Master's Thesis in Theoretical and Mathematical Physics

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LSZ Reduction in QFT and Lattice Systems

Simon Bürger

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Supervisor:
Dr. Wojciech Dybalski (TU Munich)

Second Referee:
Prof. Sven Bachmann (LMU Munich)

Abstract

In this thesis, we outline scattering theory for quasi particles in quantum spin systems. Following the Haag-Ruelle approach from algebraic quantum field theory we construct asymptotic scattering states, discuss their properties and define the S-matrix. Finally we develop the LSZ reduction formula for general lattice systems, which might in the future be used to prove non-triviality of the S-matrix in some concrete settings such as the Ising model. Furthermore we explore the possibility of using path integrals on coherent states to describe time evolution in spin systems.

Declaration of Authorship

I hereby declare that the thesis submitted is my own unaided work. All direct or indirect sources used are acknowledged as references. This work was not previously presented to another examination board and has not been published.

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Contents

Introduction

For a long time quantum field theory has been one of the two major pillars of fundamental physics (the other one being general relativity). Its major incarnations, QED and QCD, have been validated in countless experiments. But from a mathematical point of view, the situation remains quite unsatisfactory, because these field theories are plagued by divergences, such that the naive results obtained are infinite more often than not. This is typically remedied by a variety of renormalization and regularization schemes which produce finite results for physical quantities. These results are in great agreement with experimental data but mathematically they are mostly not well defined and often look more like an ad hoc solution to "just subtract the infinite term and use the remaining part".

In the sixties, R. Haag and D. Kastler [13] tried to improve this situation by starting from a few mathematical axioms for their local quantum field theory, and derive the rest of the theory in a mathematical rigorous way. But even though many results from ordinary quantum field theory can be translated into this rigorous framework, it is still an open problem to find a formulation of a real physical theory (like QED or QCD) in terms of Haag-Kastler axioms. In fact, all models of the framework analyzed so far have either been non-interacting, or only possible in a non-physical 1 or 2 dimensional world.

A major problem with rigorous construction of QFT models in physical space-time are so called ultraviolet divergences. These are infinities introduced by considering arbitrarily small distances (or equivalently high energies). The usual way to handle this is to introduce a UV-cutoff: Simply constrict the relevant integrals by an unphysical lower bound on the distances and try to take this limit to zero after all other calculations are done. In a mathematically rigorous setting we can not do this (as it boils down to incorrectly swapping the order of limits and integrals), but we can introduce a minimal distance right from the axiomatic start. Instead of modeling space as continuous $\mathbb{R}^3$, we use $\mathbb{Z}^3$ and imagine the spacing between the discrete sites as small.

Of course such a lattice system will not be a satisfactory fundamental theory of the real world. In particular because there can't be Lorentz invariance in a rectangular grid. But we can still hope to find a model in physical space-time dimension and prove rigorously the non-triviality of the scattering matrix. Then this might give us some hints as to how to find a good model for the actual continuous framework. Recently it was shown that at least in some cases, a quantum spin system satisfying Lieb-Robinson bounds is sufficient to establish scattering states and define the S-matrix. [4] This thesis is heavily motivated by that progress.

One useful tool for actually computing the scattering matrix is the so called "LSZ reduction formula", named after H. Lehmann, K. Symanzik and W. Zimmermann [15]. It expresses the amplitude of a scattering process as vacuum expectation values of time-ordered products of certain observables. And in a concrete model these correlation functions are often computable, at least perturbatively. In this thesis we develop such a reduction formula for the lattice framework (theorem 3.25), which

in the end looks very similar to the long known continuous case. Though actually evaluating the formula turns out to be a challenge because there is a non-trivial dispersion relation coming into play, and also because a perturbative approach is not possible in a straightforward manner (see discussion in section 3.6). Therefore we are looking for alternatives.
A common way to express expectation values of time-ordered products of operators are path-integrals. In [6], rigorous path-integral formulas are derived for a single spin. In this thesis, we heuristically derive a similar formula for spin systems. This is achieved by replacing the discrete set of spin-states $|-1/2\rangle$, $|+1/2\rangle$ with the continuous set of "coherent states" $|z\rangle$ with $z \in \mathbb{C}$. This enables expression like

Conjecture *(4.5).* *The time evolution of coherent states can be written as*

$$\langle z^f | e^{-itH} | z^i \rangle = \int_{\substack{z(0)=z^i \\ z(t)=z^f}} D\mu(z) \exp\left(-i \int_0^t d\tau L(z(\tau), \dot{z}(\tau))\right) \tag{1}$$

with

$$L(z, \dot{z}) = h(z, z) + is \sum_{x \in \Gamma} \frac{\overline{\dot{z}_x} z_x - \dot{z}_x \overline{z_x}}{1 + |z_x|^2}. \tag{2}$$

which looks roughly familiar to the path-integral expressions from ordinary QFT. But there are critical differences in L which takes the role of a Lagrangian even though it is not the result of a Legendre transformation but comes out of the definition of the (non-orthogonal) coherent states. Furthermore actually computing this expression is a challenge, mostly because this is not a Gaussian integral. At the very least, the expression of L can be used in a stationary phase method to get a semi-classical approximation of the spin system, which is interesting in its own right.

Overview

In chapter one, we lay out briefly the mathematical background needed for an algebraic approach to quantum theories. That is we introduce C*-Algebras with their states and representations and state some central results from their mathematical theory. In chapter two we introduce the Haag-Kastler framework of algebraic quantum field theory and discuss scattering theory and the LSZ reduction formula in that framework. Chapter three finally sets up a general framework for lattice systems and then tries to repeat everything from chapter two in this new setting. This culminates in the LSZ reduction formula for lattice systems, which is a new result of this thesis.
Chapter four on path integrals for quantum spin systems is not mathematically rigorous as everything before. It is more of a exploratory study on how to formulate path integrals for spin systems which might help in evaluating formulas such as the LSZ in concrete models. Lastly chapter five is on the Ising model and how to integrate it into our framework as a basic example.

1 Algebraic Setup

Usually, quantum theories are formulated using two mathematical constructs. First, we have a Hilbert space which contains the possible states of the system. Second, one needs to consider operators acting on this Hilbert space. In particular, self-adjoint operators correspond to physical observables, as those precisely lead to real (as in "non complex") measurements. The idea of the "algebraic approach" is, to turn this around. I.e. we start with the observables (or more precisely the algebra they are contained in), and only later consider possible states (which as a whole do not need to form a Hilbert space anymore). In and of itself, the operators we mostly want to consider form a C*-algebra, the basic theory of which we will lay out in this chapter. All of this can be found in standard literature like [21].

Definition 1.1 *(C* Algebra).* *A (complex)* C*-Algebra *is a Banach Algebra A together with a anti-linear, anti-multiplicative, involutive map $^* : A \to A$ such that for all $a \in A$ it is*

$$\|a^*a\| = \|a\|^2 \tag{3}$$

The prime example of such an algebra is the space of bounded operators on a Hilbert space $B(H)$. Therefore the algebra $B(H)$ (and sub-algebras thereof) are sometimes called "concrete C*-algebras", in contrast to "abstract C*-Algebras" defined purely by their axioms.

Definition 1.2 *(States).* *Let $\mathcal{A}$ be a C*-Algebra.*

1. *A state is a linear functional $\omega : \mathcal{A} \to \mathbb{C}$ such that $\|\omega\| = 1$ and $\omega(a^*a) \geqslant 0$ for all $a \in \mathcal{A}$.*
2. *The set of all states is denoted as $S(\mathcal{A}) \subset \mathcal{A}^*$. It is convex and weak-*-compact, iff $\mathcal{A}$ contains a unit.*
3. *For any representation π of $\mathcal{A}$, we can build states using density matrices, i.e.*

$$S_\pi(\mathcal{A}) = \{a \mapsto \operatorname{tr}(\rho a) \mid 0 \leqslant \rho = \rho^* \leqslant 1 \text{ and } \operatorname{tr}(\rho) = 1\} \subset S(\mathcal{A}) \tag{4}$$

Definition 1.3 *(*-Representations).* *A* representation *of the C*-Algebra $\mathcal{A}$ is a *-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$.*

1. *π is* unital *if $\mathcal{A}$ contains an identity and $\pi(1) = 1$*
2. *π is* non-degenerate *if $\pi(a)v = 0\ \forall a \in \mathcal{A} \Rightarrow v = 0$*
3. *π is* faithful *if it is injective.*
4. *π is* cyclic *if there is a vector $\psi \in \mathcal{H}$ such that $\pi(\mathcal{A})\psi \subset \mathcal{H}$ is dense.*

5. *A subspace $\mathcal{H}' \subset \mathcal{H}$ is called* invariant *if for any operator $a \in \mathcal{H}$, it is $\pi(a)\mathcal{H}' \subset \mathcal{H}'$.*

6. *π is* irreducible *if for any closed subspace $H' \subset H$ which is closed under the action of $\pi(\mathcal{A})$, it*

Note that any representation can be split into the direct sum of a non-degenerate representation and a trivial zero-representation. Furthermore, any non-degenerate representation is unital

Lemma 1.4. *Let $\mathcal{A}$ be a C*-Algebra, $a, b \in \mathcal{A}$, ω be a state, $\pi : \mathcal{A} \to B(\mathcal{H})$ a representation.*

1. *There is a net of elements $0 \leqslant 1_n \leqslant 1$ such that $\lim 1_n a = \lim a 1_n = a$*

2. *$\omega(a^*) = \overline{\omega(a)}$*

3. *Cauchy-Schwarz-Inequality: $|\omega(a^*b)|^2 \leqslant \omega(a^*a)\omega(b^*b)$*

4. *The representation is contracting (and thus continuous): $\|\pi(a)\| \leqslant \|a\|$*

Theorem 1.5 *(GNS construction).* *For any state ω on a C*-Algebra A, there is a Hilbert space $\mathcal{H}$, a representation $\pi : A \to B(\mathcal{H})$ and a vector $\Omega \in \mathcal{H}$ such that for all $a \in A$*

$$\omega(a) = \langle \Omega | \pi(a) \Omega \rangle_{\mathcal{H}}. \tag{5}$$

Definition 1.6 *(W* Algebra).* *A C* Algebra $\mathcal{A}$ is a* W* Algebra *(or von Neumann Algebra) if, when viewed as a Banach space, it it the dual of some other Banach space, which is called the* predual *of $\mathcal{A}$, written as $\mathcal{A}_*$.*

Theorem 1.7 *(Von Neumann Bicommutant Theorem).* *Let $\mathcal{A} \subset B(\mathcal{H})$ be a unital *-Algebra. Then the closure of $\mathcal{A}$ in weak and strong topology is equal to $\mathcal{A}''$.*

Theorem 1.8 *(by Sakai [21]).* *Let $\mathcal{A} \subset B(H)$ be a C*-Algebra of bounded operators on a Hilbert space. Then $\mathcal{A}$ is a W*-Algebra if and only if $1 \in \mathcal{A}$ and $\mathcal{A}'' = \mathcal{A}$.*

Example 1.9. *$B(H)$ itself is a W*-Algebra. Its predual consists of the trace class operators $\rho \in S_1(H)$ with the prescription*

$$\|\rho\|_{S_1} = \operatorname{tr}(|\rho|) \tag{6}$$

$$\rho(A) = \operatorname{tr}(\rho A) \ \forall A \in \mathcal{A} \tag{7}$$

Note that the dual of $B(H)$ is strictly larger than its predual.

2 Haag-Ruelle Scattering Theory in QFT

2.1 Haag-Kastler Axioms

In this work we will use the so called *algebraic approach to local quantum field theory.* Two things in this description need explanation. First, "algebraic approach" means that we start with the observables of the system we wish to study, and not - as is common in conventional quantum mechanics - with the states of the system. And these observables should be part of a $*$-algebra. States only enter the picture in the second step as a dual space, or even later when we choose specific representations of the abstract algebra. One advantage of this approach is that all observables are bounded operators, which is a technical simplification. Some unbounded operators will be discussed, but only as part of a representation on some concrete Hilbert space. They are not part of the actual observable algebra.
The second term that should be explained is "local". This means we do not attempt to describe observables located in the whole infinite space-time. Instead we start at bounded sub-regions of space time, and consider operators which are in some sense located in that region. And even if one takes the limit of such operators located in increasingly large regions, it is not possible to construct truly global operators. Their influence at space-time points outside some region will always be small in a sufficiently strong sense. Now we will formalize this idea of local operator algebras.

Definition 2.1 *(Haag-Kastler Net).* *Let*

1. $\mathcal{A}$ *be a C*-Algebra*
2. $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$ *be a W*-algebra for any open, bounded region of space-time* $\mathcal{O} \subset \mathbb{R}^{d+1}$
3. *a group of automorphisms isomorphic to the connected part of the Poincaré group* $\alpha : \mathcal{P}_0 \to \operatorname{Aut} \mathcal{A}$

This is called a Haag-Kastler net if the following conditions are met

1. *Isotony: If we have two space-time regions, one inside the other, the algebra of the smaller region should be contained in that of the larger one. I.e. if* $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathbb{R}^{d+1}$*, it should be* $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$
2. *Totality: The whole algebra is simply the completion (in norm topology) of all local algebras:*
$$\mathcal{A} = \overline{\mathcal{A}_{loc}}$$
where
$$\mathcal{A}_{loc} = \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$$
is the (incomplete) algebra of strictly local observables.

3. *Locality: If the two regions $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^{d+1}$ are separated by space-like distances, their local algebras should commute, i.e.*

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$$

4. *Covariance: The automorphism α should respect the local structure of algebras.*

$$\alpha_g(\mathcal{A}(\mathcal{O})) = \mathcal{A}(g \cdot \mathcal{O})$$

Note that it is actually possible to start with only the local algebras, and take the axiom of totality as the definition of the whole observable algebra, in the sense of an inductive limit of C*-algebras. Secondly note that we do *not* require the whole algebra to be W*. If we did that (by taking a weak closure for example), the algebra would contain many more non-local elements, which our goal was to avoid. Requiring norm-closedness by the definition of a C*-algebra is in this sense a compromise between nice mathematical properties of $\mathcal{A}$ and its "localness".

Now the main tool to study such an algebra are representations. This is the first step to get from an abstract observable algebra to much more concrete and familiar physical objects. We will be using the usual convention in relativistic theories to denote d-dimensional spatial vectors with in boldface, and $d+1$-dimensional vectors in normal font, i.e. $x = (t, \boldsymbol{x})$ for position and $p = (E, \boldsymbol{p})$ for energy-momentum.

Definition 2.2 *(Vacuum representation).* *Let $(\mathcal{A}, \alpha)$ be a Haag-Kastler net. A representation $\pi : \mathcal{A} \to B(\mathcal{H})$ is called* vacuum representation *if α is unitarily and strongly continuously implemented with translation generators H and $P_1 \dots P_d$. These generators commute, so we can define their joint spectrum $Sp(H, \boldsymbol{P})$, which will be called "EM spectrum" from now on. Furthermore we assume*

1. *Stability: The EM spectrum is contained inside the future light cone $\overline{V_+} = \{(E, \boldsymbol{p}) \in \mathbb{R}^{d+1} \mid E \geqslant \|\boldsymbol{p}\|\}$.*

2. *Uniqueness of vacuum: There is a unique (up to a phase factor) translation invariant unit vector, called the vacuum $\Omega \in \mathcal{H}$.*

3. *Irreducibility: π is an irreducible representation.*

In order to define one-particle states and corresponding creation operators (see definition 2.12), we will also need the following assumption.

4 *Mass Shell: The EM spectrum can be written as a disjoint union of three closed sets $Sp(H, \boldsymbol{P}) = \{0\} \cup H_m \cup G$ where*

   - *0 is an isolated eigenvalue (corresponding to the vacuum Ω)*
   - *H_m is the graph of the dispersion relation $\omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$ with $m > 0$.*
   - *G is a subset of $\{(E, \boldsymbol{p}) \mid E \geqslant \sqrt{\boldsymbol{p}^2 + (2m)^2}\}$*

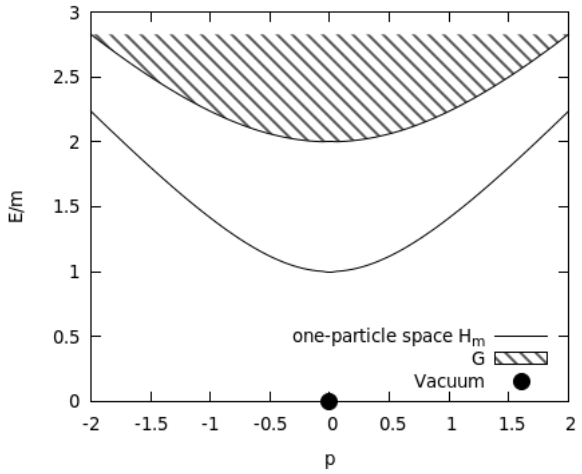

Figure 1: the EM spectrum

Note that the form of the dispersion relation is actually already fixed by the properties of the Poincaré group. Though a general theory may contain more than one mass-shell with different m.
It is well possible to study the "massless" $m = 0$ case [10], but then the mass-shell H_m is separated from neither the vacuum nor the "multi-particle" spectrum G, which makes the construction of creation operators somewhat harder [8].

2.2 Almost Local Observables and EM Transfer

Definition 2.3 *(translation and smearing of operators).* *Let $B \in \mathcal{A}$ be an observable and $f \in S(\mathbb{R}^{d+1})$ be a Schwartz function. Then we can define the smeared operator $B(f)$ as*

$$B(f) = \int d^{d+1}x f(x) B(x) \in \mathcal{A} \tag{8}$$

$$\text{with } B(x) = \alpha_x(B) \tag{9}$$

understanding that the integral only converges in weak topology.

Lemma 2.4. *For any observable $B \in \mathcal{A}$ and function $f \in \S(\mathbb{R}^{d+1})$, it is $B(f) \in \mathcal{A}$.*

Proof. Let $B' \in \mathcal{A}_{loc}$ and $f' \in C_0^\infty$ such that $\|B - B'\| \leqslant \epsilon$ and $\|f - f'\|_1 \leqslant \epsilon$. The local algebras are weakly-closed (von Neumann bicommutant theorem), so $B'(f') \in \mathcal{A}_{loc}$. Furthermore we can estimate

$$\begin{aligned} \big\|B'(f') - B(f)\big\| &\leqslant \big\|B'(f') - B'(f)\big\| + \big\|B'(f) - B(f)\big\| && (10)\\ &\leqslant \big\|B'\big\| \big\|f' - f\big\|_1 + \|f\|_1 \big\|B' - B\big\| && (11)\\ &\leqslant const \cdot \epsilon, && (12) \end{aligned}$$

so $B(f) \in \mathcal{A}$ by norm-closedness of the whole algebra $\mathcal{A}$. ∎

Now that we know how to convolute (or "smear") an operator with a function, we are tempted to do some Fourier transform by trying to compute expressions like $B(e^{ikx})$. The problem is that this operation transforms a strictly local operator into a global one, which we do not want. Therefore we will introduce an algebra of "almost-local" operators, which lies between $\mathcal{A}$ and $\mathcal{A}_{loc}$, which is closed under smearing with Schwartz functions. Note that this is a very similar situation to basic analysis, where we note that the Fourier transform of a function with compact support will never have compact support. Therefore the class of Schwartz functions is introduced as a compromise between compact support and arbitrary support.

Definition 2.5 *(Region Extension).* *Let $\mathcal{O} \subset \mathbb{R}^{d+1}$ be a region of space-time. Then for any $r \in \mathbb{R}$ we define the* extended region $\mathcal{O}^{(r)}$ *as*

$$\mathcal{O}^{(r)} = \{x \in \mathbb{R} \mid d(x, \mathcal{O}) < r\}$$

Definition 2.6 *(Almost local observables).* *$B \in \mathcal{A}$ is almost-local if for some open, bounded region of space-time $\mathcal{O}$ there is a sequence $B_r \in \mathcal{A}(\mathcal{O}^{(r)})$ such that*

$$\|B_r - B\| \in O(r^{-\infty}) \tag{13}$$

where $O(r^{-\infty})$ means decreasing faster than any polynomial in r.

This definition means that the localization region of an almost-local observable might be unbounded, but its influence at distant points decreases rapidly. This is very similar to the decay-properties of Schwartz functions. In fact this new algebra of almost local observables is closed under convolution with Schwartz functions (which $\mathcal{A}_{loc}$ was not), and the commutator of space-like separated observables decays faster than is the case for arbitrary operators from $\mathcal{A}$:

Lemma 2.7. *Let $B, C \in \mathcal{A}_{a-loc}$. Then*

1. *$\alpha_g(B) \in \mathcal{A}_{a-loc}$ for any $g \in \mathcal{P}_0$ in the Poincaré group.*
2. *$B(f) \in \mathcal{A}_{a-loc}$ for any Schwartz function $f \in S(\mathbb{R}^{d+1})$.*
3. *$\|[B, C(tx)]\| \in O(t^{-\infty})$ if $x \in \mathbb{R}^{d+1}$ is a space-like vector.*

Proof. ad (1): trivial
ad (2): As f is a Schwartz function, we can take a sequence of functions $f_r \in S(\mathbb{R}^{d+1})$ such that f_r is supported in a ball of radius r, and $\|f_r - f\|_1 \in O(r^{-\infty})$.
Then for any $B_r \in \mathcal{A}(\mathcal{O}^{(r)})$ it is $B_r(f_r) \in \mathcal{A}(\mathcal{O}^{(2r)})$, and we can estimate

$$\begin{aligned} \|B_r(f_r) - B(f)\| &\leqslant \|B_r(f_r) - B_r(f)\| + \|B_r(f) - B(f)\| && (14) \\ &\leqslant \|B_r\| \|f_r - f\|_1 + \|f\|_1 \|B_r - B\| \in O(r^{-\infty}) && (15) \end{aligned}$$

Note in particular that B_r converges to B in norm, so in particular $\|B_r\|$ is bounded by some number independent of r.

ad (3): We can estimate

$$\|[B, C(tx)]\| \leqslant \|[B_r, C_r(tx)]\| + O(r^{-\infty}). \tag{16}$$

Now choose $r = \epsilon t$ with constant $\epsilon > 0$ so small that the (anti-)commutator vanishes completely. ■

Definition 2.8 *(Arveson spectrum).* *The energy-momentum transfer of an operator $B \in \mathcal{A}$ is*

$$Sp_B\alpha = \operatorname{supp} \check{B} \tag{17}$$

$$\text{where } \check{B}(p) = \int \frac{d^{d+1}x}{(2\pi)^{(d+1)/2}} e^{ip\cdot x} B(x) \tag{18}$$

is the inverse Fourier transform of $B(x)$. Note that $\check{B}$ is only defined as a distribution.

Theorem 2.9 *(EM transfer relation).* *Let E be the spectral projection of $(H, \boldsymbol{P})$ and $\Delta \subset Sp(H, \boldsymbol{P})$ be a measurable subset of the EM spectrum. Then*

$$BE(\Delta) = E(\overline{\Delta + Sp_B\alpha})BE(\Delta)$$

which is equivalent to saying that for any vector $\psi \in E(\Delta)\mathcal{H}$, it holds

$$B\psi \in E(\overline{\Delta + Sp_B\alpha})\mathcal{H}$$

Proof. See [3]. ■

It is worth noting that the EM spectrum of a strictly local observable will always be unbounded, which makes the transfer relation useless for such operators. This is the reason we had to introduce the almost-local observables.

2.3 Construction of Asymptotic States

In this section we construct so called asymptotic states, which should be understood as scattering states. We do this by using the special form of the EM spectrum to define creation and annihilation operators, which then allow to define n-particle states. In this setting this is called Haag-Ruelle scattering theory [12][20].

Definition 2.10 *(Wave-Packets).* *Let $\hat{g} \in C_0^\infty(\mathbb{R}^d)$ be a smooth function with compact support. Then the following solution to the Klein-Gordon equation $(\Box + m^2)g = 0$ is called a wave-packet.*

$$g_t(\boldsymbol{x}) = \int \frac{d^d\boldsymbol{p}}{(2\pi)^{d/2}} \hat{g}(\boldsymbol{p}) \exp(-i\omega(\boldsymbol{p})t + i\boldsymbol{p}\cdot\boldsymbol{x}) \tag{19}$$

We define its velocity support *as*

$$V(g) = \{\nabla\omega(\boldsymbol{p}) \mid \boldsymbol{p} \in \mathrm{supp}(\hat{g})\}. \tag{20}$$

At first sight the definition of the velocity support might seem arbitrary, but it can be easily explained: With the usual relativistic dispersion relation, we have $\nabla\sqrt{\boldsymbol{p}^2+m^2} = \frac{\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2+m^2}}$ which is precisely the velocity of a relativistic particle with momentum $\boldsymbol{p}$. In the quantum field theory setting we are interested in here, a wave-packet can not be expected to have a sharply defined velocity. For that reason we are talking about velocity *support* here as a range of velocities that occur inside the wave packet g_t.
The following lemma makes precise the idea of a finite propagation speed of such packets, which further justifies the interpretation of $V(g)$ as velocity.

Lemma 2.11. *Let g_t be a wave-packet velocity support $V(g)$. Let $\chi_{V(g)}$ be an smooth approximate characteristic function of $V(G) \subset \mathbb{R}^d$. I.e. let χ be 1 in the region $V(g)$, and zero outside of a slightly larger open set. Then*

$$\int d^d x \ \left|(1-\chi_{V(g)}(\boldsymbol{x}/t))g_t(\boldsymbol{x})\right| \in O(t^{-\infty}),$$

i.e. outside of the spatial region $t \cdot V(g)$, the wave-function g_t is small.

Proof. See [19, corollary to theorem XI.14]. ■

Now we are ready to define creation operators. The goal here is, that applying a creation operator to the vacuum should result in a one-particle state. I.e. $B^*\Omega \in \mathcal{H}_1 = E(H_m)\mathcal{H}$. Using the EM transfer relation 2.9, this can be accomplished simply by putting conditions on the Arveson spectrum of B^*.

Definition 2.12 *(Creation Operators).* *An almost local observable $B^* \in \mathcal{A}_{a-loc}$ is called a* creation operator *if*

1. *Its Arveson spectrum is compact and lies inside the forward light cone $Sp_{B^*}\alpha \subset \overline{V_+}$.*

2. *This spectrum only intersect with the mass-shell, not with the other parts of the EM spectrum. I.e. $Sp_{B^*}\alpha \cap Sp(H, \boldsymbol{P}) \subset H_m$.*

If we additionally have a wave-packet g_t, we call the convoluted operator $B^*_t(g_t)$ a Haag-Ruelle creation operator. *Note that it depends on a time-parameter t. The convolution with g_t is only spatial, i.e. d-dimensional.*
Lastly an annihilation operator *is simply defined as the adjoint of a creation operator. As these are all bounded operators, there are no questions about domains and such.*

Example 2.13. *One way to explicitly construct a HR creation operator is to start from an arbitrary strictly local operator $B_0^* \in \mathcal{A}_{loc}$ and convolute it with a Schwartz function $f \in S(\mathbb{R}^{d+1})$. We can calculate*

$$\widetilde{B_0^*(f)}(p) = \int \frac{d^{d+1}x\ d^{d+1}y}{(2\pi)^{(d+1)/2}} e^{ip\cdot x} f(y) B_0^*(x+y) \tag{21}$$

$$= \int \frac{d^{d+1}x\ d^{d+1}y}{(2\pi)^{(d+1)/2}} e^{ip\cdot x - ipy} f(y) B_0^*(x) \tag{22}$$

$$= \int d^{d+1}x\ e^{-ip\cdot x} \hat{f}(p) B_0^*(x) \tag{23}$$

$$= (2\pi)^{(d+1)/2} \hat{f}(p) \hat{B}_0^*(p) \tag{24}$$

So by choosing f such that $\hat{f}$ has compact support around the mass shell, $B^* = B_0^*(f)$ will have the desired Arveson spectrum, and according to lemma 2.7 it is almost local. Note however that in order to get something non-trivial (i.e. $B^*\Omega \neq 0$), we need to start from a sufficiently non-trivial B_0^*.

Note that using this construction, an actual creation operator $B_t^*(g_t)$ includes two convolutions. One with $f \in S(\mathbb{R}^{d+1})$ and one with $g_t \in S(\mathbb{R}^d)$.

Lemma 2.14. *Let $B_t^*(g_t)$, $C_t^*(f_t)$, $D_t^*(h_t)$ be a creation operator. Then*

1. *The action on the vacuum is independent of t:*

$$\partial_t (B_t^*(g_t)\Omega) = 0 \tag{25}$$

2. *If the velocity supports $V(f)$ and $V(g)$ are disjoint (and $V(h)$ is arbitrary), the following commutators vanish asymptotically:*

$$\|[B_t^*(g_t), C_t^*(f_t)]\| \in O(t^{-\infty}) \tag{26}$$

$$\|[B_t^*(g_t), [C_t^*(f_t), D_t^*(h_t)]]\| \in O(t^{-\infty}) \tag{27}$$

$$\|[\partial_t B_t^*(g_t), C_t^*(f_t)]\| \in O(t^{-\infty}) \tag{28}$$

   These bounds stay valid if some or all of the creation operators are replaced with annihilation operators.

3. *The adjoint of a creation operator annihilates the vacuum (so calling it an annihilation operator is justified):*

$$B_t(\bar{g}_t)\Omega = 0 \tag{29}$$

   (The bar simply denotes complex conjugation here)

4. *Acting with both a creation and an annihilation operator on the vacuum results again in a multiple of the vacuum state*

$$B_t(\bar{g}_t)C_t^*(f_t)\Omega = \Omega\langle\Omega|B_t(\bar{g}_t)C_t^*(f_t)|\Omega\rangle \tag{30}$$

Proof. ad (1):

$$\partial_t B_t^*(g_t)\Omega = \partial_t \int d^d\boldsymbol{x}\ g_t(\boldsymbol{x})B_t^*(\boldsymbol{x})\Omega \tag{31}$$

$$= \partial_t \int d^d\boldsymbol{x}\ \int \frac{d^d\boldsymbol{p}}{(2\pi)^{d/2}}\hat{g}(\boldsymbol{p})\exp(-i\omega(\boldsymbol{p})t + i\boldsymbol{p}\cdot\boldsymbol{x})U(t,\boldsymbol{x})B^*\Omega \tag{32}$$

$$= \partial_t(2\pi)^{d/2}\hat{g}(\boldsymbol{P})\exp(-i\omega(\boldsymbol{P})t + i\boldsymbol{P}\cdot\boldsymbol{x})U(t,\boldsymbol{x})B^*\Omega \tag{33}$$

$$= (2\pi)^{d/2}\hat{g}(\boldsymbol{P})\exp(-i\omega(\boldsymbol{P})t + i\boldsymbol{P}\cdot\boldsymbol{x})U(t,\boldsymbol{x})(-i\omega(\boldsymbol{P}) + iH)B^*\Omega \tag{34}$$

Now $B^*\Omega \in E(H_m)\mathcal{H}$, so $\omega(\boldsymbol{P})B^*\Omega = HB^*\Omega$, so the expression vanishes.
ad (2):

$$\|[B_t^*(g_t), C_t^*(f_t)]\| = \left\| \iint d^d\boldsymbol{x}\ d^d\boldsymbol{y}\ g_t(\boldsymbol{x})f_t(\boldsymbol{y})\ [B^*(t,\boldsymbol{x}), C^*(t,\boldsymbol{y})] \right\| \tag{35}$$

$$\leqslant \iint d^d\boldsymbol{x}\ d^d\boldsymbol{y}\ |g_t(\boldsymbol{x})||f_t(\boldsymbol{y})|\ \|[B^*(\boldsymbol{x}), C^*(\boldsymbol{y})]\| \tag{36}$$

Now let $k \in \mathbb{N}$ be arbitrary. Then by almost-locality of B^* and C^*, there is a constant c_k (independent of t) such that

$$\|[B_t^*(g_t), C_t^*(f_t)]\| \leqslant \iint d^d\boldsymbol{x}\ d^d\boldsymbol{y}\ |g_t(\boldsymbol{x})||f_t(\boldsymbol{y})|\ \frac{c_k}{\|\boldsymbol{x}-\boldsymbol{y}\|^k} \tag{37}$$

$$= t^{2d-k}\iint d^d\boldsymbol{v}\ d^d\boldsymbol{w}\ |g_t(\boldsymbol{v}t)||f_t(\boldsymbol{w}t)|\ \frac{c_k}{\|\boldsymbol{v}-\boldsymbol{w}\|^k} \tag{38}$$

Now let $\tilde{V}(g)$ and $\tilde{V}(f)$ be disjoint open sets that contain the velocity supports $V(g)$ and $V(f)$. By lemma 2.11, we know that for $\boldsymbol{v} \notin \tilde{V}(g)$, the integral vanishes as $O(t^{-\infty})$, and similarly for $w \notin \tilde{V}(f)$. Thus we can estimate

$$\|[B_t^*(g_t), C_t^*(f_t)]\| \leqslant t^{2d-k}\int_{\tilde{V}(g)} d^d\boldsymbol{v}\ \int_{\tilde{V}(f)} d^d\boldsymbol{w}\ |g_t(\boldsymbol{v}t)||f_t(\boldsymbol{w}t)|\ \frac{c_k}{\|\boldsymbol{v}-\boldsymbol{w}\|^k} + O(t^{-\infty}) \tag{39}$$

$$\leqslant \frac{c_k}{\operatorname{dist}(\tilde{V}(g),\tilde{V}(f))^k}t^{2d-k}\int_{\tilde{V}(g)} d^d\boldsymbol{v}\ \int_{\tilde{V}(f)} d^d\boldsymbol{w}\ |g_t(\boldsymbol{v}t)||f_t(\boldsymbol{w}t)| + O(t^{-\infty}) \tag{40}$$

$$\leqslant \frac{c_k\|g_t\|_1\|f_t\|_1}{\operatorname{dist}(\tilde{V}(g),\tilde{V}(f))^k}t^{-k} + O(t^{-\infty}) = O(t^{-k}) \tag{41}$$

As k was arbitrary, this proves the first asymptotic bound.

For the second equation split $h = h_1 + h_2$, such that $V(h_1)$ is disjoint from $V(g)$ and $V(h_2)$ is disjoint from $V(f)$, Then you can use the Jacobi identity to split the double-commutator into two. Using the previous bound twice, the second bound follows.
For the third equation we can write

$$\partial_t B_t(g_t) = B_t(\dot{g}_t) + \dot{B}_t(g_t) \tag{42}$$

And note that $\dot{g}_t$ is again a KG-solution, and $\dot{B}^*$ is again a creation operator. Therefore both term are HR-creation operators and we can use (2).
ad (3): This is a direct result of the EM transfer relation (lemma 2.9): $Sp_B\alpha = -Sp_{B^*}\alpha$, so the Arveson spectrum of an annihilation operator is contained in the past light cone. And by assumption, the EM spectrum is fully contained in the future light-cone.
ad (4): Again, this follows directly from the transfer relation and the specific form of the EM spectrum. ■

Definition 2.15 *(In-/Outgoing states).* *For one particle states $\psi_i = B^*_{i,t}(g_{i,t})\Omega$, we define the asymptotic* outgoing *n-particle state as*

$$\psi_1 \times_{out} \ldots \times_{out} \psi_n = \lim_{t\to\infty} B^*_{1,t}(g_{1,t}) \ldots B^*_{n,t}(g_{n,t})\Omega \tag{43}$$

and similarly the asymptotic ingoing *state*

$$\psi_1 \times_{in} \ldots \times_{in} \psi_n = \lim_{t\to-\infty} B^*_{1,t}(g_{1,t}) \ldots B^*_{n,t}(g_{n,t})\Omega \tag{44}$$

In order for this notation to make sense we need to verify:

1. *The limit converges.*
2. *It is symmetric under permutation.*
3. *It does not depend on B^*_i or g_i, but only on ψ_i.*

Proof. We will only consider the outgoing states now, as the ingoing are completely analogous.
ad (1): We define $\Psi_t = B^*_{1,t}(g_{1,t}) \ldots B^*_{n,t}(g_{n,t})\Omega$ and show that the derivative w.r.t. to t goes to 0 sufficiently fast so that the limit must exist

$$\partial_t \Psi_t = \sum_k B^*_{1,t}(g_{1,t}) \ldots (\partial_t B^*_{k,t}(g_{k,t})) \ldots B^*_{n,t}(g_{n,t})\Omega \tag{45}$$

$$= \sum_{k=1}^{n} \sum_{i=k+1}^{n} B^*_{1,t}(g_{1,t}) \ldots [(\partial_t B^*_{k,t}(g_{k,t})), B^*_{i,t}(g_{i,t})] \ldots B^*_{n,t}(g_{n,t})\Omega \tag{46}$$

$$+ \sum_{k=1}^{n} B^*_{1,t}(g_{1,t}) \ldots B^*_{n,t}(g_{n,t}) \underbrace{(\partial_t B^*_{k,t}(g_{k,t}))\Omega}_{=0} \tag{47}$$

Now the commutators go to zero sufficiently fast (lemma 2.14) so that we can estimate

$$\|\partial_t \Psi_t\| \leqslant \sum_{k=1}^{n} \sum_{i=k+1}^{n} \underbrace{\left\|B^*_{1,t}(g_{1,t})\right\|}_{\in O(t^{d/2})} \dots \underbrace{\left\|[(\partial_t B^*_{k,t}(g_{k,t})), B^*_{i,t}(g_{i,t})]\right\|}_{\in O(t^{-\infty})} \dots \underbrace{\left\|B^*_{n,t}(g_{n,t})\right\|}_{\in O(t^{d/2})} \tag{48}$$

$$\in O(t^{(n-2)d/2}) \sum_{i \neq j} O(t^{-\infty}) = O(t^{-\infty}) \tag{49}$$

I.e. the derivative $\partial_t \Psi_t$ tends to zero faster than any polynomial. This implies the existence of the $t \to \infty$ limit by the Cook's method.

ad (2): By 2.14, the creation operators $B^*_{k,t}(g_{k,t})$ commute in the limit $t \to \infty$.

ad (3): By equation (25) the parameter t in the right-most creation operator is irrelevant, so clearly the asymptotic state can only depend on ψ_n and not on B^*_n and g_n. But according to (2) the whole expression is symmetric under permutation, so it cannot depend on any B^*_i or g_i, but only on the one particle states ψ_i. ■

The next step now is to analyze the structure of these asymptotic states. In particular, we need to compute the scalar product between two ingoing (or two outgoing) states, which will establish a Fock space structure. This enables us to define the wave operators and finally the scattering operator.

Lemma 2.16. *Let $B'^*_{1,t}(g'_{1,t}), \dots, B'^*_{n,t}(g'_{n',t})$ be creation operators and $B_t(\bar{g}_t)$ an annihilation operator. Let the velocity supports of $V(g'_i)$ be disjoint and $V(g)$ arbitrary. Then*

$$\lim_{t \to \pm\infty} B_t(\bar{g}_t) B'^*_{1,t}(g'_{1,t}) \dots B'^*_{n,t}(g'_{n,t}) |\Omega\rangle \tag{50}$$

$$= \lim_{t \to \pm\infty} \sum_{k=1}^{n} B'^*_{1,t}(g'_{1,t}) \dots \widehat{B'^*_{k,t}(g_{k,t})} \dots B'^*_{n,t}(g'_{n,t}) |\Omega\rangle\langle\Omega| B_t(\bar{g}_t) B'^*_{k,t}(g'_{k,t}) |\Omega\rangle \tag{51}$$

where the big hat denotes omission of the k'th creation operator.

Proof. We simply start at the left hand side and move the annihilation operator all the way to the right, introducing a bunch of commutators

$$l.h.s. = \lim_{t \to \pm\infty} \sum_{k=1}^{n'} B'^*_{1,t}(g'_{1,t}) \dots [B_{1,t}(\bar{g}_{1,t}), B'^*_{k,t}(g'_{k,t})] \dots B'^*_{n,t}(g'_{n',t}) |\Omega\rangle \tag{52}$$

These commutators now can be moved all the way to the right, without introducing new double-commutators, as those vanish in the $t \to \pm\infty$ limit according to lemma

2.14.2. Secondly, we can use lemma 2.14.3 and 2.14.4 to compute

$$l.h.s. = \lim_{t\to\pm\infty} \sum_{k=1}^{n} B'^{*}_{1,t}(g'_{1,t}) \dots B'^{*}_{n,t}(g'_{n',t})[B_{1,t}(\bar{g}_{1,t}), B'^{*}_{k,t}(g'_{k,t})]|\Omega\rangle \tag{53}$$

$$= \lim_{t\to\pm\infty} \sum_{k=1}^{n'} B'^{*}_{1,t}(g'_{1,t}) \dots B'^{*}_{n,t}(g'_{n,t}) B_{1,t}(\bar{g}_{1,t}) B'^{*}_{k,t}(g'_{k,t})|\Omega\rangle \tag{54}$$

$$= \lim_{t\to\pm\infty} \sum_{k=1}^{n'} B'^{*}_{1,t}(g'_{1,t}) \dots B'^{*}_{n,t}(g'_{n,t})|\Omega\rangle\langle\Omega|B_{1,t}(\bar{g}_{1,t}) B'^{*}_{k,t}(g'_{k,t})|\Omega\rangle \tag{55}$$

■

Corollary 2.17. *The scalar product between two outgoing (or two ingoing) states can be reduced to the scalar product between their respective one-particle states. (This motivates calling them asymptotically* free*).*

$$\big\langle \psi_1 \times_{out} \dots \times_{out} \psi_n \big| \psi'_1 \times_{out} \dots \times_{out} \psi'_{n'} \big\rangle = \delta_{nn'} \sum_{\pi} \prod_{i} \big\langle \psi_i \big| \psi_{\pi(i)} \big\rangle \tag{56}$$

Proof. First assume $n = n'$ and write the scalar product of states using their creation operators:

$$\big\langle \Psi \big| \Psi' \big\rangle = \lim_{t\to\infty} \big\langle \Omega \big| B_{n,t}(\bar{g}_{n,t}) \dots B_{1,t}(\bar{g}_{1,t}) B'^{*}_{1,t}(g'_{1,t}) \dots B'^{*}_{n,t}(g'_{n,t}) \Omega \big\rangle \tag{57}$$

Now we simply use lemma 2.16 to get rid of the right most annihilation operator B_1. By induction we arrive at the right hand side expression.
If on the other hand $n \neq n'$, we will at some point encounter a vacuum expectation value of only creation- or only annihilation operators, which is zero. ■

Definition 2.18 *(Asymptotic Completeness).* *A model is called* asymptotically complete *if the in/out-going n-particle states are dense in the whole state space.*

Note that while it is relatively easy to show that the one particle states $B^*_t(g_t)\Omega$ are dense in the one particle Hilbert space, there are only very few models which are known to be asymptotically complete even in the multi-particle space[11]. This might be a desirable property because it means that the whole structure of the model can be understood by the scattering matrix (which will be defined in the next chapter).

2.4 Scattering Matrix and LSZ Reduction

The goal of finding asymptotic ingoing and outgoing states was of course not to establish their respective structure (which is trivial as we just proved), but to analyze the interaction between ingoing and outgoing states. This idea is captured in the scattering matrix.

Definition 2.19 *(Scattering matrix).* *Let $\psi_1, \dots, \psi_n$ and $\psi'_1, \dots, \psi'_{n'}$ be one-particle states. Then the scattering matrix element*

$$S(\psi_1, \dots, \psi_n; \psi'_1, \dots, \psi'_{n'}) = \left\langle \psi_1 \times_{out} \dots \times_{out} \psi_n \middle| \psi'_1 \times_{in} \dots \times_{in} \psi'_{n'} \right\rangle \tag{58}$$

describes the amplitude of an n' to n particle scattering process.

Remark 2.20. *This very direct definition does not really explain why we would call this object a scattering* matrix *(or scattering* operator*). This comes from a slightly more involved approach using wave operators, which we will briefly explain now. Let $\mathcal{H}_m = E(H_m) \subset \mathcal{H}$ be the one-particle state. Then we can easily construct the symmetric Fock space*

$$\mathcal{F}(\mathcal{H}_m) = \overline{\oplus_{n=0}^{\infty} \otimes_{i=1}^{n} \mathcal{H}_m}$$

with creation and annihilation operators $a^*(\psi)$ and $a(\psi)$ for $\psi \in \mathcal{H}_m$ and vacuum state Ω_F. Because of corollary 2.17 we can define the unitary wave operators

$$W_{out} : \mathcal{F}(\mathcal{H}_m) \to \mathcal{H} \tag{59}$$

$$a^*(\psi_1) \dots a^*(\psi_n)\Omega_F \mapsto \psi_1 \times_{out} \dots \times_{out} \psi_n \tag{60}$$

and similarly for W_{in}. Then the scattering matrix $S = (W_{out})^* W_{in}$ actually is an operator on the Fock space.

The definition of the scattering matrix as it stands looks very reasonable at first, because it directly relates to the transition amplitude of a process one might expect to measure in an experiment. But in practice it is often more natural to work with the *connected part* of this quantity. This means subtracting from the n-particle scattering process all contributions from "lower order" scatterings of subsets of the particles. This is analogous to only considering connected Feynman diagrams in a perturbative approach to QFT. But before precisely define this we need some new notation to keep the formulas somewhat concise.

Notation 2.21.

1. *Motivated by the interpretation of the scattering matrix as operator on the Fock space, we introduce notation for the identity operator on that Fock space as*

$$\mathbb{1}(\psi_1, \dots, \psi_n; \psi'_1, \dots, \psi'_{n'}) = \delta_{nn'} \sum_{\pi} \prod_{i} \left\langle \psi_i \middle| \psi'_{\pi(i)} \right\rangle \tag{61}$$

   which is precisely the r.h.s. of corollary 2.17.

2. *Let the one-particle states $\psi_1, \dots, \psi_n$ and $\psi'_1, \dots, \psi'_{n'}$ be fixed. Then for any $A = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ and $A' = \{i'_1, \dots, i'_{k'}\} \subset \{1, \dots, n'\}$ we denote*

with $S(A;A')$ the scattering matrix element.

$$S(A;A') = S(\psi_{i_1},\dots,\psi_{i_k};\psi'_{i'_1},\dots,\psi'_{i'_{k'}}). \tag{62}$$

Similar notation will be used for other matrices, like $\mathbb{1}(A;A')$ and $S_c(A;A')$ (defined below).

As a quick exercise to get familiar with $\mathbb{1}$ and also the new notation, we prove the following purely combinatorical lemma, which we will need at a much later stage.

Lemma 2.22 *(Combinatorical Lemma).* *Let A, A' be fixed index sets. Then*

$$\sum_{\substack{M\subset A\\ M'\subset A'}} (-1)^{|M|}\mathbb{1}(M;M')\mathbb{1}(A\backslash M;A'\backslash M') = \begin{cases} 1 & A = A' = \varnothing \\ 0 & else \end{cases} \tag{63}$$

Proof. First note that everything is zero unless $|A| = |A'|$ and $|M| = |M'|$, simply by the definition of $\mathbb{1}$. Now let us split the sum with respect to the size of M.

$$\sum_{k=0}^{|A|}(-1)^k \sum_{\substack{M\subset A\\ M'\subset A'\\ |M|=|M'|=k}} \mathbb{1}(M;M')\mathbb{1}(A\backslash M;A'\backslash M') \tag{64}$$

Now consider the definition of $\mathbb{1}(M;M')$: It is a sum over all possible pairings of indices in M with indices in M'. Similar for $\mathbb{1}(A\backslash M;A'\backslash M')$. This means that in (64), there is a sum over pairings of all indices in A with all indices in A', but only those which obey the partition of $A = M \cup (A\backslash M)$. Now we note that there is also a sum over all such partitions. Therefore, eventually, all pairings of A with A' will arise. And the sum over all pairings can then be written as $\mathbb{1}(A;A')$. So the left hand side is equal to

$$\sum_{k=0}^{|A|}(-1)^k \binom{|A|}{k} \mathbb{1}(A;A'). \tag{65}$$

Where the binomial coefficient comes from the fact that a single pairing will satisfy more than one partition. More precisely, there are $\binom{|A|}{k}$ choices for M. And given a full A, A' pairing, this fixes M' uniquely. Now we can use the well-known formula

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases} \tag{66}$$

to prove the lemma.

■

Definition 2.23 *(Connected Part of the Scattering matrix).* *The* connected part *of the scattering matrix is*

$$S_c(\psi_1,\dots,\psi_n;\psi'_1,\dots,\psi'_{n'}) = \sum_k \sum_{\substack{\{A_i\}\\ \{A'_i\}}} (-1)^{k+1}(k-1)! \prod_{i=1}^{k} S(A_i;A'_i) \tag{67}$$

Where the second summation is over all partitions of $I = \{1,\dots,n\}$ and $I' = \{1,\dots,n'\}$ into k disjoint non-empty subsets $A = A_1 \sqcup \dots \sqcup A_k$ and $A' = A'_1 \sqcup \dots \sqcup A'_k$.

Remark 2.24.

1. *The combinatorics naturally extends to the case $n = 0$ using the usual convention that an empty sum is equal to zero, and an empty product is equal to one. In fact we have $S_c = S = 0$ if either $n = 0$ or $n' = 0$, and $S_c = S = 1$ if both $n = n' = 0$.*

2. *For $n = n' = 2$, we get the expected*

$$S_c(\psi_1,\psi_2;\psi'_1,\psi'_2) = S(\psi_1,\psi_2;\psi'_1,\psi'_2) - S(\psi_1;\psi'_1)S(\psi_2;\psi'_2) - S(\psi_1;\psi'_2)S(\psi_2;\psi'_1) \tag{68}$$

3. *While the combinatorical prefactor in equation (67) might not be obvious, this leads to the very nice inverse relation [2][eq. 5.75]*

$$S(\psi_1,\dots,\psi_n;\psi'_1,\dots,\psi'_{n'}) = \sum_k \sum_{\substack{\{A_i\}\\ \{A'_i\}}} \prod_{i=1}^{k} S_c(A_i;A'_i) \tag{69}$$

As a first try at scattering matrix calculations, we show that there is no "$1 \to n$" scattering.

Lemma 2.25. *Let $n = 1$. Then*

$$S(\psi_1;\psi'_1,\dots,\psi'_{n'}) = S_c(\psi_1;\psi'_1,\dots,\psi'_{n'}) = \begin{cases} \langle\psi_1|\psi'_1\rangle & n' = 1 \\ 0 & \textit{else} \end{cases}. \tag{70}$$

A similar formula holds for $n' = 1$ of course.

Proof. The first equality is simply a combinatorical consequence of equation (67), as for $n = 1$, the only partition of $\{1\}$ into non-empty subsets is the trivial one. The second equation is a consequence of the Fock space structure of ingoing states (lemma 2.17) and the fact that for a single particle, there is no difference between ingoing and outgoing (see lemma 2.14). ■

Using the definition of the scattering matrix as scalar product is not very useful for actual calculation because of the $t \to \pm\infty$ limits in the definition of asymptotic states. Therefore we now develop the LSZ reduction formula (named after Lehmann, Symanzik and Zimmermann) which essentially allows replacing the limits with time-integrals. The price to pay for this is the introduction of a *time-ordering operator*, which in ordinary quantum field theory is simply written as

$$T(A_t(\boldsymbol{x}), B_u(\boldsymbol{y})) = \begin{cases} A_t(\boldsymbol{x})B_u(\boldsymbol{y}) & t > u \\ B_u(\boldsymbol{y})A_t(\boldsymbol{x}) & t < u \end{cases} \tag{71}$$

This definition is obviously not continuous, so for a precise statement we have to regularize it. In the end, the result does not depend on the details of the regularization, but just for the sake of completeness we give one possible definition here which is identical to the ordinary time ordering whenever all t are at least ϵ apart.

Definition 2.26 *(Time-ordering).* *Let $\theta_\epsilon : \mathbb{R} \to \mathbb{R}$ be a regularized step function, i.e. a smooth function such that $\theta_\epsilon(x) = 0$ for $x \leqslant -\epsilon$ and $\theta_\epsilon(x) = 1$ for $x \geqslant \epsilon$. Then we can define a (regularized) time ordering of the operators $Q^i(t_i)$ as*

$$T_\epsilon(Q^1(t_1), \dots, Q^n(t_n)) = \sum_{\pi \in S_n} Q^{\pi(1)}(t_{\pi(1)}) \dots Q^{\pi(n)}(t_{\pi(n)}) \prod_{1 \leqslant i < j \leqslant n} \theta_\epsilon(t_{\pi(i)} - t_{\pi(j)}) \tag{72}$$

In a similar fashion to going from the full scattering matrix to its connected part, we can introduce the "truncated" vacuum expectation value of a (time-ordered) operator product.

Definition 2.27 *(Truncated Expectation Value).* *For operators $B_1, \dots, B_n$ and space-time points $x_1, \dots, x_n$, we define the* truncated vacuum expectation value

$$\langle \Omega | T(B_1(x_1) \dots B_n(x_n)) | \Omega \rangle_T = \sum_k (-1)^{k+1} (k-1)! \sum_{\{A_i\}} \prod_{i=1}^k \langle \Omega | T(B_{A_i}) | \Omega \rangle \tag{73}$$

where the second sum is over all partitions $\{1, \dots, n\} = A_1 \sqcup \dots \sqcup A_k$ into non-empty A_i. Also, for $A_i = \{j_1, \dots, j_l\}$, we defined $B_{A_i} = B_{j_1}(x_{j_1}) \dots B_{j_l}(x_{j_l})$. The order of operators does not matter as they are time-ordered anyway. Just as with the connected scattering matrix this leads to the nice inverse relation

$$\langle \Omega | T(B_1(x_1) \dots B_n(x_n)) | \Omega \rangle = \sum_k \sum_{\{A_i\}} \prod_{i=1}^k \langle \Omega | T(B_{A_i}) | \Omega \rangle_T \tag{74}$$

Lemma 2.28 *(lemma 5.7 in [2]).* *Let $F_t(\boldsymbol{x})$ be a operator valued functions such that*

1. *$F_t(\boldsymbol{x})$ is twice continuously differentiable in both x and t.*

2. *$\|F_t(\boldsymbol{x})\|$ grows at most polynomially in $\boldsymbol{x}$ for any fixed t.*

3. *There are operators Q, C_1, and C_2 such that for some time T*

$$F_t(\boldsymbol{x}) = \begin{cases} C_1 Q_t(\boldsymbol{x}) & t < -T \\ Q_t(\boldsymbol{x}) C_2 & t > T \end{cases}. \tag{75}$$

where $Q_t(\boldsymbol{x})$ is just Q, translated by $(t, \boldsymbol{x})$.

Furthermore let g be a Klein-Gordon-solution, i.e. $(\Box + m^2)g = 0$, and assume that $g_t(\boldsymbol{x}) \in O(|\boldsymbol{x}|^{-\infty})$ for any fixed t. Then

$$\begin{aligned} &\int dt \int d^d\boldsymbol{x}\; g_t(\boldsymbol{x})(\Box + m^2)\langle\Psi|F_t(\boldsymbol{x})|\Psi'\rangle \\ &\qquad = \lim_{t\to-\infty}\langle\Psi|C_1 Q_t\left(\overleftrightarrow{\partial_t}\, g_t\right)|\Psi'\rangle - \lim_{t\to\infty}\langle\Psi|Q_t\left(\overleftrightarrow{\partial_t}\, g_t\right)C_2|\Psi'\rangle \end{aligned} \tag{76}$$

for any states Ψ, Ψ' for which the limits on the right side exist.

Proof. For the sake of brevity we omit the states Ψ, Ψ' in the following calculation. But note that we are only interested in weak convergence of the integrals and limits on specific vectors.

$$\int dt \int d^d\boldsymbol{x}\; g_t(\boldsymbol{x})(\Box + m^2)F_t(\boldsymbol{x}) \tag{77}$$

$$= \int dt \int d^d\boldsymbol{x}\; \left(g_t(\boldsymbol{x})\ddot{F}_t(\boldsymbol{x}) - g_t(\boldsymbol{x})(\Delta - m^2)F_t(\boldsymbol{x})\right) \tag{78}$$

$$\overset{\text{PI}}{=} \int dt \int d^d\boldsymbol{x}\; \left(g_t(\boldsymbol{x})\ddot{F}_t(\boldsymbol{x}) - F_t(\boldsymbol{x})(\Delta - m^2)g_t(\boldsymbol{x})\right) \tag{79}$$

$$\overset{\text{KG}}{=} \int dt \int d^d\boldsymbol{x}\; \left(g_t(\boldsymbol{x})\ddot{F}_t(\boldsymbol{x}) - \ddot{g}_t(\boldsymbol{x})F_t(\boldsymbol{x})\right) \tag{80}$$

$$= \int dt \int d^d\boldsymbol{x}\; \partial_t\left(g_t(\boldsymbol{x})\dot{F}_t(\boldsymbol{x}) - \dot{g}_t(\boldsymbol{x})F_t(\boldsymbol{x})\right) \tag{81}$$

$$\begin{aligned} &= \lim_{t\to\infty}\int d^d\boldsymbol{x}\left(g_t(\boldsymbol{x})\dot{F}_t(\boldsymbol{x}) - \dot{g}_t(\boldsymbol{x})F_t(\boldsymbol{x})\right) \\ &\qquad - \lim_{t\to-\infty}\int d^d\boldsymbol{x}\left(g_t(\boldsymbol{x})\dot{F}_t(\boldsymbol{x}) - \dot{g}_t(\boldsymbol{x})F_t(\boldsymbol{x})\right) \end{aligned} \tag{82}$$

$$\begin{aligned} &= \lim_{t\to\infty}\int d^d\boldsymbol{x}\left(g_t(\boldsymbol{x})\dot{Q}_t(\boldsymbol{x}) - \dot{g}_t(\boldsymbol{x})Q_t(\boldsymbol{x})\right)C_2 \\ &\qquad - \lim_{t\to-\infty}\int d^d\boldsymbol{x}\; C_1\left(g_t(\boldsymbol{x})\dot{Q}_t(\boldsymbol{x}) - \dot{g}_t(\boldsymbol{x})Q_t(\boldsymbol{x})\right) \end{aligned} \tag{83}$$

$$= \lim_{t\to\infty}\int d^d\boldsymbol{x}\left(g_t(\boldsymbol{x})\overleftrightarrow{\partial_t} Q_t(\boldsymbol{x})\right)C_2 - \lim_{t\to-\infty}\int d^d\boldsymbol{x}\; C_1\left(g_t(\boldsymbol{x})\overleftrightarrow{\partial_t} Q_t(\boldsymbol{x})\right) \tag{84}$$

$$= \lim_{t\to-\infty} C_1 Q_t\left(\overleftrightarrow{\partial_t}\, g_t\right) - \lim_{t\to\infty} Q_t\left(\overleftrightarrow{\partial_t}\, g_t\right)C_2 \tag{85}$$

In the partial integration, boundary terms vanish because $g_t(\boldsymbol{x}) \in O(|\boldsymbol{x}|^{-\infty})$ and

$F_t(\boldsymbol{x})$ grows at most polynomially in $\boldsymbol{x}$ for fixed t. Note that there is no such assumption about growth in t, so partial integration w.r.t t is not possible. ■

The goal of this lemma is of course to take as F some time-ordered operator-product, and $Q_t = B_t^*$ some HR-creation operator. But looking at the formula, we got the strange expression $Q_t(\overleftrightarrow{\partial_t} g_t)$ instead of the more familiar $Q_t(g_t)$. The next little lemma shows that this is not a real problem, and in fact we can translate both expressions into one another.

Lemma 2.29. *Let g be a Klein-Gordon solution, i.e. (see definition 2.10)*

$$g_t(\boldsymbol{x}) = \int \frac{d^d\boldsymbol{p}}{(2\pi)^{d/2}} \hat{g}(\boldsymbol{p}) \exp(-i\omega(\boldsymbol{p})t + i\boldsymbol{p}\cdot\boldsymbol{x}) \tag{86}$$

for some $\hat{g} \in C_0^\infty(\mathbb{R}^d)$. Let f be another Klein-Gordon solution defined by

$$f_t(\boldsymbol{x}) = \frac{i}{2} \int \frac{d^d\boldsymbol{p}}{(2\pi)^{d/2}} \frac{\hat{g}(\boldsymbol{p})}{\omega(\boldsymbol{p})} \exp(-i\omega(\boldsymbol{p})t + i\boldsymbol{p}\cdot\boldsymbol{x}). \tag{87}$$

Furthermore let B^* be a HR-creation operator. Then

$$B_t^*(\overleftrightarrow{\partial_t} f_t) = B_t^*(\dot{f}_t) - \dot{B}_t^*(f_t) \tag{88}$$

is a creation operator which results in the same one particle state as the creation operator $B_t^*(g_t)$, i.e.

$$\lim_{t\to\pm\infty} B_t^*(\overleftrightarrow{\partial_t} f_t)\Omega = \lim_{t\to\pm\infty} B_t^*(g_t)\Omega. \tag{89}$$

Note that the relation between f and g can be written as $\partial_t f_t(\boldsymbol{x}) = \frac{1}{2} g_t(\boldsymbol{x})$, and this relation is invertible.

Proof.

$$\lim_{t\to\infty} B_t^*(\overleftrightarrow{\partial_t} f_t)\Omega = \lim_{t\to\infty} \left(B_t^*(\dot{f}_t) - \dot{B}_t^*(f_t)\right)\Omega \tag{90}$$

$$= \lim_{t\to\infty} \left(2B_t^*(\dot{f}_t) - \partial_t(B_t^*(f_t))\right)\Omega \tag{91}$$

$$\overset{2.14.1}{=} \lim_{t\to\infty} B_t^*(2\dot{f}_t)\Omega \tag{92}$$

$$= \lim_{t\to\infty} B_t^*(g_t)\Omega \tag{93}$$

■

Now we are finally able to state and prove the LSZ reduction formula in its full glory. At this point we will settle for the 2-particle case. The full n-particle formula can be found in [2, theorem 5.8].

Theorem 2.30 *(LSZ Reduction Formula).* *The connected part of the scattering matrix for a 2-particle scattering can be written as*

$$S_c(\psi_1,\psi_2;\psi_3,\psi_4) = \int dt_1\ dt_2\ dt_3\ dt_4 \int d^d\boldsymbol{x}_1\ d^d\boldsymbol{x}_2\ d^d\boldsymbol{x}_3\ d^d\boldsymbol{x}_4\ \overline{f_1(x_1)f_2(x_2)}f_3(x_3)f_4(x_4) \tag{94}$$
$$(\Box_{x_1}+m^2)(\Box_{x_2}+m^2)(\Box_{x_3}+m^2)(\Box_{x_4}+m^2)\langle\Omega|T(B_1(x_1)B_2(x_2)B_3^*(x_3)B_4^*(x_4))|\Omega\rangle_T$$

where

1. $\psi_i = \lim_{t\to\infty} B_{i,t}^*(g_{i,t})\Omega$ *are one particle states.*
2. T *is a (regularized) time ordering*
3. *The* g*'s and* f*'s are related as in lemma 2.29.*
4. *The time integrals are to be performed after all spatial integrations.*

Remark 2.31. *It is interesting to note that the truncation on the right side does not actually change the result in this 2-particle case. But we keep it here because for the general n-particle case it is necessary. The truncation of S on the left is always necessary.*

Proof. First let us compute the right hand side *without the truncation of the vacuum expectation value*. We note that the regularized time ordering fulfills all properties of F in the previous lemma 2.28, so we can use it to successively remove all integrals. First we apply it to the x_3 and x_4 and note that the second term of (76) vanishes in both cases because a creation operator hits the vacuum from the right. Thus we get

$$\int dt_1\ dt_2 \int d^d\boldsymbol{x}_1\ d^d\boldsymbol{x}_2\ \overline{f_1(x_1)f_2(x_2)}(\Box_{x_1}+m^2)(\Box_{x_2}+m^2)$$
$$\lim_{t\to\infty}\langle\Omega|T(B_1(x_1)B_2(x_2))B_{3,-t}^*(g_{3,-t})B_{4,-t}^*(g_{4,-t})|\Omega\rangle \tag{95}$$

Now we want to apply the same lemma to the x_1 and x_2 integrations. But now both terms of (76) actually contribute, so we end up with 4 terms. They are

$$\begin{aligned}\lim_{t\to\infty}(&\langle\Omega|B_{1,-t}(g_{1,-t})B_{2,-t}(g_{2,-t})B_{3,-t}^*(g_{3,-t})B_{4,-t}^*(g_{4,-t})|\Omega\rangle\\ &+\langle\Omega|B_{2,t}(g_{2,t})B_{1,t}(g_{1,t})B_{3,-t}^*(g_{3,-t})B_{4,-t}^*(g_{4,-t})|\Omega\rangle\\ &-\langle\Omega|B_{1,t}(g_{1,t})B_{2,-t}(g_{2,-t})B_{3,-t}^*(g_{3,-t})B_{4,-t}^*(g_{4,-t})|\Omega\rangle\\ &-\langle\Omega|B_{2,t}(g_{2,t})B_{1,-t}(g_{1,-t})B_{3,-t}^*(g_{3,-t})B_{4,-t}^*(g_{4,-t})|\Omega\rangle).\end{aligned} \tag{96}$$

Now we use the fact that the action of a creation operator on the vacuum is independent of t to switch the sign of the first t in the 3'rd and 4'th line. Finally writing

everything in the notation of asymptotic states gives us

$$\begin{aligned}
&\langle \psi_1 \times_{in} \psi_2 | \psi_3 \times_{in} \psi_4 \rangle + \langle \psi_2 \times_{out} \psi_1 | \psi_3 \times_{in} \psi_4 \rangle \\
&- \langle \psi_1 \times_{in} \psi_2 | \psi_3 \times_{in} \psi_4 \rangle - \langle \psi_2 \times_{in} \psi_1 | \psi_3 \times_{in} \psi_4 \rangle && (97)\\
&= S(\psi_1, \psi_2; \psi_3, \psi_4) - S(\psi_1; \psi_3) S(\psi_2; \psi_4) - S(\psi_1; \psi_4) S(\psi_2; \psi_3) && (98)\\
&= S_c(\psi_1, \psi_2; \psi_3, \psi_4) && (99)
\end{aligned}$$

which is precisely the left side of the theorem.
Finally we need to handle the truncation of the vacuum expectation value and show it does not change the result. According to lemma 2.25, the only additional terms that could potentially contribute are those with exactly one in- and one outgoing particle. For these we can do the same steps as before and compute

$$\int dt_1\, dt_3 \int d^d\boldsymbol{x}_1\, d^d\boldsymbol{x}_3\, \overline{f_1(x_1)} f_3(x_3) (\Box_{x_1} + m^2)(\Box_{x_3} + m^2) \langle \Omega | T(B_1(x_1) B_3^*(x_3)) | \Omega \rangle \tag{100}$$

$$= \int dt_1 \int d^d\boldsymbol{x}_1\, \overline{f_1(x_1)} (\Box_{x_1} + m^2) \langle \Omega | B_1(x_1) B^*_{3,-t}(g_{3,-t}) | \Omega \rangle \tag{101}$$

$$= \langle \Omega | B_{1,-t}(g_{1,-t}) B^*_{3,-t}(g_{3,-t}) | \Omega \rangle - \langle \Omega | B_{1,t}(g_{1,t}) B^*_{3,-t}(g_{3,-t}) | \Omega \rangle = 0 \tag{102}$$

■

Remark 2.32 *(LSZ Reduction Formula in momentum space).* *Formally setting* $f_i(x) = \frac{1}{(2\pi)^{(d+1)/2}} e^{ip_i x}$ *as a plane wave, we obtain the LSZ reduction in momentum space.*

$$S(p_1, p_2; p_3, p_4) = -(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)(p_4^2 - m^2) \Gamma(p_1, p_2; -p_3, -p_4) \tag{103}$$

where

$$\begin{aligned}
\Gamma(p_1, p_2; p_3, p_4) =& \frac{1}{(2\pi)^{2(d+1)}} \int d^{d+1}x_1 \int d^{d+1}x_2 \int d^{d+1}x_3 \int d^{d+1}x_4 && (104)\\
& e^{-ip_1x_1} e^{-ip_2x_2} e^{-ip_3x_3} e^{-ip_4x_4} \langle \Omega | T(B_2(x_2) B_1(x_1) B_3^*(x_3) B_4^*(x_4)) \Omega \rangle && (105)
\end{aligned}$$

is the Fourier transform of the correlation function. As plane waves are not Klein-Gordon solutions, this should only be regarded as a symbolic expression and will not be used further. We just mention it because in the QFT literature, this version is more commonly found than the rigorous formula we are using.

3 Lattice Systems

The previous chapter demonstrated how to establish scattering theory in continuous space-time $\mathbb{R}^{d+1}$. From now on we instead consider discretized space. This is done by replacing $\mathbb{R}^d$ with $\mathbb{Z}^d$, while keeping time continuous and using Lieb-Robinson bounds to construct the dynamics of the system. While many properties from the Haag-Kastler setting have to be reconsidered, it turns out that enough structure is left to establish the LSZ formula. In this section we mostly follow [4], though the LSZ formula for lattice systems is a new result of this thesis.
As now the theory becomes non-relativistic, we abandon the relativistic notation of space-time vectors like $x = (t, \boldsymbol{x})$, and instead always write time $t \in \mathbb{R}$ and lattice-position $x \in \Gamma = \mathbb{Z}^d$ separately.

3.1 Quasi-Local Algebra and the Global Dynamics

Definition 3.1 *(Quasilocal algebra).* *Let $\mathcal{A}_x$ be a finite dimensional C^*-algebra for every $x \in \Gamma$, where Γ will be called the* lattice. *For any finite subset $\Lambda \subset \Gamma$ we define the local algebra in Λ as*

$$\mathcal{A}(\Lambda) = \otimes_{x\in\Lambda}\mathcal{A}_x. \tag{106}$$

If $\Lambda_1 \subset \Lambda_2$, there is a natural embedding $\mathcal{A}(\Lambda_1) \hookrightarrow \mathcal{A}(\Lambda_2)$ so we write $\mathcal{A}(\Lambda_1) \subset \mathcal{A}(\Lambda_2)$ and we can define the quasi-local algebra as norm-closure of the local ones.

$$\mathcal{A}_{loc} = \cup_\Lambda \mathcal{A}(\Lambda) \tag{107}$$
$$\mathcal{A} = \overline{\mathcal{A}_{loc}} \tag{108}$$

Definition 3.2 *(Interaction).* *A function $\Phi : P_{loc}(\Gamma) \to \mathcal{A}$ is a family of* local interactions *if for any finite set $\Lambda \subset \Gamma$ it holds*

1. *$\Phi(\Lambda) \in \mathcal{A}(\Lambda)$*
2. *$\Phi(\Lambda)$ is self-adjoint.*

Then we can define a family of local Hamiltonians *and corresponding* local dynamics *as*

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X) \tag{109}$$
$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \tag{110}$$

Formally, the Hamiltonian of the whole system would be $H = \sum_{X\subset\Gamma} \Phi(X)$. But in general this sum obviously does not converge, so a global Hamiltonian does not exist. But using Lieb-Robinson bounds it is possible to construct a global dynamics $\tau_t = \text{s-lim}_{\Lambda\to\Gamma}\, \tau_t^\Lambda$, at least for certain interactions which are "small" in the sense of the following (somewhat technical) definition.

Definition 3.3 *(Norm of Interactions).* *Let $d : \Gamma \times \Gamma \to \mathbb{R}$ be a metric, and $F : \mathbb{R} \to [0, \infty)$ a non-increasing function such that*

1. $\|F\| = \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x, y)) < \infty$
2. $C = \sup_{x,y\in\Gamma} \sum_{z\in\Gamma} \frac{F(d(x,z))F(d(y,z))}{F(d(x,y))} < \infty$

Then for any $\lambda \geqslant 0$ the function $F_\lambda(d) = e^{-\lambda d}F(d)$ satisfies $\|F_\lambda\| \leqslant \|F\|$ and $C_\lambda \leqslant C$. Finally we define $\mathcal{B}_\lambda(\Gamma)$ as the set of interactions Φ satisfying

$$\|\Phi\|_\lambda = \sup_{x,y\in\Gamma} \sum_{Z\ni x,y} \frac{\|\Phi(Z)\|}{F_\lambda(d(x, y))} < \infty \tag{111}$$

Example 3.4. *As an elementary example one can choose $\Gamma = \mathbb{Z}^d$ as the regular d-dimensional lattice with euclidean metric $d(x, y) = |x - y|$. It is easy to verify that the choice $F(d) = (1 + d)^{-d-\epsilon}$ satisfies the conditions for any $\epsilon > 0$. Also, then we have for translation invariant interactions*

$$\|\Phi\|_\lambda = \sup_{x\in\Gamma} e^{\lambda|x|}(1 + |x|)^{d+\epsilon} \sum_{Z\ni x,0} \|\Phi(Z)\| \tag{112}$$

It is clear by the definition of the quasi-local algebra, that for $A \in \mathcal{A}(X)$, $B \in \mathcal{A}(Y)$ with $A \cap B = \varnothing$,

$$[A, B] = 0. \tag{113}$$

But when we evolve one of these operators in time, it becomes non-local. So even for small times t and a large distance between X and Y, the commutator $[\tau_t(A), B]$ will not vanish. In this sense the propagation speed of the theory is infinite. But it turns out that outside of a propagation cone of a finite velocity, the commutator becomes exponentially small. And this in turn can be used to prove the convergence of the global dynamics.

Theorem 3.5 *(2.1 in [17]).* *Let $\lambda \geqslant 0$ and $\Lambda \subset \Gamma$ be finite. Let $\Phi \in \mathcal{B}_\lambda(\Gamma)$ be an interaction. Then for any pair of local observables $A \in \mathcal{A}(X)$, $B \in \mathcal{A}(Y)$ with finite $X, Y \subset \Lambda$, we have*

$$\left\|[\tau_t^\Lambda(A), B]\right\| \leqslant 2\|A\|\|B\|C_{\lambda,X,Y} \begin{cases} e^{2\|\Phi\|_\lambda C_\lambda|t|} - 1 & d(X, Y) > 0 \\ e^{2\|\Phi\|_\lambda C_\lambda|t|} & else \end{cases} \tag{114}$$

$$where\ C_{\lambda,X,Y} = \frac{1}{C_\lambda} \sum_{x\in X} \sum_{y\in Y} F_\lambda(d(x, y)) \tag{115}$$

Note in particular that for fixed A, B this bound is independent of Λ, so it also holds in the limit $\Lambda \to \Gamma$, once its existence is proved.

Proof. Consider the function

$$f(t) = [\tau_t^\Lambda(A), B]. \tag{116}$$

Using its derivative

$$\begin{aligned} f'(t) &= i[\tau_t^\Lambda([H_\Lambda, A]), B] && (117)\\ &= i[\tau_t^\Lambda([H_X, A]), B] && (118)\\ &= i[\tau_t^\Lambda(H_X), [\tau_t^\Lambda(A), B]] + i[[\tau_t^\Lambda(H_X), B], \tau_t^\Lambda(A)] && (119)\\ &= i[\tau_t^\Lambda(H_X), f(t)] + i[[\tau_t^\Lambda(H_X), B], \tau_t^\Lambda(A)] && (120) \end{aligned}$$

and noting that the first term of (120) is norm-preserving it can be estimated as

$$\left\|[\tau_t^\Lambda(A), B]\right\| \leqslant \|[A, B]\| + 2\|A\| \int_0^t ds \, \left\|[\tau_s^\Lambda(H_X), B]\right\|. \tag{121}$$

Furthermore for any (finite) $X \subset \Gamma$, we consider the function

$$C_B(X, t) = \sup_{A \in \mathcal{A}(X)} \frac{\|[\tau_t^\Lambda(A), B]\|}{\|A\|} \tag{122}$$

$$\tag{123}$$

which can - using (121) - be recursively estimated as

$$\begin{aligned} C_B(X,t) &\leqslant C_B(X,0) + 2\int_0^t ds \, \left\|[\tau_s^\Lambda(H_X), B]\right\| && (124)\\ &\leqslant C_B(X,0) + 2 \sum_{Z \subset \Lambda, Z \cap X \neq \varnothing} \int_0^t ds \, \left\|[\tau_s^\Lambda(\Phi(Z)), B]\right\| && (125)\\ &\leqslant C_B(X,0) + 2 \sum_{Z \subset \Lambda, Z \cap X \neq \varnothing} \|\Phi(Z)\| \int_0^t ds \, C_B(Z,s) && (126) \end{aligned}$$

This can be iterated to obtain

$$C_B(X,t) \leqslant C_B(X,0) + 2\sum_{\substack{Z\subset\Lambda\\ Z\cap X\neq\emptyset}} \|\Phi(Z)\| \int_0^t ds\, \Big(C_B(Z,0) + 2\sum_{\substack{Y\subset\Lambda\\ Y\cap Z\neq\emptyset}} \|\Phi(Y)\| \int_0^s du\, C_B(Y,u)\Big)$$

(127)

$$= C_B(X,0) + 2t\sum_{\substack{Z\subset\Lambda\\ Z\cap X\neq\emptyset}} \|\Phi(Z)\| C_B(Z,0)$$

$$+ 4t\sum_{\substack{Z\subset\Lambda\\ Z\cap X\neq\emptyset}} \|\Phi(Z)\| \sum_{\substack{Y\subset\Lambda\\ Y\cap Z\neq\emptyset}} \|\Phi(Y)\| \int_0^t ds\, C_B(Y,s) \tag{128}$$

$$\vdots$$

$$\leqslant \sum_{n=0}^{\infty} (2t)^n \sum_{\substack{Z_1\ldots Z_n\subset\Lambda\\ Z_i\cap Z_{i+1}\neq\emptyset}} \left(\prod_{k=1}^{n} \|\Phi(Z_n)\|\right) C_B(Z_n,0) \tag{129}$$

$$\leqslant \sum_{n=0}^{\infty} 2\|B\|(2t)^n \underbrace{\sum_{\substack{Z_1\ldots Z_n\subset\Lambda\\ Z_i\cap Z_{i+1}\neq\emptyset}} \left(\prod_{k=1}^{n} \|\Phi(Z_k)\|\right)}_{=a_n} \tag{130}$$

Assuming $\Phi \in \mathcal{B}_\lambda(\Gamma)$ for some $\lambda \geqslant 0$, this can be approximated as

$$a_n \leqslant \sum_{\substack{x_0\in X\\ x_1\ldots x_{n-1}\in\Lambda\\ x_n\in Y}} \prod_{k=1}^{n} \underbrace{\sum_{\substack{Z_k\in\Lambda\\ x_{k-1},x_k\in Z_k}} \|\Phi(Z_k)\|}_{\leqslant \|\Phi\|_\lambda F_\lambda(d(x_{k-1},x_k))} \tag{131}$$

$$\leqslant \|\Phi\|_\lambda^n \sum_{\substack{x_0\in X\\ x_1\ldots x_{n-1}\in\Lambda\\ x_n\in Y}} \prod_{k=1}^{n} F_\lambda(d(x_{k-1},x_k)) \tag{132}$$

$$\leqslant \|\Phi\|_\lambda^n C_\lambda^{n-1} \sum_{x\in X}\sum_{y\in Y} F_\lambda(d(x,y)) \tag{133}$$

putting this back into the bound of $C_B(X,t)$ yields

$$C_B(X,t) \leqslant \sum_{n=0}^{\infty} 2\|B\|(2t)^n \|\Phi\|_\lambda^n C_\lambda^{n-1} \sum_{x\in X}\sum_{y\in Y} F_\lambda(d(x,y)) \tag{134}$$

$$= \frac{2\|B\|}{C_\lambda} \exp(2t\|\Phi\|_\lambda C_\lambda) \sum_{x\in X}\sum_{y\in Y} F_\lambda(d(x,y)) \tag{135}$$

which immediately gives the second version of the theorem (without the "-1"). For the other version one can go back into the calculation and verify that the first term of the exponential series vanishes (i.e. $a_0 = 0$) when $d(X,Y) > 0$. ■

Corollary 3.6 *(Lieb-Robinson Bound (equation 2.17 in [17])).* *For any interaction Φ for which $\|\Phi\|_\lambda$ is finite for some $\lambda > 0$, it is*

$$\left\|\tau_t^\Lambda(A), B]\right\| \leqslant \|A\|\|B\|\frac{2\|F\|}{C_\lambda}\min(|X|,|Y|)\exp(-\lambda(d(X,Y)-\frac{2\|\Phi\|_\lambda}{\lambda}t)) \quad (136)$$

where $v_\lambda = \frac{2\|\Phi\|_\lambda}{\lambda}$ is called the Lieb-Robinson velocity *and can be viewed as the (finite) propagation speed of the system.*

Theorem 3.7 *(Existance of Global Dynamics (theorem 2.2 in [17])).* *Let $\Phi \in \mathcal{B}_\lambda(\Gamma)$ be an interaction. Then*

$$\tau_t(A) = \lim_{\Lambda\to\Gamma}\tau_t^\Lambda(A) \quad (137)$$

converges for any $A \in \mathcal{A}$. In particular, τ_t is a strongly continuous group of automorphisms on $\mathcal{A}$.

Proof. Strictly local observables are by definition dense in the quasi-local algebra, so w.l.o.g. we can assume that $A \in \mathcal{A}(Y)$ for some finite set $Y \subset \Gamma$. Then we show that the expression on the right is a Cauchy net. For $\Lambda_1 \subset \Lambda_2$ we can calculate

$$\begin{aligned}
\left\|\tau_t^{\Lambda_1}(A) - \tau_t^{\Lambda_2}(A)\right\| &= \left\|\int_0^t ds\ \partial_s\left(\tau_s^{\Lambda_1}\circ\tau_{t-s}^{\Lambda_2}(A)\right)\right\| && (138)\\
&= \left\|\int_0^t ds\ \tau_s^{\Lambda_1}[H_{\Lambda_1}-H_{\Lambda_2}, \tau_{t-s}^{\Lambda_2}(A)]\right\| && (139)\\
&= \int_0^t ds\ \left\|[H_{\Lambda_2}-H_{\Lambda_1}, \tau_s^{\Lambda_2}(A)]\right\| && (140)\\
&= \sum_{X\subset\Lambda_2, X\not\subset\Lambda_1}\int_0^t ds\ \left\|[\Phi(X), \tau_s^{\Lambda_2}(A)]\right\|. && (141)
\end{aligned}$$

This can now be estimated using the Lieb-Robinson bounds and the fact that if $X \not\subset \Lambda_1$, there must be a point $x \in X$ that is not in Λ_1.

$$\begin{aligned}
\left\|\tau_t^{\Lambda_1}(A) - \tau_t^{\Lambda_2}(A)\right\| &\leqslant \sum_{x\in\Lambda_2\setminus\Lambda_1}\sum_{X\ni x}\int_0^t ds\ \left\|[\Phi(X), \tau_s^{\Lambda_2}(A)]\right\| && (142)\\
&\leqslant 2\|A\|\sum_{x\in\Lambda_2\setminus\Lambda_1}\sum_{X\ni x}\|\Phi(X)\|C_{\lambda,X,Y}\int_0^t ds\ \exp(2s\|\Phi\|_\lambda C_\lambda) && (143)\\
&\leqslant \frac{\|A\|}{\|\Phi\|_\lambda C_\lambda}\sum_{x\in\Lambda_2\setminus\Lambda_1}\sum_{X\ni x}\|\Phi(X)\|C_{\lambda,X,Y}\left(\exp(2t\|\Phi\|_\lambda C_\lambda)-1\right) && (144)\\
& && (145)
\end{aligned}$$

Now we plug in the definition of $C_{\lambda,X,Y}$ to get

$$\left\|\tau_t^{\Lambda_1}(A) - \tau_t^{\Lambda_2}(A)\right\| \tag{146}$$

$$\leqslant \frac{\|A\|}{\|\Phi\|_\lambda C_\lambda^2}\left(\exp(2t\|\Phi\|_\lambda C_\lambda) - 1\right) \sum_{x\in\Lambda_2\setminus\Lambda_1} \sum_{X\ni x} \|\Phi(X)\| \sum_{z\in X}\sum_{y\in Y} F_\lambda(d(z,y)) \tag{147}$$

$$= \frac{\|A\|}{\|\Phi\|_\lambda C_\lambda^2}\left(\exp(2t\|\Phi\|_\lambda C_\lambda) - 1\right) \sum_{x\in\Lambda_2\setminus\Lambda_1} \sum_{z\in\Gamma} \sum_{X\ni x,z} \|\Phi(X)\| \sum_{y\in Y} F_\lambda(d(z,y)) \tag{148}$$

$$\leqslant \frac{\|A\|}{C_\lambda^2}\left(\exp(2t\|\Phi\|_\lambda C_\lambda) - 1\right) \sum_{x\in\Lambda_2\setminus\Lambda_1} \sum_{z\in\Gamma} F_\lambda(d(x,z)) \sum_{y\in Y} F_\lambda(d(z,y)) \tag{149}$$

$$\leqslant \frac{\|A\|}{C_\lambda}\left(\exp(2t\|\Phi\|_\lambda C_\lambda) - 1\right) \sum_{x\in\Lambda_2\setminus\Lambda_1} \sum_{y\in Y} F_\lambda(d(x,y)) \tag{150}$$

Now the sum goes to zero for $\Lambda_1 \to \Gamma$, because F_λ is uniformly integrable. Therefore $\tau_t^\Lambda(A)$ is a Cauchy net in Λ so it converges in the closed algebra $\mathcal{A}$. ■

The Lieb-Robinson bound stated that the commutator of a time-evolved observable and one located outside of its propagation cone is exponentially small. Another way to formulate the finite propagation speed is to look for an approximate time-evolved observable, with an exponentially small error, that is strictly contained inside its propagation cone. So that this approximation exactly commutes with observables outside. This idea is formulated in the following proposition.

Prop. 3.8 *(Time evolution of local observables, prop C.2 of [4]). Let $\Lambda \subset \Gamma$ be finite and $A \in \mathcal{A}(\Lambda)$ be a local observable. Then for any $\epsilon > 0$ and $t \in \mathbb{R}$, there exists a local observable $A_{t,\epsilon} \in \mathcal{A}(\Lambda^{v_\lambda|t|+\epsilon})$ with*

$$\|\tau_t(A) - A_{t,\epsilon}\| \leqslant C(A,\lambda)e^{-\lambda\epsilon} \tag{151}$$

where λ, c_λ are as in 3.6.

3.2 Vacuum Representation

From this point we will assume a regular lattice, i.e. $\Gamma = \mathbb{Z}^d$ with a translation-invariant interaction, so that τ_t and τ_x commute.

Definition 3.9 *(Vacuum Representation). Let ω be a state of the quasi-local algebra $\mathcal{A}$. Consider the GNS representation $(\mathcal{H}, \pi, \Omega)$ of this state, and the (unitary) representation $U(t,x)$ of $\tau_t \circ \tau_x$. The subgroup of time-translations is strongly continuous, so it has a generator H. This is called a vacuum representation of the algebra if*

1. *H is positive*
2. *$\omega \circ \tau_t = \omega = \omega \circ \tau_x$ for all $t \in \mathbb{R}$ and $x \in \Gamma$*

3. 0 *is an isolated simple eigenvalue of* H *(corresponding to* Ω*)*

Definition 3.10 *(Spectrum).* *Let* $\mathcal{A}, \mathcal{H}, \pi, \Omega, U$ *be a vacuum representation. Then by the Stone-Neumark-Ambrose-Godement theorem there is a spectral measure* dP *on* $\mathbb{R} \times \hat{\Gamma}$ *such that*

$$U^{(d)}(x) = \int_{\mathbb{R}\times\hat{\Gamma}} e^{iEt-ipx} dP(E,p). \tag{152}$$

Here $\hat{\Gamma}$ *is the d-dimensional torus of size* 2π*, which is the dual lattice ("momentum space") of* $\Gamma = \mathbb{Z}^d$*. The spectrum of* U *now is*

$$SpU = \operatorname{supp} dP \tag{153}$$

Similar to the QFT case we can again use the group of space-time translation to convolute elements of $\mathcal{A}$ with functions. Only this time we restrict ourselves to L^1 functions.

Lemma 3.11. *Let* $A \in \mathcal{A}$*, then*

$$\tau_f^{(d+1)}(A) = (2\pi)^{-\frac{d+1}{2}} \sum_{x\in\Gamma} \int dt\ \tau_{t,x}(A) f(t,x) \text{ for } f \in L^1(\mathbb{R}\times\Gamma) \tag{154}$$

$$\tau_f^{(1)}(A) = (2\pi)^{-\frac{1}{2}} \int dt\ \tau_t(A) f(t) \text{ for } f \in L^1(\mathbb{R}) \tag{155}$$

$$\tau_f^{(d)}(A) = (2\pi)^{-\frac{d}{2}} \sum_{x\in\Gamma} \tau_x(A) f(x) \text{ for } f \in L^1(\Gamma) \tag{156}$$

are again elements of the algebra.

Next we define the Arveson spectrum of an operator in order to formulate the energy-momentum transfer relation.

Definition 3.12 *(Arveson spectrum).*

$$Sp_A\tau = \operatorname{supp} \check{A} \tag{157}$$

$$\text{where } A(E,p) = \sum_{x\in\Gamma} \int dt\ e^{itE-ipx} \tau_{t,x}(A) \tag{158}$$

is the (inverse) Fourier transform of $A(t,x) = \tau_{t,x}(A)$*.*

Note that for any $f \in L^1(\mathbb{R}\times\Gamma)$ whose Fourier transform $\hat{f}$ is supported outside if $Sp_A\tau$ this implies $\tau_f(A) = 0$.

Lemma 3.13 *(EM-transfer, proposition 3.6 in [4]).* *Let $A \in \mathcal{A}$, P be the spectral projection as in 3.10, and $\Delta \subset Sp_U$ be measurable, then*

$$\pi(A)P(\Delta) = P(\overline{\Delta + Sp_A\tau})\pi(A)P(\Delta) \tag{159}$$

$$\psi \in P(\Delta)\mathcal{H} \Rightarrow \pi(A)\psi \in P(\overline{\Delta + Sp_A\tau})\mathcal{H} \tag{160}$$

3.3 Almost Local Operators

Just like in the continuous case, it turns out that strictly local operators cannot have a compact spectrum. So in order to find creation operators with localized momentum we need to consider a slightly wider range of operators:

Definition 3.14. *An operator $A \in \mathcal{A}$ is said to be* almost local *if there is a sequence of local operators $A_n \in A(X_n)$ such that*

$$\|A - A_n\| \in O((\operatorname{diam} X_n)^{-\infty}) \tag{161}$$

This class of operators is closed under convolution with Schwarz-functions, so we state analogous to lemma 2.7 (though the proof in this case is somewhat more involved):

Lemma 3.15. *Let $A, B \in \mathcal{A}_{a-loc}$. Then*

1. *$\|[A, \tau_x(B)]\| \in O(|x|^{-\infty})$ $(x \in \Gamma)$*
2. *$\tau_{t,x}(A) \in \mathcal{A}_{a-loc}$ for any $t \in \mathbb{R}$, $x \in \Gamma$*
3. *$\tau_f(A) \in \mathcal{A}_{a-loc}$ for any $f \in S^1(\mathbb{R} \times \Gamma)$*

Proof. ad (1):

$$\|[A, \tau_x(B)]\| = \|[A_n, \tau_x(B_n)]\| + O((\operatorname{diam} X_n)^{-\infty}) \tag{162}$$

And for sufficiently large x, the commutator vanishes.
ad (2): For purely spatial translations this is trivial, so we only need to consider time evolution, i.e. $x = 0$. As A is almost local we can take its local approximation $A_n \in \mathcal{A}(X_n)$. Then we use proposition 3.8 to get $A_{n,t,\epsilon}$ as an approximation of $\tau_t(A_n)$. Setting $\epsilon = \operatorname{diam}(X_n)$ gives us

$$\|\tau_t(A) - A_{n,t,\epsilon}\| \leqslant \|\tau_t(A) - \tau_t(A_n)\| + \|\tau_t(A_n) - A_{n,t,\epsilon}\| \tag{163}$$

$$\leqslant O(\operatorname{diam}(X_n)^{-\infty}) + const \cdot e^{-\lambda\epsilon} \tag{164}$$

$$\leqslant O(\operatorname{diam}(X_n)^{-\infty}) \tag{165}$$

Furthermore the approximation is localized as $A_{n,t,\epsilon} \in \mathcal{A}(X_n^{v_\lambda|t|+\epsilon})$, and

$$\operatorname{diam}(X_n^{v_\lambda|t|+\epsilon}) \leqslant 2\operatorname{diam}(X_n) + v_\lambda|t| \tag{166}$$

So the domain is only linearly larger than the domain of A_n, which - renaming X_n - proves the almost-locality of $\tau_t(A)$.
ad (3): First assume A is strictly local, i.e. $A \in \mathcal{A}(\Lambda)$ for some finite $\Lambda \subset \Gamma$. Let $A_{t,\epsilon}$ be as in proposition 3.8, and $r > 0$ arbitrary. Then we find

$$\tau_f(A) = \int dt \sum_{x\in\Gamma} \tau_{t,x}(A) f(t,x) \tag{167}$$

$$= \int_{|t|\leqslant r} dt \sum_{x\in\Gamma,|x|\leqslant r} \tau_{t,x}(A) f(t,x) + O(r^{-\infty}) \tag{168}$$

$$= \underbrace{\int_{|t|\leqslant r} dt \sum_{x\in\Gamma,|x|\leqslant r} \tau_x(A_{t,\epsilon}) f(t,x)}_{\equiv \tilde{A}} + O(r^{-\infty}) + O(e^{-\lambda\epsilon}) \tag{169}$$

Note that $\tilde{A}$ is a local observable with $\tilde{A} \in \mathcal{A}(\underbrace{\Lambda^{v_\lambda r+\epsilon+r}}_{\equiv X_r})$. So by choosing $\epsilon = r$ we found a $A_r = \tilde{A} \in \mathcal{A}(X_r)$ with

$$\left\|\tau_f(A) - A_r\right\| \leqslant O(r^{-\infty}) = O(\mathrm{diam}(X_r)^{-\infty}) \tag{170}$$

so $\tau_f(A)$ is almost local.
Finally for general (almost local) observables A we can simply consider its local approximation, because we always have the estimate

$$\left\|\tau_f(A) - \tau_f(B)\right\| \leqslant \|A - B\| \|f\|_1 \tag{171}$$

■

3.4 Construction of Asymptotic States

Before we can define asymptotic states, which contain n particles, we will need to define one particle states and their creation operators. Just like in continuous QFT we will need an isolated mass-shell in the spectrum of U which should be given as the graph of a dispersion relation. Unlike QFT however we do not strive for a relativistic theory, so we can allow a much wider range of dispersion relations than just "$\omega(p) = \sqrt{p^2 + m^2}$". Also, in a lattice system, the momentum p is bounded by the inverse lattice size[1]. More precisely, $\omega : \hat{\Gamma} \to \mathbb{R}$ where $\hat{\Gamma}$ is the dual of the lattice, i.e. a torus.

Definition 3.16 *(Mass shell, One-particle space).* *Let $(\mathcal{A}, \tau)$ be a quasi-local algebra in a vacuum representation. Let $\Delta \subset \hat{\Gamma}$ be open and $\omega : \Delta \to \mathbb{R}$ be a smooth function. Then $h = \mathrm{graph}(\omega) \subset \hat{\Gamma} \times \mathbb{R}$ is called a* mass-shell *(and ω its* dispersion relation*) if*

- *h is contained in the spectrum $h \subset SpU$*

[1] which implies an effective UV-cutoff, which is an important reason to consider lattice systems in the first place

- *The set* $\{p \in \Delta | D^2\omega(p) = 0\}$ *has measure zero, where* $D^2\omega$ *is the Hessian matrix of* ω*, i.e. its second derivative.*

Additionally, if $(h - h) \cap SpU = \{0\}$ *the mass-shell is called* pseudo relativistic. *With such a mass shell we define the one-particle space as* $\mathcal{H}_\omega = P(h)\mathcal{H}$.

Note that in the QFT case, the pseudo-relativistic condition is equivalent to saying that the difference of two time-like vectors is space-like or zero.
The next ingredient on our way to creation operators are the wave packets. This is analogous to definition 2.10 with generalized dispersion relation

Definition 3.17 *(wave packets). For any smooth function* $\hat{g} \in C^\infty(\hat{\Gamma})$ *we define its wave-packet as*

$$g_t(x) = \int_{\hat{\Gamma}} \frac{d^d p}{(2\pi)^{d/2}} e^{-i\omega(p)t + ip\cdot x} \hat{g}(p) \tag{172}$$

and its velocity support as

$$V(g) = \{\nabla\omega(p) | p \in \operatorname{supp} \hat{g}\} \tag{173}$$

Note that in contrast to the continuous case (definition 2.10), there is no explicit restriction on the support of $\hat{g}$, because $\hat{\Gamma}$ is compact anyway.

Definition 3.18 *(Creation operators). Let* $B^* \in \mathcal{A}_{a-loc}$ *be an almost local observable such that its Arveson spectrum* $Sp_{B^*}\tau$ *is compact and* $Sp_{B^*}\tau \cap SpU \subset h$ *for some isolated mass-shell* h*. Then* B^* *is a* creation operator *and*

$$B_t^*(g_t) = (2\pi)^{-d/2} \sum_{x \in \Gamma} \tau_{t,x}(B^*) g_t(x) \tag{174}$$

is a Haag-Ruelle scattering operator*, where* g_t *is a wave packet for the dispersion relation of* h*.*

Next we prove some properties of these operators we would expect for bosonic creation, analogous to lemma 2.14

Lemma 3.19. *Let* $B_t^*(g_t)$ *and* $C_t^*(g_t)$ *be HR-creation operators with disjoint velocity supports. Then*

1. $\partial_t B_t^*(g_t)\Omega = 0$
2. $\|[B_t^*(g_t), C_t^*(f_t)]\| \in O(t^{-\infty})$
3. $\|[B_t^*(g_t), [C_t^*(f_t), D_t^*(h_t)]]\| \in O(t^{-\infty})$
4. $B_t(\overline{g}_t)C_t^*(f_t)\Omega = \Omega\langle B_t^*(g_t)\Omega | C_t^*(f_t)\Omega\rangle$ *if the mass-shell is pseudo relativistic.*

Proof. Same as in 2.14. Only that for (4) we need the explicit assumption that the mass-shell is pseudo-relativistic, which was automatic in the QFT case. ■

Now we are ready to define asymptotic scattering states.

Definition 3.20 *(Asymptotic states).* *For $\Psi_i = B^*_{i,t}(g_{i,t})\Omega$ be one particle states, define*

$$\Psi_1 \times_{out} \dots \times_{out} \Psi_n = \Psi^+ = \lim_{t\to\infty} B^*_{1,t}(g_{1,t}) \dots B^*_{n,t}(g_{n,t})\Omega \tag{175}$$

The proof of existence of these asymptotic states is completely analogous to the continuous case (2.15) and relies heavily on lemma 3.19. Therefore we skip it here. The same goes for the following lemma (cf. lemma 2.16)

Lemma 3.21. *Let $B'^*_{1,t}(g'_{1,t}), \dots, B'^*_{n,t}(g'_{n',t})$ be creation operators and $B_t(\bar{g}_t)$ an annihilation operator. Let the velocity supports of $V(g'_i)$ be disjoint and $V(g)$ arbitrary. Then*

$$\lim_{t\to\pm\infty} B_t(\bar{g}_t) B'^*_{1,t}(g'_{1,t}) \dots B'^*_{n,t}(g'_{n,t})|\Omega\rangle \tag{176}$$

$$= \lim_{t\to\pm\infty} \sum_{k=1}^{n} B'^*_{1,t}(g'_{1,t}) \dots \widehat{B'^*_{k,t}(g_{k,t})} \dots B'^*_{n,t}(g'_{n,t})|\Omega\rangle\langle\Omega|B_t(\bar{g}_t)B'^*_{k,t}(g'_{k,t})|\Omega\rangle \tag{177}$$

where the big hat denotes omission of the k'th creation operator.

3.5 Scattering Matrix and LSZ Reduction

The goal if this work is to arrive at a LSZ reduction formula similar to theorem 2.30, but for the setting of a lattice system instead of continuous QFT. We already defined wave-packets $g_t(x)$ for this system as well as creation operators and their asymptotic states. Also by matter of analogy we can expect to simply replace the spatial integrals if the LSZ formula by sums over the lattice. The only part missing is the differential operator $(\Box + m^2)$ in the kernel of which the QFT-wave-packets (i.e. "Klein-Gordon-solutions") lie. In particular, our wave function here is only defined on lattice sites, so we do not have spatial derivatives.
The obvious approach is to use a discrete gradient, like (for $d = 1$)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2) \tag{178}$$

where h is the lattice spacing. This formula is very common in numerical mathematics. But in our setting there is no small parameter for lattice spacing. Also, we want to arrive a a non-perturbative formula that is correct in all cases. Therefore the solution we came up with is somewhat more involved.

Definition 3.22 *(Lattice Gradient).* *The Lattice-Gradient is an operator defined as*

$$\tilde{\nabla} : L^2(\Gamma) \to L^2(\Gamma) \tag{179}$$

$$\tilde{\nabla} g = i\mathcal{F}^{-1} p \mathcal{F} g \tag{180}$$

where $\mathcal{F} : L^2(\Gamma) \to L^2(\hat{\Gamma})$ is the Fourier transform, and p is the momentum-multiplication operator where we fixed the Brillouin zone by identifying $\hat{\Gamma} \cong [-\pi, \pi)$.

Note that this lattice gradient is not a "local operator" in the following sense: For the usual gradient acting on continuous functions, the value of $(\nabla f)(x)$ only depends on the values of f on an arbitrarily small neighborhood of x. For a lattice function on the other hand, $(\tilde{\nabla} f)(x)$ on one lattice point $x \in \Gamma$ really depends on the whole function f. In a sense, this destroys a local interpretation of the theory.

Lemma 3.23 *(Properties of the Lattice Gradient).* *Let ω be a dispersion relation. Let $g_t(x)$ be a wave-packet as in 3.17, and let $f : \mathbb{R} \times \Gamma \to \mathbb{R}$ be a function such that for fixed t, $f_t \in L^2(\Gamma)$. Then*

1. *$\omega^2(-i\tilde{\nabla})$ is a bounded self-adjoint operator on $L^2(\Gamma)$*
2. *$(\partial_t^2 + \omega^2(-i\tilde{\nabla}))g_t(x) = 0$*
3. *$\sum_{x\in\Gamma} g_t(x)\omega^2(-i\tilde{\nabla})f_t(x) = -\sum_{x\in\Gamma} \ddot{g}_t(x)f_t(x)$*

Proof. ad (1): the Fourier transform $\mathcal{F}$ is unitary between $L^2(\Gamma)$ and $L^2((-\pi, \pi))$, and the multiplication operator p on $L^2((-\pi, \pi))$ is bounded and self-adjoint. Therefore $-i\tilde{\nabla}$ is also self-adjoint and bounded and by functional calculus, $\omega^2(-i\tilde{\nabla})$ is well-defined.
ad (2): This is trivial by the definition of the wave-packet.
ad (3): Note that (2) in particular implies that $\ddot{g}_t \in L^2(\Gamma)$ for fixed t. Then this is just a simple calculation

$$\sum_{x\in\Gamma} g_t(x)\omega^2(-i\tilde{\nabla})f_t(x) = \left\langle \overline{g_t} \middle| \omega^2(-i\tilde{\nabla})f_t \right\rangle_{L^2(\Gamma)} \tag{181}$$

$$= \left\langle \omega^2(-i\tilde{\nabla})\overline{g_t} \middle| f_t \right\rangle_{L^2(\Gamma)} \tag{182}$$

$$= -\left\langle \overline{\ddot{g}_t} \middle| f_t \right\rangle_{L^2(\Gamma)} \tag{183}$$

$$= -\sum_{x\in\Gamma} \ddot{g}_t(x)f_t(x) \tag{184}$$

■

Lemma 3.24 *(Lattice version of 2.28).* *Let $F_t(x)$ be an operator valued function (with $t \in \mathbb{R}$ and $x \in \Gamma$) such that*

1. *$F_t(x)$ is twice continuously weakly differentiable in t.*
2. *For fixed t, $\|F_t(x)\|$ grows at most polynomially in x.*
3. *There are operators Q, C_1, and C_2 such that for some time T*

$$F_t(x) = \begin{cases} C_1 Q_t(x) & t < -T \\ Q_t(x) C_2 & t > T \end{cases}. \tag{185}$$

where $Q_t(x)$ is just Q, translated by (t,x).

Furthermore let g be a wave-packet, i.e. $(\partial_t^2+\omega^2(-i\tilde{\nabla}))g_t(x)=0$. Then

$$\int dt\ \sum_{x\in\Gamma} g_t(x)(\partial_t^2+\omega^2(-i\tilde{\nabla}))\langle\Psi|F_t(x)|\Psi'\rangle$$
$$= \lim_{t\to-\infty}\langle\Psi|C_1Q_t\left(\overleftrightarrow{\partial_t}\,g_t\right)|\Psi'\rangle - \lim_{t\to\infty}\langle\Psi|Q_t\left(\overleftrightarrow{\partial_t}\,g_t\right)C_2|\Psi'\rangle \tag{186}$$

for any states Ψ,Ψ' for which the limits on the right side exist.

Proof. For the sake of brevity we omit the states Ψ,Ψ' in the following calculation. But note that we are only interested in weak convergence of the integrals and limits on these specific vectors.

$$\int dt\ \sum_{x\in\Gamma} g_t(x)(\partial_t^2+\omega^2(-i\tilde{\nabla}))F_t(x) \tag{187}$$
$$= \int dt\ \sum_{x\in\Gamma}\left(g_t(x)\ddot{F}_t(x)+g_t(x)\omega^2(-i\tilde{\nabla})F_t(x)\right) \tag{188}$$
$$=^{3.23}\int dt\ \sum_{x\in\Gamma}\left(g_t(x)\ddot{F}_t(x)-F_t(x)\ddot{g}_t(x)\right) \tag{189}$$
$$= \int dt\ \sum_{x\in\Gamma}\partial_t\left(g_t(x)\dot{F}_t(x)-\dot{g}_t(x)F_t(x)\right) \tag{190}$$
$$= \lim_{t\to\infty}\sum_{x\in\Gamma}\left(g_t(x)\dot{F}_t(x)-\dot{g}_t(x)F_t(x)\right) - \lim_{t\to-\infty}\sum_{x\in\Gamma}\left(g_t(x)\dot{F}_t(x)-\dot{g}_t(x)F_t(x)\right) \tag{191}$$
$$= \lim_{t\to\infty}\sum_{x\in\Gamma}\left(g_t(x)\dot{Q}_t(x)-\dot{g}_t(x)Q_t(x)\right)C_2 - \lim_{t\to-\infty}\sum_{x\in\Gamma}C_1\left(g_t(x)\dot{Q}_t(x)-\dot{g}_t(x)Q_t(x)\right) \tag{192}$$
$$= \lim_{t\to\infty}\sum_{x\in\Gamma}\left(g_t(x)\overleftrightarrow{\partial_t}Q_t(x)\right)C_2 - \lim_{t\to-\infty}\sum_{x\in\Gamma}C_1\left(g_t(x)\overleftrightarrow{\partial_t}Q_t(x)\right) \tag{193}$$
$$= \lim_{t\to-\infty}C_1Q_t\left(\overleftrightarrow{\partial_t}\,g_t\right) - \lim_{t\to\infty}Q_t\left(\overleftrightarrow{\partial_t}\,g_t\right)C_2 \tag{194}$$

Note that lemma 3.23 can be used here because on a lattice any function $g_t(x)$ that vanishes sufficiently fast for large x is in $L^2(\Gamma)$ (for fixed t). ■

Theorem 3.25 *(LSZ Reduction Formula for lattice systems).* *The connected part of the scattering matrix for a n-particle scattering for $n,n'\geqslant 2$ can be written as*

$$S_c(\psi_1,\dots,\psi_n;\psi'_1,\dots,\psi'_{n'}) = (-1)^n\int dt_1\dots dt_n\ \int dt'_1\dots dt'_{n'}\ \sum_{\substack{x_1,\dots,x_n\in\Gamma\\ x'_1,\dots,x'_{n'}\in\Gamma}}$$
$$\overline{f_1(t_1,x_1)}\dots\overline{f_n(t_n,x_n)}f'_1(t'_1,x'_1)\dots f'_n(t'_{n'},x'_{n'}) \tag{195}$$
$$K_1\dots K_n\ K'_1\dots K'_{n'}\langle\Omega|T(B_1(t_1,x_1)\dots B_n(t_n,x_n)B'^*_1(t'_1,x'_1)\dots B'^*_n(t'_{n'},x'_{n'}))|\Omega\rangle_T$$

where

1. $\psi_i = \lim_{t\to\infty} B^*_{i,t}(g_{i,t})\Omega$ *and* $\psi'_i = \lim_{t\to\infty} B'^*_{i,t}(g'_{i,t})\Omega$ *are one particle states.*
2. T *is a (regularized) time ordering*
3. K *are operators* $K_i = (\partial^2_{t_i} + \omega^2(-i\tilde{\nabla}_i))$
4. *The g's and f's are related as in lemma 2.29.*

Proof. This proof is split into three steps. First, we use the previous lemma to remove all integrals from the right hand side of the formula. This yields a version of the LSZ formula without any truncation. Secondly, this formula will be inverted by purely combinatorical arguments (lemma 2.22). Third, we compare the result with the definition of the *connected* scattering matrix (equation 69).

Step 1.
We define $R(\psi_1, \dots, \psi_n; \psi'_1, \dots, \psi'_{n'})$ as the right hand side of the equation *without the truncation of the vacuum expectation value*, and start our calculation from there.

We note that our regularized time ordering fulfills all properties of F in the previous lemma 3.24, so we can use it to successively remove all integrals from our expression here. First we apply it to the x' summation/integration and note that the second term of (185) vanishes because a creation operator hits the vacuum from the right. Therefore we get

$$\begin{aligned} R =& (-1)^n \int dt_1 \dots dt_n \sum_{x_1,\dots,x_n \in \Gamma} \overline{f_1(x_1)} \dots \overline{f_n(x_n)} K_1 \dots K_n \\ & \lim_{t\to\infty} \langle \Omega | T(B_1(x_1) \dots B_n(x_n)) B'^*_{1,-t}(g'_{1,-t}) \dots B'^*_{n,-t}(g'_{n',-t}) | \Omega \rangle \end{aligned} \tag{196}$$

Now we want to apply the same lemma to all of the remaining x integrations/summations. But now both terms of (76) actually contribute, so we end up with 2^n terms. Noting that the order of creation operators with the same sign does not matter in the t-limit (lemma 2.14), we can write the result as

$$R = \lim_{t\to\infty} \sum_M (-1)^{n+|M|} \langle \Omega | \prod_{i\in M} B_{i,t}(\overline{g_{i,t}}) \prod_{j\notin M} B_{j,-t}(\overline{g_{j,-t}}) \prod_{k=1}^{n'} B'^*_{k,-t}(g'_{k,-t}) | \Omega \rangle \tag{197}$$

where the sum is over all subsets $M \subset \{1, \dots, n\}$. Now we need to deal with the negative-time annihilation operators in the middle. Using lemma 3.21 we can write this as

$$R = \lim_{t\to\infty} \sum_{\substack{M\\M'}} (-1)^{n+|M|} \langle\Omega| \prod_{i\in M} B_{i,t}(\overline{g_{i,t}}) \prod_{j\in M'} B'^{*}_{j,-t}(g'_{j,-t})|\Omega\rangle$$
$$\times \langle\Omega| \prod_{l\notin M} B_{l,-t}(\overline{g_{l,-t}}) \prod_{k\notin M'} B'^{*}_{k,-t}(g'_{k,-t})|\Omega\rangle \tag{198}$$
$$= \sum_{\substack{M\\M'}} (-1)^{n+|M|} \left\langle \times^{out}_{i\in M}\psi_i \middle| \times^{in}_{j\in M'}\psi'_j \right\rangle \left\langle \times^{in}_{l\notin M}\psi_l \middle| \times^{in}_{k\notin M'}\psi'_k \right\rangle \tag{199}$$
$$= \sum_{\substack{M\\M'}} (-1)^{|M^c|} S(M;M')\mathbb{1}(M^c;M'^c) \tag{200}$$

where the sum is over all subsets $M \subset \{1,\dots,n\}$ and $M' \subset \{1,\dots,n'\}$, and M^c denotes the complement of the index set M. Writing this relation for arbitrary index sets A, A' we get

$$R(A;A') = \sum_{\substack{M\subset A\\M'\subset A'}} (-1)^{|A\backslash M|} S(M;M')\mathbb{1}(A\backslash M; A'\backslash M') \tag{201}$$

(for the notations used here see 2.21).

Step 2.
Next, we can invert this relation between the (non-truncated) R and S. Keep in mind that $\mathbb{1}$ is only non-zero if both its input sets have the same cardinality.

$$\sum_{\substack{M\subset A\\M'\subset A'}} R(M;M')\mathbb{1}(A\backslash M; A'\backslash M') \tag{202}$$
$$= \sum_{\substack{M\subset A\\M'\subset A'}} \sum_{\substack{N\subset M\\N'\subset M'}} (-1)^{|M\backslash N|} S(N;N')\mathbb{1}(M\backslash N, M'\backslash N')\mathbb{1}(A\backslash M; A'\backslash M') \tag{203}$$

In this formula the index set A is split into three parts ($A\backslash M$, $M\backslash N$ and N), and the sum is over all possible splits. So we can rewrite the sum by exchanging the meaning of $A\backslash M$ and N. Doing the same for the primed sets, we get

$$\sum_{\substack{M\subset A\\M'\subset A'}} R(M;M')\mathbb{1}(A\backslash M; A'\backslash M') \tag{204}$$
$$= \sum_{\substack{M\subset A\\M'\subset A'}} \sum_{\substack{N\subset M\\N'\subset M'}} (-1)^{|N|} S(A\backslash M; A'\backslash M')\mathbb{1}(M\backslash N; M'\backslash N')\mathbb{1}(N;N') \tag{205}$$

Now we can use lemma 2.22 for the N,N' summation. We see that the only surviving term is the one with $M = M' = \varnothing$, so the right hand side is simply equal to $S(A, A')$

and we have

$$S(A, A') = \sum_{\substack{M \subset A \\ M' \subset A'}} R(M; M') \mathbb{1}(A \backslash M; A' \backslash M') \tag{206}$$

$$= \sum_{\substack{M \subset A \\ M' \subset A'}} R(M; M') \mathbb{1}(A \backslash M; A' \backslash M') \tag{207}$$

Now we introduce truncation for the R term, i.e. we plug in the defining equation (74) for R:

$$S(A, A') = \sum_{\substack{M \subset A \\ M' \subset A'}} R(M, M') \mathbb{1}(A \backslash M, A' \backslash M') \tag{208}$$

$$= \sum_{\substack{M \subset A \\ M' \subset A'}} \sum_{k} \sum_{\substack{\{A_i\} \\ \{A'_i\}}} \mathbb{1}(M, M') \prod_{i=1}^{k} R_T(A_i; A'_i) \tag{209}$$

Where the sum is over all partitions $A \backslash M = A_1 \sqcup \ldots \sqcup A_k$ with $A_1, \ldots, A_k$ non-empty, and similarly for A' (note that M, M' are allowed to be empty). Note that both $R(A_i; A'_i)$ and $R_T(A_i; A'_i)$ are zero unless either $A_i = A'_i = \varnothing$, or $|A_i|, |A'_i| \geqslant 2$.

Step 3.
Now we consider the truncation of the S-matrix. We start from the defining equation of S_c (see (69))

$$S(A; A') = \sum_{k} \sum_{\substack{\{A_i\} \\ \{A'_i\}}} \prod_{i=1}^{k} S_c(A_i; A'_i) \tag{210}$$

Using lemma 2.25, we see that there are only two cases in which $S_c(A_i; A'_i)$ may be non-zero. Either $|A_i| = |A'_i| = 1$, or $|A_i|, |A'_i| \geqslant 2$. In the former case, we can replace $S_c(A_i; A'_i)$ with a scalar product between 1-particle states. Furthermore, we can collect all these singleton sets together and write the result using $\mathbb{1}$. This gives

$$S(A, A') = \sum_{\substack{M \subset A \\ M' \subset A'}} \sum_{k} \sum_{\substack{\{A_i\} \\ \{A'_i\}}} \mathbb{1}(M, M') \prod_{i=1}^{k} S_c(A_i; A'_i) \tag{211}$$

where the sum is over all those partitions $A \backslash M = A_1 \sqcup \ldots \sqcup A_k$, that have $|A_i|, |A'_i| \geqslant 2$. Finally, by comparing equations (209) and (211), we conclude $R_T(A; A') = S_c(A; A')$ for $|A|, |A'| \geqslant 2$, which concludes the theorem. ■

3.6 The Problem with Perturbation Theory

The natural next step now would be to choose a concrete model and try to evaluate the LSZ formula, at least perturbatively. Many quantum spin systems use only "nearest-neighbor interaction, i.e. the Hamiltonian has the form[2]

$$H = \sum_{x \in \Gamma} H_x^0 + \epsilon \sum_{\substack{x,y \in \Gamma \\ |x-y|=1}} H_{x,y}^{int} \tag{212}$$

where $H_x^0 \in \mathcal{A}(\{x\})$ is the free evolution at the site x, and $H_{x,y}^{int} \in \mathcal{A}(\{x,y\})$ is some interaction. As we will see in Chapter 5, at least for some systems and small ϵ it is possible to analyze the EM-spectrum of such a system. In particular, a mass-shell with some dispersion relation ω_ϵ exists and asymptotic states can be constructed. The most common approach to then analyze the scattering of such states in ordinary QFT is perturbation theory. I.e. one does a series expansion with respect to a (small) coupling constant.
But what is the coupling constant in this setting? It is not ϵ as one might think. Setting $\epsilon = 0$ does not yield a model of freely propagating waves. Instead it yields a model of individual lattice sites evolving independent of each other, and no waves propagating at all. In particular the dispersion relation of such a system will always be constant. This means that the velocity support of any wave-packet will always be $\{0\}$, so asymptotic states are not possible (which need multiple disjoint velocity supports). More formally, the assumptions of definition 3.16 can never be fulfilled. This means we are unable to describe a free system, which would be necessary as a starting point for perturbation theory.

[2]This expression is only symbolical, as a global Hamiltonian does not exist in the observable algebra. But the translation into an interaction Φ should be obvious, so we keep to this notation.

4 Path Integrals and Semi-Classics

It is often said that there is no classical analog to the concept of spin, aside from the crude picture of an elementary particle rotating around itself. In this section we will see one way to give meaning to concepts of "classical trajectory" of a spin degree of freedom and how to use this to build path-integral formulas for quantum spin system as well es semi-classical equation of motions for the system, which - one can hope - lead to some insight into the actual quantum system. An intuitive introduction to the idea can be found in [18]. Our discussion mostly follows the notations from [14] which discusses spin-systems in a statistical setting and to some degree [6] which achieves rigorous results for a single spin. Neither paper attempts to use path-integrals for a time-evolution operator. This we will try in this section.

4.1 Coherent States

Let Γ be a lattice and $s \in \frac{1}{2}\mathbb{N}$ be the spin. At each site $x \in \Gamma$, suppose we have a $(2s+1)$ dimensional Hilbert space spanned by the orthonormal vectors $|-s\rangle_x, |-s+1\rangle_x, \ldots, |s\rangle_x$. Furthermore let the spin operators S_x^+, S_x^-, S_x^3 act as

$$S_x^3|m\rangle_x = m|m\rangle_x \tag{213}$$
$$S_x^\pm|m\rangle_x = c_m^\pm|m\pm 1\rangle_x \tag{214}$$
$$S_x^\pm|\pm s\rangle_x = 0 \tag{215}$$

And the constants can be fixed as

$$c_m^\pm = \sqrt{s(s+1) - m(m\pm 1)} \tag{216}$$
$$= \sqrt{(s\mp m)(s\pm m+1)} \tag{217}$$

So the Hilbert space at each lattice site is finite dimensional. On one hand that makes the situation very easy because all operators are bounded, so there are no problems with domains and such. On the other hand there is no classical analog for such a spin system as trajectory of a classical system is always something continuous. One way to artificially construct such a "classical" trajectory is the following.

Definition 4.1 *(Coherent states).* *For any vector of complex numbers $z \in \mathbb{C}^\Gamma$ we define the* coherent state *as*

$$|z\rangle = \exp\left(\sum_x z_x S_x^-\right)|s\rangle \tag{218}$$
$$\langle z| = \langle s|\exp\left(\sum_x \overline{z_x} S_x^+\right) \tag{219}$$

where $|s\rangle = \bigotimes_{x\in\Gamma}|s\rangle_x$ is the state with all lattice sites in their highest state.

Note that

1. The exponential is actually a polynomial of degree $2s$, because $(S^-)^{2s+1}|s\rangle = 0$.

2. The map $z \mapsto |z\rangle$ is continuous for finite Γ.

The QFT analog of these coherent states are the states of fixed position $|q\rangle$ or fixed momentum $|p\rangle$. Note however that the Hilbert space these states live in is still finite-dimensional (per lattice site), even though there are infinitely many of them (for any complex number z). So the coherent states cannot be orthonormal. Still we do have sufficiently simple formulas for doing similar operations. There are just some extra factors:

Lemma 4.2 *(Properties of coherent states).*

1. *Scalar product of coherent states with pure spin states ($k \in \{-s, \dots, s\}^\Gamma$, $z \in \mathbb{C}^\Gamma$):*

$$\langle k|z\rangle = \prod_x z_x^{s-k_x} \binom{2s}{s-k_x}^{1/2} \tag{220}$$

2. *Scalar product of two coherent states ($w, z \in \mathbb{C}^\Gamma$):*

$$\langle w|z\rangle = \prod_x (1 + \overline{w_x} z_x)^{2s} \tag{221}$$

3. *Partition of unity:*

$$1 = \prod_x \int_{\mathbb{C}} \frac{d\mu(z_x)}{(1+|z_x|^2)^{2s}} |z\rangle\langle z| \tag{222}$$

Where the measure is

$$d\mu(z_x) = \frac{2s+1}{(1+|z_x|^2)^2} \frac{d^2 z_x}{\pi} \tag{223}$$

Remark 4.3. *Note that the measure $d\mu(z_x)$ is chosen in such a way that the path-integral formulas in the end will look nice. Furthermore, the measure is not translation invariant, which will have consequences when trying to solve path-integrals later on.*

Proof. We will do the calculations only for a single lattice site (i.e. $w, z \in \mathbb{C}$ and $k \in \{-s, \dots, s\}$). The product over all sites in the general case should be clear.

ad (1): Using that the spin states $|k\rangle$ are orthonormal to each other we calculate

$$\langle k|z\rangle = \langle k| \exp\left(zS^-\right)|s\rangle \tag{224}$$

$$= \sum_n \frac{z^n}{n!} \langle k| \left(S^-\right)^n |s\rangle \tag{225}$$

$$= \frac{z^{s-k}}{(s-k)!} \left(\prod_{l=k+1}^{s} \underbrace{(s+l)}_{\to (2s)!/(s+k)!} \underbrace{(s-l+1)}_{\to (s-k)!} \right)^{1/2} \tag{226}$$

$$= z^{s-k} \left(\frac{(2s)!}{(s+k)!(s-k)!} \right)^{1/2} \tag{227}$$

$$= z^{s-k} \binom{2s}{s-k}^{1/2} \tag{228}$$

ad (2): Using the partition of unity into pure spin states and (1), we get

$$\langle w|z\rangle = \sum_{k=-s}^{s} \langle w|k\rangle \langle k|z\rangle \tag{229}$$

$$= \sum_{k=-s}^{s} \overline{w^{s-k}} \binom{2s}{s-k}^{1/2} z^{s-k} \binom{2s}{s-k}^{1/2} \tag{230}$$

$$= \sum_{k=0}^{2s} \binom{2s}{k} (\overline{w}z)^k \tag{231}$$

$$= (1+\overline{w}z)^{2s} \tag{232}$$

ad (3): Define the operator

$$O = \int \frac{d\mu(z)}{(1+|z|^2)^{2s}} |z\rangle\langle z| \tag{233}$$

Then for any fixed spin states $|k\rangle, |k'\rangle$ it is

$$\langle k|O|k'\rangle = \frac{2s+1}{\pi} \int d^2z \; \frac{\langle k|z\rangle\langle z|k'\rangle}{(1+|z|^2)^{2s+2}} \tag{234}$$

$$= \binom{2s}{s-k}^{\frac{1}{2}} \binom{2s}{s-k'}^{\frac{1}{2}} \frac{2s+1}{\pi} \int d^2z \; \frac{z^{s-k}\overline{z^{s-k'}}}{(1+|z|^2)^{2s+2}} \tag{235}$$

$$\tag{236}$$

Now we choose polar coordinates $z = re^{i\phi}, d^2z = rdrd\phi$

$$\langle k|O|k'\rangle = \binom{2s}{s-k}^{\frac{1}{2}} \binom{2s}{s-k'}^{\frac{1}{2}} \frac{2s+1}{\pi} \int_0^\infty dr\ \frac{r^{2s-k-k'+1}}{(1+r^2)^{2s+2}} \underbrace{\int_0^{2\phi} d\phi\ e^{i(k-k')}}_{=2\pi\delta_{kk'}} \tag{237}$$

$$= 2\delta_{kk'} \binom{2s}{s-k} (2s+1) \int_0^\infty dr\ \frac{r^{2s-2k+1}}{(1+r^2)^{2s+2}} \tag{238}$$

$$\overset{t=r^2}{=} \delta_{kk'} \binom{2s}{s-k} (2s+1) \int_0^\infty dt\ \frac{t^{s-k}}{(1+t)^{2s+2}} \tag{239}$$

$$= \delta_{kk'} \binom{2s}{s-k} (2s+1) \frac{\Gamma(s+k+1)\Gamma(s-k+1)}{\Gamma(2s+2)} \tag{240}$$

$$= \delta_{kk'} \tag{241}$$

This proves that O is indeed the identity operator. ■

4.2 Time Evolution and Operators

We assume that the time-evolution of both operators and states is given by a (time-independent) Hamiltonian H:

$$|z,t\rangle = e^{itH}|z\rangle \tag{242}$$

$$O(t) = e^{itH} O e^{-itH} \tag{243}$$

$$\langle w,t|O(t)|w,t\rangle = \langle w|O|z\rangle \tag{244}$$

Now in order to work with a a "classical trajectory", we need "classical observables", which are not operators, but simply functions depending on the continuous variable z. In the context of coherent states, such functions are called "symbols".

Definition 4.4 *(Symbol).* *Let O be an operator of the observable algebra on a lattice. We then define its* symbol *as a function $h : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$*

$$O(w,z) = \frac{\langle w|O|z\rangle}{\langle w|z\rangle} \tag{245}$$

Note that for any reasonable operator O (including all Hamiltonians we are interested in), this function is continuous. For example the symbols of the ladder operators are easy to calculate

$$s_x^-(w,z) = \frac{2s\overline{w_x}}{1+\overline{w_x}z_x} \tag{246}$$

$$s_x^+(w,z) = \frac{2sz_x}{1+\overline{w_x}z_x}. \tag{247}$$

And in components this is

$$s_x^1(w,z) = s\frac{z_x + \overline{w_x}}{1 + \overline{w_x} z_x} \tag{248}$$

$$s_x^2(w,z) = \frac{s}{i}\frac{z_x - \overline{w_x}}{1 + \overline{w_x} z_x} \tag{249}$$

$$s_x^3(w,z) = s\frac{1 - \overline{w_x} z_x}{1 + \overline{w_x} z_x} \tag{250}$$

Using these it is possible to write many Hamiltonians we might want to consider (including the Ising model later).

4.3 Path Integral Expressions

Conjecture 4.5 *(Path integral formula for time evolution).* *The time evolution from one coherent state to another can be written as*

$$\langle z^f | e^{-itH} | z^i \rangle = \int_{\substack{z(0)=z^i \\ z(t)=z^f}} D\mu(z) \exp\left(-i \int_0^t d\tau L(z(\tau), \dot{z}(\tau))\right) \tag{251}$$

where

$$L(z, \dot{z}) = h(z,z) + is \sum_{x \in \Gamma} \frac{\overline{\dot{z}_x} z_x - \dot{z}_x \overline{z_x}}{1 + |z_x|^2} \tag{252}$$

takes the role of the classical Lagrangian and the path-integral measure is defined as

$$\int_{\substack{z(0)=z^i \\ z(t)=z^f}} D\mu(z)(\cdot) = \prod_x (1 + |z_x^i|^2)^s (1 + |z_x^f|^2)^s \lim_{n\to\infty} \int \prod_{k=1}^{n-1} d\mu(z_x^k)(\cdot) \tag{253}$$

with the understanding that $z(\frac{tk}{n}) = z^k$. Note that the prefactor of the measure is such that one path integral can be split into two as

$$\int_{\substack{z(0)=z^i \\ z(t)=z^f}} D\mu(z)(\cdot) = \int \prod_x \frac{d\mu(w_x)}{(1 + |w_x|^2)^{2s}} \int_{\substack{z(0)=z^i \\ z(s)=w}} D\mu(z) \int_{\substack{z(s)=w \\ z(t)=z^f}} D\mu(z)(\cdot) \tag{254}$$

which agrees nicely with the formula for the partition of unity (lemma 4.2).

Remark 4.6. *While this looks similar to the usual path integral expressions of quantum field theory, there are important differences:*

1. *$L = h(z,z) + is\sum_x \frac{\overline{\dot{z}_x} z_x - \dot{z}_x \overline{z_x}}{1+|z_x|^2}$, which takes the role of a Lagrangian density is not the result of a Legendre-transformation of the Hamiltonian. But it is manifestly real and has the Hamiltonian in it so we will use it.*

2. *The path-integral measure is not translation invariant as one might be used*

to from ordinary QFT. There is only a rotational symmetry $z \to e^{i\phi}z$. This makes actual evaluation on the level of path integrals very tricky as we will discuss later.

Symbolic Derivation. First, consider an infinitesimal time evolution (so that we neglect terms $O(dt^2)$ and higher).

$$\langle w|e^{-idtH}|z\rangle = \langle w|(1-idtH)|z\rangle \tag{255}$$
$$= (1-idth(w,z))\langle \overline{w}|z\rangle \tag{256}$$
$$= \exp\left(-idth(w,z) + 2s\sum_x \log(1+\overline{w_x}z_x)\right) \tag{257}$$

Now a finite time evolution can be expressed as a limit of infinitesimal evolutions. We use $dt = t/n$ and plug in $n-1$ partitions of unity to get

$$\langle z^f|e^{-itH}|z^i\rangle \tag{258}$$
$$= \lim_{n\to\infty}\langle z^f|(1-\frac{it}{n}H)^n|z^i\rangle \tag{259}$$
$$= \lim_{n\to\infty}\prod_{k=1}^{n-1}\left(\prod_x\int\frac{d\mu(z_x^k)}{(1+|z_x^k|^2)^{2s}}\right) \tag{260}$$
$$\times\langle z^f|(1-\frac{it}{n}H)|z^{n-1}\rangle\langle z^{n-1}|(1-\frac{it}{n}H)|z^{n-2}\rangle\ldots\langle z^1|(1-\frac{it}{n}H)|z^i\rangle \tag{261}$$

Now we use the infinitesimal result and the convention $z^0 = z^i$ and $z^n = z^f$.

$$\langle z^f|e^{-itH}|z^i\rangle \tag{262}$$
$$= \lim_{n\to\infty}\prod_{k=1}^{n-1}\left(\prod_x\int d\mu(z_x^k)\right)\exp\left(-\frac{it}{n}\sum_{k=0}^{n-1}h(z^{k+1},z^k)\right) \tag{263}$$
$$\times\exp\left(2s\sum_x\left(\sum_{k=0}^{n-1}\log(1+\overline{z_x^{k+1}}z_x^k) - \sum_{k=1}^{n-1}\log(1+\overline{z_x^k}z_x^k)\right)\right)$$
$$= \lim_{n\to\infty}\prod_{k=1}^{n-1}\left(\prod_x\int d\mu(z_x^k)\right)\exp\left(-\frac{it}{n}\sum_{k=0}^{n-1}h(z^{k+1},z^k)\right)\prod_x(1+\overline{z_x^1}z_x^0)^s(\overline{z_x^n}z_x^{n-1})^s$$
$$\times\exp\left(s\sum_x\sum_{k=1}^{n-1}\left(\log(1+\overline{z_x^{k+1}}z_x^k)+\log(1+\overline{z_x^k}z_x^{k-1})-2\log(1+\overline{z_x^k}z_x^k)\right)\right) \tag{264}$$

Now assuming the difference $\Delta z = z^{k+1} - z^k$ is small, we can approximate the term $\log(\ldots)+\log(\ldots)-2\log(\ldots)$ in first order of Δz to get

$$\langle z^f|e^{-itH}|z^i\rangle \tag{265}$$
$$= \lim_{n\to\infty}\prod_{k=1}^{n-1}\left(\prod_x\int d\mu(z_x^k)\right)\prod_x(1+\overline{z_x^1}z_x^0)^s(\overline{z_x^n}z_x^{n-1})^s \tag{266}$$
$$\times\exp\left(-\frac{it}{n}\sum_{k=0}^{n-1}h(z^{k+1},z^k) + s\sum_x\sum_{k=1}^{n-1}\left(\frac{\overline{(z_x^{k+1}-z_x^k)}z_x^k - \overline{z_x^k}(z^k-z_x^{k-1})}{1+|z_x^k|^2}+O(\Delta z^2)\right)\right)$$

Finally we can pass to the continuous limit using $z^k = z(\tau_k), \tau_k = kt/n$ and assuming (without any rigorous justification) that $\tau \mapsto z(\tau)$ is continuous, i.e. $\Delta z \approx 0$ and $z^k \approx z^{k+1}$. This means that the first line is precisely our path-integral measure. This justifies the substitutions

$$\frac{1}{n}\sum_{k=0}^{n-1} h(z^{k+1}, z^k) \to \int_0^t d\tau\ h(z(\tau), z(\tau)) \tag{267}$$

$$(z_x^{k+1} - z_x^k) \to \frac{t}{n}\dot{z}_x(\tau_k), \tag{268}$$

which immediately lead to the path integral expression we claimed. ■

One nice thing about such a path integral expression is, that it can easily be used to express expectation values of time-ordered operator products. As such, it can potentially be used to evaluate the right hand side of an LSZ reduction formula.

Conjecture 4.7 *(Path integral for time ordered products of operators).* *Let $O_k(t_k), k = 1 \dots n$ be operators located at some times t_k. Defining the (non-regularized) time-ordering of such operators as usual as*

$$T(O_1(t_1)O_2(t_2)) = \begin{cases} O_1(t_1)O_2(t_2) & t_1 > t_2 \\ O_2(t_2)O_1(t_1) & t_1 < t_2 \end{cases} \tag{269}$$

and assuming the initial (final) time t^i (t^f) is smaller (larger) than all operator times, we get the following path integral expression

$$\langle z^f, t^f | T(O_n(t_n) \dots O_1(t_1)) | z^i, t^i \rangle \tag{270}$$

$$= \int_{\substack{z(t^i)=z^i \\ z(t^f)=z^f}} D\mu(z)\ o_1(\overline{z(t_1)}, z(t_1)) \dots o_n(\overline{z(t_n)}, z(t_n)) \tag{271}$$

$$\times \exp\left(-i\int_{t^i}^{t^f} d\tau \left(h(z,z) + is\sum_{x\in\Gamma} \frac{\overline{\dot{z}_x} z_x - \dot{z}_x \overline{z_x}}{1+|z_x|^2}\right)\right) \tag{272}$$

where the o_i are the symbols of the operators O_i.

Symbolic Derivation. We assume that the times are already correctly ordered, i.e. $t^i < t^1 < \dots < t^n < t^f$. For this we can derive the expression on the right hand side. Then we note that the path-integral expression does not depend on the order of operators, because the symbols o_k are just functions. This proves that the expression must in fact be true for the time-ordered product. The idea of the calculation in the same as in the previous theorem which had not operators in between the initial and final state. The only difference is in equation (261) where we replace some of the scalar product with the operators $O_k(t_k)$. ■

4.4 Remarks on the Measure

The first difficulty one encounters when trying to actually evaluate the path integral we derived for some system, is the non-standard measure

$$d\mu(z_x^k) = \frac{2s+1}{\pi}\frac{d^2 z_x^k}{(1+|z_x^k|^2)^2}. \tag{273}$$

The fundamental reason for this additional normalizing factor is that complex numbers $\mathbb{C}$, over which z is integrated should not be thought of as an infinite plane, but as a sphere on which a spin degree-of-freedom can rotate. The factor $\frac{1}{(1+|z|^2)^2}$ then simply comes from a stereographic projection of the sphere onto the plane.
Still, it might be worthwhile to get rid of this additional factor in the metric. Compared to the well-known path-integrals from quantum mechanics there is an additional factor $\frac{2s+1}{\pi}\frac{1}{(1+|z_x|^2)^2}$ added for each intermediate time $z_x^k = z_x(kt/n)$. In the continuous limit, there are infinitely many such times, so we can calculate

$$\int D\mu(z) e^{-i\int dt\ L} = \lim_{n\to\infty}\int \prod_{k=1}^{n-1}\prod_{x\in\Gamma} d\mu(z_x^k) e^{-i\int dt\ L} \tag{274}$$

$$= \lim_{n\to\infty}\int \prod_{k=1}^{n-1}\prod_{x\in\Gamma} d^2 z_x^k \frac{2s+1}{\pi}\frac{1}{(1+|z_x^k|^2)^2} e^{-i\int dt\ L} \tag{275}$$

$$= \lim_{n\to\infty}\int \prod_{k=1}^{n-1}\prod_{x\in\Gamma} d^2 z_x^k \exp\left(-\log\left(\frac{\pi(1+|z_x^k|^2)^2}{2s+1}\right)\right) e^{-i\int dt\ L} \tag{276}$$

$$= \lim_{\nu\to\infty}\int Dz \exp\left(-\nu\sum_{x\in\Gamma}\int d\tau \log\left(\frac{\pi(1+|z_x(\tau)|^2)^2}{2s+1}\right)\right) e^{-i\int dt\ L}. \tag{277}$$

This means we can express the path-integral formula from 4.5 with the standard measure at the cost of an additional regularizing term in the Lagrangian:

$$\langle z^f | e^{-itH} | z^i\rangle = \lim_{\nu\to\infty}\int_{\substack{z(0)=z^i\\ z(t)=z^f}} Dz \exp\left(-i\int_0^t d\tau L^\nu(z(\tau),\dot z(\tau))\right) \tag{278}$$

where

$$L^\nu(z,\dot z) = h(z,z) + is\sum_{x\in\Gamma}\frac{\overline{\dot z_x} z_x - \dot z_x \overline{z_x}}{1+|z_x|^2} - i\nu\sum_{x\in\Gamma}\log\left(\frac{\pi}{2s+1}(1+|z_x|^2)^2\right) \tag{279}$$

This version of the path-integral symbolically comes close to results like [6, equation 5]. The result in there is mathematically rigorous (using a stochastic integral and the Itô formula), but is only valid for a single spin, i.e. not a lattice system. Still it might strengthen the case for our formula, even though we only showed a symbolic derivation. We will not use this formula any further. In particular because the way we derived it here, the path-integral seems likely to either diverge or vanish for $\nu \to \infty$.

4.5 Semi-Classical Analysis using Euler-Lagrange

One easy way to analyze a path integral is the "stationary phase approximation": We realize that (ignoring the intricacies of the path-integral measure), the largest contribution in the path-integral should be from those paths, where the variation $\delta_z \int d\tau L(z, \dot{z})$ vanishes. This leads to the Euler-Lagrange equations, which we can further evaluate for our spin system. Note that we will be using the simpler version of the Lagrangian

$$L(z, \dot{z}) = h(z, z) + is \sum_{x \in \Gamma} \frac{\overline{\dot{z}_x} z_x - \dot{z}_x \overline{z_x}}{1 + |z_x|^2} \tag{280}$$

and not the regularized Lagrangian L^ν. Also for this calculation we can treat z and $\overline{z}$ as independent variables. Otherwise we would need to do Euler-Lagrange separately for real and imaginary part of z, which would lead to the same result in the end but is much less elegant.

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{z}_x} - \frac{\partial L}{\partial z_x} && (281)\\
&= -is \frac{\mathrm{d}}{\mathrm{d}t} \frac{\overline{z_x}}{1 + |z_x|^2} - \frac{\partial h}{\partial z_x} - is \frac{\overline{\dot{z}_x}}{1 + |z_x|^2} + is\overline{z_x} \frac{\overline{\dot{z}_x} z_x - \dot{z}_x \overline{z_x}}{(1 + |z_x|^2)^2} && (282)\\
&= is\overline{z_x} \frac{\overline{\dot{z}_x} z_x + \dot{z}_x \overline{z_x}}{(1 + |z_x|^2)^2} - \frac{\partial h}{\partial z_x} - 2is \frac{\overline{\dot{z}_x}}{1 + |z_x|^2} + is\overline{z_x} \frac{\overline{\dot{z}_x} z_x - \dot{z}_x \overline{z_x}}{(1 + |z_x|^2)^2} && (283)\\
&= -\frac{\partial h}{\partial z_x} - 2is \frac{\overline{\dot{z}_x}}{1 + |z_x|^2} + 2is\overline{z_x} \frac{\overline{\dot{z}_x} z_x}{(1 + |z_x|^2)^2} && (284)\\
&= -\frac{\partial h}{\partial z_x} - \frac{2is\overline{\dot{z}_x}}{(1 + |z_x|^2)^2} && (285)
\end{aligned}$$

Note that

1. h generally depends on all z_x simultaneously, so this is actually a system of infinitely many coupled differential equations.

2. In contrast to ordinary quantum mechanics, this differential equation is only first order, not second. This might make it easier to solve but also means we can only implement one boundary condition. I.e. we can fix an initial or final state, but not both. We will come back to this problem in the discussion of the Ising model.

5 Example: The Ising model

5.1 Definition and Global Dynamics

The most basic example of a lattice spin system is probably the Ising model, first discussed by Ernst Ising in 1924. It is often used to describe ferromagnetism in statistical mechanics, but it will also serve here as a simple example of lattice scattering.

Definition 5.1 *(Ising Model).* *Choose the lattice $\Gamma = \mathbb{Z}^d$, and at each site $x \in \Gamma$ the local algebra $\mathcal{A}_x = \mathbb{C}^{2\times 2}$. Then for any finite volume $\Lambda \subset \Gamma$, define the local Hamiltonian as*

$$H_{\Lambda,\epsilon} = -\frac{1}{2}\sum_{x\in\Lambda}\left(\sigma_x^{(3)} - 1\right) - \epsilon \sum_{x,y\in\Lambda,|x-y|=1} \sigma_x^{(1)}\sigma_y^{(1)} \tag{286}$$

where $\sigma_x^{(i)}$ is the i-th Pauli matrix located at site x.

Note that setting $\epsilon = 0$ means the sites are not interacting at all, so the spectrum is simply

$$Sp(H_{\Lambda,0}) = -\frac{1}{2}\sum_{x\in\Lambda}\left(Sp(\sigma_x^{(3)}) - 1\right) \tag{287}$$

$$= \sum_{x\in\Lambda}\{0,1\} = \{0,1,2,3,\ldots\} \tag{288}$$

with a unique, isolated and translation-invariant ground state $\Psi_{\Lambda,0} = \bigotimes_{x\in\Lambda}|+\rangle_x$ with energy $E_{\Lambda,0} = 0$. The following results show that for sufficiently small ϵ, the EM spectrum does not change too much, in particular, there exists a unique ground state which is separated by a mass-gap from the remainder of the spectrum.

Theorem 5.2 *(Special case of Theorem 1 and 2 in [24]).* *There are $\epsilon_0 > 0, c > 0$ such that for all $\epsilon \in (0,\epsilon_0)$*

1. *$H_{\Lambda,\epsilon}$ has a unique ground state $\psi_{\Lambda,\epsilon}$ with energy $E_{\Lambda,\epsilon}$*
2. *The spectrum of the local Hamiltonians has the following band-structure*

$$Sp(H_{\Lambda,\epsilon} - E_{\Lambda,\epsilon}) \subset \bigcup_{n\in H_{\Lambda,0}\subset\mathbb{N}_0} \{z \mid |z-n| \leqslant cn\epsilon\}. \tag{289}$$

   In particular, the mass gap is $\geqslant 1 - c\epsilon$ which is positive for sufficiently small ϵ.

3. *The unique [23], translation invariant, pure ground state $\omega_\epsilon = \lim_{\Lambda\to\Gamma}\left\langle \Psi_{\Lambda,\epsilon}\right|\cdot\left.\Psi_{\Lambda,\epsilon}\right\rangle$ exists as a weak-* limit.*

In order to define one-particle states which are needed to establish asymptotic states and all the theory outlined in chapter 3, we need a mass-shell which is guaranteed to exist by the following result.

Theorem 5.3 *(Theorem 4 in [24]). For ϵ sufficiently small, there is a real and analytic dispersion relation $\omega : \hat{\Gamma} \to \mathbb{R}$ such that its mass-shell is an isolated part of the EM spectrum.*

5.2 Free Ising Model

Consider the Hamiltonian of the non-interacting Ising model (so the spin is fixed to $s = 1/2$ from now on)

$$H = -\frac{1}{2}\sum_x \left(\sigma_x^{(3)} - 1\right) \tag{290}$$

The algebra of Pauli-matrices is equivalent to that of the spin-operators, though the conventional normalization is slightly different. We can identify $\sigma_x^{(i)} = 2S_x^i$ and use relation (250) to get the symbol of H:

$$h(w, z) = \sum_x \frac{\overline{w_x} z_x}{1 + \overline{w_x} z_x} \tag{291}$$

Now we will compute the matrix elements of this time-evolution in two different ways.

direct calculation Just using the well-known exponential of Pauli-matrices we can calculate

$$\langle z^f | e^{-itH} | z^i \rangle = \langle z^f | \prod_x \exp(\frac{it}{2}(\sigma_x^{(3)} - 1)) | z^i \rangle \tag{292}$$

$$= \prod_x \langle z_x^f | \exp(\frac{-it}{2}) \exp(\frac{it}{2}\sigma_x^{(3)}) | z_x^i \rangle \tag{293}$$

$$= \prod_x \langle z_x^f | \exp(\frac{-it}{2})(\cos(t/2) + i\sigma_x^{(3)} \sin(t/2)) | z_x^i \rangle \tag{294}$$

Now we substitute the symbol of $\sigma^{(3)} = 2S^3$ from (250) and also the scalar product formula from lemma 4.2 to get

$$\langle z^f | e^{-itH} | z^i \rangle = \prod_x \exp(\frac{-it}{2})(\cos(t/2) + i\frac{1 - \overline{z_x^f} z_x^i}{1 + \overline{z_x^f} z_x^i} \sin(t/2)) \left\langle z_x^f \middle| z_x^i \right\rangle \tag{295}$$

$$= \prod_x \exp(\frac{-it}{2})((1 + \overline{z_x^f} z_x^i)\cos(t/2) + i(1 - \overline{z_x^f} z_x^i)\sin(t/2)) \tag{296}$$

$$= \prod_x \exp(\frac{-it}{2})(e^{it/2} + \overline{z_x^f} z_x^i e^{-it/2}) \tag{297}$$

$$= \prod_x (1 + \overline{z_x^f} z_x^i e^{-it}) \tag{298}$$

semi-classical approach Now we try to get the same result using path-integrals. Simply put in our expression of the Hamiltonian symbol into the equation of motion (285) to get

$$0 = \partial_{z_x} h(z,z) + \frac{2is\overline{\dot{z}_x}}{(1+|z_x|^2)^2} \tag{299}$$

$$= \frac{i\overline{\dot{z}_x}}{(1+|z_x|^2)^2} + \frac{\overline{z_x}}{1+|z_x|^2} - \frac{\overline{z_x}|z_x|^2}{(1+|z_x|^2)^2} \tag{300}$$

$$= \frac{i\overline{\dot{z}_x} + \overline{z_x}}{(1+|z_x|^2)^2} \tag{301}$$

$$\tag{302}$$

the solution of which is the classical trajectory $z(\tau) = Ae^{-i\tau}$. For some constant A. Note that we only have one constant (because the equation of motion was first order). Therefore it is in general impossible to implement both boundary conditions $z(0) = z^i$ and $z(t) = z^f$. But remembering that the states $|z\rangle$ are not actually orthogonal to each other , we can get a reasonable result: we implement the initial boundary condition by setting $A = z(0) = z^i$, so this will evolve to $z(t) = z^i e^{-it}$. Which means that semi-classically,

$$e^{-itH}|z^i\rangle = \text{const} \cdot |e^{-it}z^i\rangle = const \cdot \bigotimes_x |e^{-it}z_x^i\rangle_x \tag{303}$$

And without even looking at the path integral measure we know that the constant must be equal to one, simply by time-evolution being unitary. This then implies for the matrix element:

$$\langle z^f|e^{-itH}|z^i\rangle = \left\langle z^f \middle| e^{-it}z^i \right\rangle = \prod_x (1 + e^{-it\overline{z_x^f}z_x^i}) \tag{304}$$

which agrees perfectly with the direct calculation from before.

5.3 Interacting Ising Model

Now we add a nearest-neighbor interaction to our Hamiltonian:

$$H = -\frac{1}{2}\sum_x \left(\sigma_x^{(3)} - 1\right) - \epsilon \sum_{|x-y|=1} \sigma_x^{(1)}\sigma_y^{(1)} \tag{305}$$

Just as in the free case we use the identification $\sigma_x^{(i)} = 2S_x^i$ and equations (248)-(250) to get the symbol of H:

$$h(w,z) = \sum_x \frac{\overline{w_x}z_x}{1+\overline{w_x}z_x} - \epsilon \sum_{|x-y|=1} \frac{z_x + \overline{w_x}}{1+\overline{w_x}z_x}\frac{z_y + \overline{w_y}}{1+\overline{w_y}z_y} \tag{306}$$

Now, the equations of motion become

$$0 = \partial_{z_x} h(z,z) + \frac{2is\overline{\dot{z}_x}}{(1+|z_x|^2)^2} \tag{307}$$

$$= \frac{i\overline{\dot{z}_x} + \overline{z_x}}{(1+|z_x|^2)^2} - 2\epsilon \left(\frac{1}{1+|z|^2} - \frac{\overline{z_x}^2 + |z_x|^2}{(1+|z_x|^2)^2} \right) \sum_{\substack{y\in\Gamma \\ |x-y|=1}} \frac{z_y + \overline{z_y}}{1+|z_y|^2} \tag{308}$$

$$= \frac{1}{(1+|z_x|^2)^2} \left(i\overline{\dot{z}_x} + \overline{z_x} - 2\epsilon(1-\overline{z_x}^2) \sum_{\substack{y\in\Gamma \\ |x-y|=1}} \frac{z_y + \overline{z_y}}{1+|z_y|^2} \right) \tag{309}$$

which is a system of complex ODEs of first order. We are not aware of any analytic solution, though it might well be possible, at least in low dimension. Two final comments apply

1. In principle it is easy to solve this system of ODEs iteratively in orders of ϵ, which works as follows. Take the free solution $z_x^0(t) = z_x(0)e^{-it}$ and plug it into the interaction terms of the e.o.m. This gives a ODE of the form

$$0 = i\overline{\dot{z}_x(t)} + \overline{z_x(t)} - \epsilon f(t) \tag{310}$$

where f is an explicitly known function. This can be solved easy enough (with a computer at least), to get the first-order solution $z_x^1(t)$. This process can be iterated to get higher order solutions. But there is reason to believe that this route can not produce any useful results for scattering theory, because as previously discussed expansion w.r.t. ϵ is not the same as perturbation theory w.r.t. some coupling constant.

2. Assuming we could find a solution of the ODEs, translating it back to the spin system will always give us a state of the form

$$|z(t)\rangle = \bigotimes_{x\in\Gamma} |z_x(t)\rangle_x \tag{311}$$

which does not have any correlation between the lattice sites. This is not unexpected of course, as the semi-classical analysis explicitly neglects quantum effects. On the one hand this means that this route can not give precise results. But on the other hand it seems reasonable that this semi-classical system is fundamentally easier to solve than the full quantum theory.

Outlook

In this thesis we have seen how to do scattering theory following the Haag-Ruelle approach in lattice systems. It turns out to be remarkably similar to algebraic quantum field theory following the Haag-Kastler approach. We succeeded in developing a LSZ reduction formula for this setting. Furthermore we derived path-integral expressions and semi-classical approximations for quantum spin systems, even though we did not prove these (because the theory of mathematically rigorous path-integrals is quite involved).

In a future project, we would propose to finally use our new LSZ formula to compute the scattering matrix of a simple system such as the Ising model. An exact computation is not necessary, just a proof of $S \neq \mathbb{1}$ would be very welcome. Such a proof could conceivably be done in a perturbative way, though one would need to overcome the problems discussed in section 3.6.

A second proposal might be further investigation of path-integrals in quantum spin systems. One could try to develop rigorous formulas using stochastic integrals similar to [6]. On the other hand these formulas will only be useful if one finds practical methods of computing the non-Gaussian path-integral expressions that appear.

References

[1] C. Albanese. On the spectrum of the Heisenberg Hamiltonian. *Journal of Statistical Physics*, 55(1):297–309, 1989.

[2] H. Araki. *Mathematical Theory of Quantum Fields*. International series of monographs on physics. Oxford University Press, 1999.

[3] W. Arveson. The harmonic analysis of automorphism groups. *Operator Algebras and Applications*, 38:199–269, 1982.

[4] S. Bachmann, W. Dybalski, and P. Naaijkens. Lieb-Robinson bounds, Arveson spectrum and Haag-Ruelle scattering theory for gapped quantum spin systems, 2014.

[5] J. C. A. Barata and K. Fredenhagen. Particle scattering in Euclidean lattice field theories. *Comm. Math. Phys.*, 138(3):507–519, 1991.

[6] B. Bodmann, H. Leschke, and S. Warzel. A rigorous path integral for quantum spin using flat-space Wiener regularization. *Journal of Mathematical Physics*, 40:2549–2559, June 1999.

[7] D. Buchholz. Collision theory for massless Fermions. *Comm. Math. Phys.*, 42(3):269–279, 1975.

[8] D. Buchholz. Collision theory for massless bosons. *Comm. Math. Phys.*, 52(2):147–173, 1977.

[9] M. Duell. Scattering in quantum field theories without mass gap, Araki-Haag approach. *MSc Thesis Technische Universität München*, 2013.

[10] W. Dybalski. Haag-Ruelle scattering theory in presence of massless particles. *Letters in Mathematical Physics*, 72:27–38, April 2005.

[11] W. Dybalski and C. Gérard. A criterion for asymptotic completeness in local relativistic QFT. *Commun. Math. Phys.*, 332(3):1167–1202, 2014.

[12] R. Haag. Quantum field theories with composite particles and asymptotic conditions. *Phys. Rev.*, 112:669–673, Oct 1958.

[13] R. Haag and D. Kastler. An algebraic approach to quantum field theory. *Journal of Mathematical Physics*, 5:848–861, July 1964.

[14] N. Karchev. Path integral representation for spin systems. *ArXiv e-prints*, November 2012.

[15] H. Lehmann, K. Symanzik, and W. Zimmermann. Zur Formulierung quantisierter Feldtheorien. *Il Nuovo Cimento (1955-1965)*, 1(1):205–225, 1955.

[16] E. Lieb and D. Robinson. The finite group velocity of quantum spin systems. *Comm. Math. Phys.*, 28(3):251–257, 1972.

[17] B. Nachtergaele, Y. Ogata, and R. Sims. Propagation of Correlations in Quantum Lattice Systems. *Journal of Statistical Physics*, 124(1):1–13, 2006.

[18] H. Nielsen and D. Rohrlich. A path integral to quantize spin. *Nucl. Phys.*, B299:471–483, 1988.

[19] M. Reed and B. Simon. *III: Scattering Theory.* Methods of Modern Mathematical Physics. Elsevier Science, 1979.

[20] D. Ruelle. On asymptotic condition in quantum field theory. *Helvetica Physica Acta*, 35(3):147, 1962.

[21] S. Sakai. *C*-algebras and W*-algebras.* Springer, 1971.

[22] E. Seidewitz. Avoiding Haag's theorem with parameterized quantum field theory. *ArXiv:1501.05658*, January 2015.

[23] D. A. Yarotsky. Uniqueness of the ground state in weak perturbations of non-interacting gapped quantum lattice systems. *Journal of Statistical Physics*, 118(1):119–144.

[24] D. A. Yarotsky. Quasi-particles in weak perturbations of non-interacting quantum lattice systems. *ArXiv:math-ph/0411042*, November 2004.