One-loop Einstein-Hilbert term in minimally Supersymmetric type-IIB $T^6/\mathbb{Z}_N$ orientifolds

Tailin Li

München 2016

One-loop Einstein-Hilbert term in minimally Supersymmetric type-IIB $T^6/\mathbb{Z}_N$ orientifolds

Tailin Li

Master Thesis
Theoretical and Mathematical Physics
Department of Physics
Ludwig–Maximilians–Universität
München

presented by
Tailin Li

München, 10/10/2016

Declaration of Authorship

I hereby declare that the thesis submitted is my own work. All direct or indirect sources used are acknowledged as references.

I am aware that the thesis in digital form can be examined for the use of unauthorized aid and in order to determine whether the thesis as a whole or parts incorporated in it may be deemed as plagiarism. For the comparison of my work with existing sources I agree that it shall be entered in a database where it shall also remain after examination, to enable comparison with future theses submitted. Further rights of reproduction and usage, however, are not granted here.

This paper was not previously presented to another examination board and has not been published.

City, Date and Signature

Primary Supervisor: Dr. Michael Haack

Second Supervisor: Prof. Ilka Brunner

Date of Defense: 11/10/2016

Abstract

One-loop contributions to the Einstein-Hilbert term in toroidal minimally supersymmetric type-IIB $\mathbb{Z}_N$ orientifolds play an important role in the quantum correction to the metric of the scalars, which might improve our understanding of the low energy effective theory of String Orientifold models. The main purpose of this thesis is to evaluate string one-loop contributions to the Einstein-Hilbert term in all tadpole-free toroidal minimally supersymmetric type-IIB $\mathbb{Z}_N$ orientifolds with D-branes based on the work of [16]. The construction of the partition function of the theory is reviewed. The classification and calculation of the contributions from the one-loop surfaces are discussed in details as well. Moreover, the concept and procedure of the calculation of the corrections to the Einstein-Hilbert term are recapped. Last but not least, a new type of integral in app.B.3 is evaluated.

Contents

1 Introduction

1.1 String Perturbation Theory

In string perturbation theory, as an example, general closed bosonic string n-point amplitudes can be computed as the following path integral[1]:

$$\begin{aligned} A_n &= \sum_{g=0}^{\infty} A_n^{(g)} \\ &= \sum_{g=0}^{\infty} C_{\Sigma_g} \int \mathcal{D}h\mathcal{D}X^{\mu} \int d^2z_1 \dots d^2z_n V_1(z_1,\bar{z}_1)\dots V_n(z_n,\bar{z}_n) e^{-S[X,h]}, \end{aligned} \tag{1}$$

where we sum over all topologies of the world-sheet and integrate over the insertion points of the vertex operators. C_{Σ_g} is a weight factor which depends only on the topology of the world-sheet. $A_0^{(g)}$ is called the genus g partition function.

The generalization to the open oriented string is straightforward: we simply have to allow for surfaces with boundaries. Two-dimensional surfaces with boundary can be obtained from surfaces without boundary by removing disks. The scattering amplitude of open and closed strings is associated with Riemann surfaces with boundaries where the asymptotic open strings are realized as vertex operator insertions at a boundary component and closed strings are realized as vertex operator insertions in the bulk of the surfaces.

For the perturbation theory of unoriented strings we also need to consider non-orientable world-sheets. On a non-orientable world-sheet there are closed noncontractible paths such that if one parallel transports a pair of vectors around they change their relative orientation. A simple example with boundary is the Möbius strip. It arises if we consider the propagation of an open string. We can glue the two ends of the strip to form a annulus or we can glue up to an orientation reversal Ω transformation which results in a Möbius strip. Non-orientable world-sheets can be obtained from orientable world-sheets by adding crosscaps. A crosscap is obtained if one removes a disk and identifies opposite points on the boundary.

The Euler number of a non-orientable surface with g handles, b holes and c crosscaps is

$$\chi = 2 - 2g - b - c. \tag{2}$$

The Euler number is a topological invariant. The Euler number, the number of boundaries and orientability completely specify the topology of a two-dimensional (connected) manifold.

The Gauss-Bonnet theorem states that

$$\chi(\Sigma) = \frac{1}{4\pi}\int_{\Sigma} \sqrt{h} R d^2\sigma + \frac{1}{2\pi}\int_{\partial\Sigma} k ds, \tag{3}$$

where R is the curvature scalar of h and k the trace of the extrinsic curvature on the boundary which, in general, consists of several components. But these are precisely the terms which we can add to the Polyakov action without changing the equations of motion. If we include in S_P the term $-\lambda\chi$ and define $g_s = e^{\lambda}$, then each term in the perturbation series (1) will be weighted by

$$g_s^{-2+2g+b+c+n_c+\frac{1}{2}n_o} = g_s^{-\chi+n_c+\frac{1}{2}n_o}, \tag{4}$$

where $n_c(n_o)$ is the number of closed (open) string vertex operator insertions. The dependence on χ is clear. The factors arising from the vertex operator insertions are also easily understood as follows. Consider any world-sheet and add a handle to it, changing $g \to g+1$. This corresponds to the emission and reabsorption of a closed string, i.e. to the insertion of two closed string vertex operators, each of which contributes g_s. Consider now a world-sheet with boundary and add a handle to the boundary. This changes $b \to b+1$ and is equivalent to the insertion of two open

[1] This subsection is cited from [8]

string vertex operators. In other words, we attach to each closed string vertex operator a factor g_s and to each open string vertex operator a factor $g_s^{1/2}$.

In this thesis, we will focus on $\chi = 0$ surfaces, i.e. Torus, Annulus, Klein bottle and Möbius strip, because they are of the first order in the perturbation theory (1).

1.2 Orientifold

The approach of this part of introduction is to illustrate the main ingredients of the general procedure.[2]

1.2.1 Motivation of Orientifold

We have two main motivations for studying orientifolds.

1) New dualities: Orientifolds are particularly useful tools to establish dualities with less Supersymmetry. This is because orientifolds are built based on theories with more supersymmetry, and orientifold action can decrease the number of supersymmetry in the theory.

2) New Compactifications: Orientifolds have been proved to be useful for exploring different parts of the compactified moduli space that were not accessible before as perturbative string vacua. These new compactifications are often nonperturbatively connected with known compactifications and have interesting duals in M and F theory.

1.2.2 Type-IIB String

The world-sheet action of the Type-II String in the light cone(l.c) gauge is given by

$$S_{l.c.} = -\frac{1}{2\pi}\int d\sigma d\tau (\partial_+ X^i \partial_- X^i - i\psi^i \partial_- \psi^i - i\bar{\psi}^i \partial_+ \bar{\psi}^i). \tag{5}$$

The bosons satisfy periodic boundary condition. Fermions can be either periodic(Ramond sector) or antiperiodic(Neveu-Schwarz sector) on the left and on the right. In each sector one has to perform the GSO projection to obtain the superstring. A state with oscillator number N has mass M which satisfies the mass-shell condition

$$\alpha' M^2 = 4(N - \frac{1}{2}). \tag{6}$$

The fermionic oscillators are defined by $\sqrt{2}b_m = \psi^{2m-1} + i\psi^{2m}$, $m = 1, \ldots, 4$, which satisfy the usual anticommutation relations

$$\{b_m, b_n^\dagger\} = \delta_{mn}, \qquad \{b_m, b_n\} = 0, \qquad \{b_m^\dagger, b_n^\dagger\} = 0. \tag{7}$$

There is a NS and R sector for both of the left and right-movers. GSO projection in the R sector keeps only one of the two spinors. The relative choice of the GSO projection for the right-movers and for the left-movers is significant: we can keep either fermions of the same chirality or of opposite chirality in the two sectors. Depending on the choice, we get either Type-IIA theory or Type-IIB theory:

$$\begin{aligned} \text{Type-IIA :} \quad & (\mathbf{8v} \oplus \mathbf{8s}) \otimes (\mathbf{8v} \oplus \mathbf{8c}) \\ \text{Type-IIB :} \quad & (\mathbf{8v} \oplus \mathbf{8c}) \otimes (\mathbf{8v} \oplus \mathbf{8c}). \end{aligned} \tag{8}$$

1.2.3 D-Branes

To describe a p-dimensional soliton, consider a p-dimensional hyperplane (Dp-brane) along the directions X^1,...,X^p. Take the longitudinal coordinates X^μ, $\mu = 0, \ldots, p$ to satisfy NN boundary conditions, and the transverse coordinates X^m, $m = p{+}1, \ldots, 9$ to satisfy DD boundary conditions.

[2] This whole subsection was cited from [10].

Open strings are allowed to end on the p-dimensional hyperplane which can be viewed as a p-brane at a location determined by the zero mode of the coordinates X^m. This configuration, called a Dirichlet p-brane, behaves in every respect like a BPS soliton.

If there are n identical parallel D-branes, then the open string can begin on a D-brane labeled by i and end on one labeled by j. The label of the D-brane is called the Chan-Paton index at each end. Let us denote a general state in the open string sector by $|\psi, ij\rangle\lambda_{ij}$. Here i, j are Chan-Paton indices, λ_{ij} is the Chan-Paton factor, ψ is the state of the world-sheet fields, and by reality of the string wave function, $\lambda^\dagger = \lambda$. The massless excitations of the open string give rise to a supersymmetric $U(n)$ gauge theory on the worldvolume.

1.2.4 Orbifolds

Given a manifold $\mathcal{M}$ with a discrete symmetry G, one can construct an orbifold $\mathcal{M}' = \mathcal{M}/G$. If the symmetry acts freely on $\mathcal{M}$, i.e. without any fixed points, then $\mathcal{M}'$ is also a smooth manifold. If there are fixed points then $\mathcal{M}'$ is singular near the fixed points. If we now consider strings moving on a target space $\mathcal{M}$, then we are naturally led to the concept of orbifolds in conformal field theory.

Consider a theory A with a discrete symmetry group G. One can construct a new theory $A' =$ orbifold of A by G, $A' = A/G$.

In point particle theory, we simply take Hilbert space of A and keep only those states that are invariant under G to obtain the Hilbert space of A'. However, the particle propagation would be singular near the fixed points of G. In closed string theory, we must also add the "twisted sectors" that are localized near the fixed points. In twisted sectors, the string is closed only up to an action by an element of the group G. What is surprising is that after the inclusion of twisted sectors, string propagation on the orbifold is nonsingular even near the fixed points.

1.2.5 General Remarks about Orientifolds

In general, a symmetry operation of a string theory A can be a combination of target spacetime symmetry and orientation-reversal on the world-sheet. The group of symmetry can then be written as a union

$$G = G_1 \cup \Omega G_2. \tag{9}$$

Given such a symmetry of A, one can construct a new theory $A' = A/G$. If G_2 is non-empty, the resulting theory A' is called an "orientifold" of A. In most examples discussed recently, one starts typically with a $\mathbb{Z}_N$ orbifold of toroidally compactified Type-IIB theory and then orientifolds it further by a symmetry $\mathbb{Z}_2 = \{1, \Omega\}$. If the orbifold group $\mathbb{Z}_N$ is generated by the element θ, then the total orientifold symmetry is $G = \{1, \theta, \ldots, \theta^{N-1}, \Omega, \Omega\theta, \ldots, \Omega\theta^{N-1}\}$ or symbolically, $G = \mathbb{Z}_N \cup \Omega\mathbb{Z}_N$, cf. (60). We describe below some general features of the orientifold construction.

(1) Unoriented Surfaces:
An orientifold is obtained, like an orbifold, by gauging the symmetry G. A non-empty ΩG_2 means that orientation reversal, accompanied by an element of G_2, is a local gauge symmetry; a string and its orientation reversed image are gauge equivalent and must be identified. Therefore, the string perturbation theory of the orientifold includes unoriented surfaces like the Klein bottle.

(2) Closed String Sector:
The closed string sector of the theory A consists of states in the Hilbert space of A that are invariant under G and which survive the orientifold projection. It is completely analogous to the untwisted sector of an orbifold after the projection. Typically, starting with oriented closed strings, one gets unoriented closed strings after the projection.

Besides, orbifolds also give rise the the so called twisted sector: twisted sector states are closed on the quotient manifold but not on the original manifold.

(3) Tadpole Cancellation and Orientifold Planes:
Orientifolds often but not always have open strings in addition to the closed strings. The open string sector in orientifolds is analogous to, but not exactly the same as, the twisted sectors in orbifolds. In the case of orbifolds, twisted sectors are necessitated by the requirement of modular

invariance. In the case of orientifolds, the one-loop diagrams in string perturbation theory include unoriented and open surfaces for which there is no analog of the modular group. There is, however, a consistency requirement for these surfaces that is analogous to the requirement of modular invariance for the torus. This is the requirement of tadpole cancellation. These loop diagrams can have a divergence in the tree channel corresponding to a tadpole of a massless particle. Cancellation of all tadpoles is necessary for obtaining a stable string vacuum. This requirement is very restrictive and it more or less completely determines when and how the open string should be added.

Physically, nonzero tadpoles imply that the equations of motion of some massless fields are not satisfied. They occur for the following reason. The planes that are left invariant by an order 2 symmetry is called the orientifold plane. Like a D-brane, an orientifold plane is a p-dimensional hyperplane which couples to an R-R (p+1)-form which we generically refer to as A_{p+1}. The charge of the orientifold plane can be calculated by looking at the R-R tadpole, i.e. emission of an R-R closed string state in the zero momentum limit. If the orientifold plane has a nonzero charge then it acts as a source term in the equations of motion for the (p+1)-form field A_{p+1}:

$$dH_{p+2} = *J_{7-p} \qquad d * H_{p+2} = *J_{p+1}, \tag{10}$$

where H_{p+2} is the (p+2)-form field strength of A_{p+1}, J_{p+1} and J_{7-p} are the 'electric' and 'magnetic' sources.

Consistency of the field equations requires that $\int_{\sigma_k} *J_{10-k} = 0$, for all surfaces σ_k without a boundary. In particular, there can be no net charge on a compact space. This is the analog of Gauss law in electrodynamics. The field lines emanating from a charge must either escape to infinity or end on an opposite charge. In a compact space, the field lines have nowhere to go to and hence must end on an equal and opposite charge. The only way the negative charge of a p-dimensional orientifold plane in a compact transverse space can be neutralized is by adding the right-number of Dirichlet p-branes so that Gauss law is satisfied and all tadpoles cancel.

(4) Open String Sector and Surfaces with Boundaries:
D-branes are hyperplanes where open strings can end. Inclusion of D-branes introduces the open string sector in the theory. The action of the group G is represented in the D-brane sector by some matrices, which we denote by γ. The γ matrices act on the Chan-Paton indices:

$$g: \qquad |\psi, ij\rangle\lambda_{ij} \to |\hat{g}(\psi), ij\rangle\lambda'_{ij}; \quad \lambda \to \lambda' = \gamma_g^{-1}\lambda\gamma_g \tag{11}$$

$$\Omega h: \qquad |\psi, ij\rangle\lambda_{ij} \to |\hat{\Omega h}(\psi), i'j'\rangle\lambda'_{ij}; \quad \to \lambda' = \gamma_{\Omega h}^{-1}\lambda\gamma_{\Omega h}. \tag{12}$$

Tadpole cancellation together with the requirement that the matrices furnish a representation of the symmetry G in the D-brane sector determine not only the number of D-branes but also the form of the γ matrices. When n D-branes coincide, the worldvolume gauge group is $U(n)$. After the projection onto G-invariant states, we are left with a subgroup of $U(n)$. The group as well as the representations are usually uniquely determined by the consistency requirements discussed above.

1.2.6 Orientifold Group and Spectrum of Type-I

Type-I theory is an orientifold of Type-IIB theory with orientifold symmetry group

$$\mathbb{Z}_2 = \{1, \Omega\}. \tag{13}$$

<u>Closed String Sector:</u>

The closed string sector of Type-I theory contains unoriented strings that are invariant under orientation-reversal. The massless states are simply the states of Type-IIB that are invariant under Ω. We know that only g_{ij}, ϕ, B'_{ij}(R-R 2-form), and a symmetric combination of the two gravitini survive the projection. We should notice that

<u>Open String Sector:</u>

Open string sector arises from the addition of D-branes that are required to cancel the charge of the orientifold plane. Orientation reversal is a purely world-sheet symmetry, so it leaves the entire nine-dimensional space invariant. Thus, the orientifold plane is a 9-plane. It turns out to have -32 units of charge with respect to the 10-form non-propagating field from the R-R sector. This charge can be canceled by adding 32 Dirichlet 9-branes which each have unit charge. The world-volume theory of the D_9-branes gives rise to gauge group $U(32)$ but only an $SO(32)$ subgroup is invariant under the action of Ω.

Type-I supergravity super Yang-Mills theory is anomaly free only if the gauge group is $SO(32)$ or $E_8 \times E_8$. It is satisfying that the spectrum determined by the requiring world-sheet consistency is automatically anomaly free.

1.2.7 Loop Channel and Tree Channel

A massless tadpole leads to a divergence in tree channel. For calculating tadpoles it is useful to keep a field theory example in mind. Let us consider a very massive charged particle in field theory with charge Q. At low momentum, the charge acts as a stationary source for a massless photon. One can calculate the charge Q of the particle by calculating the amplitude for vacuum going into a single photon in the background of this charge. Alternatively, one can calculate the interaction between two particles each of charge Q at zero momentum exchange. The Feynman diagram has $1/q^2$ where q is momentum exchange and the residue is proportional to Q^2. If we write $1/q^2$ as $\int_0^\infty dl \exp(-q^2 l)$, then the zero momentum divergence corresponds to the divergence of this integral coming from very long propagation times l.

D-branes and orientifold planes can be treated similarly. A D-brane is like a very massive charged particle. The interaction between the i-th D-brane and the j-th D-brane due to closed string exchanges between the two branes can be computed by evaluating a annulus diagram with one boundary on the i-th brane and the other boundary on the j-th brane. In string theory, unlike in particle theory, because of conformal invariance the tree channel and loop channel diagrams are related. For example, the tree channel annulus diagram can also be viewed as a loop-channel diagram that evaluates the loop of an open string with one end stuck at the i-th brane and the other end at the j-th brane. Similarly, the interaction between an orientifold plane and the i-th D-brane is given by the Möbius strip diagram which has one boundary stuck at the i-th brane and one crosscap stuck at the orientifold plane. Recall that a crosscap is a circular boundary with opposite points on the boundary identified. Because some of the elements of the orientifold group leave the orientifold plane invariant, the closed string that emanates from the plane has further identifications under the symmetry and it looks like a crosscap.

To summarize, we can imagine that a crosscap is stuck at the orientifold plane and the boundary is stuck at a D-brane. With an orientifold with charge Q and with N D-branes of unit charge, the total charge is $(Q+N)^2$, which can be written as Q^2+N^2+2QN. The term N^2 is proportional to the interaction between the D-branes and is computed by the annulus diagram, the interaction $2QN$ between the D-branes and orientifold planes is computed by the Möbius strip diagram and the interaction between orientifold planes Q^2 is computed by the Klein bottle diagram. An efficient way to evaluate these diagrams is to compute them in loop channel and then factorize them in tree channel.

The loop-counting parameter in string theory is the Euler character. A k-th order term in string perturbation theory which goes as the k-th power of λ corresponds to Riemann surfaces with Euler character $k-1$, where $e^\lambda = g_s$.

A surface with no crosscaps is orientable, otherwise it is nonorientable. We are interested in the first quantum correction, i.e. Riemann surfaces with $\chi = 0$. There are four surfaces that contribute: a torus (one handle), a Klein Bottle (two crosscaps), a Möbius strip (one boundary, one crosscap), and a Annulus (two boundaries) and we follow the convention[3] of [2]. These surfaces can be defined as quotients of tori under different involutions (cf. fig. 1)

$$I_{\mathcal{A}}(z) = I_{\mathcal{M}}(z) = 1 - \bar{z}, \qquad I_{\mathcal{K}}(z) = 1 - \bar{z} + \tau/2, \tag{14}$$

where $\tau = \tau_1 + i\tau_2$ is the modular parameter of the defining torus. The fundamental cells of the

[3] The following paragraph is also cited from [2]

involutions can be chosen as follows:

$$\mathcal{A}: \quad z \in [0, \frac{1}{2}] \times [0, \tau_2] \qquad \mathcal{M}: \quad z \in [\frac{1}{2}, 1] \times [0, \tau_2] \qquad \mathcal{K}: \quad z \in [0, 1] \times [0, \tau_2/2]. \tag{15}$$

The open string boundaries, corresponding to the loci of fixed points, are drawn as thick lines in fig.1. There are no fixed points for the Klein bottle representing the evolution and orientation flip of a closed string. Notice also that the three covering tori plus the torus case are characterized by different modular parameters:

$$\tau_{\mathcal{T}} = it, \qquad \tau_{\mathcal{K}} = 2it, \qquad \tau_{\mathcal{A}} = \frac{it}{2}, \qquad \tau_{\mathcal{M}} = \frac{1}{2} + \frac{it}{2}. \tag{16}$$

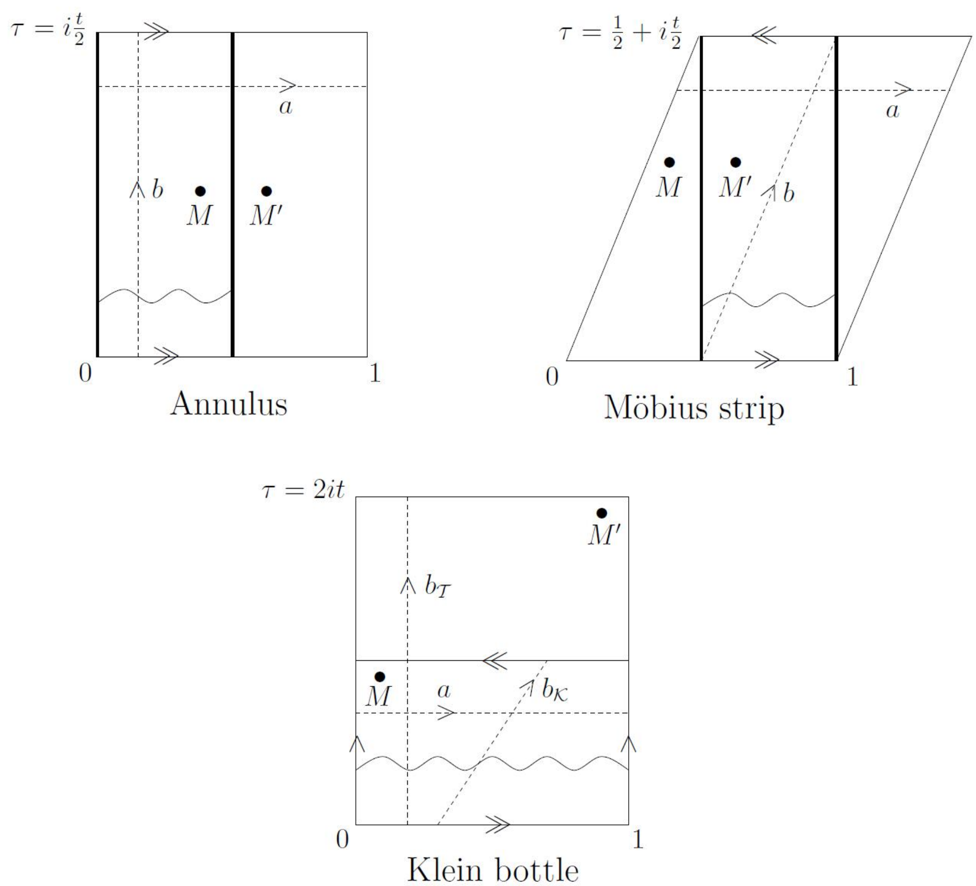

Figure 1: Covering tori and fundamental cells for the three one-loop surfaces $\sigma = \mathcal{A}, \mathcal{M}, \mathcal{K}$. Cited from figs. 1 of [2]

For now, if we transform the tree channel to loop channel according to fig.1, we can easily get the relations

$$\begin{aligned} \text{Annulus}: \quad & t = \frac{1}{l} \\ \text{Klein bottle}: \quad & t = \frac{1}{4l} \\ \text{Möbius}: \quad & t = \frac{1}{4l} \end{aligned} \tag{17}$$

1.3 Minimal Supersymmetry

Minimal supersymmetry in 4D means supersymmetry with $\mathcal{N}=1$ in 4D, i.e. only one supercharge. The $D=4$ supersymmetry algebra must be $\mathcal{N}=1$ because the gauge-couplings in the Standard Model are chiral. Otherwise it will lead to non-chiral theory which is in contrast to Standard Model Phenomenology. This is the phenomenological reason for us to be interested in models with minimal supersymmetry.

Therefore the phenomenological requirement of 4D $\mathcal{N}=1$ minimal supersymmetry gives us another important reason for studying orientifold. String theories compactified on Calabi-Yau manifolds can preserve 1/4 of the original supersymmetry. To further reduce the number of supersymmetry to $\mathcal{N}=1$, orientifold is a very good tool, thus makes it particular important.

1.4 Motivation of the corrections to the Einstein-Hilbert term

This section is cited from app. B of [19].

In the string low energy approximation, the low energy effective action depends on three functions: the superpotential $W(\Phi)$; an arbitrary holomorphic function $f_{ab}(\Phi)$ replacing the gauge coupling g_a^{-2}; the Kähler potential $K(\Phi,\Phi^*)$ which is a general function of the superfields. The purely bosonic part of the Lagrangian density is

$$\begin{aligned}\frac{\mathcal{L}_{bos}}{(-G)^{\frac{1}{2}}} = \frac{1}{2\kappa^2}R - K_{,\bar{i}j}D_\mu\phi^{i*}D^\mu\phi^j - \frac{1}{4}\Re(f_{ab}(\phi))F^a_{\mu\nu}F^{b\mu\nu} \\ -\frac{1}{8}\Im(f_{ab}(\phi))\epsilon^{\mu\nu\sigma\rho}F^a_{\mu\nu}F^b_{\sigma\rho} - V(\phi,\phi^*).\end{aligned} \tag{18}$$

The potential is

$$V(\phi,\phi^*) = \exp(\kappa^2 K)(K^{\bar{i}j}W^*_{;i}W_{;j} - 3\kappa^2 W^*W) + \frac{1}{2}f_{ab}D^aD^b. \tag{19}$$

Here $K^{\bar{i}j}$ is the inverse matrix to $\partial_{\bar{i}}\partial_j K$ and

$$W_{;i} = \partial_i W + \kappa^2\partial_i K W \tag{20}$$

$$\Re(f_{ab}(\phi))D^b = -2\xi_a - K_{,i}t^a_{ij}\phi^j. \tag{21}$$

The negative term proportional to κ^2 in $V(\phi,\phi^*)$ is a supergravity effect.

The kinetic term for the scalars is field-dependent. The second derivative

$$K_{,\bar{i}j} = \frac{\partial^2 K(\phi,\phi^*)}{\partial\phi^{i*}\partial\phi^j} \tag{22}$$

plays the role of a metric for the space of scalar fields.

We know that the one-loop contributions to the Einstein-Hilbert term in toroidal minimally supersymmetric type-IIB orientifolds with D-branes have potential applications to the determination of quantum corrections to the moduli Kähler metric in these models. We can directly see this through (116) in sec. 4.1. And we can see from above that the Kähler metric and Kähler potential show up in most terms of the low energy effective Lagrangian. Thus we can conclude that the correction to the Einstein-Hilbert term may play a potential role in the low energy effective action, which might be phenomenologically important. This is the motivation of this thesis: Try to complete the calculations of the correction in all tadpole-free $\mathbb{Z}_N$ models.

2 General Structure

All the notations and main calculations follow [8]. For Γ orbifolds, we have $SO(D)$ generators θ and twist vector $\vec{v}$. We always use the light-cone gauge.

First we directly give the general partition function for the world-sheet σ:

$$\begin{aligned} \langle \text{1-loop} \rangle_\sigma = Z_\sigma &= Tr_{NS,R}^{U+T\ or\ D-\text{branes}}[\frac{1+\Omega}{2} \cdot P \cdot \frac{1+(-1)^F}{2} e^{-2\pi t(L_0+\bar{L}_0-c/12)}] \\ &= \frac{V_{10-D}}{2\cdot(10-D)N(4\pi^2\alpha')^{(10-D)/2}} \int_0^\infty \frac{dt}{t^{D/2}} \sum_{k,\ell} \sum_{s=\text{even}} Z_\sigma[\theta^k,\theta^\ell](\tau,s) \end{aligned} \tag{23}$$

with

$$P = \frac{1}{N}\sum_{\ell=0}^{N-1}\theta^\ell, \tag{24}$$

$$Z_{\mathcal{A}}[\theta^\ell](\tau_{\mathcal{A}},s) = Z_{99}[\theta^\ell](\tau_{\mathcal{A}},s) + Z_{55}[\theta^\ell](\tau_{\mathcal{A}},s) + Z_{95}[\theta^\ell](\tau_{\mathcal{A}},s), \tag{25}$$

$$Z_{\mathcal{M}}[\theta^\ell](\tau_{\mathcal{M}},s) = Z_9[\theta^\ell](\tau_{\mathcal{M}},s) + Z_5[\theta^\ell](\tau_{\mathcal{M}},s), \tag{26}$$

$$Z_{\mathcal{K}}[\theta^\ell](\tau_{\mathcal{A}},s) = Z_{\text{untwisted}}[\mathbb{1},\theta^\ell](\tau_{\mathcal{K}},s) + \sum_{k=1}^{N-1} Z_{\text{twisted}}[\theta^k,\theta^\ell](\tau_{\mathcal{K}},s), \tag{27}$$

$$Z_{\mathcal{T}}[\theta^\ell](\tau_{\mathcal{A}},s) = Z_{\text{untwisted}}[\mathbb{1},\theta^\ell](\tau_{\mathcal{T}},s) + \sum_{k=1}^{N-1} Z_{\text{twisted}}[\theta^k,\theta^\ell](\tau_{\mathcal{T}},s). \tag{28}$$

Here τ_σ is defined in (16). $\frac{1+\Omega}{2}$ is the orientifold projection and P is the $\mathbb{Z}_N$ symmetry projection. Spin structures can be expressed in (α,β) or s, cf. table 1. And we should also notice that there is no twisted sectors for $\mathcal{A}$ and $\mathcal{M}$, because both of the two surfaces can be considered as open string in loop-channel, thus have no twisted sector.

We are considering here 1-loop amplitudes, i.e. Euler Number $\chi = 0$ surfaces. Thus we need to consider σ as Torus, Annulus, Klein bottle and Möbius strip. For the Torus and Klein bottle, we use $U+T$ to label the untwisted/twisted sectors. As we know, while Annulus and Möbius strip are the propagators of the closed strings propagating between two D-branes, they are also equivalent to closed 1-loop amplitudes of open strings with end-points on the two D-branes. In this sense, we can calculate the amplitude using open string theory. We use D-branes to label where the open strings are attached.

2.1 Bosonic partition function

We will compute the bosonic partition function of type-II string compactified on a toroidal $\mathbb{Z}_N$ orbifold first.

2.1.1 Non-compact dimension

For non-compact dimension, the computation is standard. We have the partition function

$$Z = \frac{1}{\sqrt{\tau_2}\eta\bar{\eta}}$$

for each non-compact dimension.

2.1.2 Compact dimension

Mode expansion We use the complexified coordinates

$$Z^j = \frac{1}{\sqrt{2}}(X^{2j-1} + iX^{2j})$$
$$Z^{*j} = \frac{1}{\sqrt{2}}(X^{2j-1} - iX^{2j})$$

and we have:

$$\theta^\ell Z^j \theta^{-\ell} = e^{2\pi i \ell v_j} Z^j$$
$$\theta^\ell Z^{*j} \theta^{-\ell} = e^{-2\pi i \ell v_j} Z^j \tag{29}$$

v_j is the twist vector which is determined by the crystallographical structure.
The mode expansions are

$$Z^j(\sigma^0, \sigma^1) = z_0^j + \alpha' \frac{M^j}{R}\sigma^0 + N^j R \sigma^1 + i\sqrt{\frac{\alpha'}{2}} \sum_s \frac{\alpha_s^j}{s} e^{-is(\sigma^0 - \sigma^1)} + i\sqrt{\frac{\alpha'}{2}} \sum_t \frac{\bar{\alpha}_t^j}{t} e^{-it(\sigma^0 + \sigma^1)}. \tag{30}$$

M^j and N^j are complexified internal momenta and winding numbers respectively. W.l.o.g. we consider the right-mover. We can find that

$$\theta^\ell \alpha_n^j \theta^{-\ell} = e^{2\pi i \ell v_j} \alpha_n^j$$
$$\theta^\ell \bar{\alpha}_n^j \theta^{-\ell} = e^{2\pi i \ell v_j} \bar{\alpha}_n^j \tag{31}$$

for Z^j and

$$\theta^\ell \alpha_n^{*j} \theta^{-\ell} = e^{-2\pi i \ell v_j} \alpha_n^{*j},$$
$$\theta^\ell \bar{\alpha}_n^{*j} \theta^{-\ell} = e^{-2\pi i \ell v_j} \bar{\alpha}_n^{*j} \tag{32}$$

for Z^{*j}.

Imposing

$$Z^j(\sigma^0, \sigma^1 + 2\pi) = e^{2\pi i k v_j} Z^j(\sigma^0, \sigma^1), \tag{33}$$

which is valid for a complex boson in the k-th twisted sector, fixes the frequencies of the mode expansion to $s = n + kv_j$ and $t = n - kv_j$ with n integer. Furthermore, z_0^j must satisfy $(1 - e^{2\pi i k v_j}) z_0^j = 0 \quad mod \quad 2\pi\Lambda$ (Λ is the torus coordinates lattice), i.e. it must be a fixed point of the orbifold action and, therefore, states in the twisted sectors are localized at the fixed points.
For the complex conjugate Z^{*j} there is an analogous expansion with coefficients $\alpha^{*j}_{n-kv_j} = (\alpha^j_{-n+kv_j})^\dagger$ for the right-movers, $\bar{\alpha}^{*j}_{n+kv_j} = (\bar{\alpha}^j_{-n-kv_j})^\dagger$ for the left-movers and $z_0^{*j} = (z_0^j)^\dagger$ for the center-of-mass position. Canonical quantization results in the following commutator relations for the oscillators

$$[\alpha^i_{m+kv_i}, \alpha^{*j}_{n-kv_j}] = (m + kv_i)\delta^{ij}\delta_{m+n,0},$$
$$[\bar{\alpha}^i_{m-kv_i}, \bar{\alpha}^{*j}_{n+kv_j}] = (m - kv_i)\delta^{ij}\delta_{m+n,0}. \tag{34}$$

The creation operators are $\alpha^j_{-n+kv_j}$, $n > 0$ and $\alpha^{*j}_{-n-kv_j}$, $n \geq 0$ for the right-movers and $\bar{\alpha}^j_{-n-kv_j}$, $n > 0$ and $\bar{\alpha}^{*j}_{-n+kv_j}$, $n \geq 0$ for the left-movers. Here we consider the case where $0 < kv_j < 1$. The occupation number operators are

$$N_R^j = \sum_{n=-\infty}^{\infty} : \alpha^j_{n+kv_j} \alpha^{*j}_{-n-kv_j} :,$$
$$N_L^j = \sum_{n=-\infty}^{\infty} : \bar{\alpha}^j_{n+kv_j} \bar{\alpha}^{*j}_{-n-kv_j} :,$$

with normal ordering : :. Note that the eigenvalues of N_L and N_R in the twisted sectors are multiples of $1/N$.

$Z_B[\mathbb{1},\mathbb{1}]$ **untwisted sector** For untwisted sector ($k=0$)

$$L_0^j(\mathbb{1}) = \frac{1}{2}(p_R^j)^2 + N_R^j(k=0)$$
$$\bar{L}_0^j(\mathbb{1}) = \frac{1}{2}(p_L^j)^2 + N_L^j(k=0),$$

p_L^j and p_R^j are the Kaluza-Klein momenta for the left and right movers on the (compact) j-th dimensions. L_0 without j is just the sum of L_0^j over j.

The bosonic partition function is

$$Z_{\text{bosonic}}^{\text{untwisted}} = Z_B[\mathbb{1},\mathbb{1}] = Tr(q^{L_0-\frac{1}{12}}\bar{q}^{\bar{L}_0-\frac{1}{12}})$$
$$= \frac{1}{|\eta(\tau)|^{2D}} \sum_{\mathbf{m}_R,\mathbf{m}_L\in\Lambda^*} \sum_{\mathbf{n}_R,\mathbf{n}_L\in\Lambda} q^{\frac{1}{2}(\mathbf{m}+\frac{1}{2}\mathbf{n})^2}\bar{q}^{\frac{1}{2}(\mathbf{m}-\frac{1}{2}\mathbf{n})^2},$$

$\mathbf{m}$ is quantized momentum and $\mathbf{n}$ is winding number. Λ^* is the dual lattice of the torus coordinates lattice.

$\mathbb{Z}_N$ **projection** For $\ell \neq 0$ twisted sectors, i.e. for complex bosons which satisfy the boundary conditions

$$Z^j(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) = e^{2\pi i l v_j} Z^j(\sigma^0,\sigma^1), \tag{35}$$

we need to evaluate the trace with an θ^ℓ insertion. Since we assume that θ^ℓ leaves no directions unrotated, thus neither quantized momenta nor windings survive the trace. θ^ℓ is $\mathbb{Z}_N$ group element insertion. We only need to consider states obtained from the Fock vacuum by acting with creation operators for which the complex coordinates are eigenvectors of θ^l. The Fock vacuum is defined to be invariant under θ

$$|n_1^j, n_2^j, \ldots, n_1^{*j}, n_2^{*j}, \ldots\rangle := (\alpha_{-1}^j)^{n_1}(\alpha_{-2}^j)^{n_2}\ldots(\alpha_{-1}^{*j})^{n_1^*}(\alpha_{-2}^{*j})^{n_2^*}\ldots|0\rangle.$$

$Z[\mathbb{1},\theta^\ell]$ **sector** Then, for instance, for the right movers in Z^j, using (31) and (32), we find the contribution

$$Tr(\theta^\ell q^{L_0^j(\mathbb{1})-\frac{1}{12}}) = q^{-\frac{1}{12}} \sum_{n_m^j, n_m^{*j}} \langle n_1^j, n_2^j, \ldots, n_1^{*j}, n_2^{*j}, \ldots|\theta^\ell q^{L_0^j(\mathbb{1})}|n_1^j, n_2^j, \ldots, n_1^{*j}, n_2^{*j}, \ldots\rangle$$
$$= q^{-\frac{1}{12}}(1 + qe^{2\pi i\ell v_j} + qe^{-2\pi i\ell v_j} + \ldots) \tag{36}$$

where the first term is the contribution from the vacuum, the second and third terms from states obtained by acting with α_{-1}^j and α_{-1}^{*j} on the vacuum, and so on. It is not hard to see that the whole expansion can be cast into the form

$$Tr(\theta^\ell q^{L_0^j(\mathbb{1})-\frac{1}{12}}) = q^{-\frac{1}{12}} \sum_{all\, n_m, n_m^*} \Big(\prod_m (q^m e^{2\pi i\ell v_j})^{n_m}(q^m e^{-2\pi i\ell v_j})^{n_m^*}\Big)$$
$$= q^{-\frac{1}{12}} \prod_m \Big(\sum_a (q^m e^{2\pi i\ell v_j})^a \sum_b (q^m e^{-2\pi i\ell v_j})^b\Big)$$
$$= q^{-\frac{1}{12}} \prod_{m=1}^{\infty} (1-q^m e^{2\pi i\ell v_j})^{-1}(1-q^m e^{-2\pi i\ell v_j})^{-1}$$
$$= -2\sin(\ell\pi v_j)\frac{\eta(\tau)}{\vartheta\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2}-\ell v_j \end{bmatrix}(\tau)}. \tag{37}$$

The last step is derived by using the definitions of ϑ and η functions, cf. (189) and(193).

Taking into account left and right-movers for all compact coordinates we obtain

$$Z[\mathbb{1},\theta^\ell] = Tr^U(\theta^\ell q^{L_0-\frac{1}{12}}\bar{q}^{\bar{L}_0-\frac{1}{12}}) = \chi(\theta^\ell)\left|\prod_{j=1}^{D/2}\frac{\eta(\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}-\ell v_j\end{bmatrix}(\tau)}\right|^2$$

$\mathbb{1}$ means untwisted and θ^l means $\mathbb{Z}_N$ element inserted.

Since P defined in (24) must act crystallographically on the torus lattice and since $\mathbf{L} = n_i\mathbf{e}_i$ with integer coefficients n_i, in the lattice basis θ must be a matrix of integers. Hence the quantities

$$Tr\theta^\ell = \sum_{j=1}^{D/2} 2\cos(2\pi\ell v_j) \quad \text{and} \quad \chi(\theta^l) = \prod_{j=1}^{D/2} 4\sin^2(\pi\ell v_j) \tag{38}$$

must be integers. In fact, by the Lefschetz fixed point theorem, $\chi(\theta^\ell)$ is the number of fixed points of θ^ℓ, and this can be explained as the result of the crystallographical structure.

General Bosonic Partition Function Use modular transformations of ϑ and η functions, we can get the partition functions of twisted sectors

$$S : \tau \to -\frac{1}{\tau}, \tag{39}$$

$$\begin{aligned}
S\Big(Z[\mathbb{1},\theta^k]\Big) &= \chi(\theta^k)\left|\prod_{j=1}^{D/2}\frac{\eta(-\frac{1}{\tau})}{\vartheta\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}-kv_j\end{bmatrix}(-\frac{1}{\tau})}\right|^2 \\
&= \chi(\theta^k)\left|\prod_{j=1}^{D/2}\frac{\eta(\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}+kv_j\\ \frac{1}{2}\end{bmatrix}(\tau)}\right|^2 \\
&= \chi(\theta^k)(q\bar{q})^{-\frac{D}{24}+E_k}\left|\prod_{j=1}^{D/2}\prod_{n=1}^{\infty}(1-q^{n-1+\{kv_j\}})^{-1}(1-q^{n-\{kv_j\}})^{-1}\right|^2 \\
&= Z[\theta^k,\mathbb{1}],
\end{aligned}$$

where $Z[\theta^k,\mathbb{1}]$ means θ^k twisted sector and no $\mathbb{Z}_N$ element inserted, and (cf. (10.166) in [8])

$$E_k^j = \frac{1}{2}\{kv_j\}(1-\{kv_j\}), \tag{40}$$

$$E_k = \sum_{j=1}^{D/2}\frac{1}{2}\{kv_j\}(1-\{kv_j\}) \tag{41}$$

is the vacuum expectation value of L_0 in the twisted Fock vacuum which is annihilated by all positive oscillator modes. We define $0 \le \{x\} < 1$ as the fractional value of $x : \{x\} = x - \lfloor x \rfloor$. (cf. p.304-305 of [8])

We can continue generating pieces of the partition function by employing modular transformations (198)-(201). The general result can be easily found to be

$$Z[\theta^k,\theta^\ell] = \chi(\theta^k,\theta^\ell)\left|\prod_{j=1}^{D/2}\frac{\eta(\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}+kv_j\\ \frac{1}{2}+\ell v_j\end{bmatrix}(\tau)}\right|^2, \qquad (k\ell v_j \notin \mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2}) \tag{42}$$

$\chi(\theta^k, \theta^\ell)$ is the number of simultaneous fixed points of θ^k and θ^ℓ. This formula is valid when θ^k leaves no fixed directions, otherwise a sum over momenta and windings could appear. In addition, $\chi(\theta^k, \theta^l)$ should be replaced by $\tilde{\chi}(\theta^k, \theta^l)$, the number of fixed points in the sub-lattice effectively rotated by θ^k. χ and $\tilde{\chi}$ differ because when kv_j=integer, the expansion of $\vartheta[^{\frac{1}{2}+kv_j}_{\frac{1}{2}-lv_j}]/\eta$ has a prefactor $(2\sin\pi\ell v_j)$, as follows from the product representation of the ϑ-function. Thus the actual coefficient in the expansion of (42) is $\tilde{\chi}(\theta^k, \theta^l) = \chi(\theta^k, \theta^l)/\prod_{j,kv_j\in\mathbb{Z}} 4\sin^2\pi\ell v_j$.

Summary The bosonic piece of the partition function of the type-II string compactified on a symmetric $\mathbb{Z}_N$ orbifold is:

$$Z_B[\theta^k, \theta^\ell] = \left(\frac{1}{\sqrt{\tau_2}\eta\bar{\eta}}\right)^{8-D} \tilde{\chi}(\theta^k, \theta^\ell) \left| \prod_{j=1}^{D/2} \frac{\eta(\tau)}{\vartheta\left[\begin{array}{c} \frac{1}{2}+kv_j \\ \frac{1}{2}+\ell v_j \end{array}\right](\tau)} \right|^2, \qquad (k\ell v_j \notin \mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2}) \tag{43}$$

D is the number of compact dimension.

2.1.3 Number of Fixed points χ and $\tilde{\chi}$

From (A.4) of [13] we know

$$\begin{gathered} \tilde{\chi}(1, \theta^n) = 1, \qquad \tilde{\chi}(\theta^m, \theta^n) = \chi(\theta^m, \theta^n) \quad \text{if } \chi(\theta^m) \neq 0, \\ \tilde{\chi}(\theta^m, \theta^n) = \hat{\chi}(\theta^m, \theta^n) = \chi(\theta^m, \theta^n) / \prod_{j, mv_j\in\mathbb{Z}} 4\sin^2\pi n v_j \quad \text{if } \chi(\theta^m) = 0, \end{gathered} \tag{44}$$

where $\chi(\theta^m, \theta^n)$ is the number of simultaneous fixed points of θ^m and θ^n. If θ^m leaves fixed tori, i.e. $\chi(\theta^m) = 0$, we must use $\hat{\chi}(\theta^m, \theta^n)$ which is the number of simultaneous fixed points in the subspace actually rotated by θ^m. This is the same as we discussed above.

As we will see in sec. 2.3, only $\theta^{N/2}$-twisted sector will survive, thus we are only interested in $\chi(\theta^{N/2}, \theta^n)$ cases.

From p.4 in [15], we see that the $\mathbb{Z}_N$ orbifold group action is generated by

$$\theta : z^j \to e^{2\pi i v_j} z^j, \tag{45}$$

with twist vector $\vec{v}$ (cf. app. C).

From p.301 on [11], we can conclude that (using $\chi_{g,h}$ to represent arbitrary $\chi(\theta^m, \theta^n)$) if e is the identity element of $\mathbb{Z}_N$,

$$\chi_{e,g} = \chi(F_g) = \det(1-g) = \chi(\theta^\ell) = \prod_{j=1}^{D/2} 4\sin^2(\pi\ell v_j). \tag{46}$$

Since x is a fixed point of gh, if it is a fixed point of g and a fixed point of h, one sees that

$$\chi_{g,h} = \chi_{g,gh}. \tag{47}$$

Similarly,

$$\chi_{g,h} = \chi_{g^{-1},h} \tag{48}$$

since the fixed point sets of g and g^{-1} are identical. This is also true for h and h^{-1}, thus we have

$$\chi_{g,h} = \chi_{g,h^{-1}}. \tag{49}$$

Moreover, the number is symmetric under exchanging g and h, so we have

$$\chi_{g,h} = \chi_{h,g}. \tag{50}$$

Using all these facts we can evaluate all terms of the form χ_{θ^m,θ^n}.

2.2 Fermionic partition function

Now we come to the fermionic part. Since the Torus compactification has no action on fermionic degrees of freedom, we don't have to distinguish compact or non-compact dimensions. Also be aware that the twist vectors of fermion v_i is different from the twist vectors of compactified bosons v_j, because fermions are not compactified thus they are in different dimension than the bosonic case. However, the twist vectors of fermions won't change the uncompactified dimensions of fermions, therefore we take those components of the twist vectors to be 0.

2.2.1 Fermion

We now compute the one-loop partition function of a complex fermion with twisted boundary conditions.

We define $\psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2)$ and $\bar{\psi} = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2)$. W.l.o.g, we observe the action of the right-mover

$$S = \frac{i}{\pi}\int d^2\sigma \bar{\psi}\partial_+\psi \tag{51}$$

with energy-momentum tensor

$$T = \frac{i}{2}(\bar{\psi}\partial_-\psi + \psi\partial_-\bar{\psi}). \tag{52}$$

Again, using mode expansion and canonical quantization, we can get the Hamiltonian $H = L_0 - \frac{c}{24}$ with

$$L_0 = \sum_{m=1}^{\infty}\left\{\left(m+\alpha-\frac{1}{2}\right)\bar{b}_{-m-\alpha+\frac{1}{2}}b_{m+\alpha-\frac{1}{2}} + \left(m-\alpha-\frac{1}{2}\right)b_{-m+\alpha+\frac{1}{2}}\bar{b}_{m-\alpha-\frac{1}{2}}\right\} + \frac{\alpha^2}{2} \tag{53}$$

and $c = 1$ for one complex fermion. α is the parameter of the twisted boundary condition defined in below.

Then we impose the twisted boundary conditions. For torus spatial direction

$$\psi(\sigma^0, \sigma^1 + 2\pi) = -e^{+2\pi i\alpha}\psi(\sigma^0, \sigma^1),$$
$$\bar{\psi}(\sigma^0, \sigma^1 + 2\pi) = -e^{-2\pi i\alpha}\psi(\sigma^0, \sigma^1).$$

For torus time direction

$$\psi(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) = -e^{+2\pi i\beta}\psi(\sigma^0, \sigma^1),$$
$$\bar{\psi}(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) = -e^{-2\pi i\beta}\psi(\sigma^0, \sigma^1).$$

The minus signs correspond to path-integral with anti-periodic boundary conditions. If we want periodic boundary conditions we have to insert $(-1)^F$. $\alpha, \beta \in \{0, \frac{1}{2}\}$ are spin structures, namely α stands for NS or R sectors, and β stands for $(-1)^F$ inserted or not. But we still need to implement the β-twist (i.e. GSO projection) on operators, i.e. we look for an operator P_{GSO} which satisfies

$$P_{\text{GSO}} b_{n+\alpha+\frac{1}{2}} P_{\text{GSO}}^{-1} = e^{2\pi i\beta} b_{n+\alpha+\frac{1}{2}},$$
$$P_{\text{GSO}} \bar{b}_{n+\alpha+\frac{1}{2}} P_{\text{GSO}}^{-1} = e^{-2\pi i\beta} \bar{b}_{n+\alpha+\frac{1}{2}},$$

and thus the GSO projection is implemented by P_{GSO}.
This operator is easily found to be

$$P_{\text{GSO}} = e^{2\pi i\beta(N-\bar{N})},$$

where $N, \bar{N}$ are the number operators

$$N = \sum_{\eta>0} b_{-\eta}\bar{b}_{\eta}, \qquad \bar{N} = \sum_{\eta>0} \bar{b}_{-\eta}b_{\eta}. \tag{54}$$

The partition function in the α, β sector is

$$\begin{aligned} Z[^{\alpha}_{\beta}](\tau) &= Tr(P_{\text{GSO}} q^{L_0 - \frac{1}{24}}) \\ &= Tr(e^{2\pi i \beta(N-\bar{N})} q^{L_0 - \frac{1}{24}}) \\ &= q^{\frac{\alpha^2}{2} - \frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n+\alpha-\frac{1}{2}} e^{-2\pi i \beta})(1 + q^{n-\alpha-\frac{1}{2}} e^{+2\pi i \beta}) \\ &= e^{2\pi i \alpha\beta} \frac{\vartheta\left[\begin{array}{c} \alpha \\ -\beta \end{array}\right](\tau)}{\eta(\tau)}, \end{aligned} \tag{55}$$

cf. (187) for the definition of q. We can get the full result by adding the left-mover part.

2.2.2 Partition function

Now we consider the orbifold symmetry, which imposes additional boundary conditions

$$\begin{aligned} \psi^j(\sigma^0, \sigma^1 + 2\pi) &= -e^{+2\pi i \alpha} e^{2\pi i k v_i} \psi^j(\sigma^0, \sigma^1), \\ \psi^j(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) &= -e^{+2\pi i \beta} e^{2\pi i \ell v_i} \psi^j(\sigma^0, \sigma^1). \end{aligned} \tag{56}$$

And the partition function on the j-th complex compact dimension is

$$Z_F^j[\theta^k, \theta^\ell] = Tr_{(NS\oplus R)\otimes(NS\oplus R)}\Big(P_{\text{GSO}} \theta^\ell q^{L_0^j(\theta^k) - \frac{1}{24}} \bar{q}^{\bar{L}_0^j(\theta^k) - \frac{1}{24}}\Big). \tag{57}$$

The trace is over the left and right NS and R sectors for the fermions. This is equivalent to summing over $\alpha \in \{0, \frac{1}{2}\}$. Similarly, the GSO projection amounts to summing over $\beta \in \{0, \frac{1}{2}\}$.

Using the result from 2.2.1, we get the partition function of fermion

$$Z_F[\theta^k, \theta^\ell] = \frac{1}{4} \left| \sum_{\alpha,\beta} s_{\alpha\beta}(k,\ell) \prod_{j=1}^{4} \frac{\vartheta\left[\begin{array}{c} \alpha + k v_j \\ -\beta - \ell v_j \end{array}\right]}{\eta} \right|^2, \tag{58}$$

$s_{\alpha\beta}(k,\ell)$ is the spin structure coefficients. By convention we take $s_{00}(k,\ell) = 1$. Imposing modular invariance, notice that $\sum v_i = 0$, we check

$$s_{00}(k,\ell) = -s_{\frac{1}{2}0}(k,\ell) = 1, s_{0\frac{1}{2}}(k,\ell) = -e^{i\pi k \sum v_i} = -1 = \mp s_{\frac{1}{2}\frac{1}{2}}(k,\ell) \tag{59}$$

leads to a modular invariant partition function.

Note that $k = N$ should give the same solution as $k = 0$. This gives, once more, the condition $\sum v_i = 0$. Note further that the sign of $s_{\frac{1}{2}\frac{1}{2}}(k,\ell)$ is not fixed by modular invariance. Choosing opposite(equal) signs in the left and right-movers corresponds to orbifold compactifications of Type-IIA(B) strings (as one can see by looking at the $k = \ell = 0$ sector).

2.3 Orientifold Ω symmetry

There are two distinct orientifold groups possible:

$$Y_N = \{1, \Omega, \theta^k, \Omega_k\}, \quad k = 1, 2, \ldots, N, \quad \theta^k \equiv e^{2\pi i k/N}, \quad \Omega_k \equiv e^{2\pi i k/N}\Omega \tag{60}$$

and

$$W_N = \{1, \theta^{2k-2}, \Omega_{2k-1}\}, \quad k = 1, 2, \ldots, \frac{N}{2}, \quad N \,\text{even}. \tag{61}$$

Ω action and CP factors All conventions follow sec.2 of [1]. We now elaborate the action of the orientifold groups on the states in the open string sector, on D-branes. A generic state can be written as $\lambda_{ij}|X, ij\rangle$ where i, j label the end points of the open strings, λ is a CP matrix, and X collectively labels the world-sheet oscillators that are involved in that state.

The orientifold elements have two possible actions on a generic D-brane state. In addition to the obvious action on the oscillator states, they also act on the CP indices with a matrix representation of the orientifold group. It is generated via matrices γ_θ

$$\theta^k : |X, ij\rangle \to \epsilon_k(\gamma_k)_{ii'}|\theta^k \cdot X, i'j'\rangle(\gamma_k^{-1})_{j'j}, \tag{62}$$

$$\Omega_k : |X, ij\rangle \to \epsilon_{\Omega_k}(\gamma_{\Omega_k})_{ii'}|\theta^k \cdot X, j'i'\rangle(\gamma_{\Omega_k}^{-1})_{j'j}, \tag{63}$$

where $\epsilon_k, \epsilon_{\Omega_k}$ are signs. Note that the Ω_k elements interchange also the string end points. The group property $\theta^k = (\theta_1)^k$ and $\theta_N = 1$ implies

$$\gamma_k = \pm(\gamma_1)^k, \quad (\gamma_k)^N = \pm 1. \tag{64}$$

Furthermore, the condition that Ω^2

$$\Omega^2 : |X, ij\rangle \to \epsilon_\Omega^2(\gamma_\Omega(\gamma_\Omega^T)^{-1})_{ii'}|X, i'j'\rangle(\gamma_\Omega^T\gamma_\Omega^{-1})_{j'j}, \tag{65}$$

is equal to the identity requires that

$$\gamma_\Omega = \zeta\gamma_\Omega^T, \quad \zeta^2 = 1. \tag{66}$$

Note that the adjoint action on the CP indices implies that the representation of the orientifold group on the CP sector is defined up to a sign.

To evaluate the trace of partition functions under Ω, we require the action of the orientation reversal on the bosonic oscillators

$$\Omega\alpha_k^\mu\Omega^{-1} = \bar{\alpha}_k^\mu, \qquad \Omega\bar{\alpha}_k^\mu\Omega^{-1} = \alpha_k^\mu, \qquad \text{(Closed String)} \tag{67}$$

$$\Omega\alpha_k^\mu\Omega^{-1} = (-1)^k\alpha_k^\mu, \qquad \Omega\bar{\alpha}_k^\mu\Omega^{-1} = (-1)^k\bar{\alpha}_k^\mu, \qquad \text{(NN boundary condition for Open String)} \tag{68}$$

$$\Omega\alpha_k^\mu\Omega^{-1} = (-1)^{k+1}\alpha_k^\mu, \qquad \Omega\bar{\alpha}_k^\mu\Omega^{-1} = (-1)^{k+1}\bar{\alpha}_k^\mu, \qquad \text{(DD boundary condition for Open String)}, \tag{69}$$

and Ω also transforms ND boundary conditions to DN ones.

For the fermionic ones, we have

$$\Omega\psi_r\Omega^{-1} = \bar{\psi}_r, \qquad \Omega\bar{\psi}_r\Omega^{-1} = -\psi_r, \qquad \text{(Closed String)} \tag{70}$$

$$\Omega\psi_r\Omega^{-1} = (-1)^r\psi_r, \qquad \Omega\bar{\psi}_r\Omega^{-1} = (-1)^r\bar{\psi}_r, \qquad \text{(NN boundary condition for Open String)} \tag{71}$$

$$\Omega\psi_r\Omega^{-1} = (-1)^{r+1}\psi_r, \qquad \Omega\bar{\psi}_r\Omega^{-1} = (-1)^{r+1}\bar{\psi}_r, \qquad \text{(DD boundary condition for Open String)}. \tag{72}$$

The extra minus sign in (70) is inserted in order for the product $\psi_r\bar{\psi}_r$ to be orientation invariant. This choice does not affect the GSO-invariant states.

Moreover, we should notice that only the left-right symmetric sectors (NS-NS and R-R) survive the Ω projection.

Lattice Sum on T^D under Ω We only have the lattice sum in the case of that there is fixed tori, i.e. $\chi(\theta^m) = 0$, or equivalently, ℓv_j is integer or half-integer. Otherwise there is no windings nor momenta in the compactified dimensions. And we need to compute the traces of the lattice states, which is what we are going to do to here: Lattice Sum. We use complex torus coordinates to represent the coordinates of the compact dimensions, thus we complexify the momenta and windings

$$M_j = m_{2j-1} + im_{2j} \qquad j = 1 \dots \frac{D}{2}, \tag{73}$$

$$N_j = n_{2j-1} + in_{2j} \qquad j = 1 \dots \frac{D}{2}. \tag{74}$$

This is allowed because if we observe the mode expansion of X^i

$$X^i(\sigma,\tau) = x^i + \alpha' p^i \tau + LR\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}(\alpha_n^i e^{-in(\tau-\sigma)} + \bar{\alpha}_n^i e^{-in(\tau+\sigma)}), \tag{75}$$

we see that the momenta $m_i = p_i \cdot R$ and windings $n_i = L$ follow the same θ^ℓ transformation as X^i. Thus there will be no problem to complexify those parameters.

The orientation reversal acts on momenta and windings as

$$\Omega|M_j, N_j\rangle = |M_j, -N_j\rangle, \tag{76}$$

then only momenta survive the trace when no $\mathbb{Z}_N$ element θ^ℓ is inserted

$$\langle M_j, N_j|\Omega|M_j, N_j\rangle = \prod_{i=1}^{D} \delta_{N_j,0}. \tag{77}$$

On the other hand, due to (31) and (32), we can get

$$\theta^\ell|M_j, N_j\rangle = |e^{2\pi i\ell v_j} M_j, e^{2\pi i \ell v_j} N_j\rangle, \tag{78}$$

we observe that the state survives the θ^ℓ action after trace only when ℓv_j is integer, because m_i and n_i have to be integers.

Furthermore,

$$\Omega\theta^\ell|M_j, N_j\rangle = |e^{2\pi i\ell v_j} M_i, e^{2\pi i(\ell v_j - \frac{1}{2})} N_i\rangle. \tag{79}$$

We can easily see that the state survives the $\Omega\theta^\ell$ action after trace only when ℓv_j is integer or half-integer. However, momenta and windings will not simultaneously survive the $\Omega\theta^\ell$ action after trace. If ℓv_j is integer, then momentum survives. If ℓv_j is half-integer, then winding number survives.

Chapter 4.18.5 "Multiple compact scalars" of [17] gives the details of the calculation. The j_0 current of L_0 is changed due to the toroidal compactification, which results in a lattice sum over the internal momenta and windings, cf. section 4.2.2 [9]. The general result is

$$Z^{\text{torus}}_{\text{lattice}} = \frac{\sqrt{g}}{\ell_s^2(\sqrt{\tau_2}\eta)^2} \sum_{\vec{m},\vec{n}} e^{[\pi(g_{ij}+B_{ij})/\tau_2\ell_s^2](m^i+n^i\tau)(m^j+n^j\bar{\tau})}, \tag{80}$$

g_{ij} is the metric of the 2-torus in the target space, B_{ij} is antisymmetric constant background value of the two-index antisymmetric tensor over the 2-torus. We won't consider B in our calculation, thus set $B_{ij} = 0$. We define $V_j = \sqrt{g}$ to be the regularized volume of the torus. j stands for the j-th coordinate of the torus. G is the determinant of the metric g_{ij}.

Since in the following sections, momentum and winding won't simultaneously appear in the partition function. After performing a Poisson re-summation, we summarize and rewrite the momentum/winding sum along the j-th torus with volume V_j and metric $g^{[j]}_{ab}$ from (80) as

$$\mathcal{L}^{[j,M]} = \frac{V_j}{4\pi^2\alpha' t} \sum_{m^1,m^2} e^{-\frac{\pi}{t} m^a m^b g^{[j]}_{ab}}, \tag{81}$$

$$\mathcal{L}^{[j,W]} = \frac{4\pi^2\alpha'}{V_j t} \sum_{n^1,n^2} e^{-\frac{\pi}{t} n_a n_b g^{[j]ab}}. \tag{82}$$

These sums are expressed in the closed string channel. Details could be found in (8.2.9) of [18].

Twisted Sectors Here we need to consider the insertion of the Orientifold element Ω. We know that only left-right symmetric states will survive the Ω insertion after trace. Using the results from 2.1.2, we can easily see that only when $s = n + kv_j$ and $t = n - kv_j$ (k is the k-th twisted sector) are the same index set, the state is left-right symmetric. This is equivalent to requiring kv_j is integer or half-integer for all j. However, this could only be possible for $k = 0$ or $\frac{N}{2}$. Then we know that for twisted sectors of Klein bottles, only the $\frac{N}{2}$-th twisted sector survives.

2.4 D-branes on $T^D/\mathbb{Z}_N$

Refer to 9.14.3 of [17] and Section 2.2 of [1]

The tadpole of Klein bottle amplitudes will be canceled by the insertion of D_9-branes filling all ten dimensions. Through T-duality, we can further see the existence of D_5-branes because T-duality transforms D_9-branes to D_5-branes. And the tadpole must be canceled by the addition of D_5-branes as well. After that, we will also see that O-planes cancel D-brane charges over compact space.

D_5-branes will be stretching in the non-compact dimensions. The orbifold now acts on the transverse positions of the branes. Therefore, there are two main options to consider.

We may consider a group of branes sitting at a fixed point of the orbifold action. In such a case there is no further restriction on the transverse position. We may also consider a group of branes at a generic position x^i on T^D. Orbifold invariance imposes that we also include a mirror brane group at the position $-x^i$.

In the orientifold we are considering, the D_5-branes will have vanishing twisted tadpoles and therefore will not be fractional. Fractional means branes which are fixed to the orbifold fixed points. This means we won't have to worry about those fixed branes.

In order to accommodate the orbifold action on the CP factors of D_9-and D_5-branes we must introduce matrices $\gamma_{\theta/\Omega,9}$ and $\gamma_{\theta/\Omega,5}$. They satisfy the constraints (64)-(66) coming from the orbifold group property.

For the trace of the CP factors, using (63) we may evaluate the trace as in (5.3.24) of [17]

$$\sum_{ij} \langle i,j|\Omega|i,j\rangle = \sum_{iji'j'} \langle i,j|j',i'\rangle (\gamma_\Omega)_{ii'} (\gamma_\Omega^{-1})_{j'j} = Tr[\gamma_\Omega^T \gamma_\Omega^{-1}]. \tag{83}$$

And we have similar results for θ

$$\sum_{ij} \langle i,j|\theta^k|i,j\rangle = \sum_{iji'j'} \langle i,j|j',i'\rangle (\gamma_{\theta^k})_{ii'} (\gamma_{\theta^k}^{-1})_{j'j} = Tr[\gamma_{\theta^k}^T \gamma_{\theta^k}^{-1}]. \tag{84}$$

Fixing signs According to the detailed discussion in section 7.3 of [17], in the NS sector there is an ϵ phase for each of the 9-9 and 5-5 strings as follows

$$\Omega|9-9,p;ij\rangle_{NS} = \epsilon_{99} (\gamma_{\Omega,9})_{ii'} |9-9,p;j'i'\rangle_{NS} (\gamma_{\Omega,9})^{-1}_{j'j}, \tag{85}$$

$$\Omega|5-5,p;ij\rangle_{NS} = \epsilon_{55} (\gamma_{\Omega,5})_{ii'} |5-5,p;j'i'\rangle_{NS} (\gamma_{\Omega,5})^{-1}_{j'j}. \tag{86}$$

Similar arguments as in section 7.3 of [17] fix

$$\epsilon_{99}^2 = \epsilon_{55}^2 = -1, \qquad \gamma_{\Omega,5/9} = \zeta_{5/9} \gamma_{\Omega,5/9}^T, \qquad \zeta_5^2 = \zeta_9^2 = 1. \tag{87}$$

In the 5-9, 9-5 sectors, however, we may write

$$\Omega|5-9,p;ij\rangle_{NS} = \epsilon_{59} (\gamma_{\Omega,5})_{ii'} |9-5,p;j'i'\rangle_{NS} (\gamma_{\Omega,9})^{-1}_{j'j}, \tag{88}$$

$$\Omega|9-5,p;ij\rangle_{NS} = \epsilon_{59} (\gamma_{\Omega,9})_{ii'} |5-9,p;j'i'\rangle_{NS} (\gamma_{\Omega,5})^{-1}_{j'j}, \tag{89}$$

Imposing $\Omega^2 = 1$ we obtain

$$\epsilon_{59}^2 \zeta_5 \zeta_9 = 1. \tag{90}$$

The phase ϵ_{59} captures the transformation properties under Ω of the $SO(D)$ twisted spinor as well of the NS open string vacuum. If two 9-5 states interact, they may produce a 5-5 or a 9-9 state. Therefore, a nontrivial coupling of two 9-5 states to the massless 9-9 or 5-5 states should be allowed. This implies that $\epsilon_{59}^2 = -1$. Thus from (90), the CP projection is opposite for D_5-branes compared to that of D_9-branes,

$$\zeta_5\zeta_9 = -1. \tag{91}$$

Boundary conditions We have to notice that in the case of effective open string surfaces of Annulus and Möbius strips, due to the boundary conditions, we have the general properties: NN directions have only momenta, DD directions have only windings, and DN have none of both.

D_5**-branes** Due to tadpole cancellation, only $\mathbb{Z}_{even}$ type-IIB orbifold has D_9-branes filling the space and D_5-branes transversal to 1-st and 2-nd tori and parallel to(wrapped around) 3-rd torus. So it means D_5-branes only exist for $\mathbb{Z}_{even}$ models.

3 Analysis of the 4 Euler Number $\chi = 0$ surfaces

Since the calculation of the partition function of the surfaces is related to the twist vector v_j of a certain $\mathbb{Z}_N$ group, we'll give the general idea first, then give the examples in detailed orientifolds in the following sections.

From now on we'll concentrate on phenomenally interesting $D = 4$ case.

3.1 Partition Function

This section follows closely to sec.3 of [16].

Using the results of partition functions we derived in section 2, we get the general partition functions of the 3 different $\chi = 0$ surfaces except torus

$$Z_\sigma^{(\ell)}(\tau_\sigma, s) = (-2\pi) CP_\sigma \tilde{\chi}_\sigma (-2\sin(\pi\gamma_3)) \left(\prod_{j=1}^{2} f(\gamma_j) \right) Z_s^\vartheta(\gamma_i, h_i, g_i) \tag{92}$$

with $Z_s^\vartheta(\gamma_i, h_i, g_i)$ being the ϑ-dependent part of the partition function given by

$$Z_s^\vartheta(\gamma_i, h_i, g_i) = \eta_{\alpha\beta} \frac{\vartheta\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \vartheta\begin{bmatrix} \alpha + h_1 \\ \beta + \gamma_1 + g_1 \end{bmatrix} \vartheta\begin{bmatrix} \alpha + h_2 \\ \beta + \gamma_2 + g_2 \end{bmatrix} \vartheta\begin{bmatrix} \alpha \\ \beta + \gamma_3 \end{bmatrix}}{\vartheta'\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \vartheta\begin{bmatrix} \frac{1}{2} + h_1 \\ \frac{1}{2} + \gamma_1 + g_1 \end{bmatrix} \vartheta\begin{bmatrix} \frac{1}{2} + h_2 \\ \frac{1}{2} + \gamma_2 + g_2 \end{bmatrix} \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} + \gamma_3 \end{bmatrix}}, \tag{93}$$

where the spin structure relation between s and (α, β) can be found in Table 1. And $\vartheta'[\frac{1}{2}, \frac{1}{2}] \equiv -2\pi\eta^3$, cf. (193). σ stands for the surfaces of Klein bottle $\mathcal{K}$, Annulus $\mathcal{A}$ and Möbius strip $\mathcal{M}$, with world-sheet parameters $\tau_{\mathcal{K}} = 2it$, $\tau_{\mathcal{A}} = \frac{it}{2}$, $\tau_{\mathcal{M}} = \frac{1}{2} + \frac{it}{2}$. More details can be found in [2]. CP_σ stands for the corresponding Chan-Paton factor of the open string world-sheets and $CP = 1$ for the Klein bottle, cf. section 2.3. Values for CP_σ, $\tilde{\chi}_\sigma$, γ_i, $f(\gamma_j)$, h_i and g_i can be found in Table 2. Formula (92) holds for all tadpole-free Z_N type-IIB orientifolds. Orientifolds with even N have D_5-branes wrapped around the third torus leading to the distinction of γ_3 in (92). And therefore the 3-rd torus always has NN boundary condition no matter whether it is attached to D_9 or D_5-branes.

We choose

$$tr(\gamma_{\Omega_\ell,5}^{-1} \gamma_{\Omega_\ell,5}^T) = -tr\gamma_{2\ell,5} \tag{94}$$

and

$$tr(\gamma_{\Omega_\ell,9}^{-1} \gamma_{\Omega_\ell,9}^T) = tr\gamma_{2\ell,9}. \tag{95}$$

The minus sign is due to the GP[4] action of Ω, cf. sec. 2.3 and eq.(2.41) of [1].

s	1	2	3	4
$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$
η_s	-1	-1	$+1$	-1

Table 1: Spin structures

3.1.1 $\mathcal{N} \geq 2$ sectors

In these cases $(-2\sin(\pi\gamma_3)) \prod_{j=1}^{2} f(\gamma_j)$ vanishes. $\mathcal{N} = 2$ sectors are characterized by that along exactly one i_{th}-torus, h_i vanishes and $\gamma_i + g_i$ is integer. $\mathcal{N} = 4$ sectors are characterized by that along all three torus, all three h_i vanish and all three $\gamma_i + g_i$ are integer. In these cases, (92)

[4] Gimon and Polchinski

σ	CP	$\tilde{\chi}$	γ_i	$f(\gamma_i)$ (i=1 or 2)	h_1	h_2	g_1	g_2
$\mathcal{K}_u$	1	1	$2\ell v_i$	$-2\sin(\pi\gamma_i)$	0	0	0	0
$\mathcal{K}_t$	1	$\tilde{\chi}(\theta^{N/2},\theta^\ell)$	$2\ell v_i$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\mathcal{A}_{99}$	$(tr\gamma_{\ell,9})^2$	1	ℓv_i	$-2\sin(\pi\gamma_i)$	0	0	0	0
$\mathcal{A}_{55}$	$(tr\gamma_{\ell,5})^2$	1	ℓv_i	$-2\sin(\pi\gamma_i)$	0	0	0	0
$\mathcal{A}_{95}$	$(tr\gamma_{\ell,9})(tr\gamma_{\ell,5})$	2	ℓv_i	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\mathcal{M}_9$	$tr\gamma_{2\ell,9}$	-1	ℓv_i	$-2\sin(\pi\gamma_i)$	0	0	0	0
$\mathcal{M}_5$	$tr\gamma_{2\ell,5}$	-1	ℓv_i	$2\cos(\pi\gamma_i)$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$

Table 2: Refer to [16]. $\mathcal{K}_u$ and $\mathcal{K}_t$ denote the Klein bottle contributions with untwisted and $\theta^{N/2}$-twisted closed strings running in the loop. $\tilde{\chi}(\theta^{N/2},\theta^\ell)$ denotes the number of simultaneous fixed points of $\theta^{N/2}$ and θ^ℓ. The CP factors corresponding to the D_5-branes assume that all D_5-branes are sitting at the fixed point at the origin of the compact transverse space, details cf. sec. 2.3 of [1]. Derivation of these constants in the table will be explained in the following subsections.

has a well defined limit $\frac{1}{\eta^2}$ of singular part, but one has to include internal momenta or windings, therefore we should substitute these singular part with momentum/winding lattice sum (81) and (82).

For $\mathcal{A}$ and $\mathcal{M}$ the momentum sum $\mathcal{L}^{[j,M]}$ appears if the j-th torus is parallel to the branes whereas the winding sum $\mathcal{L}^{[j,W]}$ appears if the j-th torus is transversal to the branes, and this actually is related to the boundary conditions of the open strings attached to the D-branes. For $\mathcal{K}$ the situation is as follows: If γ_j is even, the corresponding torus is not reflected. The orientation reversal Ω, however, reverses the winding modes. Thus only the momentum modes survive. On the other hand, if γ_j is odd, the corresponding torus is reflected (i.e. kv_j is half-integer). Combined with Ω, this leaves the winding modes along this torus invariant. The terms "momentum" and "winding" are used here referring to the open string channel.

3.2 Torus

Topologically Torus is the 1-loop closed string amplitude, without Orientifold symmetry Ω action.

This part is just the type-IIB orbifold thus is trivial and has no tadpole.

3.3 Klein bottle

Topologically Klein bottle is the 1-loop closed string amplitude, with Orientifold symmetry Ω action.

In the operator form, the amplitude of Klein bottle is

$$\Lambda_{\mathcal{K}} = \int_0^\infty \frac{dt}{2t} Tr^{U+T}\left[\frac{\Omega}{2}\cdot\frac{1}{N}\sum_{\ell=0}^{N-1}\theta^\ell\cdot\frac{1+(-1)^F}{2}e^{-2\pi(2it)(L_0-c/24)}\right] \tag{96}$$

Be aware that Ω can act on bosonic and fermionic oscillators as described in (67)-(72). Ω projects out NS-R and R-NS sectors. The action of Ω on the bosonic and fermionic oscillators results in a nonzero contribution in the trace only if the state has the same left and right oscillators. This effectively sets $L_0+\bar{L}_0\to 2L_0$ for such symmetric states and causes the final amplitude to have a modular parameter 2τ instead of τ.

Also, since Ω exchanges θ^k with θ^{N-k}, we only have twisted strings with $k=0$ and $k=\frac{N}{2}$, N even.

CP factors Since Klein bottle is not attached to D-branes, thus the CP factor is 1.

γ_i Due to the Ω action, $L_0 + \bar{L}_0 \to 2L_0$ will also double the γ_i. This can be easily seen from the calculation of (37).

3.3.1 Untwisted sector

$\tilde{\chi}$ and $f(\gamma_i)$ Since Ω action leaves only left-right symmetric states, from (37) we can see that we no longer have $4\sin^2(\pi\ell v_j)$ for $f(\gamma_i)$, but only have $-2\sin(2\pi\ell v_j)$.

Lattice sum cf. para. 2 in sec. 3.1.1

$$\gamma_i = \text{even-integer}, \quad i = 1,2,3: \quad \frac{-2\sin\pi\gamma_i}{\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2}+\gamma_i \end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[j,M]} \tag{97}$$

$$\gamma_i = \text{odd-integer}, \quad i = 1,2,3: \quad \frac{-2\sin\pi\gamma_i}{\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2}+\gamma_i \end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[j,W]} \tag{98}$$

3.3.2 Twisted sector

From the para. "Twisted Sectors" in sec. 2.3 we know that only $\frac{N}{2}$-twisted sector is allowed.

h_i $\mathcal{K}_t$ is $\theta^{N/2}$-twisted, thus kv_j = half integer. And this is equivalent to shifting the α of ϑ functions in the T^4 direction(1-st and 2-nd tori) by h_i, cf. (42).

$\tilde{\chi}$ and $f(\gamma_i)$ As we discussed after (42), here $\frac{N}{2}\cdot v_j$ is integer, thus we have $\tilde{\chi}(\theta^{N/2}, \theta^\ell)$ for $\tilde{\chi}$.

Lattice sum cf. para. 2 in sec. 3.1.1

$$\gamma_3 = \text{even-integer} \quad : \quad \frac{-2\sin\pi\gamma_3}{\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2}+\gamma_3 \end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[i,M]} \tag{99}$$

$$\gamma_3 = \text{odd-integer} \quad : \quad \frac{-2\sin\pi\gamma_3}{\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2}+\gamma_3 \end{bmatrix}} \to \frac{1}{\eta^3}\mathcal{L}^{[i,W]} \tag{100}$$

3.4 Annulus

Annulus surface represents closed string propagates between two D-branes, without Orientifold symmetry Ω action. Topologically and effectively we can consider it as the 1-loop open string amplitude, without Orientifold symmetry Ω action.

In the operator form, the amplitude is

$$\Lambda_{\mathcal{A}} = \int_0^\infty \frac{dt}{2t} Tr_{NS,R}^{99+55+95+59}\left[\frac{1}{2}\cdot\frac{1}{N}\sum_{\ell=0}^{N-1}\cdot\frac{1+(-1)^F}{2}e^{-2\pi(\frac{it}{2})(L_0-c/24)}\right] \tag{101}$$

Now we need to consider D-branes. According to earlier discussion about tadpole cancellation in section 2.4, we know that we would only consider D_9 and D_5-branes. Follow the discussion in section 2.4 and section 9.14.3 of [17], we have non-trivial CP factors in the partition function for Annulus.

Recall that open string boundary conditions on compactified dimensions have the results: NN directions have only momenta. DD only windings, and DN none of the above.

3.4.1 $\mathcal{A}_{99}$

CP factors $\mathcal{A}_{99}$ is attached to two D_9-branes. Therefore we have the CP factor as square of $tr\gamma_{9,k}$.

Lattice sum Here we have NN boundary conditions in the T^4 directions of $\mathcal{A}_{99}$, and also NN boundary conditions in the 3-rd torus. Then the compact directions have only momenta. And we need to substitute

$$\gamma_i = \text{integer},\ i=1,2,3: \quad \frac{-2\sin(\pi\gamma_i)\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i\end{bmatrix}} \to \frac{1}{\eta^2}\mathcal{L}^{[i,M]} \tag{102}$$

3.4.2 $\mathcal{A}_{55}$

CP factors $\mathcal{A}_{55}$ is attached to two D_5-branes. Therefore we have the CP factor as square of $tr\gamma_{5,k}$.

Lattice sum Here we have DD boundary conditions in the T^4 directions of $\mathcal{A}_{55}$, and NN boundary conditions in the 3-rd torus. Then the T^4 compact directions have only windings. And we need to substitute

$$\gamma_i = \text{integer},\ i=1,2: \quad \frac{-2\sin(\pi\gamma_i)\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i\end{bmatrix}} \to \frac{1}{\eta^2}\mathcal{L}^{[i,W]} \tag{103}$$

$$\gamma_3 = \text{integer}: \quad \frac{-2\sin(\pi\gamma_3)\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{bmatrix}} \to \frac{1}{\eta^2}\mathcal{L}^{[3,M]} \tag{104}$$

3.4.3 $\mathcal{A}_{95}$

CP factors $\mathcal{A}_{95}$ is attached to one D_5-brane and one D_9-brane. Therefore we have the CP factor as the product of $tr\gamma_{5,k}$ and $tr\gamma_{9,k}$.

h_i **and** $f(\gamma_i)$ $\mathcal{A}_{95}$ has Dirichlet-Neumann boundary conditions along 1-st and 2-nd torus. And the presence of 4 DN directions effectively $\mathbb{Z}_2$-twist the T^4 space(1-st and 2-nd torus), cf. sec. 13.4 in [19]. This is equivalent to the $\theta^{N/2}$-twisted sector in 3.3.2. Therefore we have the same h_i and $f(\gamma_i)$ as in 3.3.2.

$\tilde{\chi}$ $\mathcal{A}_{95}$ actually has two orientation, which are $\mathcal{A}_{95}$ and $\mathcal{A}_{59}$. Thus this contribute a factor of 2 to the partition function.

Lattice sum Here we have ND boundary conditions in the T^4 directions of $\mathcal{A}_{55}$, and NN boundary conditions in the 3-rd torus. Then the T^4 compact directions have no momentum or windings. And we need to substitute

$$\gamma_3 = \text{integer}: \quad \frac{-2\sin(\pi\gamma_3)\eta}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{bmatrix}} \to \frac{1}{\eta^2}\mathcal{L}^{[3,M]} \tag{105}$$

3.5 Möbius strip

Möbius strip surface represents closed string propagates between D-brane and orientifold plane, with Orientifold symmetry Ω action. Topologically and effectively we can consider it as the 1-loop open string amplitude, with Orientifold symmetry Ω action.

In the operator form, the amplitude is

$$\Lambda_{\mathcal{M}} = \int_0^\infty \frac{dt}{2t} Tr^{9+5}_{NS,R}\left[\frac{\Omega}{2}\cdot\frac{1}{N}\sum_{\ell=0}^{N-1}\cdot\frac{1+(-1)^F}{2}e^{-2\pi(\frac{1}{2}+\frac{it}{2})(L_0-c/24)}\right]. \tag{106}$$

Be aware that Ω in the $Tr[\Omega q^{L_0-c/24}]$ is equivalent to adding a minus sign to q because of the action of Ω on L_0, cf. (67)-(72). This is equivalent to substitute the torus parameter τ in the partition functions with the half-shifted torus parameter

$$\tau_M = \frac{1}{2} + \frac{it}{2}, \tag{107}$$

as we have mentioned before about world-sheet parameters, cf. (16).

Since Ω changes the orientation of the string, 9-5 strings do not contribute to the trace. For the same reason, only strings starting and ending on the same D_5-brane contribute after $\mathbb{Z}_2$ projection.

3.5.1 $\mathcal{M}_9$

Lattice sum Open strings on $\mathcal{M}_9$ has NN boundary condition, thus only K-K momentum states survive.

$$\gamma_i = \text{integer}, \quad i = 1,2,3: \quad \frac{-2\sin\pi\gamma_i}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i\end{smallmatrix}\right]} \to \frac{1}{\eta^3}\mathcal{L}^{[i,M]} \tag{108}$$

CP factors $\mathcal{M}_9$ is attached to D_9-branes, and it has Ω action, thus we have $CP_{\mathcal{M}_9} = tr(\gamma^{-1}_{\Omega_\ell,9}\gamma^T_{\Omega_\ell,9}) = tr\gamma_{2\ell,9}$, cf. (2.36) of [1].

$\tilde{\chi}$ Due to the Ω action on the fermionic state for NN boundary condition(cf. (71)) and the Ω action on the vacuum states(cf. (7.3.10) and (7.3.16) in [17]), we have $\Omega(\psi^\mu_{\frac{1}{2}}|0\rangle) \propto -\psi^\mu_{\frac{1}{2}}|0\rangle$, i.e. we have an extra minus sign in $\tilde{\chi}$, also cf. (3.11) and (3.12) [14].

3.5.2 $\mathcal{M}_5$

Lattice sum Open strings on $\mathcal{M}_5$ has DD boundary condition, thus only winding states survive.

$$\gamma_i = \text{half-integer}, \quad i = 1,2: \quad \frac{2\cos\pi\gamma_i}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_i+g_i\end{smallmatrix}\right]} \to \frac{(-1)^i}{\eta^3}\mathcal{L}^{[i,W]} \tag{109}$$

$$\gamma_3 = \text{integer}: \quad \frac{-2\sin\pi\gamma_3}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_3\end{smallmatrix}\right]} \to \frac{1}{\eta^3}\mathcal{L}^{[3,M]} \tag{110}$$

g_i Because now we have DD boundary conditions for T^4 directions, according to (72), the T^4 directions have an extra minus sign. This is equivalent to an insertion of $\theta^{N/2}$ element in the trace, and thus equivalent to shifting the β in ϑ functions in the T^4 direction(1-st and 2-nd tori) by g_i.

$f(\gamma_i)$ Due to the insertion of $\theta^{N/2}$, this will shift the $\sin(\pi\gamma_j)$ in $f(\gamma_j)$ for $\pi/2$, or equivalently shift γ_j to $\gamma_j + g_j$, and thus turns $-\sin$ into cos function for each of the 1-st and 2-nd tori.

CP factors and $\tilde{\chi}$ $\mathcal{M}_5$ is attached to D_5-branes, and it has Ω action, thus we have $CP_{\mathcal{M}_5} = tr(\gamma^{-1}_{\Omega_\ell,5}\gamma^T_{\Omega_\ell,5}) = -tr\gamma_{2\ell,5}$, cf. (2.41) of [1]. But here we take $CP_{\mathcal{M}_5} = tr\gamma_{2\ell,5}$, thus we move the minus sign to $\tilde{\chi}$, which means we get $\tilde{\chi} = -1$.

4 One-loop corrections to Einstein-Hilbert term

This section follows very closely to [16].

4.1 Effective field theory

In this section we review how the quantum corrections to the Einstein-Hilbert term influence the form of the low energy effective action of string compactifications. cf. sec. 2 in [4].

The quantum corrected kinetic term of tree level modulus $\tau^{(0)}$ coupled to gravity in string frame and up to 1-loop order is given by

$$S_4 = \frac{1}{\kappa_4^2}\int d^4x\sqrt{-g}\Big[(e^{-2\Phi_4}+\delta E)\frac{1}{2}R + \Big(\tilde{G}^{(0)}+\tilde{G}^{(1)}\Big)\partial_\mu\tau^{(0)}\partial^\mu\tau^{(0)}\Big] + \dots, \tag{111}$$

where δE denotes the correction to the Einstein-Hilbert term, including tree level α' corrections and corrections from 1-loop. $\tilde{G}^{(0)}$ stands for the tree level metric including α' corrections and $\tilde{G}^{(1)}$ stands for the contributions to the string frame metric arising at 1-loop level. Here we choose τ as an example for concreteness. Furthermore,

$$\kappa_4^{-2} = (2\pi\sqrt{\alpha'})^6\kappa_{10}^{-2} = (\pi\alpha')^{-1} \tag{112}$$

and

$$e^{-2\Phi_4} \equiv e^{-2\Phi_{10}}t_1t_2t_3 = \sqrt{\sigma^{(0)}\tau_1^{(0)}\tau_2^{(0)}\tau_3^{(0)}}, \tag{113}$$

where $e^{-2\Phi_{10}}$ is the ten dimensional dilaton and

$$\sigma^{(0)} = e^{-\Phi_{10}}t_1t_2t_3 \qquad , \qquad \tau_i^{(0)} = e^{-\Phi_{10}}t_i. \tag{114}$$

Here the t_i are the dimensionless torus volumes measured with the string frame metric. The definition of the Kähler variables in general gets quantum corrected

$$\tau = \tau^{(0)} + \delta\tau, \tag{115}$$

where $\delta\tau$ is a moduli dependent function.

Starting from (111) and performing a Weyl transformation to go to the Einstein frame, we see that the quantum correction to the metric of the quantum corrected Kähler modulus T (with imaginary part τ), is given, up to 1-loop order, by

$$\begin{aligned} G^{(1)}_{T\bar{T}}(T) =& e^{2\Phi_4}\tilde{G}^{(1)}(\tau) + 12\left(\frac{\partial\Phi_4}{\partial\tau^{(0)}}\right)^2\delta E e^{2\Phi_4} + 6\frac{\partial\Phi_4}{\partial\tau^{(0)}}\frac{\partial\delta E}{\partial\tau^{(0)}}e^{2\Phi_4} \\ &- \delta E e^{4\Phi_4}\tilde{G}^{(0)}(\tau) + \frac{1}{2\tau^3}\delta\tau - \frac{1}{2\tau^2}\frac{\partial\delta\tau}{\partial\tau} + \dots . \end{aligned} \tag{116}$$

We can see that δE showed up in different terms. Therefore we can conclude that δE does play an important role to the quantum correction to the metric.

4.2 General Analysis of Graviton 1-loop 2-point function

In this section we derive some general formulas needed for computing the 1-loop correction to the Planck mass in $\mathcal{N}=1$ type-IIB toroidal orientifolds. And these will be applied to tadpole-free $\mathbb{Z}_6$, $\mathbb{Z}_7$ and $\mathbb{Z}_{12}$ models (tadpole-free condition is discussed in [1]). Here we follow closely to the sec. 3 in [16].

Begin with the amplitude of two gravitons(with momenta p_i and polarization tensors ε_i)

$$\langle V_g(p_1,\varepsilon_1)V_g(p_2,\varepsilon_2)\rangle = \sum_{\sigma\in\{\mathcal{T},\mathcal{K},\mathcal{A},\mathcal{M}\}}\langle V_g(p_1,\varepsilon_1)V_g(p_2,\varepsilon_2)\rangle_\sigma, \tag{117}$$

where the vertex operators are given by

$$V_g(p,\varepsilon) = -\frac{2g_c}{\alpha'}\varepsilon_{\mu\nu}\Big(i\partial X^\mu + \frac{\alpha'}{2}p\cdot\psi\psi^\mu\Big)\Big(i\bar\partial X^\nu + \frac{\alpha'}{2}p\cdot\tilde\psi\tilde\psi^\nu\Big)e^{ip\cdot X} \tag{118}$$

with $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = 1$. Using the on-shell, transversality and tracelessness conditions

$$p_1^2 = p_2^2 = p_1\cdot p_2 = p_{1\mu}\varepsilon_1^{\mu\nu} = p_{2\mu}\varepsilon_2^{\mu\nu} = \eta_{\mu\nu}\varepsilon_1^{\mu\nu} = \eta_{\mu\nu}\varepsilon_2^{\mu\nu} = 0, \tag{119}$$

the amplitude (117) has to be proportional to the only remaining contraction, i.e.

$$\langle V_g(p_1,\varepsilon_1)V_g(p_2,\varepsilon_2)\rangle = AiV_4 g_c^2 p_2^\mu \varepsilon_{1\mu\nu}\eta^{\nu\lambda}\varepsilon_{2\lambda\rho}p_1^\rho + \mathcal{O}(p^4). \tag{120}$$

We have to compare this to the relevant term in the action which leads to the linearized Einstein equations. We read off

$$S = \frac{M_P^2}{2}\int d^4x\Big(-\frac{1}{2}h_{\mu\nu,\rho}h^{\nu\rho,\mu}\Big), \tag{121}$$

where

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{122}$$

for a symmetric fluctuation $h_{\mu\nu}$. $h_{\mu\nu}$ and $\varepsilon_{\mu\nu}$ have the relation in momentum space showed by the vertex operator (118)

$$h_{\mu\nu} = -4\pi g_c \varepsilon_{\mu\nu}e^{ip\cdot X}. \tag{123}$$

Using (111), we have

$$M_P^2 = \frac{1}{\kappa_4^2}(e^{-2\Phi_4} + \delta E). \tag{124}$$

Thus we compare (120) with

$$-\frac{1}{4}\kappa_4^2\int d^4x\,\delta E\, h_{\mu\nu,\rho}h^{\nu\rho,\mu}. \tag{125}$$

And we get

$$\delta E = \frac{\kappa_4^2}{8\pi^2}A = \frac{\alpha'}{8\pi}A. \tag{126}$$

The amplitude A gets contributions from all 1-loop surfaces, i.e. $\mathcal{T}$, $\mathcal{K}$, $\mathcal{A}$, $\mathcal{M}$.

4.3 Torus and Sphere contribution

We read off the torus contribution from eq. (5.3) in [3]. Including also the α' correction to the Planck mass from the sphere it gives

$$(\delta E)_{S_2+\mathcal{T}} = \frac{\chi}{(2\pi)^3}\Big(2\zeta(3)\frac{e^{-2\Phi_4}}{\mathcal{V}} + \frac{\pi^2}{3}\Big), \tag{127}$$

where $\mathcal{V}$ is the overall volume (in units of $(2\pi\sqrt{\alpha'})^6$) and, due to the orientifold projection, we added a factor of 1/2.

4.4 Contributions from $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$

Here we closely follow the calculation in [12]. Neglecting the momentum conservation δ function arising from the bosonic zero mode integration we have

$$\begin{aligned} A_\sigma = -\frac{1}{8N(4\pi^2\alpha')^2} \sum_{s=\text{even}} \int_0^\infty \frac{dt}{t^3} \sum_{k=0}^{N-1} Z_\sigma^{(\ell)}(\tau_\sigma, s) \int_\sigma d^2\nu_1 \int_\sigma d^2\nu_2 \\ \Big(\langle \bar\partial X_1 \bar\partial X_2 \rangle_\sigma (\langle \psi_2 \psi_1 \rangle_\sigma^s)^2 + \langle \partial X_1 \bar\partial X_2 \rangle_\sigma (\langle \psi_2 \tilde\psi_1 \rangle_\sigma^s)^2 \\ + \langle \bar\partial X_1 \partial X_2 \rangle_\sigma (\langle \tilde\psi_2 \psi_1 \rangle_\sigma^s)^2 + \langle \partial X_1 \partial X_2 \rangle_\sigma (\langle \tilde\psi_2 \tilde\psi_1 \rangle_\sigma^s)^2 \Big) \end{aligned} \tag{128}$$

where σ stands for the different world-sheet topologies $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$, with world-sheet parameters $\tau_\mathcal{K} = 2it$, $\tau_\mathcal{A} = \frac{it}{2}$, $\tau_\mathcal{M} = \frac{1}{2} + \frac{it}{2}$. $Z_\sigma^{(\ell)}(\tau_\sigma, s)$ is the contribution (92) to the partition function from the θ^ℓ element inserted sector. The spin structure sum only runs over the even spin structures s. Note that there is no contribution to A_σ from eight fermion terms, cf. sec. 3.4 in [4].

From [12], we use

$$(\langle \psi_2(\nu)\psi_1(0)\rangle_\sigma^s)^2 = -\partial_\nu^2 \ln \vartheta_1(\nu,\tau) + \partial_v^2 \frac{\vartheta_s(v,\tau)}{\vartheta_s(0,\tau)}\bigg|_{v=0}. \tag{129}$$

It is the sum of a spin structure independent term with a spin structure dependent term. The contribution to A_σ involving the first term in (129) (the spin structure independent term) does not survive the sum over spin structures in the super-symmetric case. On the other hand, the spin structure dependent term does not depend on the vertex operator position and, thus can be taken out of the ν integrals. Besides, provided that it does depend on the vertex operator position, this is the same for $(\langle\psi_2\psi_1\rangle_\sigma^s)^2$, $(\langle\psi_2\tilde\psi_1\rangle_\sigma^s)^2$, $(\langle\tilde\psi_2\psi_1\rangle_\sigma^s)^2$ and $(\langle\tilde\psi_2\tilde\psi_1\rangle_\sigma^s)^2$. Take care of the relative minus signs arising from conventions, the resulting ν integral can be solved using [2]

$$\int_\sigma d^2\nu_1 \int_\sigma d^2\nu_2 \Big(\langle \bar\partial X_1 \bar\partial X_2 \rangle_\sigma - \langle \partial X_1 \bar\partial X_2 \rangle_\sigma - \langle \bar\partial X_1 \partial X_2 \rangle_\sigma + \langle \partial X_1 \partial X_2 \rangle_\sigma \Big) = \frac{\alpha'\pi\Im(\tau_\sigma)}{2}. \tag{130}$$

Taking into account (126), we finally achieve

$$\begin{aligned} (\delta E)_\sigma &= -\frac{\alpha'}{8\pi}\frac{1}{8N(4\pi^2\alpha')^2}\partial_v^2 \sum_{s=\text{even}} \int_0^\infty \frac{dt}{t^3} \sum_{\ell=0}^{N-1} Z_\sigma^{(\ell)}(\tau_\sigma,s) \frac{\vartheta_s(v,\tau_\sigma)}{\vartheta_s(0,\tau_\sigma)} \frac{\alpha'\pi\Im(\tau_\sigma)}{2}\bigg|_{v=0} \\ &= -\frac{(\alpha')^2}{8\pi}\frac{1}{8N(4\pi^2\alpha')^2} \int_0^\infty \frac{dt}{t^3}\frac{\pi\Im(\tau_\sigma)}{2} \sum_{\ell=0}^{N-1} \partial_v^2 \sum_{s=\text{even}} Z_\sigma^{(\ell)}(\tau_\sigma,s)\frac{\vartheta_s(v,\tau_\sigma)}{\vartheta_s(0,\tau_\sigma)}\bigg|_{v=0} \\ &= -\frac{(\alpha')^2}{8\pi}\frac{1}{8N(4\pi^2\alpha')^2} \int_0^\infty \frac{dt}{t^3}\frac{\pi\Im(\tau_\sigma)}{2} \sum_{\ell=0}^{N-1} \sum_{s=\text{even}} Z_\sigma^{(\ell)}(\tau_\sigma,s)\frac{\vartheta_s''(0,\tau_\sigma)}{\vartheta_s(0,\tau_\sigma)}. \end{aligned} \tag{131}$$

The lattice sums can be done after performing the spin-structure summation. Thus the sum over spin-structure in (131) can be performed using (92) and (93) for the partition function. Then we need the formula (cf. eq.(130) in [5])

$$\sum_{s=\text{even}} Z_s^{(\ell)} \frac{\vartheta_s''(0)}{\vartheta_s(0)} = \sum_{i=1}^{3} \frac{\vartheta'\begin{bmatrix} \frac{1}{2}+h_i \\ \frac{1}{2}+\gamma_i+g_i \end{bmatrix}(0)}{\vartheta\begin{bmatrix} \frac{1}{2}+h_i \\ \frac{1}{2}+\gamma_i+g_i \end{bmatrix}(0)}. \tag{132}$$

With this, (131) reads

$$(\delta E)_\sigma = -\frac{\pi(\alpha')^2}{32N(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^2}\frac{\Im(\tau_\sigma)}{t}\sum_{\ell=0}^{N-1} CP_\sigma\tilde{\chi}_\sigma \sin(\pi\gamma_3) \cdot\left(\prod_{j=1}^{2} f(\gamma_j)\right)\sum_{i=1}^{3}\frac{\vartheta'\begin{bmatrix}\frac{1}{2}+h_i\\ \frac{1}{2}+\gamma_i+g_i\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}+h_i\\ \frac{1}{2}+\gamma_i+g_i\end{bmatrix}(0)}. \tag{133}$$

4.5 $\mathcal{N}=1$ sectors

Following secs. 3.8-3.11 of [4], $\mathcal{N}=1$ sectors contribution to the Planck mass is

$$(\delta E)^{(\mathcal{N}=1)} = \sum_\sigma (\delta E)_\sigma^{(\mathcal{N}=1)} = -\frac{\pi(\alpha')^2}{64N(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^2}\sum_\sigma \sum_{\ell\in\{\mathcal{N}=1\}} CP_\sigma \sigma^{(\ell)}. \tag{134}$$

Here

$$\sigma^{(\ell)} = \tilde{e}_\sigma \tilde{\chi}_\sigma \sin(\pi\gamma_3)\left(\prod_{j=1}^2 f(\gamma_j)\right)\hat{\sigma}^{(\ell)} \qquad \text{for } \ell\in\{\mathcal{N}=1\} \tag{135}$$

with

$$\tilde{e}_\sigma = \begin{cases} 1 & \text{for } \mathcal{A},\mathcal{M} \\ 4 & \text{for } \mathcal{K} \end{cases} \tag{136}$$

and

$$\hat{\sigma}^{(\ell)} = \sum_{i=1}^3 \frac{\vartheta'\begin{bmatrix} \frac{1}{2}+h_i \\ \frac{1}{2}+\gamma_i+g_i \end{bmatrix}(0)}{\vartheta\begin{bmatrix} \frac{1}{2}+h_i \\ \frac{1}{2}+\gamma_i+g_i \end{bmatrix}(0)}. \tag{137}$$

For later use, we also introduce

$$e_\sigma = \begin{cases} 1 & \text{for } \mathcal{A} \\ 4 & \text{for } \mathcal{M},\mathcal{K} \end{cases}, \tag{138}$$

From (136)-(138) and table 2, we have

$$\begin{aligned}
\mathcal{K}_u^{(\ell)} &= 16\sin(2\pi\ell v_3)\sin(2\pi\ell v_1)\sin(2\pi\ell v_2)\hat{\mathcal{K}}_u^{(\ell)}, \\
\mathcal{K}_t^{(\ell)} &= 4\tilde{\chi}(\theta^{N/2},\theta^\ell)\sin(2\pi\ell v_3)\hat{\mathcal{K}}_t^{(\ell)}, \\
\mathcal{A}_{99}^{(\ell)} &= 4\sin(\pi\ell v_3)\sin(\pi\ell v_1)\sin(\pi\ell v_2)\hat{\mathcal{A}}_{99}^{(\ell)}, \\
\mathcal{A}_{55}^{(\ell)} &= 4\sin(\pi\ell v_3)\sin(\pi\ell v_1)\sin(\pi\ell v_2)\hat{\mathcal{A}}_{55}^{(\ell)}, \\
\mathcal{A}_{95}^{(\ell)} &= 2\sin(\pi\ell v_3)\hat{\mathcal{A}}_{95}^{(\ell)}, \\
\mathcal{M}_9^{(\ell)} &= -4\sin(\pi\ell v_3)\sin(\pi\ell v_1)\sin(\pi\ell v_2)\hat{\mathcal{M}}_9^{(\ell)}, \\
\mathcal{M}_5^{(\ell)} &= -4\sin(\pi\ell v_3)\cos(\pi\ell v_1)\cos(\pi\ell v_2)\hat{\mathcal{M}}_5^{(\ell)}.
\end{aligned} \tag{139}$$

Note that for odd N there is no contribution from $\mathcal{K}_t$, $\mathcal{A}_{55}$, $\mathcal{A}_{95}$ and $\mathcal{M}_5$.

Making use of (202) and the fact that the even/odd spin structure ϑ functions are even/odd functions of their argument, together with the super-symmetry condition $\sum_i v_i = 0$, we can get

$$\begin{aligned}
\hat{\sigma}^{(qN\pm\ell)} &= \pm\hat{\sigma}^{(\ell)} \qquad \text{for all } \sigma, \qquad \hat{\sigma}^{(\frac{qN}{2}\pm\ell)} = \pm\hat{\sigma}^{(\ell)} \qquad \text{for } \mathcal{K}, \\
\sigma^{(qN\pm\ell)} &= \sigma^{(\ell)} \qquad \text{for all } \sigma, \qquad \sigma^{(\frac{qN}{2}\pm\ell)} = \sigma^{(\ell)} \qquad \text{for } \mathcal{K}.
\end{aligned} \tag{140}$$

q is an arbitrary integer. These identities allow the individual sectors to be related to each other.

For $\mathcal{N}=1$ sectors with $h_i=0$, the t-integral in (134) can be performed using (115)-(117) of [4], i.e.(assuming $0<\gamma<1$ for $\mathcal{A}$ and $\mathcal{K}$, and $0<\gamma<1/2$ for $\mathcal{M}$)

$$\begin{aligned}
I_{\mathcal{A}/\mathcal{K}}(\gamma) &= \int_{\frac{1}{e_\sigma\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_1'(\gamma,\tau_\sigma)}{\vartheta_1(\gamma,\tau_\sigma)} \\
&= e_\sigma\pi(1-2\gamma)\Lambda^2 + e_\sigma\frac{\pi}{24}[\psi'(\gamma)-\psi'(1-\gamma)],
\end{aligned} \tag{141}$$

$$\begin{aligned}
I_{\mathcal{M}}(\gamma) &= \int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_1'(\gamma,\frac{1}{2}+\frac{it}{2})}{\vartheta_1(\gamma,\frac{1}{2}+\frac{it}{2})} \\
&= 8\pi(1-4\gamma)\Lambda^2 + \frac{\pi}{12}[\psi'(\gamma)-\psi'(1-\gamma)-\frac{1}{2}\psi'(\frac{1}{2}+\gamma)+\frac{1}{2}\psi'(\frac{1}{2}-\gamma)].
\end{aligned} \tag{142}$$

Here $\psi'(x)$ denotes the trigamma function, i.e. the derivative of the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$.

The t-integral of terms with $h_i = \pm 1/2$, appearing in $\mathcal{K}_t$ and $\mathcal{A}_{95}$, is computed in app.B.1 where we find (for $0 < \gamma < 1$)

$$\tilde{I}_{\mathcal{A}/\mathcal{K}}(\gamma) = \int_{\frac{1}{e_\sigma \Lambda}}^{\infty} \frac{dt}{t^2} \frac{\vartheta_4'(\gamma, \tau_\sigma)}{\vartheta_4(\gamma, \tau_\sigma)} = e_\sigma \pi (1 - 2\gamma)\Lambda^2 - e_\sigma \frac{\pi}{48}[\psi'(\gamma) - \psi'(1-\gamma)]. \tag{143}$$

Furthermore, the t-integral for $\mathcal{M}$ when $\gamma > \frac{1}{2}$ is computed in app.B.3 where we find (for $\frac{1}{2} < \gamma < 1$)

$$\begin{aligned} \tilde{I}_{\mathcal{M}}(\gamma) &= \int_{\frac{1}{4\Lambda}}^{\infty} \frac{dt}{t^2} \frac{\vartheta_1'(\gamma, \frac{1}{2} + \frac{it}{2})}{\vartheta_1(\gamma, \frac{1}{2} + \frac{it}{2})} \\ &= 8\pi(3 - 4\gamma)\Lambda^2 - \frac{\pi}{24}\Big[\psi'(\gamma - \frac{1}{2}) - \psi'(\frac{3}{2} - \gamma) + 2\psi'(1-\gamma) - 2\psi'(\gamma)\Big]. \end{aligned} \tag{144}$$

4.6 $\mathcal{N} \geq 2$ sectors

$\mathcal{N} = 2$ sectors are characterized by the fact that along exactly one torus (say the n-th torus) h_n vanishes and $\gamma_n + g_n$ is integer. Thus we need to take the limit of (133)

$$(-2\sin\pi(\gamma_n + g_n))\frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_n+g_n\end{bmatrix}(0)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_n+g_n\end{bmatrix}(0)} \to \frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\gamma_n+g_n\end{bmatrix}(0)}{\eta^3}\mathcal{L}^{[n,M/W]}$$
$$= (-2\pi)(-1)^{\gamma_n+g_n}\mathcal{L}^{[n,M/W]}. \tag{145}$$

To summarize, the $\mathcal{N} = 2$ sector contribution is given by

$$(\delta E)^{(\mathcal{N}=2)} = \sum_\sigma (\delta E)^{(\mathcal{N}=2)}_\sigma = -\frac{\pi(\alpha')^2}{64N(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^2}\sum_\sigma \sum_{\ell\in\{\mathcal{N}=2\}} CP_\sigma \sigma^{(\ell)}. \tag{146}$$

Here

$$\sigma^{(\ell)} = \pi\tilde{e}_\sigma\tilde{\chi}_\sigma D^{(\ell)}_\sigma \mathcal{L}^{[n,M/W]} \quad \text{for } k \in \{\mathcal{N}=2\}, \tag{147}$$

and the constant factor $D^{(\ell)}_\sigma$ is given by

$$D^{(\ell)}_\sigma = (-1)^{\gamma_n+g_n}\prod_{i\neq n}^{3} f(\gamma_i) \tag{148}$$

with $f(\gamma_3) = -2\sin\pi\gamma_3$. n depends on ℓ and σ.

Let us express (81) and (82) collectively as

$$\mathcal{L}^{[n,M/W]} = \frac{C^{[n,M/W]}}{t}\sum_{m^1,m^2} e^{-\frac{\pi}{t}m^a m^b g^{[n,M/W]}_{ab}}, \tag{149}$$

where

$$C^{[n,M/W]} = \begin{cases}\frac{V_n}{4\pi^2\alpha'} & \text{for M (momentum sum)}\\ \frac{4\pi^2\alpha'}{V_n} & \text{for W (winding sum)}\end{cases} \tag{150}$$

and

$$g^{[n,M/W]}_{ab} = \begin{cases} g^{[n]}_{ab} & \text{for M (momentum sum)}\\ g^{[n]ab} & \text{for W (winding sum)}\end{cases}, \tag{151}$$

i.e. $g^{[n,W]}_{ab}$ is the inverse matrix of $g^{[n,M]}_{ab}$.

Now we split $\mathcal{L}^{[n,M/W]}$ as

$$\mathcal{L}^{[n,M/W]} = \frac{C^{[n,M/W]}}{t}\left(1 + \sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} e^{-\frac{\pi}{t}m^a m^b g^{[n,M/W]}_{ab}}\right)$$
$$= \frac{C^{[n,M/W]}}{t} + \mathcal{L}'^{[n,M/W]} \tag{152}$$

with

$$\mathcal{L}'^{[n,M/W]} = \frac{C^{[n,M/W]}}{t}\sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} e^{-\frac{\pi}{t}m^a m^b g^{[n,M/W]}_{ab}}. \tag{153}$$

Then we have

$$\int_{\frac{1}{e_\sigma\Lambda}}^\infty \frac{dt}{t^2}\mathcal{L}^{[n,M/W]} = \frac{C^{[n,M/W]}e_\sigma^2\Lambda^2}{2} + \int_0^\infty \frac{dt}{t^2}\mathcal{L}'^{[n,M/W]}. \tag{154}$$

Here we set $\Lambda = \infty$ in the second term on the right hand side since it is finite in the limit $\Lambda = \infty$. It can be evaluated using (see app. B.2)

$$\Gamma^{[n,M/W]} \equiv \int_0^\infty \frac{dt}{t^3} \sum_{\vec{m}\in\mathbb{Z}^2\setminus\vec{0}} e^{-\frac{\pi}{t} m^a m^b g_{ab}^{[n,M/W]}}$$
$$= \begin{cases} \frac{(4\pi^2\alpha')^2}{\pi^2 V_n^2} E_2(U^{[n]}) & \text{for M (momentum sum)} \\ \frac{V_n^2}{\pi^2(4\pi^2\alpha')^2} E_2\Big(-\frac{1}{U^{[n]}}\Big) & \text{for W (winding sum)} \end{cases}, \tag{155}$$

where $U^{[n]}$ is the complex structure of the n-th torus and E_2 is a non-holomorphic Eisenstein series, cf. (216).

For $\mathcal{N} = 4$ sectors h_i vanish and $\gamma_i + g_i$ are integer along all three tori. Thus, the numerator of (133) has a triple zero which can not be balanced by the simple zero in the denominator. Consequently the $\mathcal{N} = 4$ sectors do not contribute.

Now we collected all the relevant formulas to evaluate the 1-loop correction to the Planck mass in explicit models. We will do the calculation for 3 specific models in the next section.

5 Examples

Here we use the techniques and methods from above sections to calculate 3 specific models $\mathbb{Z}_6$, $\mathbb{Z}_7$ and $\mathbb{Z}_{12}$.

5.1 $\mathbb{Z}_6$

The twist vector[5] of $\mathbb{Z}_6$ is $v = \frac{1}{6}(1, 1, -2)$. Since the torus lattice has to be invariant under the orbifold action, the complex structures of all the three tori are fixed. The model has both D_9 and D_5-branes wrapped around the third torus. For simplicity we assume that all the D_5-branes are sitting at the fixed point at the origin of the compact transverse space as in [16]. In the table 3 we present the volume factors of different $\mathcal{N}$ sectors of the model.

5.1.1 $\mathcal{N} = 1$ sectors

The $\mathcal{N} = 1$ sector sum is

$$\sum_{\sigma} \sum_{\ell \in \{\mathcal{N}=1\}} CP_\sigma \sigma^{(\ell)} = \sum_{\ell=1,2,4,5} \Big(\mathcal{K}_u^{(\ell)} + \mathcal{K}_t^{(\ell)} + (tr\gamma_9^\ell)^2 \mathcal{A}_{99}^{(\ell)} + (tr\gamma_5^\ell)^2 \mathcal{A}_{55}^{(\ell)} + (tr\gamma_9^\ell)(tr\gamma_5^\ell)\mathcal{A}_{95}^{(\ell)} + (tr\gamma_9^{2\ell})\mathcal{M}_9^{(\ell)} + (tr\gamma_5^{2\ell})\mathcal{M}_5^{(\ell)} \Big) \tag{156}$$

Using (139), (140), Chan-Paton traces [1]

$$\begin{aligned} tr\gamma_9^\ell &= tr\gamma_5^\ell = 0; \quad \ell = 1, 3, 5, \\ tr\gamma_9^2 &= tr\gamma_5^2 = 4, \\ tr\gamma_9^4 &= tr\gamma_5^4 = -4, \\ \gamma_9^6 &= \gamma_5^6 = -\mathbb{1}, \\ tr\gamma_9^6 &= tr\gamma_5^6 = -32, \\ tr\gamma_9^0 &= tr\gamma_5^0 = 32 \end{aligned} \tag{157}$$

and

$$\tilde{\chi}(\theta^3, \theta^1) = \tilde{\chi}(\theta^3, \theta^2) = \tilde{\chi}(\theta^3, \theta^4) = \tilde{\chi}(\theta^3, \theta^5) = 1, \tag{158}$$

$$\tilde{\chi}(\theta^3, \theta^0) = \tilde{\chi}(\theta^3, \theta^3) = 16, \tag{159}$$

[5] refers to app.C

we obtain

$$
\begin{aligned}
&\sum_{\sigma^{(\ell)}\in\{\mathcal{N}=1\}} CP_\sigma \sigma^{(\ell)} = \\
&= \sum_{\ell=1,2,4,5} \Big(\mathcal{K}_u^{(\ell)} + \mathcal{K}_t^{(\ell)}\Big) + 16\Big(\mathcal{A}_{99}^{(2)} + \mathcal{A}_{55}^{(2)} + \mathcal{A}_{95}^{(2)}\Big) \\
&\quad + 16\Big(\mathcal{A}_{99}^{(4)} + \mathcal{A}_{55}^{(4)} + \mathcal{A}_{95}^{(4)}\Big) \\
&\quad + 4\Big(\mathcal{M}_9^{(1)} + \mathcal{M}_5^{(1)} - \mathcal{M}_9^{(2)} - \mathcal{M}_5^{(2)} - \mathcal{M}_9^{(4)} - \mathcal{M}_5^{(4)} + \mathcal{M}_9^{(5)} + \mathcal{M}_5^{(5)}\Big) \\
&= 4\Big(\mathcal{K}_u^{(1)} + \mathcal{K}_t^{(1)}\Big) + 32\Big(\mathcal{A}_{99}^{(2)} + \mathcal{A}_{55}^{(2)} + \mathcal{A}_{95}^{(2)}\Big) \\
&\quad + 8\Big(\mathcal{M}_9^{(1)} + \mathcal{M}_5^{(1)}\Big) - 8\Big(\mathcal{M}_9^{(2)} + \mathcal{M}_5^{(2)}\Big) \\
&= 4\Big(16\prod_{j=1}^{3}\sin(2\pi v_j)\hat{\mathcal{K}}_u^{(1)} + 4\sin(2\pi v_3)\hat{\mathcal{K}}_t^{(1)}\Big) \\
&\quad + 32\Big(4\prod_{j=1}^{3}\sin(2\pi v_j)\hat{\mathcal{A}}_{99}^{(2)} + 4\prod_{j=1}^{3}\sin(2\pi v_j)\hat{\mathcal{A}}_{55}^{(2)} + 2\sin(2\pi v_3)\hat{\mathcal{A}}_{95}^{(2)}\Big) \\
&\quad + 8\Big(-4\prod_{j=1}^{3}\sin(\pi v_j)\hat{\mathcal{M}}_9^{(1)} - 4\sin(\pi v_3)\cos(\pi v_1)\cos(\pi v_2)\hat{\mathcal{M}}_5^{(1)}\Big) \\
&\quad - 8\Big(-4\prod_{j=1}^{3}\sin(2\pi v_j)\hat{\mathcal{M}}_9^{(2)} - 4\sin(2\pi v_3)\cos(2\pi v_1)\cos(2\pi v_2)\hat{\mathcal{M}}_5^{(2)}\Big) \\
&= -24\sqrt{3}\hat{\mathcal{K}}_u^{(1)} - 8\sqrt{3}\hat{\mathcal{K}}_t^{(1)} - 48\sqrt{3}\hat{\mathcal{A}}_{99}^{(2)} - 48\sqrt{3}\hat{\mathcal{A}}_{55}^{(2)} - 32\sqrt{3}\hat{\mathcal{A}}_{95}^{(2)} \\
&\quad + 4\sqrt{3}\hat{\mathcal{M}}_9^{(1)} + 12\sqrt{3}\hat{\mathcal{M}}_5^{(1)} - 12\sqrt{3}\hat{\mathcal{M}}_9^{(2)} - 4\sqrt{3}\hat{\mathcal{M}}_5^{(2)}
\end{aligned} \tag{160}
$$

Then we do the t-integral using (141)-(144)

$$
\begin{aligned}
&\int_0^\infty \frac{dt}{t^2}\Big[-24\hat{\mathcal{K}}_u^{(1)} - 8\hat{\mathcal{K}}_t^{(1)} - 48\hat{\mathcal{A}}_{99}^{(2)} - 48\hat{\mathcal{A}}_{55}^{(2)} - 32\hat{\mathcal{A}}_{95}^{(2)} \\
&\qquad + 4\hat{\mathcal{M}}_9^{(1)} + 12\hat{\mathcal{M}}_5^{(1)} - 12\hat{\mathcal{M}}_9^{(2)} - 4\hat{\mathcal{M}}_5^{(2)}\Big] = \\
&= -24\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{K}}_u^{(1)} - 8\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{K}}_t^{(1)} - 48\int_{\frac{1}{\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{A}}_{99}^{(2)} - 48\int_{\frac{1}{\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{A}}_{55}^{(2)} - 32\int_{\frac{1}{\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{A}}_{95}^{(2)} \\
&\quad + 4\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{M}}_9^{(1)} + 12\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{M}}_5^{(1)} - 12\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{M}}_9^{(2)} - 4\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{M}}_5^{(2)} \\
&= -24\cdot 3I_{\mathcal{K}}(\frac{1}{3}) - 8\big[2\cdot \tilde{I}_{\mathcal{K}}(\frac{1}{3}) + I_{\mathcal{K}}(\frac{1}{3})\big] - 2\cdot 48\cdot 3I_{\mathcal{A}}(\frac{1}{3}) - 32\big[2\cdot\tilde{I}_{\mathcal{A}}(\frac{1}{3}) + I_{\mathcal{A}}(\frac{1}{3})\big] \\
&\quad + 4\big[2\cdot I_{\mathcal{M}}(\frac{1}{6}) + \tilde{I}_{\mathcal{M}}(\frac{2}{3})\big] + 12\big[3\cdot\tilde{I}_{\mathcal{M}}(\frac{2}{3})\big] - 12\big[3\cdot I_{\mathcal{M}}(\frac{1}{3})\big] - 4\big[2\cdot\tilde{I}_{\mathcal{M}}(\frac{5}{6}) + I_{\mathcal{M}}(\frac{1}{3})\big] \\
&= \pi\Lambda^2\bigg(-24\cdot 4(1-2\cdot\frac{1}{3})\cdot 3 - 8\cdot 4(1-2\cdot\frac{1}{3})\cdot 3 - 2\cdot 48(1-2\cdot\frac{1}{3})\cdot 3 - 32(1-2\cdot\frac{1}{3})\cdot 3 \\
&\quad + 4\cdot 8\Big[(1-4\cdot\frac{1}{6})\cdot 2 + (3-4\cdot\frac{2}{3})\Big] + 12\cdot 8\Big[3\cdot(3-4\cdot\frac{2}{3})\Big] \\
&\quad - 12\cdot 8(1-4\cdot\frac{1}{3})\cdot 3 - 4\cdot 8\Big[(1-4\cdot\frac{1}{3})\cdot 2 + (1-4\cdot\frac{1}{3})\Big]\bigg) \\
&\quad - 72\cdot\frac{\pi}{6}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})] - 8\Big[-2\cdot\frac{\pi}{12}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})] + \frac{\pi}{6}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})]\Big] \\
&\quad - 288\cdot\frac{\pi}{24}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})] - 32\Big[-2\cdot\frac{\pi}{48}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})] + \frac{\pi}{24}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})]\Big] \\
&\quad + 4\Big[2\cdot\frac{\pi}{12}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6}) - \frac{1}{2}\psi'(\frac{2}{3}) + \frac{1}{2}\psi'(\frac{1}{3})] - \frac{\pi}{24}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6}) + 2\psi'(\frac{1}{3}) - 2\psi'(\frac{2}{3})]\Big] \\
&\quad + 36\Big[-\frac{\pi}{24}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6}) + 2\psi'(\frac{1}{3}) - 2\psi'(\frac{2}{3})]\Big] \\
&\quad - 36\cdot\frac{\pi}{12}\Big[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3}) - \frac{1}{2}\psi'(\frac{5}{6}) + \frac{1}{2}\psi'(\frac{1}{6})\Big] \\
&\quad - 4\Big[2\cdot\big[-\frac{\pi}{24}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3}) + 2\psi'(\frac{1}{6}) - 2\psi'(\frac{5}{6})]\big] \\
&\quad + \frac{\pi}{12}\big[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3}) - \frac{1}{2}\psi'(\frac{5}{6}) + \frac{1}{2}\psi'(\frac{1}{6})\big]\Big]
\end{aligned}
$$

$\sigma\backslash\ell$	0	1	2	3	4	5
$\mathcal{K}_u$	$V_1V_2V_3$			$\frac{V_3}{V_2V_1}$		
$\mathcal{K}_t$	V_3			V_3		
$\mathcal{A}_{99}$	$V_1V_2V_3$			V_3		
$\mathcal{A}_{55}$	$\frac{V_3}{V_1V_2}$			V_3		
$\mathcal{A}_{95}$	V_3			V_3		
$\mathcal{M}_9$	$V_1V_2V_3$			V_3		
$\mathcal{M}_5$	V_3			$\frac{V_3}{V_1V_2}$		

Table 3: Volume factors for the different $\mathcal{N}$ sectors of the $\mathbb{Z}_6$ orientifold. Fields with no entry correspond to $\mathcal{N}=1$ sectors, fields with a single volume factor correspond to $\mathcal{N}=2$ sectors and fields with three volume factors denote $\mathcal{N}=4$ sectors. Volumes in the numerator/denominator are accompanied by momentum/winding sums.

$$
\begin{aligned}
&= 0\cdot\pi\Lambda^2 - 30\pi\psi'(\frac{1}{3}) + 30\pi\psi'(\frac{2}{3}) - 2\pi\psi'(\frac{1}{6}) + 2\pi\psi'(\frac{5}{6})\\
&= -60\pi\psi'(\frac{1}{3}) + 40\pi^3 - 4\pi\psi'(\frac{1}{6}) + 8\pi^3\\
&= -80\pi\psi'(\frac{1}{3}) + \frac{160}{3}\pi^3\\
&= -160\sqrt{3}\pi\mathrm{Cl}_2(\frac{\pi}{3}),
\end{aligned}
\tag{161}
$$

where we used the properties of the trigamma function

$$
\psi'(\gamma+1) = \psi'(\gamma) - \frac{1}{\gamma^2}, \qquad \psi'(1-\gamma) + \psi'(\gamma) = \frac{\pi^2}{\sin^2(\pi\gamma)} \tag{162}
$$

and the special relations (cf. (81) and (82) in [16])

$$
\psi'(\frac{2}{3}) = -\psi'(\frac{1}{3}) + \frac{4}{3}\pi^2, \quad \psi'(\frac{1}{6}) = 5\psi'(\frac{1}{3}) - \frac{4}{3}\pi^2, \quad \psi'(\frac{5}{6}) = -5\psi'(\frac{1}{3}) + \frac{16}{3}\pi^2 \tag{163}
$$

and

$$
\psi'(\frac{1}{3}) = 4\sin(\frac{\pi}{3})\mathrm{Cl}_2(\frac{\pi}{3}) + \frac{2}{3}\pi^2 \tag{164}
$$

to simplify the result of the integration. Here Cl_2 is the second Clausen function and $\mathrm{Cl}_2(\frac{\pi}{3}) \approx 1.015$. We see that the UV divergences ($\propto \Lambda^2$) cancel and only finite constant is left.

The result of the $\mathcal{N}=1$ sector is (cf. (134))

$$
\begin{aligned}
(\delta E)^{(\mathcal{N}=1)} &= \sum_\sigma (\delta E)_\sigma^{(\mathcal{N}=1)} =\\
&= -\frac{\pi(\alpha')^2}{64\cdot 6(4\pi^2\alpha')^2}\cdot\sqrt{3}\Big[-160\sqrt{3}\pi\mathrm{Cl}_2(\frac{\pi}{3})\Big]\\
&= \frac{15}{192\pi^2}\mathrm{Cl}_2(\frac{\pi}{3}).
\end{aligned}
\tag{165}
$$

5.1.2 $\mathcal{N}=2$ sectors

Now we consider the $\mathcal{N}=2$ sectors. Using table 2 and table 3, the contribution is

$$\begin{aligned}\sum_{\sigma^\ell\in\{\mathcal{N}=2\}} CP_\sigma\sigma^{(\ell)} &= \sum_{\ell=0,3}\mathcal{K}_t^{(\ell)} + (tr\gamma_9^3)^2\mathcal{A}_{99}^{(3)} + (tr\gamma_5^3)^2\mathcal{A}_{55}^{(3)} \\ &\quad + \sum_{\ell=0,3}\Big[(tr\gamma_9^\ell)(tr\gamma_5^\ell)\mathcal{A}_{95}^{(\ell)}\Big] + (tr\gamma_9^6)\mathcal{M}_9^{(3)} + (tr\gamma_5^0)\mathcal{M}_5^{(0)} \\ &= \sum_{\ell=0,3}\mathcal{K}_t^{(\ell)} + (tr\gamma_9^0)(tr\gamma_5^0)\mathcal{A}_{95}^{(0)} + (tr\gamma_9^6)\mathcal{M}_9^{(3)} + (tr\gamma_5^0)\mathcal{M}_5^{(0)} \\ &= \sum_{\ell=0,3}\mathcal{K}_t^{(\ell)} + 1024\mathcal{A}_{95}^{(0)} - 32\mathcal{M}_9^{(3)} + 32\mathcal{M}_5^{(0)}. \end{aligned} \tag{166}$$

From (148) we have

$$\begin{aligned}\pi\tilde{e}_{\mathcal{K}}\tilde{\chi}_{\mathcal{K}_t}^{(0)}D_{\mathcal{K}_t}^{(0)} &= 64\pi, \\ \pi\tilde{e}_{\mathcal{K}}\tilde{\chi}_{\mathcal{K}_t}^{(3)}D_{\mathcal{K}_t}^{(3)} &= 64\pi, \\ \pi\tilde{e}_{\mathcal{A}}\tilde{\chi}_{\mathcal{A}_{95}}^{(0)}D_{\mathcal{A}_{95}}^{(0)} &= 2\pi, \\ \pi\tilde{e}_{\mathcal{M}}\tilde{\chi}_{\mathcal{M}_9}^{(3)}D_{\mathcal{M}_9}^{(3)} &= 4\pi, \\ \pi\tilde{e}_{\mathcal{M}}\tilde{\chi}_{\mathcal{M}_5}^{(0)}D_{\mathcal{M}_5}^{(0)} &= -4\pi. \end{aligned} \tag{167}$$

Using (138), (150), (154) and (155) we can easily find that

$$\begin{aligned}&\int_{\frac{1}{e_\sigma\Lambda}}^\infty \frac{dt}{t^2}\sum_{\sigma^\ell\in\{\mathcal{N}=2\}} CP_\sigma\sigma^{(\ell)} = \\ &= 128\pi\int_{\frac{1}{e_{\mathcal{K}}\Lambda}}^\infty \frac{dt}{t^2}\mathcal{L}^{[3,M]} + 2048\pi\int_{\frac{1}{e_{\mathcal{A}}\Lambda}}^\infty \frac{dt}{t^2}\mathcal{L}^{[3,M]} \\ &\quad - 128\pi\int_{\frac{1}{e_{\mathcal{M}}\Lambda}}^\infty \frac{dt}{t^2}\mathcal{L}^{[3,M]} - 128\pi\int_{\frac{1}{e_{\mathcal{M}}\Lambda}}^\infty \frac{dt}{t^2}\mathcal{L}^{[3,M]} \\ &= \frac{C^{[3,M]}\Lambda^2}{2}\Big(128e_{\mathcal{K}_t}^2 + 2048e_{\mathcal{A}_{95}}^2 - 128e_{\mathcal{M}_9}^2 - 128e_{\mathcal{M}_5}^2\Big) \\ &\quad + (128+2048-128-128)\pi C^{[3,M]}\Gamma^{[3,M]} \\ &= \frac{C^{[3,M]}\Lambda^2}{2}\cdot 0 + 1920\pi C^{[3,M]}\Gamma^{[3,M]} \\ &= \frac{7680\pi\alpha'}{V_3}E_2(U^{[3]}). \end{aligned} \tag{168}$$

We see again the UV divergences cancel as expected. Thus

$$\begin{aligned}(\delta E)^{(\mathcal{N}=2)} &= -\frac{\pi(\alpha')^2}{64\cdot 6(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^2}\sum_{\sigma^\ell\in\{\mathcal{N}=2\}} CP_\sigma\sigma^{(\ell)} \\ &= -\frac{5}{4}\frac{\alpha'}{\pi^2 V_3}E_2(U^{[3]}). \end{aligned} \tag{169}$$

Adding the contributions from the sphere and the torus (cf. (127)), $\mathcal{N}=1$ and $\mathcal{N}=2$, the final result is

$$\begin{aligned}\delta E &= (\delta E)_{S_2+\mathcal{T}} + (\delta E)^{(\mathcal{N}=1)} + (\delta E)^{(\mathcal{N}=2)} \\ &= \frac{\chi}{(2\pi)^3}\Big(2\zeta(3)\frac{e^{-2\Phi_4}}{\mathcal{V}} + \frac{\pi^2}{3}\Big) + \frac{15}{192\pi^2}\mathrm{Cl}_2(\frac{\pi}{3}) - \frac{5}{4}\frac{\alpha'}{\pi^2 V_3}E_2(U^{[3]}). \end{aligned} \tag{170}$$

The Euler number of the $\mathbb{Z}_6$ orientifold is $\chi = 2(h^{(1,1)} - h^{(2,1)}) = 48$, cf. table 20 in [7]. And the numerical value of the contribution from $\mathcal{N} = 1$ part is 0.0080345.

5.2 $\mathbb{Z}_7$

The twist vector[6] of $\mathbb{Z}_7$ is $v = \frac{1}{7}(1, 2, -3)$. And only D_9-branes exist in odd case, cf. sec. 2.2 of [1]. Moreover there are no $\mathcal{N} = 2$ sectors nor non-trivial $\mathcal{N} = 4$ sectors(i.e. $\ell = 0$) as discussed in sec. 2.4 and sec. 4.6.

The contribution to the Planck mass is determined by (cf. (134))

$$\begin{aligned}\sum_{\sigma} \sum_{\ell \in \{\mathcal{N}=1\}} CP_\sigma \sigma^{(\ell)} =& \sum_{\ell=1,\ldots,6} \left[\mathcal{K}_u^{(\ell)} + (tr\gamma_9^\ell)^2 \mathcal{A}_{99}^{(\ell)} + (tr\gamma_9^{2\ell})\mathcal{M}_9^{(\ell)}\right] \\ =& 2 \sum_{\ell=1,2,3} \left[\mathcal{K}_u^{(\ell)} + 16\mathcal{A}_{99}^{(\ell)} + 4\mathcal{M}_9^{(\ell)}\right] \\ =& -32\left(\prod_{j=1}^{3} \sin \pi v_j\right)\left(\sum_{\ell=1,2} \left[\hat{\mathcal{K}}_u^{(\ell)} + 4\hat{\mathcal{A}}_{99}^{(\ell)} - \hat{\mathcal{M}}_9^{(\ell)}\right] - \left[\hat{\mathcal{K}}_u^{(3)} + 4\hat{\mathcal{A}}_{99}^{(3)} - \hat{\mathcal{M}}_9^{(3)}\right]\right).\end{aligned} \tag{171}$$

In the second equality we used (140) and the tadpole conditions $tr\gamma_\theta \equiv tr\gamma_9 = 4 = tr\gamma_9^2 = tr\gamma_9^3 = tr\gamma_9^4 = tr\gamma_9^5 = tr\gamma_9^6$ (cf. (2.36), (2.37) and the following paragraph and sec.3.3 in [1], we choose $\gamma_9^7 = 1$). In the third equality we used (139).

[6]refers to app.C

Next we have to perform the t-integral, using (141)-(144), i.e.

$$\sum_{\ell=1,2}\int_0^\infty \frac{dt}{t^2}\Big[\hat{\mathcal{K}}_u^{(\ell)}+4\hat{\mathcal{A}}_{99}^{(\ell)}-\hat{\mathcal{M}}_9^{(\ell)}\Big]-\int_0^\infty \frac{dt}{t^2}\Big[\hat{\mathcal{K}}_u^{(3)}+4\hat{\mathcal{A}}_{99}^{(3)}-\hat{\mathcal{M}}_9^{(3)}\Big]$$

$$
\begin{aligned}
&=\sum_{\ell=1,2}\left[\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{K}}_u^{(\ell)}+4\int_{\frac{1}{\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{A}}_{99}^{(\ell)}-\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{M}}_9^{(\ell)}\right]-\left[\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{K}}_u^{(\ell)}+4\int_{\frac{1}{\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{A}}_{99}^{(\ell)}-\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\hat{\mathcal{M}}_9^{(\ell)}\right]\\
&=\sum_{\ell=1,2}\sum_{j=1,2,3}\left(\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_1'(\lfloor 2\ell v_j\rfloor),\tau_{\mathcal{K}})}{\vartheta_1(\lfloor 2\ell v_j\rfloor),\tau_{\mathcal{K}})}+4\int_{\frac{1}{\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_1'(\lfloor \ell v_j\rfloor),\tau_{\mathcal{A}})}{\vartheta_1(\lfloor \ell v_j\rfloor,\tau_{\mathcal{A}})}-\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_1'(\lfloor \ell v_j\rfloor,\tau_{\mathcal{M}})}{\vartheta_1(\lfloor \ell v_j\rfloor,\tau_{\mathcal{M}})}\right)\\
&\quad-\sum_{j=1,2,3}\left(\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_1'(\lfloor 6v_j\rfloor),\tau_{\mathcal{K}})}{\vartheta_1(\lfloor 6v_j\rfloor),\tau_{\mathcal{K}})}+4\int_{\frac{1}{\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_1'(\lfloor 3v_j\rfloor),\tau_{\mathcal{A}})}{\vartheta_1(\lfloor 3v_j\rfloor,\tau_{\mathcal{A}})}-\int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_1'(\lfloor 3v_j\rfloor,\tau_{\mathcal{M}})}{\vartheta_1(\lfloor 3v_j\rfloor,\tau_{\mathcal{M}})}\right)\\
&=\sum_{\ell=1,2}\sum_{j=1,2,3}\Big(I_{\mathcal{K}}(\lfloor 2\ell v_j\rfloor)+4I_{\mathcal{A}}(\lfloor \ell v_j\rfloor)-I_{\mathcal{M}}(\lfloor \ell v_j\rfloor)\Big)-\sum_{j=1,2,3}\Big(I_{\mathcal{K}}(\lfloor 6v_j\rfloor)+4I_{\mathcal{A}}(\lfloor 3v_j\rfloor)-I_{\mathcal{M}}(\lfloor 3v_j\rfloor)\Big)\\
&=\sum_{\ell=1,2}\Bigg(4\pi\Lambda^2(3-2\sum_{j=1,2,3}\lfloor 2\ell v_j\rfloor)+\frac{\pi}{6}\sum_{j=1,2,3}[\psi'(\lfloor 2\ell v_j\rfloor)-\psi'(1-\lfloor 2\ell v_j\rfloor)]\\
&\quad+4\pi\Lambda^2(3-2\sum_{j=1,2,3}\lfloor \ell v_j\rfloor)+\frac{\pi}{6}\sum_{j=1,2,3}[\psi'(\lfloor \ell v_j\rfloor)-\psi'(1-\lfloor \ell v_j\rfloor)]\Bigg)\\
&\quad-\Bigg(4\pi\Lambda^2(3-2\sum_{j=1,2,3}\lfloor 6v_j\rfloor)+\frac{\pi}{6}\sum_{j=1,2,3}[\psi'(\lfloor 6v_j\rfloor)-\psi'(1-\lfloor 6v_j\rfloor)]\\
&\quad+4\pi\Lambda^2(3-2\sum_{j=1,2,3}\lfloor 3v_j\rfloor)+\frac{\pi}{6}\sum_{j=1,2,3}[\psi'(\lfloor 3v_j\rfloor)-\psi'(1-\lfloor 3v_j\rfloor)]\Bigg)\\
&\quad-\sum_{\ell=1,2}\sum_{j=1,2,3}\left(\begin{cases}I_{\mathcal{M}} & \lfloor \ell v_j\rfloor<\frac{1}{2}\\ \tilde{I}_{\mathcal{M}} & \lfloor \ell v_j\rfloor>\frac{1}{2}\end{cases}\right)+\sum_{j=1,2,3}\left(\begin{cases}I_{\mathcal{M}} & \lfloor 3v_j\rfloor<\frac{1}{2}\\ \tilde{I}_{\mathcal{M}} & \lfloor 3v_j\rfloor>\frac{1}{2}\end{cases}\right)\\
&=4\pi\Lambda^2\Big[(3-2\cdot 1)+(3-2\cdot 1)-(3-2\cdot 2)\Big]+4\pi\Lambda^2\Big[(3-2\cdot 1)+(3-2\cdot 1)-(3-2\cdot 2)\Big]\\
&\quad-8\pi\Lambda^2\Big[(5-4\cdot 1)+(5-4\cdot 1)-(7-4\cdot 2)\Big]\\
&\quad+\frac{\pi}{6}\sum_{\ell=1,2}\sum_{j=1,2,3}\Big[\psi'(\lfloor 2\ell v_j\rfloor)-\psi'(1-\lfloor 2\ell v_j\rfloor)+\psi'(\lfloor \ell v_j\rfloor)-\psi'(1-\lfloor \ell v_j\rfloor)\Big]\\
&\quad-\frac{\pi}{6}\sum_{j=1,2,3}\Big[\psi'(\lfloor 6v_j\rfloor)-\psi'(1-\lfloor 6v_j\rfloor)+\psi'(\lfloor 3v_j\rfloor)-\psi'(1-\lfloor 3v_j\rfloor)\Big]\\
&\quad-\frac{\pi}{6}\Big[\psi'(\frac{1}{7})-\psi'(\frac{6}{7})-\frac{1}{2}\psi'(\frac{9}{14})+\frac{1}{2}\psi'(\frac{5}{14})\Big]-\frac{\pi}{6}\Big[\psi'(\frac{2}{7})-\psi'(\frac{5}{7})-\frac{1}{2}\psi'(\frac{11}{14})+\frac{1}{2}\psi'(\frac{3}{14})\Big]\\
&\quad+\frac{\pi}{12}\Big[\psi'(\frac{3}{7})-\psi'(\frac{4}{7})-\frac{1}{2}\psi'(\frac{12}{14})+\frac{1}{2}\psi'(\frac{1}{14})\Big]-\frac{\pi}{12}\Big[\psi'(\frac{1}{14})-\psi'(\frac{13}{14})+2\psi'(\frac{3}{7})-2\psi'(\frac{4}{7})\Big]\\
&\quad+\frac{\pi}{24}\Big[\psi'(\frac{3}{14})-\psi'(\frac{11}{14})+2\psi'(\frac{2}{7})-2\psi'(\frac{5}{7})\Big]+\frac{\pi}{24}\Big[\psi'(\frac{5}{14})-\psi'(\frac{9}{14})+2\psi'(\frac{3}{7})-2\psi'(\frac{4}{7})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= 0\cdot\Lambda^2 + \frac{\pi}{6}\Big[4\psi'(\frac{1}{7}) - 4\psi'(\frac{6}{7}) + 5\psi'(\frac{2}{7}) - 5\psi'(\frac{5}{7}) - 5\psi'(\frac{3}{7}) + 5\psi'(\frac{4}{7})\Big]\\
&\quad - \frac{\pi}{24}\Big[2\psi'(\frac{1}{7}) + 2\psi'(\frac{2}{7}) + 2\psi'(\frac{3}{7}) - 2\psi'(\frac{4}{7}) - 2\psi'(\frac{5}{7}) - 2\psi'(\frac{6}{7})\\
&\quad + \psi'(\frac{1}{14}) + \psi'(\frac{3}{14}) + \psi'(\frac{5}{14}) - \psi'(\frac{9}{14}) - \psi'(\frac{11}{14}) - \psi'(\frac{13}{14})\Big]\\
&= \frac{7}{12}\pi\psi'(\frac{1}{7}) + \frac{3}{4}\pi\psi'(\frac{2}{7}) - \frac{11}{12}\pi\psi'(\frac{3}{7}) + \frac{11}{12}\pi\psi'(\frac{4}{7}) - \frac{3}{4}\pi\psi'(\frac{5}{7}) - \frac{7}{12}\pi\psi'(\frac{6}{7})\\
&\quad - \frac{\pi}{24}\Big[\psi'(\frac{1}{14}) + \psi'(\frac{3}{14}) + \psi'(\frac{5}{14}) - \psi'(\frac{9}{14}) - \psi'(\frac{11}{14}) - \psi'(\frac{13}{14})\Big]. \qquad (172)
\end{aligned}
$$

Here $\lfloor \ell v_j \rfloor = \ell v_j + N$, $N \in \mathbb{Z}$ and $0 < \lfloor \ell v_j \rfloor < 1$. Moreover, we can see that all terms related to Λ^2 cancel each other, that is just the result of the requirement of tadpole cancellation. For $\mathcal{M}$, it is possible that $\frac{1}{2} < \gamma < 1$. Thus we used the new t-integral $\tilde{I}_{\mathcal{M}}$ (144) for this special case which is discussed in app. B.3.

Using (162), we can simplify the expression:

$$
\begin{aligned}
&\sum_{\ell=1,2,3}\int_0^\infty \frac{dt}{t^2}\Big[\hat{\mathcal{K}}_u^{(\ell)} + 4\hat{\mathcal{A}}_{99}^{(\ell)} - \hat{\mathcal{M}}_9^{(\ell)}\Big] =\\
&= \frac{7}{12}\pi\psi'(\frac{1}{7}) + \frac{3}{4}\pi\psi'(\frac{2}{7}) - \frac{11}{12}\pi\psi'(\frac{3}{7}) + \frac{11}{12}\pi\psi'(\frac{4}{7}) - \frac{3}{4}\pi\psi'(\frac{5}{7}) - \frac{7}{12}\pi\psi'(\frac{6}{7})\\
&\quad - \frac{\pi}{24}\Big[\psi'(\frac{1}{14}) + \psi'(\frac{3}{14}) + \psi'(\frac{5}{14}) - \psi'(\frac{9}{14}) - \psi'(\frac{11}{14}) - \psi'(\frac{13}{14})\Big]\\
&= \frac{7}{6}\pi\psi'(\frac{1}{7}) + \frac{3}{2}\pi\psi'(\frac{2}{7}) - \frac{11}{6}\pi\psi'(\frac{3}{7}) - \frac{7\pi^3}{12\sin^2(\frac{\pi}{7})} - \frac{3\pi^3}{4\sin^2(\frac{2\pi}{7})} + \frac{11\pi^3}{12\sin^2(\frac{3\pi}{7})}\\
&\quad - \frac{\pi}{12}\Big[\psi'(\frac{1}{14}) + \psi'(\frac{3}{14}) + \psi'(\frac{5}{14})\Big] + \frac{\pi^3}{24\sin^2(\frac{\pi}{14})} + \frac{\pi^3}{24\sin^2(\frac{3\pi}{14})} + \frac{\pi^3}{24\sin^2(\frac{5\pi}{14})}. \qquad (173)
\end{aligned}
$$

Since there is no special properties for $\psi'(\frac{\ell}{7})$, we have to leave the expression without further simplification.

Putting all factors together, the contribution from the $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$ is

$$
\begin{aligned}
&(\delta E)_{\mathcal{K}+\mathcal{A}+\mathcal{M}} = -\frac{\pi(\alpha')^2}{64\cdot 7(4\pi^2\alpha')^2}32\sin(\frac{\pi}{7})\sin(\frac{2\pi}{7})\sin(\frac{3\pi}{7})\int_0^\infty \frac{dt}{t^2}\sum_{\sigma^{(\ell)}\in\{\mathcal{N}=1\}} CP_\sigma\sigma^{(\ell)}\\
&= -\frac{\sin(\frac{\pi}{7})\sin(\frac{2\pi}{7})\sin(\frac{3\pi}{7})}{224\pi^2}\cdot\Bigg[\frac{7}{6}\pi\psi'(\frac{1}{7}) + \frac{3}{2}\pi\psi'(\frac{2}{7}) - \frac{11}{6}\pi\psi'(\frac{3}{7}) - \frac{7\pi^3}{12\sin^2(\frac{\pi}{7})} - \frac{3\pi^3}{4\sin^2(\frac{2\pi}{7})} + \frac{11\pi^3}{12\sin^2(\frac{3\pi}{7})}\\
&\quad -\frac{\pi}{12}\Big[\psi'(\frac{1}{14}) + \psi'(\frac{3}{14}) + \psi'(\frac{5}{14})\Big] + \frac{\pi^3}{24\sin^2(\frac{\pi}{14})} + \frac{\pi^3}{24\sin^2(\frac{3\pi}{14})} + \frac{\pi^3}{24\sin^2(\frac{5\pi}{14})}\Bigg]. \qquad (174)
\end{aligned}
$$

Adding the contribution from the sphere and the torus, cf. (127), the final result is

$$
\begin{aligned}
\delta E &= (\delta E)_{S_2+\mathcal{T}} + (\delta E)_{\mathcal{K}+\mathcal{A}+\mathcal{M}}\\
&= \frac{\chi}{(2\pi)^3}\Big(2\zeta(3)\frac{e^{-2\Phi_4}}{\mathcal{V}} + \frac{\pi^2}{3}\Big)\\
&-\frac{\sin(\frac{\pi}{7})\sin(\frac{2\pi}{7})\sin(\frac{3\pi}{7})}{224\pi^2}\cdot\Bigg[\frac{7}{6}\pi\psi'(\frac{1}{7}) + \frac{3}{2}\pi\psi'(\frac{2}{7}) - \frac{11}{6}\pi\psi'(\frac{3}{7}) - \frac{7\pi^3}{12\sin^2(\frac{\pi}{7})} - \frac{3\pi^3}{4\sin^2(\frac{2\pi}{7})} + \frac{11\pi^3}{12\sin^2(\frac{3\pi}{7})}\\
&\quad -\frac{\pi}{12}\Big[\psi'(\frac{1}{14}) + \psi'(\frac{3}{14}) + \psi'(\frac{5}{14})\Big] + \frac{\pi^3}{24\sin^2(\frac{\pi}{14})} + \frac{\pi^3}{24\sin^2(\frac{3\pi}{14})} + \frac{\pi^3}{24\sin^2(\frac{5\pi}{14})}\Bigg]. \qquad (175)
\end{aligned}
$$

The Euler number of the $\mathbb{Z}_7$ orientifold is $\chi = 2(h^{(1,1)} - h^{(2,1)}) = 48$, cf. table 20 in [7]. And the numerical value of the contribution from $\mathcal{N} = 1$ part is -0.0115702.

5.3 $\mathbb{Z}_{12}$

The twist vector[7] of $\mathbb{Z}_{12}$ is $v = \frac{1}{12}(1, -5, 4)$. Here we have the same procedure as in the $\mathbb{Z}_6$ case.

5.3.1 $\mathcal{N} = 1$ sectors

The $\mathcal{N} = 1$ sector sum is

$$\sum_{\sigma} \sum_{\ell \in \{\mathcal{N}=1\}} CP_\sigma \sigma^{(\ell)} = \sum_{\ell=1,2,4,5,7,8,10,11} \Big(\mathcal{K}_u^{(\ell)} + \mathcal{K}_t^{(\ell)} + (tr\gamma_9^\ell)^2 \mathcal{A}_{99}^{(\ell)} + (tr\gamma_5^\ell)^2 \mathcal{A}_{55}^{(\ell)} + (tr\gamma_9^\ell)(tr\gamma_5^\ell)\mathcal{A}_{95}^{(\ell)} + (tr\gamma_9^{2\ell})\mathcal{M}_9^{(\ell)} + (tr\gamma_5^{2\ell})\mathcal{M}_5^{(\ell)} \Big). \tag{176}$$

Using (139), (140), Chan-Paton traces [1]

$$\begin{aligned} tr\gamma_9^\ell &= tr\gamma_5^\ell = 0; \quad \ell \neq 0, 4, 8, \\ tr\gamma_9^4 &= tr\gamma_5^4 = 4, \\ tr\gamma_9^8 &= tr\gamma_5^8 = -4, \\ \gamma_9^{12} &= \gamma_5^{12} = -1, \\ tr\gamma_9^{12} &= tr\gamma_5^{12} = -32, \\ tr\gamma_9^0 &= tr\gamma_5^0 = 32 \end{aligned} \tag{177}$$

and

$$\begin{aligned} \tilde{\chi}(\theta^6, \theta^1) = \tilde{\chi}(\theta^6, \theta^2) = \tilde{\chi}(\theta^6, \theta^4) = \tilde{\chi}(\theta^6, \theta^5) = \tilde{\chi}(\theta^6, \theta^7) = \\ = \tilde{\chi}(\theta^6, \theta^8) = \tilde{\chi}(\theta^6, \theta^{10}) = \tilde{\chi}(\theta^6, \theta^{11}) = 1, \\ \tilde{\chi}(\theta^6, \theta^3) = \tilde{\chi}(\theta^6, \theta^9) = 4, \\ \tilde{\chi}(\theta^6, \theta^0) = \tilde{\chi}(\theta^6, \theta^6) = 16. \end{aligned} \tag{178}$$

we obtain

$$\begin{aligned} &\sum_{\sigma^\ell \in \{\mathcal{N}=1\}} CP_\sigma \sigma^{(\ell)} = \\ &= 4 \sum_{\ell=1,2} \Big(\mathcal{K}_u^{(\ell)} + \mathcal{K}_t^{(\ell)} \Big) + 32 \Big(\mathcal{A}_{99}^{(4)} + \mathcal{A}_{55}^{(4)} + \mathcal{A}_{95}^{(4)} \Big) \\ &\quad + 8 \Big(\mathcal{M}_9^{(2)} + \mathcal{M}_5^{(2)} \Big) - 8 \Big(\mathcal{M}_9^{(4)} + \mathcal{M}_5^{(4)} \Big) \\ &= -64 \sin\frac{\pi}{6} \sin\frac{5\pi}{6} \sin\frac{2\pi}{3} \hat{\mathcal{K}}_u^{(1)} - 64 \sin^3\frac{\pi}{3} \hat{\mathcal{K}}_u^{(2)} + 16 \sin\frac{2\pi}{3} \hat{\mathcal{K}}_t^{(1)} \\ &\quad - 16 \sin\frac{2\pi}{3} \hat{\mathcal{K}}_t^{(2)} - 128 \sin^3\frac{\pi}{3} \hat{\mathcal{A}}_{99}^{(4)} - 128 \sin^3\frac{\pi}{3} \hat{\mathcal{A}}_{55}^{(4)} - 64 \sin\frac{\pi}{3} \hat{\mathcal{A}}_{95}^{(4)} \\ &\quad + 32 \sin\frac{\pi}{6} \sin\frac{5\pi}{6} \sin\frac{2\pi}{3} \hat{\mathcal{M}}_9^{(2)} - 32 \sin\frac{2\pi}{3} \cos\frac{\pi}{6} \cos\frac{5\pi}{6} \hat{\mathcal{M}}_5^{(2)} \\ &\quad - 32 \sin^3\frac{\pi}{3} \hat{\mathcal{M}}_9^{(4)} - 32 \sin\frac{\pi}{3} \cos^2\frac{\pi}{3} \hat{\mathcal{M}}_5^{(4)} \\ &= -8\sqrt{3}\hat{\mathcal{K}}_u^{(1)} - 24\sqrt{3}\hat{\mathcal{K}}_u^{(2)} + 8\sqrt{3}\hat{\mathcal{K}}_t^{(1)} - 8\sqrt{3}\hat{\mathcal{K}}_t^{(2)} \\ &\quad - 48\sqrt{3}\hat{\mathcal{A}}_{99}^{(4)} - 48\sqrt{3}\hat{\mathcal{A}}_{55}^{(4)} - 32\sqrt{3}\hat{\mathcal{A}}_{95}^{(4)} \\ &\quad + 4\sqrt{3}\hat{\mathcal{M}}_9^{(2)} + 12\sqrt{3}\hat{\mathcal{M}}_5^{(2)} - 12\sqrt{3}\hat{\mathcal{M}}_9^{(4)} - 4\sqrt{3}\hat{\mathcal{M}}_5^{(4)}. \end{aligned} \tag{179}$$

[7] refers to C

Then we do the t-integral using (141)-(144)

$$\int_0^\infty \frac{dt}{t^2}\Big[-8\hat{\mathcal{K}}_u^{(1)} - 24\hat{\mathcal{K}}_u^{(2)} + 8\hat{\mathcal{K}}_t^{(1)} - 8\hat{\mathcal{K}}_t^{(2)} - 48\hat{\mathcal{A}}_{99}^{(4)} - 48\hat{\mathcal{A}}_{55}^{(4)} - 32\hat{\mathcal{A}}_{95}^{(4)}$$
$$+ 4\hat{\mathcal{M}}_9^{(2)} + 12\hat{\mathcal{M}}_5^{(2)} - 12\hat{\mathcal{M}}_9^{(4)} - 4\hat{\mathcal{M}}_5^{(4)}\Big] =$$
$$= -8\Big[2I_{\mathcal{K}}(\frac{1}{6}) + I_{\mathcal{K}}(\frac{2}{3})\Big] - 24\cdot 3I_{\mathcal{K}}(\frac{1}{3}) + 8\Big[2\tilde{I}_{\mathcal{K}}(\frac{1}{6}) + I_{\mathcal{K}}(\frac{2}{3})\Big] - 8\Big[2\tilde{I}_{\mathcal{K}}(\frac{1}{3}) + I_{\mathcal{K}}(\frac{1}{3})\Big]$$
$$- 48\cdot 3I_{\mathcal{A}}(\frac{1}{3}) - 48\cdot 3I_{\mathcal{A}}(\frac{1}{3}) - 32\Big[2\tilde{I}_{\mathcal{A}}(\frac{1}{3}) + I_{\mathcal{A}}(\frac{1}{3})\Big]$$
$$+ 4\Big[2I_{\mathcal{M}}(\frac{1}{6}) + \tilde{I}_{\mathcal{M}}(\frac{2}{3})\Big] + 12\Big[2\tilde{I}_{\mathcal{M}}(\frac{2}{3}) + I_{\mathcal{M}}(\frac{1}{6})\Big]$$
$$- 12\cdot 3I_{\mathcal{M}}(\frac{1}{3}) - 4\cdot 3\tilde{I}_{\mathcal{M}}(\frac{5}{6})$$
$$= \Lambda^2\cdot\Big[-8[2\cdot 4\pi(1-\frac{1}{3}) + 4\pi(1-\frac{4}{3})] - 72\cdot 4\pi(1-\frac{2}{3}) + 8[2\cdot 4\pi(1-\frac{1}{3}) + 4\pi(1-\frac{4}{3})]$$
$$- 24\cdot 4\pi(1-\frac{2}{3}) - 144\pi(1-\frac{2}{3})\cdot 2 - 96\pi(1-\frac{2}{3}) + 4[2\cdot 8\pi(1-\frac{2}{3}) + 8\pi(3-\frac{8}{3})]$$
$$+ 12[2\cdot 8\pi(3-\frac{8}{3}) + 8\pi(1-\frac{2}{3})] - 36\cdot 8\pi(1-\frac{4}{3}) - 12\cdot 8\pi(3-\frac{10}{3})\Big]$$
$$- 8\Big[2\big(\frac{\pi}{6}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6})]\big) + \big(\frac{\pi}{6}[\psi'(\frac{2}{3}) - \psi'(\frac{1}{3})]\big)\Big] - 72\Big[\frac{\pi}{6}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})]\Big]$$
$$+ 8\Big[-2\big(\frac{\pi}{12}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6})]\big) + \big(\frac{\pi}{6}[\psi'(\frac{2}{3}) - \psi'(\frac{1}{3})]\big)\Big]$$
$$- 8\Big[-2\big(\frac{\pi}{12}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})]\big) + \big(\frac{\pi}{6}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})]\big)\Big] - 288\Big[\frac{\pi}{24}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})]\Big]$$
$$- 32\Big[-2\big(\frac{\pi}{48}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})]\big) + \big(\frac{\pi}{24}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3})]\big)\Big]$$
$$+ 4\Big[2\big(\frac{\pi}{12}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6}) - \frac{1}{2}\psi'(\frac{2}{3}) + \frac{1}{2}\psi'(\frac{1}{3})]\big) - \frac{\pi}{24}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6}) + 2\psi'(\frac{1}{3}) - 2\psi'(\frac{2}{3})]\Big]$$
$$+ 12\Big[2\big(-\frac{\pi}{24}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6}) + 2\psi'(\frac{1}{3}) - 2\psi'(\frac{2}{3})\big) + \big(\frac{\pi}{12}[\psi'(\frac{1}{6}) - \psi'(\frac{5}{6}) - \frac{1}{2}\psi'(\frac{2}{3}) + \frac{1}{2}\psi'(\frac{1}{3})]\big)\Big]$$
$$- 36\Big[\frac{\pi}{12}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3}) - \frac{1}{2}\psi'(\frac{5}{6}) + \frac{1}{2}\psi'(\frac{1}{6})]\Big] + 12\Big[\frac{\pi}{24}[\psi'(\frac{1}{3}) - \psi'(\frac{2}{3}) + 2\psi'(\frac{1}{6}) - 2\psi'(\frac{5}{6})]\Big]$$

$\sigma\backslash\ell$	0	1	2	3	4	5	6	7	8	9	10	11
$\mathcal{K}_u$	$V_1V_2V_3$			V_3			$\frac{V_3}{V_1V_2}$			V_3		
$\mathcal{K}_t$	V_3			V_3			V_3			V_3		
$\mathcal{A}_{99}$	$V_1V_2V_3$			V_3			V_3			V_3		
$\mathcal{A}_{55}$	$\frac{V_3}{V_1V_2}$			V_3			V_3			V_3		
$\mathcal{A}_{95}$	V_3			V_3			V_3			V_3		
$\mathcal{M}_9$	$V_1V_2V_3$			V_3			V_3			V_3		
$\mathcal{M}_5$	V_3			V_3			$\frac{V_3}{V_1V_2}$			V_3		

Table 4: Volume factors for the different $\mathcal{N}$ sectors of the $\mathbb{Z}_{12}$ orientifold.

$$\begin{aligned}
&= 0\cdot\Lambda^2 - 3\pi\psi'(\frac{1}{6}) + 3\pi\psi'(\frac{5}{6}) - \frac{55}{2}\pi\psi'(\frac{1}{3}) + \frac{55}{2}\pi\psi'(\frac{2}{3}) \\
&= -55\pi\psi'(\frac{1}{3}) + \frac{110}{3}\pi^3 - 6\pi\psi'(\frac{1}{6}) + 12\pi^3 \\
&= -55\pi\psi'(\frac{1}{3}) + \frac{146}{3}\pi^3 - 30\pi\psi'(\frac{1}{3}) + 8\pi^3 \\
&= -85\pi\psi'(\frac{1}{3}) + \frac{170}{3}\pi^3 \\
&= -170\sqrt{3}\pi \mathrm{Cl}_2(\frac{\pi}{3}),
\end{aligned} \tag{180}$$

where as for the $\mathbb{Z}_6$ case we used (162), (163) and (164) to simplify the result. The UV divergence is absent as well.

The result of the $\mathcal{N}=1$ sector is

$$\begin{aligned}
(\delta E)^{(\mathcal{N}=1)} &= \sum_\sigma (\delta E)_\sigma^{(\mathcal{N}=1)} = \\
&= -\frac{\pi(\alpha')^2}{64\cdot 12(4\pi^2\alpha')^2}\cdot\sqrt{3}\Big[-170\sqrt{3}\pi\mathrm{Cl}_2(\frac{\pi}{3})\Big] \\
&= \frac{85}{2048\pi^2}\mathrm{Cl}_2(\frac{\pi}{3}).
\end{aligned} \tag{181}$$

5.3.2 $\mathcal{N}=2$ sectors

The contribution is

$$\begin{aligned}
&\sum_{\sigma^\ell\in\{\mathcal{N}=2\}} CP_\sigma \sigma^{(\ell)} = \\
&\quad= \sum_{\ell=3,9}\mathcal{K}_u^{(\ell)} + \sum_{\ell=0,3,6,9}\mathcal{K}_t^{(\ell)} + \sum_{\ell=3,6,9}(tr\gamma_9^\ell)^2\mathcal{A}_{99}^{(\ell)} + \sum_{\ell=3,6,9}(tr\gamma_5^\ell)^2\mathcal{A}_{55}^{(\ell)} \\
&\qquad+ \sum_{\ell=0,3,6,9}\Big[(tr\gamma_9^\ell)(tr\gamma_5^\ell)\mathcal{A}_{95}^{(\ell)}\Big] + \sum_{\ell=3,6,9}(tr\gamma_9^{2\ell})\mathcal{M}_9^{(\ell)} + \sum_{\ell=0,3,9}(tr\gamma_5^{2\ell})\mathcal{M}_5^{(\ell)} \\
&\quad= \sum_{\ell=3,9}\mathcal{K}_u^{(\ell)} + \sum_{\ell=0,3,6,9}\mathcal{K}_t^{(\ell)} + 1024\mathcal{A}_{95}^{(0)} - 32\mathcal{M}_9^{(6)} + 32\mathcal{M}_5^{(0)}.
\end{aligned} \tag{182}$$

From (148) we have

$$\begin{aligned}
\pi\tilde{e}_{\mathcal{K}}\tilde{\chi}_{\mathcal{K}_u}D^{(3)}_{\mathcal{K}_u} &= -16\pi, & \pi\tilde{e}_{\mathcal{K}}\tilde{\chi}_{\mathcal{K}_u}D^{(9)}_{\mathcal{K}_u} &= -16\pi,\\
\pi\tilde{e}_{\mathcal{K}}\tilde{\chi}_{\mathcal{K}_t}D^{(0)}_{\mathcal{K}_t} &= 64\pi, & \pi\tilde{e}_{\mathcal{K}}\tilde{\chi}_{\mathcal{K}_t}D^{(3)}_{\mathcal{K}_t} &= 16\pi,\\
\pi\tilde{e}_{\mathcal{K}}\tilde{\chi}_{\mathcal{K}_t}D^{(6)}_{\mathcal{K}_t} &= 64\pi, & \pi\tilde{e}_{\mathcal{K}}\tilde{\chi}_{\mathcal{K}_t}D^{(9)}_{\mathcal{K}_t} &= 16\pi,\\
\pi\tilde{e}_{\mathcal{A}}\tilde{\chi}_{\mathcal{A}_{95}}D^{(0)}_{\mathcal{A}_{95}} &= 2\pi, & \pi\tilde{e}_{\mathcal{M}}\tilde{\chi}_{\mathcal{M}_9}D^{(6)}_{\mathcal{M}_9} &= 4\pi,\\
\pi\tilde{e}_{\mathcal{M}}\tilde{\chi}_{\mathcal{M}_5}D^{(0)}_{\mathcal{M}_5} &= -4\pi. && \qquad (183)
\end{aligned}$$

Using (138), (150), (154) and (155) we can easily get that

$$\begin{aligned}
&\int_{\frac{1}{e_\sigma}\Lambda}^{\infty}\frac{dt}{t^2}\sum_{\sigma^\ell\in\{\mathcal{N}=2\}}CP_\sigma\sigma^{(\ell)} =\\
&= -16\pi\int_{\frac{1}{e_{\mathcal{K}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]} - 16\pi\int_{\frac{1}{e_{\mathcal{K}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]} + 64\pi\int_{\frac{1}{e_{\mathcal{K}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]}\\
&\quad + 16\pi\int_{\frac{1}{e_{\mathcal{K}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]} + 64\pi\int_{\frac{1}{e_{\mathcal{K}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]} + 16\pi\int_{\frac{1}{e_{\mathcal{K}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]}\\
&\quad + 2048\pi\int_{\frac{1}{e_{\mathcal{A}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]} - 128\pi\int_{\frac{1}{e_{\mathcal{M}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]} - 128\pi\int_{\frac{1}{e_{\mathcal{M}}}\Lambda}^{\infty}\frac{dt}{t^2}\mathcal{L}^{[3,M]}\\
&= \frac{\pi C^{[3,M]}\Lambda^2}{2}\Big[(-16-16+64+16+64+16)e^2_{\mathcal{K}} + 2048e^2_{\mathcal{A}} + (-128-128)e^2_{\mathcal{M}}\Big]\\
&\quad + (-16-16+64+16+64+16+2048-128-128)\pi C^{[3,M]}\Gamma^{[3,M]}\\
&= 0\cdot\frac{\pi C^{[3,M]}\Lambda^2}{2} + 1920\pi C^{[3,M]}\Gamma^{[3,M]}\\
&= \frac{7680\pi\alpha'}{V_3}E_2(U^{[3]}). \qquad (184)
\end{aligned}$$

The UV divergences cancel again as expected. Thus

$$\begin{aligned}
(\delta E)^{(\mathcal{N}=2)} &= -\frac{\pi(\alpha')^2}{64\cdot 12(4\pi^2\alpha')^2}\int_0^\infty\frac{dt}{t^2}\sum_{\sigma^\ell\in\{\mathcal{N}=2\}}CP_\sigma\sigma^{(\ell)}\\
&= -\frac{5}{8}\frac{\alpha'}{\pi^2V_3}E_2(U^{[3]}). \qquad (185)
\end{aligned}$$

Adding the contributions from the sphere and the torus (cf. (127)), $\mathcal{N}=1$ and $\mathcal{N}=2$, the final result is

$$\begin{aligned}
\delta E &= (\delta E)_{S_2+\mathcal{T}} + (\delta E)^{(\mathcal{N}=1)} + (\delta E)^{(\mathcal{N}=2)}\\
&= \frac{\chi}{(2\pi)^3}\Big(2\zeta(3)\frac{e^{-2\Phi_4}}{\mathcal{V}} + \frac{\pi^2}{3}\Big) + \frac{85}{2048\pi^2}\mathrm{Cl}_2(\frac{\pi}{3}) - \frac{5}{8}\frac{\alpha'}{\pi^2V_3}E_2(U^{[3]}). \qquad (186)
\end{aligned}$$

The Euler number of the $\mathbb{Z}_{12}$ orientifold is $\chi = 2(h^{(1,1)} - h^{(2,1)}) = 48$, cf. table 20 in [7]. And the numerical value of the contribution from $\mathcal{N}=1$ part is 0.0042683.

6 Conclusions

We determined the quantum corrections to the Einstein-Hilbert term in toroidal minimally supersymmetric type-IIB orientifolds at 1-loop order. And we calculated the contribution in 3 concrete models: $\mathbb{Z}_6$, $\mathbb{Z}_7$ and $\mathbb{Z}_{12}$. During the calculation there is a new kind of integral arising in the case of Möbius with $\frac{1}{2} < \gamma < 1$. Also it is worth mentioning that the $\mathcal{N} = 1$ contributions are 0.0080345 for $\mathbb{Z}_6$, -0.0115702 for $\mathbb{Z}_7$ and 0.0042683 for $\mathbb{Z}_{12}$, which all of them are almost of the same order of magnitude and are much smaller than 1. We may observe that the sum of these three $\mathcal{N} = 1$ contributions is 0.0007326 which is much smaller than any one of single contribution.

Until now, the 1-loop calculation of δE is complete because we went through all tadpole-free models. However, there are still more open questions. Observing (116), we can see that there are still correction terms like $\delta\tau$ left unsolved. Therefore, further evaluation about these terms should be fulfilled in order to finally complete the full 1-loop correction.

Acknowledgment

I thank Dr. Michael Haack for the patient and efficient supervision during the whole period of my master thesis. I have learned a lot from Michael Haack. This thesis is heavily based on his work, all those helpful discussions made me have a much better understanding of the topic I worked on. I would like to thank Garam Jeong for interesting discussions, and Mingzhen Li for support in daily life. I would also like to thank the TMP program at LMU München for giving me the opportunity to get the best theoretical physics education around the world.

A Useful formulas

Cited from [8] and [16].

$$q = e^{2\pi i\tau} \tag{187}$$

ϑ functions:

$$\vartheta[^{\vec{\alpha}}_{\vec{\beta}}](\vec{\nu}, G) = \sum_{\vec{n}\in\mathbb{Z}^N} e^{i\pi(\vec{n}+\vec{\alpha})^T G(\vec{n}+\vec{\alpha})} e^{2\pi i(\vec{\nu}+\vec{\beta})^T(\vec{n}+\vec{\alpha})}, \tag{188}$$

$$\vartheta_1 = -\vartheta[^{\frac{1}{2}}_{\frac{1}{2}}](\nu,\tau) = 2e^{\pi i\tau/4}\sin(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^n), \tag{189}$$

$$\vartheta_2 = \vartheta[^{\frac{1}{2}}_{0}](\nu,\tau) = 2e^{\pi i\tau/4}\cos(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)(1+zq^n)(1+z^{-1}q^n), \tag{190}$$

$$\vartheta_3 = \vartheta[^0_0](\nu,\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}}), \tag{191}$$

$$\vartheta_4 = \vartheta[^0_{\frac{1}{2}}](\nu,\tau) = \prod_{n=1}^{\infty}(1-q^n)(1-zq^{n-\frac{1}{2}})(1-z^{-1}q^{n-\frac{1}{2}}). \tag{192}$$

where $z = e^{2\pi i\nu}$.

η function:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \left[\frac{\partial_\nu\vartheta_1(0,\tau)}{-2\pi}\right]^{\frac{1}{3}}, \tag{193}$$

and

$$\frac{\vartheta[^\alpha_\beta](0,\tau)}{\eta(\tau)} = e^{2\pi i\alpha\beta}q^{\frac{\alpha^2}{2}-\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^{n+\alpha-\frac{1}{2}}e^{2\pi i\beta})(1+q^{n-\alpha-\frac{1}{2}}e^{-2\pi i\beta}). \tag{194}$$

Poisson re-summation:

$$\vartheta[^{\vec{0}}_{\vec{0}}](0, itG^{-1}) = \sqrt{G}t^{-N/2}\vartheta[^{\vec{0}}_{\vec{0}}](0, it^{-1}G) \tag{195}$$

Modular transformation S for annulus and Klein bottle:

$$\vartheta[^\alpha_\beta](\nu,\tau) = (-i\tau)^{-1/2}e^{2\pi i\alpha\beta-\pi i\nu^2/\tau}\vartheta[^{-\beta}_{\alpha}](\nu/\tau, -1/\tau). \tag{196}$$

Modular transformation ST^2S for Möbius:

$$\vartheta[^\alpha_\beta](\nu,\tau) = (1-2\tau)^{-1/2}e^{2\pi i\beta^2}e^{-\pi i\nu^2/(\tau-1/2)}\vartheta[^{\alpha+2\beta}_{\beta}](\frac{\nu}{1-2\tau}, \frac{\tau}{1-2\tau}). \tag{197}$$

General Modular transformation S and T for ϑ-functions and η-function:

$$\vartheta[^\alpha_\beta](\tau+1) = e^{-\pi i(\alpha^2-\alpha)}\vartheta[^{\alpha}_{\alpha+\beta-\frac{1}{2}}](\tau), \tag{198}$$

$$\vartheta[^\alpha_\beta](-\frac{1}{\tau}) = \sqrt{-i\tau}e^{2\pi i\alpha\beta}\vartheta[^{-\beta}_{\alpha}](\tau) \qquad |\arg\sqrt{-i\tau}| < \frac{\pi}{2}, \tag{199}$$

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau), \tag{200}$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau). \tag{201}$$

Shifts in characteristics:

$$\begin{aligned}\vartheta[^{\alpha+1}_{\beta}](\nu,\tau) &= \vartheta[^\alpha_\beta](\nu,\tau),\\ \vartheta[^{\alpha}_{\beta+1}](\nu,\tau) &= e^{2\pi i\alpha}\vartheta[^\alpha_\beta](\nu,\tau).\end{aligned} \tag{202}$$

ν-periodicity formula:

$$\vartheta[^\alpha_\beta](\nu+a\tau+b,\tau) = e^{-2\pi iab}e^{-\pi ia^2\tau}e^{-2\pi ia(\nu+b)}\vartheta[^{\alpha+a}_{\beta+b}](\nu,\tau). \tag{203}$$

B t-integrals

App.B.1 and B.2 are cited from app. C in [16]. App.B.3 is new.

B.1 $\mathcal{N}=1$ sector t-integral

In order to evaluate the t-integral of $\mathcal{N}=1$ sectors with $h_i \neq 0$ (i.e. for $\mathcal{K}_t$ and $\mathcal{A}_{95}$) we need the integral (assuming $0<\gamma<1$)

$$I=\int_{\frac{1}{e_\sigma \Lambda}}^{\infty}\frac{dt}{t^2}\frac{\vartheta_4'(\gamma,\tau_\sigma)}{\vartheta_4(\gamma,\tau_\sigma)} \tag{204}$$

with $\sigma=\mathcal{K},\mathcal{A}$ and $\tau_\sigma=\frac{ie_\sigma t}{2}$ (e_σ was defined in (138)). Evaluating this integral follows very closely a similar calculation in app. M of [4]. By modular transformation of the Jacobi θ function (using (196), $\vartheta_4=\vartheta[{0 \atop \frac{1}{2}}]$ and $\vartheta_2=\vartheta[{\frac{1}{2} \atop 0}]$) we have

$$\frac{\vartheta_4'(\gamma,ie_\sigma t/2)}{\vartheta_4(\gamma,ie_\sigma t/2)}=-4\pi\gamma l-2il\frac{\vartheta_2'(-2i\gamma l,2il)}{\vartheta_2(-2i\gamma l,2il)}, \tag{205}$$

where $l\equiv\frac{1}{e_\sigma t}$. Using the representation from $|\Im(z)|<\Im(\tau_\sigma)$

$$\begin{aligned}\frac{\vartheta_2'(z)}{\vartheta_2(z)}=&-\pi\tan\pi z+4\pi\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1-q^n}\sin 2\pi nz\\ =&-\pi\tan\pi z+4\pi\sum_{n,m=1}^{\infty}(-1)^n q^{nm}\sin 2\pi nz\end{aligned} \tag{206}$$

we arrive at

$$\begin{aligned}I=&\int_{\frac{1}{e_\sigma \Lambda}}^{\infty}\frac{dt}{t^2}\frac{\vartheta_4'(\gamma,\tau_\sigma)}{\vartheta_4(\gamma,\tau_\sigma)}\\ =&e_\sigma\int_0^{\Lambda}dl\Big(-4\pi\gamma l-2il\frac{\vartheta_2'(-2i\gamma l,2il)}{\vartheta_2(-2i\gamma l,2il)}\Big)\\ =&-2\pi e_\sigma\int_0^{\Lambda}dl\ l\Big(2\gamma-\tanh(2\pi\gamma l)+4\sum_{n,m=1}^{\infty}(-1)^n e^{-4\pi lnm}\sinh(4\pi n\gamma l)\Big).\end{aligned} \tag{207}$$

Let us start with the last term, which is free of UV divergences (so we can set $\Lambda=\infty$):

$$\begin{aligned}I_1=&-8\pi e_\sigma\int_0^{\infty}dll\sum_{n,m=1}^{\infty}(-1)^n e^{-4\pi lnm}\sinh(4\pi n\gamma l)\\ =&-\pi e_\sigma\sum_{n,m=1}^{\infty}\frac{(-1)^n\gamma m}{(\gamma^2-m^2)^2 n^2\pi^2}\\ =&-\pi e_\sigma\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2}\right)\left(\sum_{m=1}^{\infty}\frac{\gamma m}{(\gamma^2-m^2)^2}\right)\\ =&-e_\sigma\frac{\pi}{48}[\psi'(1+\gamma)-\psi'(1-\gamma)]\\ =&-e_\sigma\frac{\pi}{48}\left[\psi'(\gamma)-\psi'(1-\gamma)-\frac{1}{\gamma^2}\right].\end{aligned} \tag{208}$$

Here $\psi'(x)$ denotes the trigamma function as in 4.5 and in the last line we used $\psi'(1+\gamma)=\psi'(\gamma)-1/\gamma^2$.

Now let us look at the first and second term in (207):

$$I_2 = -2\pi e_\sigma \int_0^\Lambda dll(2\gamma) = -2\pi e_\sigma \gamma\Lambda^2, \tag{209}$$

$$\begin{aligned} I_3 = &\ 2\pi e_\sigma \int_0^\infty dll \tanh(2\pi\gamma l) \\ = &\ e_\sigma \left[-\frac{\pi}{48\gamma^2} + \pi\Lambda^2 + \frac{\Lambda\log(1+e^{-4\gamma\Lambda\pi})}{\gamma} - \frac{Li_2(-e^{-4\gamma\Lambda\pi})}{4\gamma^2\pi} \right] \\ \overset{\Lambda\to\infty}{=} &\ e_\sigma \left[-\frac{\pi}{48\gamma^2} + \pi\Lambda^2 \right]. \end{aligned} \tag{210}$$

Here $Li_2(z)$ is the dilogarithm function. In the second equality we used that the third and last term vanish as $\Lambda\to\infty$.

In total we obtain

$$\int_{\frac{1}{e_\sigma\Lambda}}^\infty \frac{dt}{t^2}\frac{\vartheta_4'(\gamma, ie_\sigma t/2)}{\vartheta_4(\gamma, ie_\sigma t/2)} = I_1 + I_2 + I_3 = e_\sigma\pi(1-2\gamma)\Lambda^2 - e_\sigma\frac{\pi}{48}[\psi'(\gamma) - \psi'(1-\gamma)]. \tag{211}$$

B.2 $\mathcal{N}=2$ sector t-integral

The t-integrals appearing in $\mathcal{N}=2$ sectors are very similar to those determining the $\mathcal{N}=2$ sector corrections to the Kähler metric calculated in [6]. Concretely, they are given by

$$\begin{aligned} \Gamma[n, M/W] &= \int_0^\infty \frac{dt}{t^3} \sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} e^{-\frac{\pi}{t}m^a m^b g_{ab}^{[nM/W]}} \\ &= \sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} \int_0^\infty \frac{dt}{t^3} e^{-\frac{\pi}{t}m^a m^b g_{ab}^{[nM/W]}} \\ &= \frac{1}{\pi^2} \sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} \frac{1}{\left(m^a m^b g_{ab}^{[nM/W]}\right)^2}. \end{aligned} \tag{212}$$

The metric $g_{ab}^{[nM/W]}$ is given by (151). Using (151) and the expression for $g_{ab}^{[n]}$ in terms of the complex structure $U^{[n]} = U_1^{[n]} + iU_2^{[n]}$ of n-th torus, i.e.

$$g_{ab}^{[n]} = \frac{\sqrt{\det g^{[n]}}}{U_2^{[n]}} \begin{pmatrix} 21 & U_1^{[n]} \\ U_1^{[n]} & |U^{[n]}|^2 \end{pmatrix}, \tag{213}$$

one can write

$$g_{ab}^{[n,M/W]} = \begin{cases} \frac{\sqrt{\det g^{[n]}}}{U_2^{[n]}} \begin{pmatrix} 21 & U_1^{[n]} \\ U_1^{[n]} & |U^{[n]}|^2 \end{pmatrix} & \text{for M (momentum sum)} \\ \frac{1}{U_2^{[n]}\sqrt{\det g^{[n]}}} \begin{pmatrix} 21 & \tilde{U}_1^{[n]} \\ \tilde{U}_1^{[n]} & |\tilde{U}^{[n]}|^2 \end{pmatrix} & \text{for W (winding sum)} \end{cases} \tag{214}$$

with $\tilde{U}^{[n]} = \tilde{U}_1^{[n]} + i\tilde{U}_2^{[n]} = -(U^{[n]})^{-1}$ (i.e. $\tilde{U}_1^{[n]} = -U_1^{[n]}/|U^{[n]}|^2$ and $\tilde{U}_2^{[n]} = U_2^{[n]}/|U^{[n]}|^2$). $\tilde{U}^{[n]} = -(U^{[n]})^{-1}$ follows from the fact that $g_{ab}^{[n,W]}$ is the inverse matrix of $g_{ab}^{[n,M]}$.

Then we obtain

$$\sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} \frac{1}{\left(m^a m^b g_{ab}^{[n,M/W]}\right)^2} = \begin{cases} \frac{1}{\det g^{[n]}} E_2(U^{[n]}) & \text{for M (momentum sum)} \\ \det g^{[n]} E_2(-(U^{[n]})^{-1}) & \text{for W (winding sum)} \end{cases}. \tag{215}$$

Here $E_s(U)$ is the non-holomorphic Eisenstein series

$$E_s(U) = \sum_{\vec{m}\in\mathbb{Z}^2\backslash\vec{0}} \frac{U_2^s}{|m^1 + m^2 U|^{2s}}. \tag{216}$$

Therefore, from (212) and using $\sqrt{\det g^{[n]}} = \frac{V_n}{4\pi^2\alpha'}$, we obtain

$$\Gamma^{[n,M/W]} = \begin{cases} \frac{(4\pi^2\alpha')^2}{\pi^2 V_n^2} E_2(U^{[n]}) & \text{for M (momentum sum)} \\ \frac{V_n^2}{\pi^2(4\pi^2\alpha')^2} E_2(-(U^{[n]})^{-1}) & \text{for W (winding sum)} \end{cases}. \tag{217}$$

B.3 t-integral for $\mathcal{M}$ with $\gamma > \frac{1}{2}$

When $\frac{1}{2} < \gamma < 1$ for $\mathcal{M}$, we need to do the integral

$$\tilde{I}_{\mathcal{M}} = \int_0^\infty \frac{dt}{t^2} \frac{\vartheta_1'(\gamma, \tau_{\mathcal{M}})}{\vartheta_1(\gamma, \tau_{\mathcal{M}})}, \tag{218}$$

here $\tau_{\mathcal{M}} = \frac{it}{2} + \frac{1}{2}$. We substitute $\gamma' = \gamma - \frac{1}{2}$ for γ, and this transforms the original integral to

$$\tilde{I}_{\mathcal{M}} = \int_0^\infty \frac{dt}{t^2} \frac{\vartheta_2'(\gamma', \tau_{\mathcal{M}})}{\vartheta_2(\gamma', \tau_{\mathcal{M}})}. \tag{219}$$

By following the similar calculation in app. M.2 of [4], we perform ST^2S modular transformations:

$$\tau_{\mathcal{M}} = \frac{it}{2} + \frac{1}{2} \to -\frac{1}{\tau_{\mathcal{M}}} \to -\frac{1}{\tau_{\mathcal{M}}} + 2 \to \left(\frac{1}{\tau_{\mathcal{M}}} - 2\right)^{-1} = 2il - \frac{1}{2} =: l_{\mathcal{M}}. \tag{220}$$

Here $l = \frac{1}{4t}$. The result of ST^2S modular transformation (197) is

$$\frac{\vartheta_2'(\gamma', \tau_{\mathcal{M}})}{\vartheta_2(\gamma', \tau_{\mathcal{M}})} \overset{l=\frac{1}{4t}}{=} -16\pi\gamma' l + 4il \frac{\vartheta_2'(4i\gamma' l, 2il - \frac{1}{2})}{\vartheta_2(4i\gamma' l, 2il - \frac{1}{2})}. \tag{221}$$

Using the representation of ϑ_2'/ϑ_2 for $|\Im(z)| < \Im(\tau_\sigma)$(cf. (206)) we get

$$\begin{aligned} \tilde{I}_{\mathcal{M}} &= \int_{\frac{1}{4\Lambda}}^\infty \frac{dt}{t^2} \frac{\vartheta_2'(\gamma', \tau_{\mathcal{M}})}{\vartheta_2(\gamma', \tau_{\mathcal{M}})} \\ &= 4\int_0^\Lambda dl \left(-16\pi\gamma' l + 4il \frac{\vartheta_2'(4i\gamma' l, 2il - \frac{1}{2})}{\vartheta_2(4i\gamma' l, 2il - \frac{1}{2})} \right) \\ &= -16\pi \int_0^\Lambda dl\ l \Big(4\gamma' - \tanh(4\pi\gamma' l) + 4\sum_{n,m=1}^\infty (-1)^{n(m+1)} e^{-4\pi lnm} \sinh(8\pi n\gamma' l) \Big) \end{aligned} \tag{222}$$

The integral of first and second term are the same to the term in (207). Now, let's look at the third term. Following the similar calculation as eq. (397) and (398) in [4]:

$$\begin{aligned} I_1 &= \sum_{m,n=1}^\infty \int_0^\infty dl\ l(-1)^{n(m+1)} e^{-4\pi lnm} \sinh(8\pi n\gamma' l) \\ &= \sum_{m,n=1}^\infty (-1)^{n(m+1)} \frac{m\gamma'}{4n^2\pi^2(4\gamma'^2 - m^2)^2} \\ &= \sum_{m=1}^\infty \frac{m\gamma' \mathrm{Li}_2((-1)^{m+1})}{4\pi^2(4\gamma'^2 - m^2)^2}. \end{aligned} \tag{223}$$

Note that the integral converges provided that $2|\gamma'| \leq m$ (which is true now because $\gamma' = \gamma - \frac{1}{2}$). Now we split the sum into sums over even and odd m:

$$I_1 = \sum_{k=1}^{\infty} \left[\frac{(2k)\gamma' \mathrm{Li}_2(-1)}{4\pi^2(4\gamma'^2 - (2k)^2)^2} \right] + \sum_{k=0}^{\infty} \left[\frac{(2k+1)\gamma' \mathrm{Li}_2(1)}{4\pi^2(4\gamma'^2 - (2k+1)^2)^2} \right]$$
$$= \frac{1}{1536} \Big[\psi'(1+\gamma') - \psi'(1-\gamma') \Big] + \frac{1}{768} \Big[\psi'(\frac{1}{2} - \gamma') - \psi'(\frac{1}{2} + \gamma') \Big]. \tag{224}$$

All together we arrive at

$$= 8\pi(1-4\gamma')\Lambda^2 - \frac{\pi}{24\gamma'^2} - \frac{\pi}{24} \Big[\psi'(1+\gamma') - \psi'(1-\gamma') + 2\psi'(\frac{1}{2} - \gamma') - 2\psi'(\frac{1}{2} + \gamma') \Big]$$
$$= 8\pi(3-4\gamma)\Lambda^2 - \frac{\pi}{24} \Big[\psi'(\gamma - \frac{1}{2}) - \psi'(\frac{3}{2} - \gamma) + 2\psi'(1-\gamma) - 2\psi'(\gamma) \Big]. \tag{225}$$

C $\mathbb{Z}_N$ actions in $D = 4$

In the table are the twist vectors for different $\mathbb{Z}_N$ orbifold type-IIB string models on T^6 ($D = 4$ space-time dimensions with 6 compact dimensions).

$\mathbb{Z}_3$	$\frac{1}{3}(1,1,2)$	$\mathbb{Z}'_6$	$\frac{1}{6}(1,-3,2)$	$\mathbb{Z}'_8$	$\frac{1}{8}(1,-3,2)$
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,-2)$	$\mathbb{Z}_7$	$\frac{1}{7}(1,2,-3)$	$\mathbb{Z}_{12}$	$\frac{1}{12}(1,-5,4)$
$\mathbb{Z}_6$	$\frac{1}{6}(1,1,-2)$	$\mathbb{Z}_8$	$\frac{1}{8}(1,3,-4)$	$\mathbb{Z}'_{12}$	$\frac{1}{12}(1,5,-6)$

Cited from Table 2 of [1]. Only $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}'_6$, $\mathbb{Z}_7$, $\mathbb{Z}_{12}$ models are tadpole-free, which is discussed in [1].

References

[1] G. Aldazabal, A. Font, Luis E. Ibanez, and G. Violero. D = 4, N=1, type IIB orientifolds. *Nucl. Phys.*, B536:29–68, 1998.

[2] Ignatios Antoniadis, C. Bachas, C. Fabre, H. Partouche, and T. R. Taylor. Aspects of type I - type II - heterotic triality in four-dimensions. *Nucl. Phys.*, B489:160–178, 1997.

[3] Ignatios Antoniadis, S. Ferrara, R. Minasian, and K. S. Narain. R**4 couplings in M and type II theories on Calabi-Yau spaces. *Nucl. Phys.*, B507:571–588, 1997.

[4] Marcus Berg, Michael Haack, Jin U Kang, and Stefan Sjörs. Towards the one-loop Khler metric of Calabi-Yau orientifolds. *JHEP*, 12:077, 2014.

[5] Marcus Berg, Michael Haack, and Boris Kors. Loop corrections to volume moduli and inflation in string theory. *Phys. Rev.*, D71:026005, 2005.

[6] Marcus Berg, Michael Haack, and Boris Kors. String loop corrections to Kahler potentials in orientifolds. *JHEP*, 11:030, 2005.

[7] Ralph Blumenhagen, Boris Kors, Dieter Lust, and Stephan Stieberger. Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes. *Phys. Rept.*, 445:1–193, 2007.

[8] Ralph Blumenhagen, Dieter Lüst, and Stefan Theisen. *Basic Concepts of String Theory.* Springer, 2013.

[9] Ralph Blumenhagen and Erik Plauschinn. *Introduction to Conformal Field Theory.* Springer, 2009.

[10] Atish Dabholkar. Lectures on orientifolds and duality. In *High-energy physics and cosmology. Proceedings, Summer School, Trieste, Italy, June 2-July 4, 1997*, pages 128–191, 1997.

[11] Lance J. Dixon, Jeffrey A. Harvey, C. Vafa, and Edward Witten. Strings on Orbifolds. 2. *Nucl. Phys.*, B274:285–314, 1986.

[12] Friedel T. J. Epple. Induced gravity on intersecting branes. *JHEP*, 09:021, 2004.

[13] A. Font, Luis E. Ibanez, F. Quevedo, and A. Sierra. The Construction of 'Realistic' Four-Dimensional Strings Through Orbifolds. *Nucl. Phys.*, B331:421–474, 1990.

[14] Eric G. Gimon and Joseph Polchinski. Consistency conditions for orientifolds and d manifolds. *Phys. Rev.*, D54:1667–1676, 1996.

[15] Florian Gmeiner and Gabriele Honecker. Mapping an Island in the Landscape. *JHEP*, 09:128, 2007.

[16] Michael Haack and Jin U Kang. One-loop Einstein-Hilbert term in minimally supersymmetric type IIB orientifolds. *JHEP*, 02:160, 2016.

[17] Elias Kiritsis. *String Theory in a Nutshell.* Princeton University Press, 2007.

[18] Joseph Polchinski. *String Theory: Volume 1, An Introduction to the Bosonic String*, volume 1. Cambridge University Press, 2005.

[19] Joseph Polchinski. *String Theory: Volume 2, Superstring Theory and Beyond*, volume 2. Cambridge University Press, 2005.