_\phi[g])^n}{n!}M(g)\right|_{g=\eta} \tag{4.25}$$

and use it to rewrite (4.24) up to $\mathcal{O}\left(\phi^3\right)$:

$$\begin{aligned}
-2\left(\partial_\alpha\left[D_\phi[g]M(g)|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}\right)(x)\left(\partial_\rho\phi_{\sigma\tau}\right)(x)&\\
-2M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\left(\partial_\alpha\partial_\rho\phi_{\sigma\tau}\right)(x)&\\
-2\left[D_\phi[g]M(g)|_{g=\eta}\right]^{\alpha\mu\nu\rho\sigma\tau}(x)\left(\partial_\alpha\partial_\rho\phi_{\sigma\tau}\right)(x)&\\
-\kappa\left(\partial_\alpha\phi_{\beta\gamma}\right)(x)\left.\frac{\partial M}{\partial g_{\mu\nu}}\right|_{g=\eta}^{\alpha\beta\gamma\rho\sigma\tau}\left(\partial_\rho\phi_{\sigma\tau}\right)(x)+\kappa J^{\mu\nu}(x)&=\mathcal{O}\left(h^3\right)
\end{aligned} \tag{4.26}$$

where we used that $\partial_\alpha M(\eta)=0$. This corresponds to setting $\delta S^{\text{eom}}_{(1)}+\delta S^{\text{eom}}_{(2)}+\kappa J^{\mu\nu}(x)=0$. We can further reformulate this expression by first noting that:

$$\begin{aligned}
D_\phi[g]M(g)|_{g=\eta}(x)&=-\kappa\int d^4y\,\phi_{\alpha\beta}(y)\left.\frac{\partial M}{\partial g_{\alpha\beta}}\delta(x-y)\right|_{g=\eta}\\
&=-\kappa\phi_{\alpha\beta}(x)\left.\frac{\partial M}{\partial g_{\alpha\beta}}\right|_{g=\eta}
\end{aligned} \tag{4.27}$$

This means that we can finally write the equations of motion including the coupling to the external source as:

$$\begin{aligned}
2\kappa\partial_\alpha\phi_{\alpha_1\beta_1}(x)\left.\frac{\partial M^{\alpha\mu\nu\rho\sigma\tau}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\left(\partial_\rho\phi_{\sigma\tau}\right)(x)-2M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\left(\partial_\alpha\partial_\rho\phi_{\sigma\tau}\right)(x)&\\
+2\kappa\phi_{\alpha_1\beta_1}(x)\left.\frac{\partial M^{\alpha\mu\nu\rho\sigma\tau}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\left(\partial_\alpha\partial_\rho\phi_{\sigma\tau}\right)(x)&\\
-\kappa\left(\partial_\alpha\phi_{\beta\gamma}\right)(x)\left.\frac{\partial M^{\alpha\beta\gamma\rho\sigma\tau}}{\partial g_{\mu\nu}}\right|_{g=\eta}\left(\partial_\rho\phi_{\sigma\tau}\right)(x)+\kappa J^{\mu\nu}(x)&=\mathcal{O}\left(\phi^3\right)
\end{aligned} \tag{4.28}$$

4.4 Iterative solution of the equations of motion

It is now possible to solve the equations of motion iteratively.

Let us start by considering the following expansion of ϕ:

$$\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \dots \tag{4.29}$$

where $\phi^{(0)}$ is proportional to κ, $\phi^{(1)}$ is proportional to κ^3, and so on.
Each iteration step corresponds to taking new diagrams into account. As we will see soon, $\phi^{(0)}$ takes into account just one coupling with the external source, $\phi^{(1)}$ considers two of them and so on. This is represented in Figure 5.

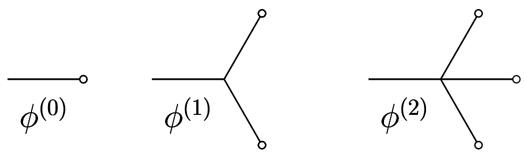

Figure 5: Each iteration step corresponds to taking new diagrams into account.

The differential equation to determine $\phi^{(0)}$ must then be:

$$-2M^{\alpha\mu\nu\rho\sigma\tau}\left(\eta\right)\left(\partial_\alpha\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x) = -\kappa J^{\mu\nu}\left(x\right) \tag{4.30}$$

i.e.

$$M^{\alpha\mu\nu\rho\sigma\tau}\left(\eta\right)\left(\partial_\alpha\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x) = \frac{\kappa}{2}J^{\mu\nu}\left(x\right) \tag{4.31}$$

Using the method of Green's function we see that the solution must be:

$$\phi^{(0)}_{\sigma\tau}\left(x\right) = \frac{\kappa}{2}\int d^4z G^{(0)}_{\sigma\tau\mu\nu}\left(x-z\right)J^{\mu\nu}\left(z\right) \tag{4.32}$$

where $G^{(0)}_{\sigma\tau\mu\nu}$ satisfies:

$$M^{\alpha\mu\nu\rho\sigma\tau}\left(\eta\right)\partial_\alpha\partial_\rho G^{(0)}_{\sigma\tau\alpha_1\beta_1}\left(x-y\right) = \delta^{\mu\nu}{}_{\alpha_1\beta_1}\delta\left(x-y\right) \tag{4.33}$$

Now we consider the equation of motion of the next order, that is:

$$\begin{aligned}2\kappa\partial_\alpha\phi^{(0)}_{\alpha_1\beta_1}(x)\left.\frac{\partial M^{\alpha\mu\nu\rho\sigma\tau}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\left(\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x)-2M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\left(\partial_\alpha\partial_\rho\phi^{(1)}_{\sigma\tau}\right)(x)&\\ +2\kappa\phi^{(0)}_{\alpha_1\beta_1}(x)\left.\frac{\partial M^{\alpha\mu\nu\rho\sigma\tau}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\left(\partial_\alpha\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x)&\\ -\kappa\left(\partial_\alpha\phi^{(0)}_{\beta\gamma}\right)(x)\left.\frac{\partial M^{\alpha\beta\gamma\rho\sigma\tau}}{\partial g_{\mu\nu}}\right|_{g=\eta}\left(\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x)&=0 \qquad (4.34)\end{aligned}$$

where we have already used the fact that $-2M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\left(\partial_\alpha\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x)-\frac{\kappa}{2}J^{\mu\nu}(x)=0$.

Solving this for $M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\,\partial_\alpha\partial_\rho\phi^{(1)}_{\sigma\tau}$ gives us:

$$\begin{aligned}M^{\alpha\mu\nu\rho\sigma\tau}(\eta)\,\partial_\alpha\partial_\rho\phi^{(1)}_{\sigma\tau}(x)&=\kappa\partial_\alpha\phi^{(0)}_{\alpha_1\beta_1}(x)\left.\frac{\partial M^{\alpha\mu\nu\rho\sigma\tau}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\left(\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x)\\ &+\kappa\phi^{(0)}_{\alpha_1\beta_1}(x)\left.\frac{\partial M^{\alpha\mu\nu\rho\sigma\tau}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\left(\partial_\alpha\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x)\\ &-\frac{1}{2}\kappa\left(\partial_\alpha\phi^{(0)}_{\beta\gamma}\right)(x)\left.\frac{\partial M^{\alpha\beta\gamma\rho\sigma\tau}}{\partial g_{\mu\nu}}\right|_{g=\eta}\left(\partial_\rho\phi^{(0)}_{\sigma\tau}\right)(x) \qquad (4.35)\end{aligned}$$

Which leads to the following expression for $\phi^{(1)}_{\sigma\tau}$:

$$\begin{aligned}\phi^{(1)}_{\sigma\tau}(x)&=\kappa\int d^4yG^{(0)}_{\sigma\tau\mu\nu}(x-y)\left(\partial_\alpha\phi^{(0)}_{\alpha_1\beta_1}\right)(y)\left.\frac{\partial M^{\alpha\mu\nu\rho\alpha_2\beta_2}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\left(\partial_\rho\phi^{(0)}_{\alpha_2\beta_2}\right)(y)\\ &+\kappa\int d^4yG^{(0)}_{\sigma\tau\mu\nu}\phi^{(0)}_{\alpha_1\beta_1}(y)\left.\frac{\partial M^{\alpha\mu\nu\rho\alpha_2\beta_2}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\left(\partial_\alpha\partial_\rho\phi^{(0)}_{\alpha_2\beta_2}\right)(y)\\ &-\frac{1}{2}\kappa\int d^4yG^{(0)}_{\sigma\tau\mu\nu}(x-y)\left(\partial_\alpha\phi^{(0)}_{\beta\gamma}\right)(y)\left.\frac{\partial M^{\alpha\beta\gamma\rho\alpha_2\beta_2}}{\partial g_{\mu\nu}}\right|_{g=\eta}\left(\partial_\rho\phi^{(0)}_{\alpha_2\beta_2}\right)(y) \qquad (4.36)\end{aligned}$$

We can write this equation in Fourier space as:

$$\begin{aligned}\phi^{(1)}_{\sigma\tau}(k_1)&=-\kappa\left.\frac{\partial M^{\alpha\mu\nu\rho\alpha_2\beta_2}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\int đ^4k_2đ^4k_3\,k_{2\,\alpha}k_{3\,\rho}G^{(0)}_{\sigma\tau\mu\nu}(k_1)\,\phi^{(0)}_{\alpha_1\beta_1}(k_2)\,\phi^{(0)}_{\alpha_2\beta_2}(k_3)\,\delta(k_1+k_2+k_3)\\ &-\kappa\left.\frac{\partial M^{\alpha\mu\nu\rho\alpha_2\beta_2}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta}\int đ^4k_2đ^4k_3\,k_{3\,\alpha}k_{3\,\rho}G^{(0)}_{\sigma\tau\mu\nu}(k_1)\,\phi^{(0)}_{\alpha_1\beta_1}(k_2)\,\phi^{(0)}_{\alpha_2\beta_2}(k_3)\,\delta(k_1+k_2+k_3)\\ &+\frac{\kappa}{2}\left.\frac{\partial M^{\alpha\beta\gamma\rho\alpha_2\beta_2}}{\partial g_{\mu\nu}}\right|_{g=\eta}\int đ^4k_2đ^4k_3\,k_{2\,\alpha}k_{3\,\rho}G^{(0)}_{\sigma\tau\mu\nu}(k_1)\,\phi^{(0)}_{\beta\gamma}(k_2)\,\phi^{(0)}_{\alpha_2\beta_2}(k_3)\\ &\times\delta(k_1+k_2+k_3) \qquad (4.37)\end{aligned}$$

4.5 Comparison with the previous results

4.5.1 Gauge fixing

At this point, we can use part of what we did in section 3.2 to fix the gauge. We need to do this in order to compute the explicit expressions for $\phi^{(0)}$ and $\phi^{(1)}$.

We already know the form of the free propagator in the harmonic gauge:

$$G^{(0)\mu\nu\rho\sigma}\left(k^2\right) = \frac{1}{2}\frac{1}{k^2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right) \tag{4.38}$$

Its inverse gives us the kinetic part of the Lagrangian after fixing the gauge:

$$\left(G^{(0)-1}\right)^{\mu\nu\rho\sigma}\left(k^2\right) = \frac{1}{2}k^2\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right) \tag{4.39}$$

Thus instead of (4.31), our 0-th order equation of motion can be rewritten after gauge fixing as:

$$M_G^{\alpha\mu\nu\rho\sigma\tau}\left(\eta\right)\left(\partial_\alpha\partial_\rho\phi^{(0)}_{\sigma\tau}\right)\left(x\right) = \frac{\kappa}{2}J^{\mu\nu}\left(x\right) \tag{4.40}$$

where

$$M_G^{\alpha\mu\nu\rho\sigma\tau}\left(\eta\right) = \frac{1}{2}\left(\eta^{\mu\tau}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\tau} - \eta^{\mu\nu}\eta^{\tau\sigma}\right)\eta^{\alpha\rho} \tag{4.41}$$

4.5.2 0-th order

Let us now proceed to solving the equation of motion (4.31) for $\phi^{(0)}$.
We use the same external source and set thus $J_{\mu\nu}\left(x\right) = T_{\mu\nu}\left(x\right)$, where $T_{\mu\nu}$ is the energy-momentum tensor defined in section 3.4.1.
This yields:

$$\phi^{(0)\,\alpha\beta}\left(x\right) = \frac{\kappa}{2}\int d^4z G^{(0)\alpha\beta\mu\nu}\left(x-z\right)T_{\mu\nu}\left(z\right) \tag{4.42}$$

In Fourier space this is simply the product of the Fourier transforms:

$$\phi^{(0)\alpha\beta}\left(k\right) = \frac{\kappa}{2}G^{(0)\alpha\beta\mu\nu}\left(k\right)T_{\mu\nu}\left(k\right) \tag{4.43}$$

This is in perfect harmony with the first term in the formula (3.73) for the vacuum expectation value obtained previously.

Thus the results will be the same, i.e.:

$$\kappa\phi^{(0)00} = 2V + \frac{3}{2}\frac{\kappa^2}{\Delta}p \tag{4.44}$$

and

$$\kappa\phi^{(0)ij} = \left(2V - \frac{\kappa^2}{2}\frac{1}{\Delta}p\right)\eta^{ij} \tag{4.45}$$

4.5.3 1-st order

We now turn to the equation (4.37) and compute $\phi^{(0)\,\sigma\tau}$ explicitly.
We can rewrite (4.37) as:

$$\begin{aligned}\phi^{(1)\,\sigma\tau}\left(k_1\right) \;=\; & \frac{\kappa^3}{8}\int đ^4k_2 đ^4k_3\left[-2\left.\frac{\partial M_G^{\alpha\mu\nu\rho\alpha_2\beta_2}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta} k_{2\,\alpha}k_{3\,\rho} - 2\left.\frac{\partial M_G^{\alpha\mu\nu\rho\alpha_2\beta_2}}{\partial g_{\alpha_1\beta_1}}\right|_{g=\eta} k_{3\,\alpha}k_{3\,\rho}\right.\\ & \left. + \left.\frac{\partial M_G^{\alpha\alpha_1\beta_1\rho\alpha_2\beta_2}}{\partial g_{\mu\nu}}\right|_{g=\eta} k_{2\,\alpha}k_{3\,\rho}\right] G^{(0)\sigma\tau}{}_{\mu\nu}\left(k_1\right)G^{(0)}{}_{\alpha_1\beta_1}{}^{\mu_1\nu_1}\left(k_2\right)G^{(0)}{}_{\alpha_2\beta_2}{}^{\mu_2\nu_2}\left(k_3\right)\\ & \times T_{\mu_1\nu_1}\left(k_2\right)T_{\mu_2\nu_2}\left(k_3\right)\delta^4\left(k_2+k_3-k_1\right)\end{aligned} \tag{4.46}$$

This looks very similar to (3.73) and is in accordance with the diagrammatical representation of Figure 5.
We can use the same integrals as the ones described in section 3.4.4 to compute the final result as a function of V and p.

We implement the same rules on a computer and obtain:

$$\kappa\phi^{(1)\,00} = -\frac{1}{\Delta}\eta_{kl}\partial^k V\partial^l V - \frac{2}{\Delta}V\Delta V \tag{4.47}$$

and

$$\kappa\phi^{(1)\,ij} = \frac{\eta^{ab}}{\Delta}\partial_a V\partial_b V\eta^{ij} - \frac{1}{\Delta}\left(\partial^i V\partial^j V\right) \tag{4.48}$$

4.5.4 Final result and comparison

Remembering that $\phi = \phi^{(0)} + \phi^{(1)} + \ldots$ we obtain the following final result:

$$\kappa\phi^{00} = \left(2V + \frac{3}{2}\frac{\kappa^2}{\Delta}p\right) - \frac{1}{\Delta}\eta_{kl}\partial^k V\partial^l V - \frac{2}{\Delta}V\Delta V \tag{4.49}$$

and

$$\kappa\phi^{ij} = \left(2V - \frac{\kappa^2}{2}\frac{1}{\Delta}p + \frac{\eta_{ab}}{\Delta}\partial^a V\partial^b V\right)\eta^{ij} - \frac{1}{\Delta}\left(\partial^i V\partial^j V\right) \tag{4.50}$$

Some more details on these calculations can be found in appendix F.
These results have the same form as the ones obtained in (3.101) and (3.112), up to a factor of 4 and a sign in the term proportional to $\frac{1}{\Delta}\left(\partial^i V\partial^j V\right)$.
The origin of the problem is still unknown, but probably hides in the computer calculations.

5 Conclusions

The goal of this thesis was to study a particular approach for the computation of the vacuum expectation value of the gravitational potential.
The mass configuration considered was the one of a perfectly spherically symmetric homogeneous density.

We first derived the classical result using the standard procedure of general relativity.

Then we considered two different approaches, both based on the formulation of gravity as a quantum field theory.
The first approach, following [1], requires the 3-vertex function , when writing the vacuum expectation value as the sum of a diagram involving one coupling to the external source and one involving two.
There are many options to calculate the 3-vertex function and even the results from the literature do not always agree.
We considered three different possibilities that led to results differing at most by a factor of 4.

Secondly, we computed the vacuum expectation value by means of writing the equations of motion in a form that can be solved iteratively. This allows us to avoid the computation of the 3-vertex function and requires only one partial derivative of a matrix with respect to the metric.

The results obtained with this method differ only by a factor of 2 from the previous ones.
The reason behind this disagreement has most probably nothing to do with the method used, but rather with some mistake in the calculations.

A Computation of Goldberg's expression for the Einstein-Hilbert Lagrangian

We start with the usual form of the Einstein-Hilbert Lagrangian:

$$\mathcal{L} = \frac{2}{\kappa^2} (-g)^{\frac{1}{2}} g^{\mu\nu} R_{\mu\nu} \tag{A.1}$$

We are using the following convention for the Ricci tensor:

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} + \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho} \tag{A.2}$$

and for the Christoffel symbol:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\gamma} \left(\partial_\gamma g_{\lambda\beta} + \partial_\beta g_{\lambda\gamma} - \partial_\lambda g_{\gamma\beta}\right) \tag{A.3}$$

Our goal is to reexpress this Lagrangian using the tensor density $\mathfrak{g}^{\alpha\beta}$:

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} \qquad \Rightarrow \qquad \det\left(\mathfrak{g}^{\alpha\beta}\right) = \det\left(\sqrt{-g} g^{\alpha\beta}\right) = \left(\sqrt{-g}\right)^n \det\left(g^{\alpha\beta}\right)$$

where n is the dimensionality of our spacetime.
Note that:

$$\underbrace{\det\left(\mathfrak{g}^{\alpha\beta}\right)}_{=(-\mathfrak{g})^{-1}} = (-g)^{\frac{n}{2}-1} = (-g)^{\frac{n-2}{2}} \tag{A.4}$$

$$\Rightarrow \qquad \mathfrak{g} = -(-g)^{\frac{2}{n-2}} \qquad \Rightarrow \qquad g = -(-\mathfrak{g})^{\frac{n-2}{2}} \tag{A.5}$$

Using the relation $\partial_\rho \left(\det\left(g^{\mu\nu}\right)\right) = g_{\alpha\beta} g^{\alpha\beta}{}_{,\rho} \det\left(g^{\mu\nu}\right)$, we obtain:

$$\begin{aligned}
\partial_\rho g &= \partial_\rho \left(-(-\mathfrak{g})^{\frac{n-2}{2}}\right) \\
&= -\partial_\rho \left(\left(-\det\left(\mathfrak{g}^{\mu\nu}\right)\right)^{\frac{2}{n-2}}\right) \\
&= \frac{2}{n-2} \left(-\det\left(\mathfrak{g}^{\mu\nu}\right)\right)^{\frac{2}{n-2}-1} \partial_\rho \det\left(\mathfrak{g}^{\mu\nu}\right) \\
&= \frac{2}{n-2} \left(-\det\left(\mathfrak{g}^{\mu\nu}\right)\right)^{\frac{2}{n-2}-1} \mathfrak{g}_{\alpha\beta} \mathfrak{g}^{\alpha\beta}{}_{,\rho} \det\left(\mathfrak{g}^{\mu\nu}\right) \\
&= \frac{2}{n-2} \underbrace{(-1)\left(-\det\left(\mathfrak{g}^{\mu\nu}\right)\right)^{\frac{2}{n-2}}}_{=g} \mathfrak{g}_{\alpha\beta} \mathfrak{g}^{\alpha\beta}{}_{,\rho} \\
&= \frac{2}{n-2} g \mathfrak{g}_{\alpha\beta} \mathfrak{g}^{\alpha\beta}{}_{,\rho}
\end{aligned} \tag{A.6}$$

Furthermore:

$$\partial_\gamma g^{\alpha\beta} = -g^{\mu\alpha} g^{\nu\beta} \left(\partial_\gamma g_{\mu\nu}\right) \tag{A.7}$$

$$\partial_\gamma g_{\alpha\beta} = -g_{\mu\alpha} g_{\nu\beta} \partial_\gamma g^{\mu\nu} \tag{A.8}$$

We can use all of this information to rewrite the Christoffel symbol:

$$\begin{aligned}
\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\lambda} \left(\partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}\right) \\
&= \frac{1}{2} (-g)^{-\frac{1}{2}} \mathfrak{g}^{\alpha\lambda} \left(-g_{\mu\lambda} g_{\nu\gamma} \partial_\beta g^{\mu\nu} - g_{\mu\lambda} g_{\nu\beta} \partial_\gamma g^{\mu\nu} + g_{\mu\beta} g_{\gamma\nu} \partial_\lambda g^{\mu\nu}\right) \\
&= \frac{1}{2} (-g)^{-\frac{1}{2}} \mathfrak{g}^{\alpha\lambda} \left(-(-g)^{\frac{1}{2}} (-g)^{\frac{1}{2}} \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\gamma} \partial_\beta \left((-g)^{-\frac{1}{2}} \mathfrak{g}^{\mu\nu}\right) - (-g)^{\frac{1}{2}} (-g)^{\frac{1}{2}} \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\beta} \partial_\gamma \left((-g)^{-\frac{1}{2}} \mathfrak{g}^{\mu\nu}\right)\right. \\
&\quad \left. + (-g)^{\frac{1}{2}} (-g)^{\frac{1}{2}} \mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} \partial_\lambda \left((-g)^{\frac{1}{2}} \mathfrak{g}^{\mu\nu}\right)\right) \\
&= \frac{1}{2} (-g)^{\frac{1}{2}} \mathfrak{g}^{\alpha\lambda} \left(-\mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\gamma} \partial_\beta \left((-g)^{-\frac{1}{2}} \mathfrak{g}^{\mu\nu}\right) - \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\beta} \partial_\gamma \left((-g)^{-\frac{1}{2}} \mathfrak{g}^{\mu\nu}\right) + \mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} \partial_\lambda \left((-g)^{\frac{1}{2}} \mathfrak{g}^{\mu\nu}\right)\right) \\
&= \frac{1}{2} (-g)^{\frac{1}{2}} \mathfrak{g}^{\alpha\lambda} \left(-\mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\gamma} \mathfrak{g}^{\mu\nu} \partial_\beta (-g)^{-\frac{1}{2}} - \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\gamma} (-g)^{-\frac{1}{2}} \partial_\beta \mathfrak{g}^{\mu\nu} - \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\beta} \mathfrak{g}^{\mu\nu} \partial_\gamma (-g)^{-\frac{1}{2}}\right. \\
&\quad \left. -\mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\beta} (-g)^{-\frac{1}{2}} \partial_\gamma \mathfrak{g}^{\mu\nu} + \mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} \mathfrak{g}^{\mu\nu} \partial_\lambda (-g)^{-\frac{1}{2}} + \mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} (-g)^{-\frac{1}{2}} \partial_\lambda \mathfrak{g}^{\mu\nu}\right)
\end{aligned} \tag{A.9}$$

In order to get rid of the derivatives of the determinant, we compute:

$$\begin{aligned}
\partial_\alpha (-g)^{-\frac{1}{2}} &= \frac{1}{2} (-g)^{-\frac{3}{2}} \partial_\alpha g \\
&= \frac{1}{2} (-g)^{-\frac{3}{2}} \frac{2}{n-2} g \mathfrak{g}_{\mu\nu} \mathfrak{g}^{\mu\nu}{}_{,\alpha} \\
&= -\frac{1}{n-2} (-g)^{\frac{1}{2}} \mathfrak{g}_{\mu\nu} \mathfrak{g}^{\mu\nu}{}_{,\alpha}
\end{aligned} \tag{A.10}$$

So we can finally compute an expression for the Christoffel symbol, that depends only on the tensor density $\mathfrak{g}^{\mu\nu}$:

$$\begin{aligned}
\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2} (-g)^{\frac{1}{2}} \mathfrak{g}^{\alpha\lambda} \left(-\mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\gamma} \mathfrak{g}^{\mu\nu} \left(-\frac{1}{n-2}\right) (-g)^{-\frac{1}{2}} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\beta} - \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\gamma} (-g)^{-\frac{1}{2}} \mathfrak{g}^{\mu\nu}{}_{,\beta}\right. \\
&\quad -\mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\beta} \mathfrak{g}^{\mu\nu} \left(-\frac{1}{n-2}\right) (-g)^{-\frac{1}{2}} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\gamma} - \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\beta} (-g)^{-\frac{1}{2}} \mathfrak{g}^{\mu\nu}{}_{,\gamma} \\
&\quad \left. +\mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} \mathfrak{g}^{\mu\nu} \left(-\frac{1}{n-2}\right) (-g)^{-\frac{1}{2}} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\lambda} + \mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} (-g)^{-\frac{1}{2}} \mathfrak{g}^{\mu\nu}{}_{,\lambda}\right) \\
&= \frac{1}{2} \mathfrak{g}^{\alpha\lambda} \left(\frac{1}{n-2} \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\gamma} \mathfrak{g}^{\mu\nu} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\beta} - \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\gamma} \mathfrak{g}^{\mu\nu}{}_{,\beta} + \frac{1}{n-2} \mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\beta} \mathfrak{g}^{\mu\nu} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\gamma}\right. \\
&\quad \left. -\mathfrak{g}_{\mu\lambda} \mathfrak{g}_{\nu\beta} \mathfrak{g}^{\mu\nu}{}_{,\gamma} - \frac{1}{n-2} \mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} \mathfrak{g}^{\mu\nu} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\lambda} + \mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} \mathfrak{g}^{\mu\nu}{}_{,\lambda}\right) \\
&= \frac{1}{2} \left(\frac{1}{n-2} \delta^{\alpha}_{\gamma} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\beta} - \mathfrak{g}_{\nu\gamma} \mathfrak{g}^{\alpha\nu}{}_{,\beta} + \frac{1}{n-2} \delta^{\alpha}_{\beta} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\gamma} - \mathfrak{g}_{\nu\beta} \mathfrak{g}^{\alpha\nu}{}_{,\gamma}\right. \\
&\quad \left. -\frac{1}{n-2} \mathfrak{g}_{\gamma\beta} \mathfrak{g}^{\alpha\lambda} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\lambda} + \mathfrak{g}_{\mu\beta} \mathfrak{g}_{\gamma\nu} \mathfrak{g}^{\alpha\lambda} \mathfrak{g}^{\mu\nu}{}_{,\lambda}\right) \\
&= -\frac{1}{2} \left(-\frac{1}{n-2} \delta^{\alpha}_{\gamma} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\beta} + \mathfrak{g}_{\mu\gamma} \mathfrak{g}^{\alpha\mu}{}_{,\beta} - \frac{1}{n-2} \delta^{\alpha}_{\beta} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\gamma} + \mathfrak{g}_{\mu\beta} \mathfrak{g}^{\alpha\mu}{}_{,\gamma}\right. \\
&\quad \left. +\frac{1}{n-2} \mathfrak{g}_{\gamma\beta} \mathfrak{g}^{\alpha\lambda} \mathfrak{g}_{\pi\sigma} \mathfrak{g}^{\pi\sigma}{}_{,\lambda} - \mathfrak{g}_{\beta\mu} \mathfrak{g}_{\gamma\nu} \mathfrak{g}^{\alpha\lambda} \mathfrak{g}^{\mu\nu}{}_{,\lambda}\right)
\end{aligned} \tag{A.11}$$

After inserting the explicit form of the Christoffel symbol in the Ricci tensor and some computations that are too long even for the appendix, we obtain the following:

$$\begin{aligned}
S_{EH} &= \frac{1}{2\kappa^2}\int d^n x 4\mathfrak{g}^{\mu\nu}R_{\mu\nu} \\
&= \frac{1}{2\kappa^2}\int d^n x\left[\frac{8-2n}{n-2}\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\alpha\beta,\nu}\mathfrak{g}^{\alpha\beta}{}_{,\mu}+\frac{4}{n-2}\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\alpha\beta}\mathfrak{g}^{\alpha\beta}{}_{,\mu\nu}+4\mathfrak{g}^{\alpha\rho}{}_{,\alpha\rho}\right. \\
&\quad -\frac{1}{n-2}\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\alpha\beta}\mathfrak{g}_{\alpha'\beta'}\mathfrak{g}^{\alpha\beta}{}_{,\mu}\mathfrak{g}^{\alpha'\beta'}{}_{,\nu}-\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\sigma\alpha}\mathfrak{g}_{\rho\alpha'}\mathfrak{g}^{\alpha\rho}{}_{,\nu}\mathfrak{g}^{\sigma\alpha'}{}_{,\mu}+4\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\mu\alpha,\rho}\mathfrak{g}^{\alpha\rho}{}_{,\nu} \\
&\quad \left.+2\mathfrak{g}_{\alpha\alpha'}\mathfrak{g}^{\rho\alpha}{}_{,\sigma}\mathfrak{g}^{\alpha'\sigma}{}_{,\rho}+\frac{4}{n-2}\mathfrak{g}^{\rho\gamma}{}_{,\rho}\mathfrak{g}_{\alpha\beta}\mathfrak{g}^{\alpha\beta}{}_{,\gamma}\right]
\end{aligned} \tag{A.12}$$

Integration by parts yields then the desired result:

$$\begin{aligned}
S_{EH} &= \frac{1}{2\kappa^2}\int d^n x\left[\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\alpha\alpha'}\mathfrak{g}^{\alpha'\beta'}{}_{,\nu}\mathfrak{g}^{\alpha\beta}{}_{,\mu}-\frac{1}{n-2}\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\alpha\beta}\mathfrak{g}_{\alpha'\beta'}\mathfrak{g}^{\alpha\beta}{}_{,\mu}\mathfrak{g}^{\alpha'\beta'}{}_{,\nu}\right. \\
&\quad \left.-2\mathfrak{g}_{\alpha\beta}\mathfrak{g}^{\alpha\rho}{}_{,\sigma}\mathfrak{g}^{\sigma\beta}{}_{,\rho}\right]
\end{aligned} \tag{A.13}$$

B Computations for the free propagator

B.1 $\overbracket{\phi^{\alpha_1\beta_1}(x)\,\phi}^{\alpha_2\beta_2}(y)$

We can now check that $\overbracket{\phi^{\alpha_1\beta_1}(x)\,\phi}^{\alpha_2\beta_2}(y) = -iG^{\alpha_1\beta_1\alpha_2\beta_2}(x-y)$:

$$\begin{aligned}
\overbracket{\phi^{\alpha_1\beta_1}(x)\,\phi}^{\alpha_2\beta_2}(y) = -iG^{\alpha_1\beta_1\alpha_2\beta_2}(x-y) &= \left\langle 0\left|T\left\{\phi^{\alpha_1\beta_1}(x)\,\phi^{\alpha_2\beta_2}(y)\right\}\right|\right\rangle \\
&= \frac{1}{Z[J]}\left(-i\frac{\delta}{\delta J_{\alpha_1\beta_1}(x)}\right)\left(-i\frac{\delta}{\delta J_{\alpha_2\beta_2}(y)}\right)Z[J]\Big|_{J=0} \\
&= -\frac{1}{Z[J]}\frac{\delta}{\delta J_{\alpha_1\beta_1}(x)}\left[-\int d^4x'd^4y' I^{\alpha_2\beta_2}{}_{\mu\nu}\,\delta(y'-y)\right. \\
&\quad \left.\times(-i)\,G^{\alpha\beta\mu\nu}(x'-y')\,J_{\alpha\beta}(x')\right]Z[J]\Big|_{J=0} \\
&= -i\frac{1}{Z[J]}\left[\int d^4x' G^{\alpha\beta\alpha_2\beta_2}(x'-y)\,I^{\alpha_1\beta_1}{}_{\alpha\beta}\right. \\
&\quad \left.\times\ \delta(x'-x)\right]Z(J)|_{J=0} \\
&= -iG^{\alpha_1\beta_1\alpha_2\beta_2}(x-y)
\end{aligned} \tag{B.1}$$

B.2 Proof that $G^{\mu\nu\rho\sigma}$ satisfies (3.36)

We now show that $G^{\mu\nu\rho\sigma}(k^2) = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\rho\nu}\eta^{\mu\sigma}-\eta^{\mu\nu}\eta^{\rho\sigma}\right)\frac{1}{k^2}$ satisfies (3.36).

Indeed in momentum space we have:

$$\begin{aligned}
&\frac{1}{8}\left(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau}-\eta^{\rho\sigma}-4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\iota\tau}+4\delta^{\rho}{}_{\kappa}\delta^{\sigma}{}_{\tau}\eta_{\iota\lambda}\right)\\
&\quad\times k_\rho k_\sigma\left(-i\right)\left(\eta^{\lambda\mu}\eta^{\tau\nu}+\eta^{\mu\tau}\eta^{\lambda\nu}-\eta^{\lambda\tau}\eta^{\mu\nu}\right)\frac{1}{k^2} = \frac{-i}{8}\left(2k^2\eta_{\lambda\iota}\eta_{\kappa\tau}-k^2\eta_{\iota\kappa}\eta_{\lambda\tau}\right.\\
&\qquad\left.-4k_\kappa k_\lambda\eta_{\iota\tau}+4k_\kappa k_\tau\eta_{\iota\lambda}\right)\\
&\qquad\times\left(\eta^{\lambda\mu}\eta^{\tau\nu}+\eta^{\mu\tau}\eta^{\lambda\nu}-\eta^{\lambda\tau}\eta^{\mu\nu}\right)\frac{1}{k^2}\\
&= \frac{-i}{8}\left(2k^2\delta^{\mu}{}_{\iota}\delta^{\nu}{}_{\kappa}+2k^2\delta^{\mu}{}_{\kappa}\delta^{\nu}{}_{\iota}-2k^2\eta_{\kappa\iota}\eta^{\mu\nu}\right.\\
&\qquad-k^2\eta_{\iota\kappa}\eta^{\mu\nu}-k^2\eta_{\iota\kappa}\eta^{\mu\nu}-4k^2\eta_{\iota\kappa}\eta^{\mu\nu}\\
&\qquad-4k_\kappa k^\mu\delta^{\nu}{}_{\iota}-4k^\kappa k^\nu\delta^{\mu}{}_{\iota}+4k_\kappa k_\iota\eta^{\mu\nu}\\
&\qquad\left.+4k_\kappa k^\nu\delta^{\mu}{}_{\iota}+4k_\kappa k^\mu\delta^{\nu}{}_{\iota}\eta^{\mu\nu}\right)\\
&= \frac{-i}{4}\left(\delta^{\mu}{}_{\iota}\delta^{\nu}{}_{\kappa}+\delta^{\mu}{}_{\kappa}\delta^{\nu}{}_{\iota}\right)\\
&= \frac{-i}{2}I^{\mu\nu}{}_{\iota\kappa}
\end{aligned}\tag{B.2}$$

C 3-graviton interaction

C.1 Computation of $\tilde{\Gamma}$

We need to compute three variations of A_G with respect to $\mathfrak{g}$.
In order to do this, we will use the `xTensor` package from `Mathematica`, contained in the `xAct` bundle. This package allows us to perform tensor calculus, inculding variations.

The desired expression can be rewritten as a function of $\tilde{\phi}$:

$$\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\left(x_1,x_2,x_3\right)=\left.\frac{\delta^3\int d^4x\kappa\mathcal{L}_G^{(1)}}{\delta\tilde{\phi}^{\alpha_1\beta_1}\left(x_1\right)\delta\tilde{\phi}^{\alpha_2\beta_2}\left(x_2\right)\delta\tilde{\phi}^{\alpha_3\beta_3}\left(x_3\right)}\right|_{\tilde{\phi}=0}\tag{C.1}$$

Remembering that

$$\begin{aligned}
\tilde{\mathcal{L}}_G^{(1)} &= \frac{1}{8}\left(-4\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\alpha}\eta_{\tau\beta}+2\eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\alpha}\eta_{\beta\tau}+2\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\eta_{\lambda\iota}\eta_{\kappa\tau}-\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}\eta_{\iota\kappa}\eta_{\lambda\tau}\right.\\
&\quad\left.+4\delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\eta_{\alpha\iota}\eta_{\tau\beta}\right)\tilde{\phi}^{\alpha\beta}\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma}
\end{aligned}\tag{C.2}$$

we leave the contraction with the Minkowski metric aside for the moment and vary just $\tilde{\phi}^{\alpha\beta}\tilde{\phi}^{\iota\kappa}{}_{,\rho}\tilde{\phi}^{\lambda\tau}{}_{,\sigma}$.
We define this as follows in `Mathematica`:

```
ϕ[f, h] ψ[d, e, -a] ψ[c, g, -b]
```

where ψ is a shorthand for $\partial\tilde{\phi}$.
After defining some variation rules, we are ready to do the actual computations:

```
ψ /: VarD[pert1[a1, b1], PD][ψ[d_, e_, -a_], rest_] :=
  mom1[-a] 1/2 (metricg[d, -a1] metricg[e, -b1] + metricg[e, -a1] metricg[d, -b1]) rest;
ψ /: VarD[pert2[a2, b2], PD][ψ[d_, e_, -a_], rest_] :=
  mom2[-a] 1/2 (metricg[d, -a2] metricg[e, -b2] + metricg[e, -a2] metricg[d, -b2]) rest;
ψ /: VarD[pert3[a3, b3], PD][ψ[d_, e_, -a_], rest_] :=
  mom3[-a] 1/2 (metricg[d, -a3] metricg[e, -b3] + metricg[e, -a3] metricg[d, -b3]) rest;
ϕ /: VarD[pert1[a1, b1], PD][ϕ[f_, h_], rest_] :=
  1/2 (metricg[f, -a1] metricg[h, -b1] + metricg[h, -a1] metricg[f, -b1]) rest;
ϕ /: VarD[pert2[a2, b2], PD][ϕ[f_, h_], rest_] :=
  1/2 (metricg[f, -a2] metricg[h, -b2] + metricg[h, -a2] metricg[f, -b2]) rest;
ϕ /: VarD[pert3[a3, b3], PD][ϕ[f_, h_], rest_] :=
  1/2 (metricg[f, -a3] metricg[h, -b3] + metricg[h, -a3] metricg[f, -b3]) * rest;
```

```
(*Compute the variations*)
res1 = VarD[pert1[a1, b1], PD][ϕ[f, h] ψ[d, e, -a] ψ[c, g, -b]];
(*first variation*)
res2 = VarD[pert2[a2, b2], PD][res1]; (*second variation*)
res3 = VarD[pert3[a3, b3], PD][res2];
(*third variation*)
```

We can then multiply this by the prefactor that we had left away to obtain the result stated in (3.44).

C.2 Computation of the vertex function in position space

The variation rules used to work in position space are:

```
metricg /: VarD[var1[m1, n1], PD][metricg[m_, n_], rest_] :=
  VarD[metricg[m1, n1], PD][metricg[m, n]] * deltaXX1 * rest
metricg /: VarD[var2[m2, n2], PD][metricg[m_, n_], rest_] :=
  VarD[metricg[m2, n2], PD][metricg[m, n]] * deltaXX2 * rest
metricg /: VarD[var3[m3, n3], PD][metricg[m_, n_], rest_] :=
  VarD[metricg[m3, n3], PD][metricg[m, n]] * deltaXX3 * rest
```

```
Ω /: VarD[var1[m1, n1], PD][Ω[a_, b_, -c_], rest_] :=
  PD[-c][VarD[metricg[m1, n1]][metricg[a, b]] deltaXX1] * rest
Ω /: VarD[var2[m2, n2], PD][Ω[a_, b_, -c_], rest_] :=
  PD[-c][VarD[metricg[m2, n2]][metricg[a, b]] deltaXX2] * rest
Ω /: VarD[var3[m3, n3], PD][Ω[a_, b_, -c_], rest_] :=
  PD[-c][VarD[metricg[m3, n3]][metricg[a, b]] deltaXX3] * rest
```

where $\Omega^{\alpha\beta}{}_{\gamma}$ is a shorthand for $\mathfrak{g}^{\alpha\beta}{}_{,\gamma}$ and `deltaXX1` stands for $\delta\left(x - x_1\right)$ and so on. The rules for integrating over d^4x are:

```
intX1 := deltaXX1 * PD[a_][deltaXX3] PD[b_][deltaXX2] →
   PD[a][deltaX3X1] PD[b][deltaX1X2]
intX2 := deltaXX2 * PD[a_][deltaXX1] PD[b_][deltaXX3] →
   PD[a][deltaX1X2] PD[b][deltaX2X3]
intX3 := deltaXX3 * PD[a_][deltaXX2] PD[b_][deltaXX1] →
   PD[a][deltaX2X3] PD[b][deltaX3X1]
```

and the rules for performing the Fourier transformation are:

```
doubleX1toMom := PD[a_][deltaX1X2] PD[b_][deltaX3X1] → -mom2[a] mom3[b];
doubleX2toMom := PD[a_][deltaX2X3] PD[b_][deltaX1X2] → -mom3[a] mom1[b];
doubleX3toMom := PD[a_][deltaX3X1] PD[b_][deltaX2X3] → -mom1[a] mom2[b];
```

where `mom2[a]` means $k_{2\alpha}$ and so on.

D Explicit computation of the vacuum expectation value

The generic rules that we will need for computing the integrals in the vacuum expectation value are

```
(*Here are first the definitions of the rules that I
 will be using to compute the factors in front of VDV
 and dVdV.*)

metrictoscalar = metricg[a_, b_] → -1;
mom2tovDv = mom2[a_] mom2[-a_] → 16 vDv;
mom3tovDv = mom3[a_] mom3[-a_] → 16 vDv;
mom32todvdv = mom3[a_] mom2[-a_] → 16 dvdv;
mom23todvdv = mom2[a_] mom3[-a_] → 16 dvdv;

(*These are the rules for setting the momenta to 0,
when the indices are 0
 (because of the delta in the mu(k) term).*)
mom3m1to0 = mom3[m1] → 0;
mom3n1to0 = mom3[n1] → 0;
mom3m2to0 = mom3[m2] → 0;
mom3n2to0 = mom3[n2] → 0;
mom3m3to0 = mom3[m3] → 0;
mom3n3to0 = mom3[n3] → 0;
mom2m1to0 = mom2[m1] → 0;
mom2n1to0 = mom2[n1] → 0;
mom2m2to0 = mom2[m2] → 0;
mom2n2to0 = mom2[n2] → 0;
mom2m3to0 = mom2[m3] → 0;
mom2n3to0 = mom2[n3] → 0;

(*This is for when m1≠
 n1 and hence the metric has to be set to zero.*)
metricto0 = metricg[m1, n1] → 0;
(*When m1 or n1 are ≠0 and a_ is 0,
the metric should also be set to 0.*)
metricm1alphato0 = metricg[m1, a_] → 0;
metricn1alphato0 = metricg[a_, n1] → 0;
metricalpham1to0 = metricg[a_, m1] → 0;
metricalphan1to0 = metricg[n1, a_] → 0;

metricto1 = metricg[m1, n1] → 1;
metricto1symm = metricg[n1, m1] → 1;
metricgenericto1 = metricg[a_, b_] → 1;

(*This is the actual substitution into terms proportional
 to diVdjV.*)
diVdjVv1 = mom2[m1] mom3[n1] → 16 diVdjV;
diVdjVv2 = mom3[m1] mom2[n1] → 16 diVdjV;
```

D.1 00-component

The implementation of the rules for calculating the factors in front of $\frac{1}{\Delta}V\Delta V$ and $\frac{1}{\Delta}\partial_i V\partial^i V$ is

```
vertexConstrained /. metrictoscalar;
% /. mom2tovDv;
% /. mom3tovDv;
% /. mom32todvdv;
% /. mom23todvdv;
% /. mom3m1to0;
% /. mom3n1to0;
% /. mom3m2to0;
% /. mom3n2to0;
% /. mom3m3to0;
% /. mom3n3to0;
% /. mom2m1to0;
% /. mom2n1to0;
% /. mom2m2to0;
% /. mom2n2to0;
% /. mom2m3to0;
% /. mom2n3to0
```

D.2 ij-component

The implementation of the rules to determine the factors in front of $\frac{1}{\Delta}\partial^i V\partial^j V$ are

```
resultdiVdjV = vertexConstrained /. metricto0;
% /. metricm1alphato0;
% /. metricn1alphato0;
% /. metricalpham1to0;
% /. metricalphan1to0;
% /. diVdjVv1;
% /. diVdjVv2;
% /. metricgenericto1
```

and for the factors in front of $\eta^{ij}\frac{1}{\Delta}\partial^k V\partial_k V$

```
vertexConstrained - resultdiVdjV /. metricto1;
% /. metricto1symm;
% /. mom2m2to0;
% /. mom2n2to0;
% /. mom2m3to0;
% /. mom2n3to0;
% /. mom3m2to0;
% /. mom3n2to0;
% /. mom3m3to0;
% /. mom3n3to0;
% /. metricgenericto1;
% /. mom23todvdv;
% /. mom32todvdv
```

E Proof of the relations stated in sections 3.4.3, 3.4.4 and 3.5.3

E.1 Proof of (3.86)

We begin by noting that

$$\begin{aligned}
V\left(x'\right) &= \frac{1}{4}\kappa^2 \int đ^4 k \frac{e^{ik\cdot x'}}{k^2}\left(2\pi\right)\delta\left(k^0\right)\int d^3 x e^{-i\vec{k}\cdot\vec{x}}\mu\left(\vec{x}\right) \\
&= \frac{1}{4}\frac{\kappa^2}{\left(2\pi\right)^3}\int d^3 x \mu\left(\vec{x}\right)\int d^3\vec{k}\frac{e^{i\vec{k}\cdot\left(\vec{x}'-\vec{x}\right)}}{\vec{k}^2} \\
&= -\frac{1}{4}\frac{\kappa^2}{\left(2\pi\right)^2}\int d^3 x \mu\left(\vec{x}\right)\int_0^\infty dk \int_{\theta=0}^{\theta=\pi} d\left(\cos\left(\theta\right)\right) e^{ik\left(\left|\vec{x}'-\vec{x}\right|\right)cos\left(\theta\right)} \\
&= -\frac{1}{4}\frac{\kappa^2}{\left(2\pi\right)^2}\int d^3 x \mu\left(\vec{x}\right)\int_0^\infty dk \frac{1}{ik\left|\vec{x}'-\vec{x}\right|}\left[e^{-ik\left|\vec{x}'-\vec{x}\right|}-e^{ik\left|\vec{x}'-\vec{x}\right|}\right] \\
&= \frac{1}{2}\frac{\kappa^2}{\left(2\pi\right)^2}\int d^3 x \mu\left(\vec{x}\right)\underbrace{\int_0^\infty dk \frac{\sin\left(k\left|\vec{x}'-\vec{x}\right|\right)}{k\left|\vec{x}'-\vec{x}\right|}}_{=\frac{\pi}{2\left|\vec{x}'-\vec{x}\right|}} \\
&= \frac{\pi}{4}\frac{\kappa^2}{\left(2\pi\right)^2}\int d^3 x \mu\left(\vec{x}\right)\frac{1}{\left|\vec{x}'-\vec{x}\right|} \\
&= \frac{\pi}{4}\frac{\kappa^2}{\left(2\pi\right)^2}\rho\int dr d\theta d\varphi r^2 \sin\left(\theta\right)\frac{\theta\left(\varepsilon - r\right)}{\left|\vec{x}'-\vec{x}\right|}
\end{aligned} \tag{E.1}$$

We can insert $|\vec{x}' - \vec{x}| = \sqrt{(\vec{x}' - \vec{x}')^2} = \sqrt{r'^2 + r^2 - 2rr'\cos(\theta)}$ in the above expression to obtain:

$$V(x') = \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\rho\int_0^\varepsilon dr\int_0^\theta d\theta\frac{r^2\sin(\theta)}{\sqrt{\underbrace{r'^2 + r^2 - 2rr'\cos(\theta)}_{=:f(\theta)}}} \tag{E.2}$$

(E.3)

Noticing that $\frac{df}{d\theta} = 2rr'\sin(\theta)$ we can go on with the calculations:

$$\begin{aligned}
V(x') &= -\frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\rho\int_0^\varepsilon dr\frac{1}{2rr'}\int_0^\theta d\theta r^2\frac{df}{d\theta}\frac{1}{[f(\theta)]^{\frac{1}{2}}} \\
&= -\frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\int_0^\varepsilon drr\left[\sqrt{r'^2 + r^2 + 2rr'} - \sqrt{r'^2 + r^2 - 2rr'}\right] \\
&= -\frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\int_0^\varepsilon drr\left[\sqrt{(r'+r)^2} - \sqrt{(r'-r)^2}\right] \\
&= -\underbrace{\frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\int_0^\varepsilon drr\sqrt{(r'+r)^2}}_{=:I_1} + \underbrace{\frac{\pi}{4}\frac{\kappa^2}{(2\pi)^2}\frac{\rho}{r'}\int_0^\varepsilon drr\sqrt{(r'-r)^2}}_{=:I_2}
\end{aligned} \tag{E.4}$$

Let us deal with the two terms separately:

$$\begin{aligned}
I_1 &= \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\int_0^\varepsilon drr(r'+r) \\
&= \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\left(\frac{r'\varepsilon^2}{2} + \frac{\varepsilon^3}{3}\right)
\end{aligned} \tag{E.5}$$

and

$$\begin{aligned}
I_2 &= \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\int_0^\varepsilon drr\sqrt{(r'-r)^2} \\
&= \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\left\{\theta(r'-\varepsilon)\int_0^\varepsilon drr(r'-r) + \theta(\varepsilon - r')\int_0^\varepsilon drr\sqrt{(r'-r)^2}\right\} \\
&= \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\left\{\theta(r'-\varepsilon)\left(\frac{\varepsilon^2 r'}{2} - \frac{\varepsilon^3}{3}\right) + \theta(\varepsilon - r')\left[\int_0^{r'} drr(r'-r) + \int_{r'}^\varepsilon drr(r-r')\right]\right\} \\
&= \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\left\{\theta(r'-\varepsilon)\left(\frac{\varepsilon^2 r'}{2} - \frac{\varepsilon^3}{3}\right) + \theta(\varepsilon - r')\left[\frac{r'^3}{2} - \frac{r'^3}{3} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^2 r'}{2} - \frac{r'^3}{3} + \frac{r'^3}{2}\right]\right\} \\
&= \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\left\{\theta(r'-\varepsilon)\left(\frac{\varepsilon^2 r'}{2} - \frac{\varepsilon^3}{3}\right) + \theta(\varepsilon - r')\left[r'^3 - \frac{2r'^3}{3} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^2 r'}{2}\right]\right\}
\end{aligned} \tag{E.6}$$

Rewriting I_1 as

$$I_1 = \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\left[\left(\frac{r'\varepsilon^2}{2} + \frac{\varepsilon^3}{3}\right)\theta(\varepsilon - r') + \left(\frac{r'\varepsilon^2}{2} + \frac{\varepsilon^3}{3}\right)\theta(r'-\varepsilon)\right] \tag{E.7}$$

we obtain the final result:

$$\begin{aligned} V\left(x'\right) &= -I_1 + I_2 \\ &= \frac{\pi}{4}\frac{\kappa^2}{(2\pi)}\frac{\rho}{r'}\left[\theta\left(\varepsilon - r'\right)\left(-r'\varepsilon^2 + r'^3 - \frac{2r'^3}{3}\right) - \theta\left(r' - \varepsilon\right)\frac{2\varepsilon^3}{3}\right] \\ &= \frac{\kappa^2\lambda}{4\,(2\pi)\,r'}\left[\theta\left(\varepsilon - r'\right)\left(-\frac{3}{4}\frac{r'}{\varepsilon} + \frac{1}{4}\frac{r'^3}{\varepsilon^3}\right) - \frac{1}{2}\theta\left(r' - \varepsilon\right)\right] \\ &= \frac{16\pi G\lambda}{4\,(2\pi)\,r'} \\ &= \frac{2G\lambda}{r'}\left[\theta\left(\varepsilon - r'\right)\left(-\frac{3}{4}\frac{r'}{\varepsilon} + \frac{1}{4}\frac{r'^3}{\varepsilon^3}\right) - \frac{1}{2}\theta\left(r' - \varepsilon\right)\right] \\ &= \left(-G\lambda\right)\left[\theta\left(\varepsilon - r'\right)\left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r'^2}{\varepsilon^3}\right) + \theta\left(r' - \varepsilon\right)\frac{1}{r'}\right] \end{aligned} \tag{E.8}$$

where we used $\rho = \frac{3}{4}\frac{\lambda}{\pi\varepsilon^3}$ and $\kappa^2 = 16\pi G$.

E.2 Proof of (3.88) and (3.89)

(3.88) is the same as

$$\frac{1}{16}\kappa^4 \int đ^3k_1 đ^3k_2 đ^3k_3 \Delta e^{i\vec{k}_1\cdot\vec{x}}\delta^3\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right)\frac{k_2^i k_2^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) = \left(V\partial^i\partial^j V\right) \tag{E.9}$$

In order to show this, we note that:

$$\Delta e^{i\vec{k}_1\cdot\vec{x}} = -\vec{k}_1^2 e^{i\vec{k}_1\cdot\vec{x}} \tag{E.10}$$

Indeed:

$$\begin{aligned}
&\frac{1}{16}\kappa^4 \int đ^3k_1 đ^3k_2 đ^3k_3 \Delta e^{i\vec{k}_1\cdot\vec{x}} \\
&\times\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\frac{k_2^ik_2^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) = -\frac{1}{16}\kappa^4\int đ^3k_1 đ^3k_2 đ^3k_3 e^{i\vec{k}_1\cdot\vec{x}}\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\frac{k_2^ik_2^j}{\vec{k}_2^2\vec{k}_3^2} \\
&\qquad\times\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&= -\frac{1}{16}\kappa^4\int đ^3k_2 đ^3k_3 e^{-i\vec{k}_2\cdot\vec{x}}e^{-i\vec{k}_3\cdot x}\frac{k_2^ik_2^j}{\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&= \underbrace{-\frac{1}{4}\kappa^2\int đ^3k_3\frac{e^{-i\vec{k}_3\cdot\vec{x}}}{\vec{k}_3^2}\mu\left(\vec{k}_3\right)}_{=V(x)}\frac{1}{4}\kappa^2\int đ^3k_2 e^{-i\vec{k}_2\cdot\vec{x}}\frac{k_2^ik_2^j}{\vec{k}_2^2}\mu\left(\vec{k}_2\right) \\
&= -V\left(x\right)\frac{1}{4}\kappa^2\int đ^3k_2 e^{-i\vec{k}_2\cdot\vec{x}}\frac{k_2^ik_2^j}{\vec{k}_2^2}\mu\left(\vec{k}_2\right) \\
&= V\left(x\right)\partial^i\partial^j\underbrace{\frac{1}{4}\kappa^2\int đ^3k_2 e^{-i\vec{k}_2\cdot\vec{x}}\frac{1}{\vec{k}_2^2}\mu\left(\vec{k}_2\right)}_{=V(\vec{x})} \\
&= V\partial^i\partial^jV
\end{aligned}\tag{E.11}$$

(3.89) is equivalent to

$$\frac{1}{16}\kappa^4\int đ^3k_1 đ^3k_2 đ^3k_3\Delta_{\vec{x}}e^{i\vec{k}_1\cdot\vec{x}}\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\frac{k_2^i\vec{k}_3^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right)=\partial^iV\partial^jV. \tag{E.12}$$

Again using (E.10) we have indeed:

$$\begin{aligned}
&\frac{1}{16}\kappa^4\int đ^3k_1 đ^3k_2 đ^3k_3\Delta_{\vec{x}}e^{i\vec{k}_1\cdot\vec{x}}\delta^3\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\frac{k_2^i\vec{k}_3^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu\left(\vec{k}_2\right)\mu\left(\vec{k}_3\right) \\
&= -\frac{1}{16}\kappa^4\int đ^3k_2 đ^3k_3\frac{e^{-i\vec{k}_2\cdot\vec{x}}}{\vec{k}_2^2}\frac{e^{-i\vec{k}_3\cdot\vec{x}}}{\vec{k}_3^2}k_2^i\mu\left(\vec{k}_2\right)k_3^j\mu\left(\vec{k}_3\right) \\
&= \underbrace{\frac{1}{4}\kappa^2\int đ^3k_2\frac{\partial^ie^{-i\vec{k}_2\cdot\vec{x}}}{\vec{k}_2^2}\mu\left(\vec{k}_2\right)}_{=\partial^iV}\underbrace{\frac{1}{4}\kappa^2\int đ^3k_3\frac{\partial^je^{-i\vec{k}_3\cdot\vec{x}}}{\vec{k}_3^2}\mu\left(\vec{k}_3\right)}_{=\partial^jV} \\
&= \partial^iV\partial^jV
\end{aligned}\tag{E.13}$$

E.3 Proof of (3.114)

The Laplacian in spherical coordinates is given by:

$$\Delta f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\left(\theta\right)}\frac{\partial}{\partial\theta}\left(\sin\left(\theta\right)\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\left(\theta\right)}\frac{\partial^2 f}{\partial\theta^2} \tag{E.14}$$

In our case the computations are made easier by the fact that $\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \varphi} = 0$.
The first partial derivative with respect to r is given by:

$$\begin{aligned}
\frac{\partial V}{\partial r} &= -G\lambda\left[-\frac{1}{r^2}\theta(r-\varepsilon) + \frac{1}{r}\frac{\partial}{\partial r}\underbrace{\theta(r-\varepsilon)}_{=\delta(r-\varepsilon)} - \frac{r}{\varepsilon^3}\theta(\varepsilon-r) + \left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\underbrace{\frac{\partial}{\partial r}\theta(\varepsilon-r)}_{=-\delta(\varepsilon-r)}\right] \\
&= -G\lambda\left[-\frac{1}{r^2}\theta(r-\varepsilon) + \frac{1}{\varepsilon}\delta(r-\varepsilon) - \frac{r}{\varepsilon^3}\theta(\varepsilon-r) - \left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{1}{\varepsilon}\right)\delta(\varepsilon-r)\right] \\
&= -G\lambda\left[-\frac{1}{r^2}\theta(r-\varepsilon) - \frac{r}{\varepsilon^3}\theta(\varepsilon-r)\right] \qquad \text{(E.15)}
\end{aligned}$$

Then

$$r^2\frac{\partial V}{\partial r} = -G\lambda\left[-\theta(\varepsilon-r) - \frac{r^3}{\varepsilon^3}\theta(\varepsilon-r)\right] \qquad \text{(E.16)}$$

Differentiating this once more with respect to r we obtain:

$$\begin{aligned}
\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) &= G\lambda\left[\frac{\partial}{\partial r}\theta(r-\varepsilon) + \frac{3r^2}{\varepsilon^3}\theta(\varepsilon-r) + \frac{r^3}{\varepsilon^3}\frac{\partial}{\partial r}\theta(\varepsilon-r)\right] \\
&= G\lambda\left[\delta(r-\varepsilon) + \frac{3r^2}{\varepsilon^3}\theta(\varepsilon-r) - \frac{r^3}{\varepsilon^3}\delta(\varepsilon-r)\right] \\
&= G\lambda\frac{3r^2}{\varepsilon^3}\theta(\varepsilon-r) \qquad \text{(E.17)}
\end{aligned}$$

and finally

$$\Delta V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = G\lambda\frac{3}{\varepsilon^3}\theta(\varepsilon-r) \qquad \text{(E.18)}$$

Then I can write down:

$$\begin{aligned}
V\Delta V &= -G^2\lambda^2\left(\frac{3}{\varepsilon^3}\theta(\varepsilon-r)\right)\left[\frac{1}{r}\theta(r-\varepsilon) + \left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\theta(\varepsilon-r)\right] \\
&= -G^2\lambda^2\left[\frac{9}{2\varepsilon^4} - \frac{3}{2}\frac{r^2}{\varepsilon^6}\right]\theta(\varepsilon-r) \qquad \text{(E.19)}
\end{aligned}$$

We can now compute the Laplacian of the right hand side of (3.114).
The Laplacian of the first term is:

$$\begin{aligned}
\Delta\left[\frac{6}{5r\varepsilon}\theta(r-\varepsilon)\right] &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\left[\frac{6}{5r\varepsilon}\theta(r-\varepsilon)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[-\frac{6}{5r^2\varepsilon}\theta(r-\varepsilon) + \frac{6}{5r\varepsilon}\delta(r-\varepsilon)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[-\frac{6}{5\varepsilon}\theta(r-\varepsilon) + \frac{6r}{5\varepsilon}\delta(r-\varepsilon)\right] \\
&= \frac{1}{r^2}\left[-\frac{6}{5\varepsilon}\delta(r-\varepsilon) + \frac{6}{5\varepsilon}\delta(r-\varepsilon) + \frac{6r}{5\varepsilon}\delta'(r-\varepsilon)\right] \\
&= \frac{6}{5r\varepsilon}\delta'(r-\varepsilon) \qquad \text{(E.20)}
\end{aligned}$$

The Laplacian of the second term is:

$$
\begin{aligned}
\Delta\left[\left(\frac{15}{8\varepsilon^2}-\frac{3r^2}{4\varepsilon^4}+\frac{3}{40}\frac{r^4}{\varepsilon^6}\right)\theta(\varepsilon-r)\right] &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\left[\left(\frac{15}{8\varepsilon^2}-\frac{3r^2}{4\varepsilon^4}+\frac{3}{40}\frac{r^4}{\varepsilon^6}\right)\theta(\varepsilon-r)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[\left(-\frac{3r}{2\varepsilon^4}+\frac{3}{10}\frac{r^3}{\varepsilon^6}\right)\theta(\varepsilon-r)-\right. \\
&\quad \left.-\left(\frac{15}{8\varepsilon^2}-\frac{3r^2}{4\varepsilon^4}+\frac{3r^4}{40\varepsilon^6}\right)\delta(\varepsilon-r)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(-\frac{3r^3}{2\varepsilon^4}+\frac{3r^5}{10\varepsilon^6}\right)\theta(\varepsilon-r)\right. \\
&\quad \left.-\left(\frac{15r^2}{8\varepsilon^2}-\frac{3r^4}{4\varepsilon^4}+\frac{3r^6}{40\varepsilon^6}\right)\delta(\varepsilon-r)\right] \\
&= \frac{1}{r^2}\left[\left(-\frac{9r^2}{2\varepsilon^4}+\frac{15r^4}{10\varepsilon^6}\right)\theta(\varepsilon-r)-\left(-\frac{3r^3}{2\varepsilon^4}+\frac{3r^5}{10\varepsilon^6}\right)\delta(\varepsilon-r)\right. \\
&\quad -\left(\frac{30r}{8\varepsilon^2}-\frac{12r^3}{4\varepsilon^4}+\frac{18r^5}{40\varepsilon^6}\right)\delta(\varepsilon-r) \\
&\quad \left.-\left(\frac{15r^2}{8\varepsilon^2}-\frac{3r^4}{4\varepsilon^4}+\frac{3r^6}{40\varepsilon^6}\right)\delta'(\varepsilon-r)\right] \qquad \text{(E.21)}
\end{aligned}
$$

Thus the Laplacian of the right hand side of (3.114) is:

$$
\begin{aligned}
G^2\lambda^2 &\left[\frac{6}{5r\varepsilon}\delta'(r-\varepsilon)-\left(\frac{15}{8\varepsilon^2}-3r^24\varepsilon^4+\frac{3r^4}{40\varepsilon^6}\right)\underbrace{\delta'(\varepsilon-r)}_{=-\delta'(r-\varepsilon)}\right. \\
&\left.+\left(-\frac{9}{2\varepsilon^4}+\frac{15r^2}{10\varepsilon^6}\right)\theta(\varepsilon-r)\right] = G^2\lambda^2\left(-\frac{9}{2\varepsilon^4}+\frac{3r^2}{2\varepsilon^6}\right)\theta(\varepsilon-r)
\end{aligned}
$$

This proves (3.114).

E.4 Proof of (3.115)

The first term on the right hand side is:

$$\begin{aligned}
\Delta\left[\left(\frac{1}{2r^2}-\frac{6}{5r\varepsilon}\right)\theta(r-\varepsilon)\right] &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[\left(-\frac{1}{r^3}+\frac{6}{5r^2\varepsilon}\right)\theta(r-\varepsilon)+\left(\frac{1}{2r^2}-\frac{6}{5r\varepsilon}\right)\delta(r-\varepsilon)\right]\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(-\frac{1}{r}+\frac{6}{5\varepsilon}\right)\theta(r-\varepsilon)+\left(\frac{1}{2}-\frac{6r}{5\varepsilon}\right)\delta(r-\varepsilon)\right]\\
&= \frac{1}{r^2}\left[\frac{1}{r^2}\theta(r-\varepsilon)+\left(-\frac{1}{r}+\frac{6}{5\varepsilon}\right)\delta(r-\varepsilon)\right.\\
&\quad\left.-\frac{6}{5\varepsilon}\delta(r-\varepsilon)+\left(\frac{1}{2}-\frac{6r}{5\varepsilon}\right)\delta'(r-\varepsilon)\right]\\
&= \frac{1}{r^4}\theta(r-\varepsilon)+\left(-\frac{1}{r^3}+\frac{6}{5\varepsilon r^2}\right)\delta(r-\varepsilon)-\frac{6}{5\varepsilon r^2}\delta(r-\varepsilon)\\
&\quad+\left(\frac{1}{2r^2}-\frac{6}{5\varepsilon r}\right)\delta'(r-\varepsilon)\\
&= \frac{1}{r^4}\theta(r-\varepsilon)-\frac{1}{r^3}\delta(r-\varepsilon)+\left(\frac{1}{2r^2}-\frac{6}{5\varepsilon r}\right)\delta'(r-\varepsilon) \qquad \text{(E.22)}
\end{aligned}$$

The other term is:

$$\begin{aligned}
\Delta\left[\left(-\frac{3}{4\varepsilon^2}+\frac{r^4}{20\varepsilon^6}\right)\theta(\varepsilon-r)\right] &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\left[\left(-\frac{3}{4\varepsilon^2}+\frac{r^4}{20\varepsilon^6}\right)\theta(\varepsilon-r)\right]\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[\frac{r^3}{5\varepsilon^6}\theta(\varepsilon-r)-\left(-\frac{3}{4\varepsilon^2}+\frac{r^4}{20\varepsilon^6}\right)\delta(\varepsilon-r)\right]\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{r^5}{5\varepsilon^6}\theta(\varepsilon-r)-\left(-\frac{3r^2}{4\varepsilon^2}+\frac{r^6}{20\varepsilon^6}\right)\delta(\varepsilon-r)\right]\\
&= \frac{1}{r^2}\left[\frac{r^4}{\varepsilon^6}\theta(\varepsilon-r)-\left(-\frac{6r}{4\varepsilon^2}+\frac{6r^5}{20\varepsilon^6}\right)\delta(\varepsilon-r)-\frac{r^5}{5\varepsilon^6}\delta(\varepsilon-r)\right.\\
&\quad\left.-\left(\frac{-3r^2}{4\varepsilon^2}+\frac{r^6}{20\varepsilon^6}\right)\delta(\varepsilon-r)\right] \qquad \text{(E.23)}\\
&= \frac{r2}{\varepsilon^6}\theta(\varepsilon-r)-\left(-\frac{6}{4\varepsilon^2 r}+\frac{3r^3}{10\varepsilon^6}\right)\delta(\varepsilon-r)-\frac{r^3}{5\varepsilon^6}\delta(\varepsilon-r)\\
&\quad-\left(-\frac{3}{4\varepsilon^2}+\frac{r^4}{20\varepsilon^6}\right)\delta'(\varepsilon-r) \qquad \text{(E.24)}
\end{aligned}$$

Now to compute the left hand side of (3.115) we can use $\frac{\partial r}{\partial x^l}=\frac{x^l}{r}$ and $\frac{\partial V}{\partial x^l}=\frac{\partial V}{\partial r}\frac{\partial r}{\partial x^l}$.
Since

$$\frac{\partial V}{\partial r}=-G\lambda\left[-\frac{1}{r^2}\theta(r-\varepsilon)+\frac{1}{r}\delta(r-\varepsilon)-\frac{r}{\varepsilon^3}\theta(\varepsilon-r)-\left(\frac{3}{2}\frac{1}{\varepsilon}-\frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\delta(\varepsilon-r)\right] \qquad \text{(E.25)}$$

we can write:

$$\begin{aligned}
\partial_l V \partial^l V &= \frac{G^2\lambda^2}{r^2} x^l x_l \left[-\frac{1}{r^2}\theta\left(r-\varepsilon\right) + \frac{1}{r}\delta\left(r-\varepsilon\right) - \frac{r}{\varepsilon^3}\theta\left(\varepsilon-r\right) - \left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\delta\left(\varepsilon-r\right)\right] \\
&\quad \times \left[-\frac{1}{r^2}\theta\left(r-\varepsilon\right) + \frac{1}{r}\delta\left(r-\varepsilon\right) - \frac{r}{\varepsilon^3}\theta\left(\varepsilon-r\right) - \left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\delta\left(\varepsilon-r\right)\right] \\
&= G^2\lambda^2 \left[\frac{1}{r^4}\theta\left(r-\varepsilon\right) + \frac{1}{r^2}\delta\left(r-\varepsilon\right) + \frac{r^2}{\varepsilon^6}\theta\left(\varepsilon-r\right) + \left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right)^2 \delta\left(\varepsilon-r\right) \right. \\
&\quad -\frac{2}{r^3}\theta\left(r-\varepsilon\right)\delta\left(r-\varepsilon\right) + \frac{2}{r\varepsilon^3}\underbrace{\theta\left(r-\varepsilon\right)\theta\left(\varepsilon-r\right)}_{=0} + \frac{2}{r^2}\left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\delta\left(\varepsilon-r\right)\theta\left(r-\varepsilon\right) \\
&\quad -\frac{2}{\varepsilon^3}\delta\left(r-\varepsilon\right)\theta\left(\varepsilon-r\right) - \frac{2}{r}\left(\frac{3}{2}\frac{1}{\varepsilon} - \frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\delta\left(r-\varepsilon\right)\delta\left(\varepsilon-r\right) \\
&\quad \left. +\frac{2r}{\varepsilon^3}\left(\frac{3}{2\varepsilon} - \frac{r^2}{2\varepsilon^3}\right)\theta\left(\varepsilon-r\right)\delta\left(\varepsilon-r\right)\right]
\end{aligned} \tag{E.26}$$

All that is now left is to compare the two sides:

$$\begin{aligned}
\frac{1}{r^4}\theta\left(r-\varepsilon\right) + \frac{1}{r^2}\delta\left(r-\varepsilon\right)& \\
+\frac{r^2}{\varepsilon^6}\theta\left(\varepsilon-r\right) + \left(\frac{9}{4\varepsilon^2} + \frac{1}{4}\frac{r^4}{\varepsilon^6} - \frac{3r^2}{2\varepsilon^4}\right)\delta\left(\varepsilon-r\right)& \\
-\frac{2}{r^3}\theta\left(r-\varepsilon\right)\delta\left(r-\varepsilon\right) + \left(\frac{3}{r^2\varepsilon} - \frac{1}{\varepsilon^3}\right)\delta\left(\varepsilon-r\right)\theta\left(r-\varepsilon\right)& \\
-\frac{2}{\varepsilon^3}\delta\left(r-\varepsilon\right)\theta\left(\varepsilon-r\right) - \left(\frac{3}{r\varepsilon} - \frac{r}{\varepsilon^3}\right)\delta\left(r-\varepsilon\right)\delta\left(\varepsilon-r\right)& \\
+\left(\frac{3r}{\varepsilon^4} - \frac{r^3}{\varepsilon^6}\right)\theta\left(\varepsilon-r\right)\delta\left(\varepsilon-r\right) &\stackrel{!}{=} \frac{1}{r^4}\theta\left(r-\varepsilon\right) - \frac{1}{r^3}\delta\left(r-\varepsilon\right) \\
&\quad + \left(\frac{1}{2r^2} - \frac{6}{5\varepsilon r}\right)\delta'\left(r-\varepsilon\right) + \frac{r^2}{\varepsilon^6}\theta\left(\varepsilon-r\right) \\
&\quad - \left(-\frac{6}{4\varepsilon^2 r} + \frac{3r^3}{10\varepsilon^6}\right)\delta\left(\varepsilon-r\right) - \frac{r^3}{5\varepsilon^6}\delta\left(\varepsilon-r\right) \\
&\quad - \left(-\frac{3}{4\varepsilon^2} + \frac{r^4}{20\varepsilon^6}\right)\delta'\left(\varepsilon-r\right)
\end{aligned}$$

The two sides are indeed equal as one can see from:

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\delta\left(r-\varepsilon\right)+\left(\frac{9}{4\varepsilon^2}+\frac{1}{4\varepsilon^2}-\frac{3}{2\varepsilon^2}\right)\delta\left(\varepsilon-r\right)\\
&-\frac{2}{\varepsilon^3}\delta\left(r-\varepsilon\right)\theta\left(r-\varepsilon\right)+\left(\frac{3}{\varepsilon^3}-\frac{1}{\varepsilon^3}\right)\delta\left(\varepsilon-r\right)\theta\left(r-\varepsilon\right)\\
&-\frac{2}{\varepsilon^3}\delta\left(r-\varepsilon\right)\theta\left(\varepsilon-r\right) = \left(-\frac{1}{\varepsilon^3}+\frac{6}{5\varepsilon^3}\right)\delta\left(r-\varepsilon\right)-\frac{6}{5\varepsilon^3}\delta\left(r-\varepsilon\right)\\
&\qquad+\left(\frac{1}{2\varepsilon^2}-\frac{6}{5\varepsilon^2}\right)\delta'\left(\varepsilon-r\right)-\frac{1}{5\varepsilon^3}\delta\left(\varepsilon-r\right)\\
&\qquad-\left(-\frac{6}{4\varepsilon^3}+\frac{3}{10\varepsilon^3}\right)\delta\left(\varepsilon-r\right)\\
&\qquad-\left(-\frac{3}{4\varepsilon^2}+\frac{1}{20\varepsilon^2}\right)\delta'\left(\varepsilon-r\right)
\end{aligned}$$

E.5 Proof of (3.116)

We want to show:

$$\frac{\kappa^4}{4}p = \Delta\left[-\frac{1}{5r\varepsilon}G^2\lambda^2\theta\left(r-\varepsilon\right)+\left(-\frac{3}{8\varepsilon^2}+\frac{r^2}{4\varepsilon^4}-\frac{3r^4}{40\varepsilon^6}\right)G^2\lambda^2\theta\left(\varepsilon-r\right)\right] \tag{E.27}$$

The left hand side can be rewritten as:

$$\frac{\kappa^2}{4}p = \frac{\kappa^2}{4}\frac{1}{24}\kappa^2\frac{9}{16\pi^2}\frac{\lambda^2}{\varepsilon^6}\left(\varepsilon^2-r^2\right)\theta\left(\varepsilon-r\right) \tag{E.28}$$

$$= \frac{3}{2}G^2\lambda^2\frac{1}{\varepsilon^6}\left(\varepsilon^2-r^2\right)\theta\left(\varepsilon-r\right) \tag{E.29}$$

The right hand side requires a bit more computations:

$$\begin{aligned}
&\Delta\left[\left(-\frac{1}{5r\varepsilon}G^2\lambda^2\theta\left(r-\varepsilon\right)\right)\right] \\
+\Delta\left[\left(-\frac{3}{8\varepsilon^2}+\frac{r^2}{4\varepsilon^4}-\frac{3r^4}{40\varepsilon^6}\right)G^2\lambda^2\theta\left(\varepsilon-r\right)\right] &= G^2\lambda^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\left(\left(-\frac{1}{5r\varepsilon}\right)\theta\left(r-\varepsilon\right)\right) \\
&\quad +G^2\lambda^2\frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(-\frac{3}{8\varepsilon^2}+\frac{r^2}{4\varepsilon^4}-\frac{3r^4}{40\varepsilon^6}\right)\theta\left(\varepsilon-r\right)\right] \\
&= G^2\lambda^2\frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{1}{5\varepsilon}\theta\left(r-\varepsilon\right)-\frac{r}{5\varepsilon}\delta\left(r-\varepsilon\right)\right] \\
&\quad +G^2\lambda^2\frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(\frac{r^3}{2\varepsilon^4}-\frac{3r^5}{10\varepsilon^6}\right)\theta\left(\varepsilon-r\right)\right. \\
&\quad \left.-\left(-\frac{3r^2}{8\varepsilon^2}+\frac{r^4}{4\varepsilon^4}-\frac{3r^6}{40\varepsilon^6}\right)\delta\left(\varepsilon-r\right)\right] \\
&= -\frac{G^2\lambda^2}{5r\varepsilon}\delta'\left(r-\varepsilon\right)+G^2\lambda^2\left[\frac{3}{2}\frac{1}{\varepsilon^6}\left(\varepsilon^2-r^2\right)\theta\left(\varepsilon-r\right)\right. \\
&\quad \left.-\frac{1}{\varepsilon^3}\frac{2}{5}\delta\left(\varepsilon-r\right)+\frac{1}{\varepsilon^2}\frac{1}{5}\delta'\left(\varepsilon-r\right)\right] \\
&= \frac{3}{2}G^2\lambda^2\frac{1}{\varepsilon^6}\left(\varepsilon^2-r^2\right)\theta\left(\varepsilon-r\right) \qquad \text{(E.30)}
\end{aligned}$$

E.6 Proof of (3.119)

We now show (3.119).
We start with the left hand side of the equation:

$$\begin{aligned}
\partial^i V\partial^j V &= \partial^i V\frac{\partial r}{\partial x_j}\partial_r\left[-G\lambda\left(\frac{1}{r}\theta\left(r-\varepsilon\right)+\left(\frac{3}{2}\frac{1}{\varepsilon}-\frac{1}{2}\frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\theta\left(\varepsilon-r\right)\right)\right] \\
&= \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\left(-G\lambda\right)^2\left[-\frac{1}{r}\theta\left(r-\varepsilon\right)+\frac{1}{r}\delta\left(r-\varepsilon\right)-\frac{r}{\varepsilon^3}\theta\left(\varepsilon-r\right)-\left(\frac{3}{2}\frac{1}{\varepsilon}-\frac{1}{2}\frac{r^2}{\varepsilon^3}\right)\delta\left(\varepsilon-r\right)\right]^2 \\
&= \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}G^2\lambda^2\frac{1}{r^4} \qquad \text{(E.31)}
\end{aligned}$$

where we used the fact that we are only interested in $r > \varepsilon$.

The derivative of the radius can be rewritten as:

$$\begin{aligned}
\frac{\partial r}{\partial x_i} &= \frac{\partial}{\partial x_i}\sqrt{\eta^{ab}x_a x_b} \\
&= \frac{x^i}{r} \qquad \text{(E.32)}
\end{aligned}$$

Thus the left hand side of (3.119) is:

$$\partial^i V\partial^j V = \frac{x^i x^j}{r^6}G^2\lambda^2 \qquad \text{(E.33)}$$

Now we take care of the right hand side:

$$\begin{aligned}
\Delta\left[G^2\lambda^2\left(\left(\frac{1}{4r^2}-\frac{2}{5r\varepsilon}\right)\eta^{ij}-\frac{x^ix^j}{4r^4}\right)\right] &= G^2\lambda^2\frac{\eta^{ij}}{r^2}\frac{\partial}{\partial r}\left(-\frac{1}{2r}+\frac{2}{5\varepsilon}\right) \\
&\quad -G^2\lambda^2\left[\frac{\eta^{ab}}{4}\partial_a\left(\frac{\delta^i{}_b x^j+\delta^j{}_b x^i}{r^4}\right)-\frac{\eta^{ab}}{4}\partial_a\left(4\frac{x^ix^jx_b}{r^6}\right)\right] \\
&= G^2\lambda^2\frac{\eta^{ij}}{2r^4}-G^2\lambda^2\left(\frac{\eta^{ij}}{2r^4}-\frac{x^ix^j}{r^6}\right) \\
&= \frac{G^2\lambda^2x^ix^j}{r^6} \qquad \text{(E.34)}
\end{aligned}$$

F Calculations leading to (4.49) and (4.50)

In order to determine the exact factors appearing in front of $\frac{1}{\Delta}\partial^k V\partial^l V$ and $\frac{1}{\Delta}V\Delta V$ in (4.49) and (4.50) we first need to compute the partial derivative of M_G, as one can see from (4.46), and then can implement again the integration rules described in appendix D.

The derivative of M_G can be defined as

```
UndefTensor[derM];
DefTensor[derM[a, m, n, r, s, t, a1, b1], M4];
derM[a_, m_, n_, r_, s_, t_, a1_, b1_] :=
 1/2 (1/2 metricg[a1, b1]) metricg[a, r]
     (metricg[m, t] metricg[n, s] + metricg[m, s] metricg[n, t] -
       metricg[m, n] metricg[t, s]) +
    1/2
     (-1/2 (metricg[r, a1] metricg[a, b1] +
         metricg[r, b1] metricg[a, a1]))
     (metricg[m, t] metricg[n, s] + metricg[m, s] metricg[n, t] -
       metricg[m, n] metricg[t, s]) +
    1/2 metricg[a, r]
     (-1/2 (metricg[m, a1] metricg[t, b1] + metricg[m, b1] metricg[t, a1])
       metricg[n, s] -
      1/2 (metricg[n, a1] metricg[s, b1] + metricg[n, b1] metricg[s, a1])
       metricg[m, t] -
      1/2 (metricg[m, a1] metricg[s, b1] + metricg[m, b1] metricg[s, a1])
       metricg[n, t] -
      1/2 (metricg[n, a1] metricg[t, b1] + metricg[n, b1] metricg[t, a1])
       metricg[m, s] +
      1/2 (metricg[m, a1] metricg[n, b1] + metricg[m, b1] metricg[n, a1])
       metricg[t, s] +
      1/2 (metricg[t, a1] metricg[s, b1] + metricg[t, b1] metricg[s, a1])
       metricg[m, n]) // ContractMetric // Simplification // Expand
```

Then all we have to do is insert this in (4.46) and reapply the integration rules seen in appendix D to get the desired result.

References

[1] M. J. Duff. Quantum tree graphs and the Schwarzschild solution. *Phys. Rev. D*, 7:2317–2326, Apr 1973.

[2] Wolfram Research, Inc. Mathematica 10.4.

[3] José M. Martín-García. xact 1.1.2.

[4] J. N. Goldberg. Conservation laws in general relativity. *Phys. Rev.*, 111:315–320, Jul 1958.

[5] Bryce S. DeWitt. Quantum theory of gravity. III. Applications of the covariant theory. *Phys. Rev.*, 162:1239–1256, Oct 1967.

[6] Wheeler Misner, Thorne. *Gravitation.* W. H. Freeman and Company, 1973.

[7] D. M. Capper, G. Leibbrandt, and M. Ramón Medrano. Calculation of the graviton self-energy using dimensional regularization. *Phys. Rev. D*, 8:4320–4331, Dec 1973.

[8] Bryce S. DeWitt. Quantum theory of gravity. II. The manifestly covariant theory. *Phys. Rev.*, 162:1195–1239, Oct 1967.

[9] Karl Schwarzschild. Über das gravitationsfeld eines massenpunktes nach der einsteinschen theorie. *Sitzungsberichten der Königlich-Preussischen Akademie der Wissenschaften*, 1916.

[10] M.E. Peskin and D.V. Schroeder. *An Introduction to Quantum Field Theory.* Advanced book classics. Addison-Wesley Publishing Company, 1995.

[11] A. Zee. *Quantum Field Theory in a Nutshell.* Princeton University Press, 2003.

[12] S. Weinberg. *Gravitation and cosmology: principles and applications of the general theory of relativity.* Wiley, 1972.

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Appendix

Declaration

I declare that I wrote this thesis by myself and listed all used sources. I agree with making this thesis publicly available.

Ottavia Balducci

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