Master Thesis

Monodromy of q-difference equations in 3D supersymmetric gauge theories

Alexander Tabler

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First Supervisor: **Prof. Dr. Peter Mayr**

Second Supervisor: **Prof. Dr. Ilka Brunner**

Ludwigs-Maximilans-Universität München

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Alexander Tabler
Matrikelnummer 11221078

"In physics we don't need to prove things,
we just need to be right."

Nathan Seiberg, January 2017

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Abstract

Localization techniques for supersymmetric quantum field theories allow one to produce non-perturbative results such as computing partition functions exactly, in stark contrast to general field theories. In many two-dimensional examples of supersymmetric theories, the path integral or partition function is related geometric invariants and appears as a solution to certain differential equations with geometric and physical interpretation. Recently a program has been initiated to "lift" these constructions from two- to three-dimensional theories. Beem, Dimofte and Pasquetti argued that the natural 3D analogue of the differential equations whose solutions determine the partition function in two-dimensions are q-difference equations, i.e. functional equations involving q-shifts. Their structure and features resemble those of differential equations, but their general theory is less developed. Nevertheless, the global behavior of their solutions is expected to be relevant and interesting for physics, as in the case of differential equations. The aim of this work is to study the monodromy of certain examples of q-difference equations and its relevance to supersymmetric gauge theories. We review the setting in which they arise in physics, and then we provide a preview of the techniques to be used by studying the monodromies of differential equations. Finally we compute the connection matrix, the q-analogue of monodromy, for two examples of q-difference equations and discuss their physical meaning.

Contents

Notation

The q-analogue of a number $z \in \mathbb{C}$ is
$$[z]_q \coloneqq \frac{1-q^z}{1-q},$$
with $q \in \mathbb{C}^* \setminus S^1$.

q-Pochhammer symbol with integer index:
$$(z;q)_n = \begin{cases} (1-z)(1-qz)\dots(1-q^{n-1}z) & n = 1,2,\dots \\ 1 & n = 0 \\ [(1-zq^{-1})(1-zq^{-2})\dots(1-zq^{n})]^{-1} & n = -1,-2,\dots. \end{cases}$$

q-Pochhammer symbol with infinite index:
$$(z;q)_\infty = \begin{cases} \prod\limits_{k=0}^{\infty}(1-zq^k), & \text{if } |q|<1 \\ \prod\limits_{k=1}^{\infty}\frac{1}{(1-zq^{-k})}, & \text{if } |q|>1 \end{cases}$$

The inversion formula
$$(z;q)_\infty = \frac{1}{(pz;p)_\infty}, \quad q = p^{-1}.$$

q-Pochhammer symbol with complex index:
$$(z;q)_\alpha = \frac{(z;q)_\infty}{(q^\alpha z;q)_\infty}, \quad |q| \gtrless 1$$
$$(z;q)_\alpha = (pz;p)_{-\alpha}^{-1}, \quad p = q^{-1}.$$

Notation for products:
$$(z_1,\dots,z_n;q)_\alpha = (z_1;q)_\alpha \dots (z_n;q)_\alpha\,,$$
which we also use for repeated functions like $\Gamma(x)\Gamma(y) = \Gamma(x,y)$.

The derivative symbol
$$\vartheta_z = z\frac{\mathrm{d}}{\mathrm{d}z}.$$

The q-shift operator:
$$\sigma_q : f(z) \mapsto f(qz), \quad \sigma_q \equiv q^{\vartheta_z}$$
also denoted as $\hat{p}$ in Part I.

The q-derivative operator:
$$\mathcal{D}_q = \frac{1}{z}\frac{\sigma_q - 1}{q-1}.$$

The (classical) binomial coefficient:
$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1,n-m+1)}, \quad n,m \in \mathbb{C}.$$

The q-binomial coefficient:
$$\binom{n}{m}_q = \frac{(q;q)_n}{(q;q)_m\,(q;q)_{n-m}} \equiv \frac{\Gamma_q(n+1)}{\Gamma_q(m+1,n-m+1)}, \quad n,m \in \mathbb{C}.$$

Throughout the text we will 'pick logarithms' in their principal branch:

$$a = q^{\alpha}, b = q^{\beta}, c = q^{\gamma}, \text{etc.}$$

Often we will parametrize

$$q = e^{\hbar}$$

with $\hbar \in \mathbb{C}$.

The Heine series or basic hypergeometric series

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \;\middle|\; q; z\right) := \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(c, q; q)_n} z^n, \quad |z| < 1$$

can be generalized to the generalized basic hypergeometric series

$$ {}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1 \dots, b_s \end{matrix} \;\middle|\; q; z\right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s, q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n.$$

1. Introduction

Quantum field theories are perhaps the most important object of study in modern theoretical physics. From the standard model of particle physics, to condensed matter physics, string theory and quantum gravity, it is the modern framework physicists use to describe the world. The interaction between physics and mathematics as experienced by a physics student usually boils down to a daily (ab)use of well-established mathematical framework to solve and understand physical questions. A shining exception to this trend are classical and quantum field theories that have also "returned the favor" to mathematics. Field theories studied in physics have become an ample source of questions and directions for pure mathematics, whose investigation has lead to breakthrough results in both mathematics and physics. This surge of results has lead to the revitalization physicists' interest in pure mathematics and mathematicians' interest in physics, which in turn has given birth to a field of study best described by the term "Physical Mathematics" coined by string theorist Greg Moore*. It is therefore easy to see why a deeper and more detailed understanding of quantum field theories is imperative, even if particular examples have little hope to ever contribute to a phenomenological application.

A particular class of very well studied quantum field theories are *supersymmetric gauge theories* and *supersymmetric conformal field theories*, both in mathematics and physics. As their name suggests, they are quantum field theories with *supersymmetry* allowing for the interchange between bosons and fermions, as well as internal *gauge* symmetries or conformal symmetry respectively. The main motivation for their study in physics comes from string theory: supersymmetric gauge theories and conformal theories describe string compactification and the low energy effective field theory. The study has lead to a number of seminal breakthroughs in the context of string theory and pure mathematics inspired by string theory, the most notable of which is *mirror symmetry*.

The extended symmetry allows supersymmetric theories to be further analyzed than their "less symmetric" counterpart field theories, mainly through the emergence of powerful computational tools like localization and deformation invariance. Despite their "distance" from experimental data, the study of supersymmetric theories has lead to deeper insight of the general structure of quantum field theories, and perhaps will pave the way for a *general, precise* definition of a quantum field theory, which is still elusive.

A fundamental object in a quantum field theory is the *path integral* without insertions or *partition function*. It serves as a generating function for *correlation functions* of operators corresponding to observables. For general non-supersymmetric field theories one can compute it only perturbatively as it involves functional integration on an infinite dimensional space. For theories with supersymmetry, or more generally a nilpotent fermionic symmetry, under assumptions of genericity the path-integral *localizes* to a finite dimensional integral which can be computed *exactly*, including non-perturbative contributions. Often, the partition functions of theories defined on specific geometries are define or are related to *invariants* associated to the geometries.

In supersymmetric field theories in two dimensions (with $\mathcal{N} = (2,2)$ supersymmetry), there has been further progress in the *systematic* study of the theories, their space of inequivalent vacua and the computation of their partition functions. In cases of interest, where the two-dimensional field theory admits a so-called geometric phase, the partition functions are defined on general grounds by differential equations-constraints, similarly to Ward identities constraining correlation functions in quantum field theories. The *global behavior* of their solutions, i.e. *monodromies* or *Stokes data*, encodes physically interesting information.

A general goal of the study of supersymmetric gauge theories, is to "lift" the established results and constructions to *three*-dimensional supersymmetric gauge theories. In a particular class of three-dimensional gauge theories which we study, one may indeed find counterparts to two-dimensional "machinery" as well

* http://www.physics.rutgers.edu/~gmoore/PhysicalMathematicsAndFuture.pdf

as novel tools to help understand the theories and in particular, compute their partition functions. In this particular class, the three-dimensional counterpart of differential equations defining/constraining the partition functions are "lifted" to *q-difference equations*.

The study of q-difference equations is somewhat lacking in progress compared to that of their "sister" differential equations. Yet, their structure and their features usually go "hand in hand": q-difference equations can be thought of as one-parameter deformation of differential equations and correspondingly, differential equations arise as a limit of q-difference equations. One thus often tries to study q-difference equations with a "compass" provided by more concrete results of differential equations. As in the case of differential equations, it becomes a mathematically and physically interesting question to ask about the global data of q-difference equations: q-monodromy and q-Stokes phenomena. Do these objects exist? What is their mathematical meaning? What physical information do they encode? Although we give no conclusive answers to these questions in this work, we hope to illuminate some aspects of these questions, in particular about the monodromy of q-difference equations. As a last note, we mention that the appearance of q-difference equations in (supersymmetric) gauge theories is one of many (possibly related) instances: q-difference equations play an intimate role in representation theory of quantum groups (q-Knizhnik-Zamolodchikov equations), quantum cohomology groups, knot invariants, the Geometric Langlands program. We expect in the future their presence in physical context to be ubiquitous and therefore we strongly support their further study, both for mathematics and for physics.

The outline of this work is as follows: Part I is devoted to background material involving the appearance of q-difference equations in gauge theories, as well as some results related to our later study of particular cases of q-difference equations. Chapters 2 and 3 are a short exposition of a three-dimensional gauge-theoretic "recipe" to compute exact partition functions given by Beem, Dimofte and Pasquetti. Chapter 4 gives an ultra-short exposition of two-dimensional gauged linear sigma models, a particular case of which will serve as a motivation for the study of a q-difference equation. In Part II, we first devote chapter 5 to the study of monodromy of *differential equations*. This investigation serves as a guide for our subsequent study of q-difference equations in chapter 6, where the bulk of our work lies. In chapter 6, we first introduce some generalities on q-difference equations, q-difference systems and their (local) solutions. We then study two particular cases of q-difference equations motivated by the physical settings presented in Part I. We study the monodromy properties of these equations, in the form of the *connection matrices*, adapting existing literature. We use techniques developed by mathematicians to derive a new connection matrix as a degenerate limit. Lastly, we end with a discussion and outlook of further directions of study.

Part I

Background

2. A perturbative calculation of 3D partition functions

In this and the following chapters we want to give a brief exposition of the existing literature that serves as the main motivation for this work. In particular we will present the main ideas and some technicalities presented in the seminal work of Beem, Dimofte and Pasquetti in [BDP14]. This work is one instance of appearance of q-difference equations with applications to gauge theory, and our study of q-difference equations in later chapters will revolve around applications as presented in this paper. This chapter is devoted to the perturbative calculation of partition functions by reduction to quantum mechanics, while the next one is reserved for the non-perturbative completion of the calculation.

2.1 Outline: A three-dimensional lift

We first give some overview of the work, as well as some general remarks about the existing literature. The main aim of our work in broad terms, as well as that in [BDP14], is to study gauge theories, compute (often topological) invariants like partition functions and indices. Of particular interest are superconformal $\mathcal{N} = 2$ gauge theories on curved three-dimensional manifolds that still preserve some supersymmetry, specifically three-manifolds that are realized as fibrations over an S^1 base. Such theories and also supersymmetric gauge theories in general have been the subject of intense research in the past years, in particular because one can under certain assumptions obtain *exact*, non-perturbative results from these theories [Tes14]. This is in stark contrast with traditional quantum field theories, where perturbative methods are the main "game in town" to obtain correlation functions, partition functions and understanding of the theory in general.

Techniques, in particular like localization, from theories on curved manifolds with *deformed* supersymmetry algebras — to accommodate the curvature — have been used to produce interesting results in low-dimensional supersymmetric field theories [Pas12; DGG11a; DGG11b]. Here, we focus on $\mathcal{N} = 2$ SCFTs and their *massive* deformations are studied. The field content is determined as follows: for every $U(1)$ subgroup of the flavor symmetry group it is possible to turn on a real mass deformation [Ton00; DT00]. In particular, one needs theories where such deformations are sufficient to induce strictly gapped vacua and one also requires that the $U(1)_R$ R-symmetry is preserved (Examples: $\mathcal{N} = 2$ SQED, SQCD, and more). The backgrounds that are studied are the *ellipsoid* S^3_b and the *twisted product* $S^2 \times_q S^1$ where $\log q$ denotes the holonomy with which the S^2 fibers over S^1. The partition function of S^3_b (of form $\sim \mathrm{tr}_{\mathcal{H}}\, e^{-\beta H}$ plus insertions) and the superconformal index of $S^2 \times_q S^1$ ($\sim \mathrm{tr}_{\mathcal{H}}(-1)^F e^{-\beta H}$ plus insertions) have been computed in the literature and can be rederived through the methods developed in [BDP14].

The main ideas in the work of Beem, Dimofte and Pasquetti can be summarized as:

1. The calculation of the partition function (or index) of a $\mathcal{N} = 2$ supersymmetric gauge theory T_M on a certain class of three-dimensional manifolds M^3 (including products and twisted products of the form $\mathcal{C} \times S^1$, where $\mathcal{C}$ is a Riemann surface) with "enough" flavor symmetry can be calculated, by the computing quantities called *"holomorphic blocks"* $B(x; q)$ instead of the partition function itself. Here, x is a general symbol for the exponentiated masses (deformations) that were introduced for $U(1)$ factor in the flavor symmetry group and q a general symbol for the *fugacities* associated to the global symmetries. The partition function is then computed as a "sum of products" of such blocks

$$\mathcal{Z}_{M^3}(\text{parameters}) = \sum_{\alpha} B^{\alpha}(x; q) B^{\alpha}(\tilde{x}; \tilde{q}) =: \|B^{\alpha}(x; q)\|^2_g \,.$$

The summation is over the label α, which as we will see labels the *discrete* classical vacua of the (effective) theory on one lower dimension, and the holomorphic blocks $B^\alpha(x;q), B^\alpha(\tilde{x};\tilde{q})$ are the *same, universal* functions. Their different arguments $x, q, \tilde{x}, \tilde{q}$ related with the parameters of the theory. The very right-hand side of the equation is indicative notation which we explain later.

This bold-looking statement is not so unfamiliar: the name "blocks" already refers to standard two-dimensional conformal field theories where one can express higher-order correlators from sums of simple products of the so-called conformal blocks of the form $\sum F_i j C_i(z) C_j(\bar{z})$, with clear holomorphic $\times$ anti-holomorphic structure. A very much related phenomenon is the main result of the so called Alday-Gaiotto-Tachikawa (AGT) correspondence [AGT10; Tes14] where the authors found that four-point functions of two-dimensional Liouville field theories (CFTs) can be expressed as an integral (sum) of products of partition functions of four-dimensional supersymmetric gauge theories. From a computational point of view, these examples have the same structure: computing a quantity as a sum of products of (generally) simpler, *universal* functions where only the arguments of said functions need to be identified appropriately. In the context of conformal blocks, the identification is simply complex conjugation. In more complicated settings, this identification scheme depends on the field content as well as the geometry, i.e. the choice of background manifold, metric etc. This apparently ubiquitous feature is called *factorization* of partition functions/indices, and the statement that partition functions are of this form is called the *factorization conjecture*.

2. The blocks $B^\alpha(x;q)$ also necessarily have interesting properties themselves. They are called holomorphic because we will demand that they satisfy certain analyticity constraints with respect to the arguments x. More interestingly, they are in fact partition functions themselves of theories (deformed from the original) on the "pieces" coming from the Heegaard splitting of the initial manifold M^3 into two handlebodies, which in our case will always be of the form of a twisted product $\mathbb{R}^2 \times_q S^1$ (twisted solid torus). This what we will refer to as "cigar" or "cigar geometry". They take the form of a BPS index counting BPS states in the massively deformed vacua $B^\alpha(x;q) \sim \mathrm{tr}_{\mathcal{H}(\mathbb{R}^2;\alpha)}(-1)^R e^{-\beta H} q^{-J+\frac{R}{2}} x^e$, where R is the $U(1)_R$ generator, J is the $U(1)$ generator of the (massive) little group in 3d, x is the exponentiated mass associated to the flavor deformation and e is the $U(1)$ charge (in fact $x \sim e^{\text{deform.}+ie\int A}$ where A is a gauged flavor symmetry field and e is the generator (charge) of the corresponding flavor symmetry). The full partition function is then computed by the "sum of products" factorization formula, which we can think of as gluing the two Heegaard pieces (and the gauge theories defined on them) along their common boundary T^2 back together. The gluing then involves an element g of the automorphism group $\mathbb{P}SL(2;\mathbb{Z})$ of the torus.

3. The blocks are computed by reducing their calculation to an amplitude computation in supersymmetric quantum mechanics on the *boundary* T^2 of the two Heegaard pieces. This involves some more steps which we will lay out later. In short, one "stretches" the geometry of each piece of $\mathbb{R}^2 \times_q S^1 \cong D^2 \times_q S^1$. After reducing the S^1 factor by Kaluza-Klein (KK) reduction, one can interpret the "stretch" as the "infinite time" limit of a supersymmetric quantum mechanical theory "living" on the boundary T^2, which in turn projects the states on the T^2 boundary $|0_q\rangle$ on the *exact* SUSY groundstates $|\alpha\rangle$ [CV91; Hor03; Wit10]. The block then can be computed precisely by the overlap of these states
$$B^\alpha(x;q) = \langle 0_q|\alpha\rangle\,,$$
while the full partition function is the overlap
$$\mathcal{Z}_{M^3} = \langle 0_q|0_{\tilde{q}}\rangle = \sum_\alpha \langle 0_q|\alpha\rangle\,\langle\alpha|0_{\tilde{q}}\rangle = \sum_\alpha B^\alpha(x;q)B^\alpha(\tilde{x};\tilde{q}) =: \|B^\alpha(x;q)\|^2_g\,. \tag{2.1}$$
The most important and useful property of the blocks is that they satisfy certain q-difference equations. We will explore the appearance and construction of these equations in detail.

4. In practice, the above methods translate to the following procedure: writing $q = e^{\hbar}$* one can "sum-up" the contributions to the quantum mechanical superpotential $\mathcal{W}^{\text{QM}} \sim \frac{1}{\hbar}[\ldots] + \mathcal{O}(\hbar^0)$, from which one can then also determine the (classical) vacua by the vacuum equations. The computation of the partition function (block) then localizes along the gradient flow lines of (the imaginary part of) the QM superpotential [Wit10] starting at each of the classical vacua α, and the path integral can be computed *peturbatively* in $\hbar$.

Supplementing the aforementioned vacuum equations with equations that determine the Fayet-Iliopoulos (FI) coupling constants (really, these are also mass deformations of the dual, topological

* This will be explained further later in subsection 3.1.1.

$U(1)$ symmetry groups), one obtains polynomial constraints $f(x;p) = 0$ in the (exponentiated) masses of the flavor group x and the effective FI parameters p. These polynomial constraints are then promoted to q-difference operators by enforcing that they obey the q-difference algebra $\hat{x}_i\hat{p}_j = q^{-\delta_{ij}}\hat{p}_j\hat{x}_i$. As it turns out, the full, non-perturbative (localized) path-integral calculation is equivalent to solving the q-difference constraints given by the "quantized" $f(\hat{x};\hat{p}) \cdot B^\alpha(x;q) = 0$ polynomials. A complete dictionary is presented to

(a) collect and sum up the relevant terms for the QM superpotential $\mathcal{W}^{\mathrm{QM}}$ from the field content,

(b) read-off the relevant q-difference constraints $f(\hat{x};\hat{p})$ from $\mathcal{W}^{\mathrm{QM}}$,

(c) systematically solve the constraints $f(\hat{x};\hat{p}) \cdot B^\alpha(x;q) = 0$ by expressing the solutions (path-integrals) as contour integrals over an integrand that is the non-perturbative completion of the perturbative integrand, and thus obtain the blocks $B^\alpha(x;q)$, as well as prescribe the non-trivial cycles over which the integrands have to be integrated.

These are the main aspects of the innovative work of the authors in [BDP14].

2.2 Setup and Geometry

2.2.1 Field content

We first discuss more explicitly the theory which we are describing, always following [BDP14]: these are three-dimensional $\mathcal{N} = 2$ supersymmetric theories that are superconformal in the infrared and admit a Lagrangian description in the ultraviolet.

The setup consists of r *gauge vector multiplets* $\{V_a\}$ and *chiral matter multiplets* $\{\Phi_I\}$, and we assume that the gauge group is Abelian. The vector multiplet in three-dimensions can be written [IS13] as a *linear* multiplet $\Sigma_\alpha = \epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta V_a$ (where $D, \bar{D}$ denote standard covariant derivatives in supersymmetry [Hor03]). Using these, the kinetic part of the Lagrangian takes the form

$$\mathcal{L}_{\text{kin.}} = \int \mathrm{d}^4\theta \Big(\sum_{a=1}^{r} \frac{1}{e_a^2}\Sigma_a^2 + \sum_I \Phi_I^\dagger \big(\sum_a Q_I^a V_a \big) \Phi_I \Big),$$

while the F-term is as usual

$$\mathcal{L}_{\mathrm{F}} = \int \mathrm{d}^2\theta W(\theta) + \text{h.c.},$$

with argument such that the $U(1)_R$ symmetry is preserved.

"Gauging" of global symmetries These terms remain invariant under some manifest global symmetries (acting explicitly on fields), as well as *topological* $U(1)$ symmetries that shift the *dual gauge fields* of the gauge multiplets. If we consider the Cartan subgroup $\prod_i^N U(1)_i$ of the flavorpromoted symmetry group, we can introduce N non-dynamical fields A_i^μ that couple to these $U(1)_i$ currents, and similarly introduce background gauge fields A_R for the R-symmetry. These can be to background vector superfields $\hat{V}_i$ that are again written as a linear multiplet $\hat{\Sigma}_i$. When the scalar components $m_i^{3\mathrm{d}}$ of these multiplets are non-zero, they are *real mass deformations* of our theory and appear as $\int \mathrm{d}^4\theta \left(\Phi^\dagger e^{m^{3\mathrm{d}}\theta\bar{\theta}}\Phi \right)$ terms, leading to terms of the form $(m^{3\mathrm{d}})^2 |\Phi|^2 + i m^{3\mathrm{d}}\epsilon^{\alpha\beta}\bar{\psi}_\alpha\psi_\beta$. The mass deformations for the *topological* $U(1)$ symmetries appear as Fayet-Iliopoulos terms for the (dynamical) gauge fields. A requirement of the setup is that these real mass deformations render *all* the vacua massive, i.e. "lift" all the flat directions in the (classical) moduli space. We also require that the theory has *enough* flavor symmetry such that the mass deformations completely "lift" all flat directions in the moduli space. In particular, after dimensional reduction on a space of the form $M^2 \times S^1$ we want to have discrete massive vacua at generic points of the mass parameters.

Lastly, we can include (Abelian) gauge-invariant Chern-Simons (CS) terms of the form

$$\mathcal{L}_{CS} = \int \mathrm{d}^4\theta \Big(\frac{1}{2}k_{ab}\Sigma_a V_b + k_{ia}\Sigma_a\hat{V}_i + \frac{1}{2}k_{ij}\hat{\Sigma}_i\hat{V}_j \Big).$$

The first term is a gauge-gauge CS term (σ_a^2 contribution), the second is a gauge-flavor CS term, which is equivalent to a Fayet-Iliopoulos term for the dynamical gauge fields, and the last term is a 'pure' background flavor-flavor CS term. The pure background term induces so-called "contact terms" in the calculation of partition functions and we will discuss them later. One thing to note here is that the levels k_{ij} sometimes have to be chosen to be fractional, due to the appearance of the so-called "parity anomaly" when fermions are integrated out: the effective CS level becomes $k_{ij}^{\text{eff.}} = k_{ij} + \frac{1}{2}\sum_{\text{ferm.}} \text{charge}_i^f \text{charge}_j^f \operatorname{sign}(m_f)$ and *must* be an integer. Note that when the VEV of the fermion changes sign, the CS levels "jump".

2.2.2 Stretching and Deforming

As mentioned in the outline, one "splits" the three-manifold into two pieces via the Heegaard splitting. In particular, the cases of interest, the "squashed 3-sphere" $S_b^3 = \{b^2 |z_1|^2 + b^{-2} |z_2|^2 = 1$ and the twisted fibration $S^2 \times_q S^1$ as the gluings of solid tori $D^2 \times_q S^1$: In the case of S^3 the gluing map is the S element of the $\mathbb{P}SL(2,\mathbb{Z})$ plus an orientation reversal, while in the case of $S^2 \times S^1$ it is just the identity map id plus orientation reversal. This also signifies the use of the notation q for the fugacity later on: one can view q as the modular parameter $q = e^{2\pi i \tau}$ of the boundary tori, and thus in the S^3 case we have $\tau \mapsto \tilde{\tau} = -S \cdot \tau = -\frac{1}{\tau}$, while in the $S^2 \times_q S^1$ case we have $\tau \mapsto \tilde{\tau} = -\operatorname{id} \cdot \tau = -\tau$ where we distinguish the two halves by a tilde in the corresponding parameters.. We investigate partition functions on the "halves" $D^2 \times_q S^1$, as the factorization conjecture asserts that they should form building blocks for partition functions and supersymmetric indices on other three-dimensional theories. Of course this cannot work in the simple, naive way: The partition functions of the original theory on a solid torus will not correspond to the blocks; the solid tori cut from different three-manifolds do not have the same metric, and more importantly, the partition function of the pieces are *not* necessarily the same. One has to *deform* the Heegaard decomposition so that one obtains the factorization (2.1) exactly, with correct relations between (x, q) and $(\tilde{x}, \tilde{q})$. This is in fact the 3-dimensional analogue of the topological/anti-topological twisting (and fusion) of Cecotti and Vafa [CV91]. This construction requires describing the S_b^3 and $S^2 \times_q S^1$ as T^2 fibrations over an interval, where the cycles of T^2 smoothly become degenerate on the boundary, and the interval is "stretched" to infinite length. Therefore, each $D^2 \times_q S^1$ which is a D^2 fibration over $S^1 \equiv S_\beta^1$ (notation which we will use later), we view the disc as an "elongated" cigar, i.e. a *degenerate* $S^1 \equiv S_\rho^1$ fibration over a semi-infinite $\mathbb{R}_+$. The $0 \in \mathbb{R}_+$ is called the "tip" of the cigar, and we obtain the "degenerate" T^2 fibration over $\mathbb{R}_+$. The resulting space consists, topologically at least, of two such "halves" $D^2 \times_q S^1$ which are glued together via some element of the modular group of the boundary torus along with an orientation reversal. One chooses a metric that preserves the $U(1)$ rotations of D^2 as isometries, and the D^2 fibration is chosen with $U(1)$ holonomy q.

Holomorphic blocks are then defined to be the partition functions on these halves $D^2 \times_q S^1$. One can also interpret these partition functions as *wavefunctions* $\langle 0_q| \in \mathcal{H}(T^2)$ in the "flat" region $T^2 \times \mathbb{R}$ of the "stretched" $D^2 \times_q S^1$. The infinite stretching projects [Hor03; CV91] this wavefunction to the space of supersymmetric ground states $|\alpha\rangle$ on T^2, i.e.

$$B^\alpha(x;q) \coloneqq \langle 0_q|\alpha\rangle \, .$$

Gluing two such blocks via some modular element, say $g = S$ or $g = \text{id}$, we recover the expression

$$\mathcal{Z} = \langle 0_q|0_{\tilde{q}}\rangle = \sum_\alpha \langle 0_q|\alpha\rangle \langle\alpha|0_{\tilde{q}}\rangle \sim \sum_\alpha B^\alpha(x;q) B^\alpha(\tilde{x};\tilde{q}).$$

Evidently, the relations between the "tildes" depends on the choice of modular element g.

The two halves can be *naturally* fused together as long as the Hilbert spaces on their (asymptotic) boundaries are *dual*. Similarly to the two-dimensional case, an infinite "stretching" of the D^2 cigars to an infinte flat region $\mathbb{R}^2 \times_q S^1$ projects any state onto the ground state of the asymptotic boundary in $\mathcal{H}_0(T^2)$. The resulting partition function is thus *quasi-topological*, i.e. invariant under all but finite deformations of the $\mathcal{N} = 2$ theory and is by construction of factorized form, as above.

The cigar geometry

The observable we are interested in is the partition function on the twisted $D^2 \times_q S^1$, which is topologically an (open) solid torus. The metric is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + f(r)^2(\mathrm{d}\varphi + \epsilon\beta\mathrm{d}\theta)^2 + \beta^2\mathrm{d}\theta, \tag{2.2}$$

where $r \in [0, \infty)$, φ, θ are 2π-periodic and $f(r) \sim r$ for small r, and $f(r) \to \rho$ for large r, where ρ is a constant, whence the notation S^1_ρ earlier. In particular, note that this cycle S^1_ρ shrinks at the "tip" $r = 0$. This metric shows that the cigar D^2 has (asymptotic) radius ρ and the "twisting" of the product consists of a rotation of the D^2 by $2\pi\beta\epsilon$ around the S^1 (denoted as S^1_β — the "non-contractible loop") parametrized by θ. In other words, the variable $z = re^{i\varphi}$ is identified as

$$(z, 0) \sim (q^{-1}z, 2\pi), \quad \text{where } q = e^{2\pi i\epsilon\beta} = e^{\hbar}.$$

This space does *not* admit covariantly constant spinors [BDP14]; however, one can *twist* the theory to preserve some fermionic symmetry still be able to use localization techniques. There are again two choices for twisting, a "topological" and an "anti-topological", corresponding to preservation of different pairs of scalar supercharges ($(Q_-, \bar{Q}_+)$ or $(Q_+, \bar{Q}_-)$ respectively) [CV91; Hor03].

Twisting

As per usual [CV91], the twist is implemented by "mixing" the geometry — here, the spin connection — with the gauge group —here, the newly introduced "background" gauge fields coming from the R and flavor symmetries. We impose for topological or anti-topological twisting respectively

$$A^R_\mu = A^R_{0\mu} + \omega_\mu, \quad A^R_\mu = A^R_{0\mu} - \omega_\mu,$$

where A^R_μ is a background gauge field associated to the R-symmetry, $A^R_{0\mu}$ denotes a non-trivial, *flat* connection with holonomy $\oint_{S^1_\beta} A^R_0 = \pi$, and ω_μ is a $\mathfrak{U}(1)$-valued spin connection associated to the metric of the cigar. We define also

$$A^{\text{flavor}}_\mu = A^{\text{flavor}}_{0\mu} + \kappa\omega_\mu,$$

where A^{flavor}_0 is again a *flat* connection for the cigar metric, $\kappa \in \mathbb{R}$ and ω is a $\mathfrak{u}(1)$-valued spin connection for the cigar metric. The A_0 connection has holonomies around $\mathcal{C} = S^1_\beta, S^1_\rho$ (for some fixed ρ)

$$\frac{1}{2\pi}\oint_{S^1_\beta} A^{\text{flavor}}_0 =: \vartheta, \qquad \oint_{S^1_\rho} A^{\text{flavor}}_0 = 0,$$

We are especially interested for the holonomies of the spin connection ω at the "tip" $(r \to 0)$ and the flat region $(r \to \infty)$ of the D^2 cigar:

$$\frac{1}{2\pi}\oint_{S^1_\beta|_{r=0}} \omega = -\beta\epsilon, \qquad \oint_{S^1_\beta|_{r\to\infty}} \omega = 0, \quad \frac{1}{2\pi}\oint_{S^1_\rho} \omega = 1.$$

We thus obtain the following holonomy table $\oint_{\mathcal{C}} w$ with cycles $\mathcal{C}$ and 1-forms w:

	S^1_β		S^1_ρ
	$r \to 0,$	$r \to \infty$	
ω	$-2\pi\beta\epsilon,$	0	2π
A^R_0	π		-
A^{flavor}_0	$2\pi\vartheta$		0

from which we can infer all the different holonomies of the introduced background gauge fields, in particular $A^{\text{flavor}} \equiv A$ (we shall drop the label "flavor" in the following).

Non-compactness of the cigar also requires some boundary conditions on the boundary S^1 $(r \to \infty)$: fields are "sitting" in the vacuum of the $\mathbb{R}^2 \times_q S^1$ theory.

In the $\rho \to \infty$ limit, the partition function can be schematically written as a BPS index, as taking the limit $\rho \to \infty$ doesn't change the partition function. We have

$$\text{tr}_{\mathcal{H}(D;\alpha)}(-1)^R e^{-2\pi\beta H} q^{-J\mp\frac{R}{2}} \exp\left(ie\oint_{S^1_\beta|_{r=0}} A_{\text{flavor}}\right),$$

where J is the $U(1)_E$ generator, R the $U(1)_R$ generator, and $e = (e_1, \ldots, e_N)$ are the flavor charges with corresponding connections $A_{\text{flavor}} = (A_1, \ldots, A_N)$.

We note that the Hamiltonian from above is *not* Q-exact. The *complexified* fugacities $x_\pm$ are introduced to render the Hamiltonian Q-exact

$$x_\pm := \exp(X_\pm) = \exp\left(2\pi\beta m^{3\text{d}} \mp i\oint_{S^1_\beta|_{r=0}} A_{\text{flavor}}\right) = \exp\left(2\pi\beta m^{3\text{d}} \mp (2\pi i\vartheta - \kappa\hbar)\right), \qquad (2.3)$$

since we set the spin connection contribution at the tip $2\pi i\epsilon\beta = \hbar$. The sign $\pm$ correspond to topological and anti-topological twisted masses respectively. We can now write

$$e^{-2\pi\beta H}\exp\left(ie\oint_{S^1_\beta|_{r=0}} A_{\text{flavor}}\right) = e^{-\beta H_+}x_+^e = e^{-\beta H_-}x_-^e,$$

where $H_\pm = 2\pi(H\pm Z) = \pi\{Q,Q^\dagger\}$. In this way, one can interpret the partition functions (stemming from Hamiltonians that are not Q-exact) as BPS indices with Q-exact Hamiltonians:

$$\begin{aligned}
\mathcal{Z}^\alpha_{\text{BPS}}(x_+;q) &= \mathrm{tr}_{\mathcal{H}(D;\alpha)}(-1)^R e^{\beta H_+} q^{-J-\frac{R}{2}} x_+^e \quad \text{(topological)},\\
\mathcal{Z}^\alpha_{\overline{\text{BPS}}}(x_-;q) &= \mathrm{tr}_{\mathcal{H}(D;\alpha)}(-1)^R e^{\beta H_-} q^{-J+\frac{R}{2}} x_-^e \quad \text{(anti-topological)}.
\end{aligned}$$

In order for these traces to converge and be (meromorphic) functions of their arguments $x_\pm, q$, we need to analytically continue q a bit inside and outside the unit circle in the q-plane. This is related to the conjecture of this work that holomorphic blocks have a *single* series expansion both inside and outside the circle, i.e. we want

$$B^\alpha(x;q) \sim \begin{cases} \mathcal{Z}^\alpha_{\overline{\text{BPS}}}(x;q) & \text{for } |q|<1\\ \mathcal{Z}^\alpha_{\text{BPS}}(x;q) & \text{for } |q|>1,\end{cases}$$

where both expressions are meromorphic functions of their arguments.

In the finite ρ region, one has to reduce the theory to supersymmetric quantum mechanics on $\mathbb{R}_+$ obtained by Kaluza-Klein reduction on the tori of the fibers. The non-compactness of the geometry (the 0 of $\mathbb{R}_+$) implies one has to pick boundary conditions at the "tip", in particular that the state $\langle 0_q|$ at the cigar is annihilated by the (preserved by twisting pairs) $(Q_-,\bar{Q}_+)$ or $(Q_+,\bar{Q}_-)$. Similarly, the state "deep" in the cigar, close to the T^2 boundary is a *unique* supersymmetric groundstate $|\alpha\rangle \in \mathcal{H}(T^2)$ to which the state $|0_q\rangle$ is projected under infinite "time" (ρ) evolution [Wit10]. The blocks are then simply the overlap of these states, i.e.

$$B^\alpha(x;q) = \begin{cases} \langle 0_q|\alpha\rangle_{\text{anti-top}} & |q|<1\\ \langle \alpha|0_q\rangle_{\text{top}} & |q|>1\end{cases}$$

and they will depend on the parameters x,q which are defined by

$$q = \exp(2\pi i\beta\epsilon) = \exp(2\pi i\,\mathrm{Re}\,\tau), \quad x = e^X = \exp(2\pi\beta m^{3\text{d}} + 2\pi i\vartheta + k\hbar),$$

where $\tau = \epsilon\beta + i\beta\rho^{-1}$ is the complex parameter of the boundary T^2.

2.2.3 Gluing

We now consider the gluing of the two halves in the two cases of interest. The fusion of blocks is the three-dimensional lift of the two-dimensional Cecotti-Vafa construction [CV91; CGV14; Hor03], where one considers the partition function on two stretched cigars D^2 glued together to a topological sphere $S^2 = D^2 \cup_\varphi D^2$ with chiral operators inserted at the tips. These partition functions satisfy a set of differential equations, the tt^* equations, imposed by the insertions which exhibit properties of *special geometry*. In three dimensions, the analogue of the chiral operator insertions will be the line-operators and the corresponding statement is that our partition functions satisfy q-difference equations coming from the *algebra of line-operators* (cf. subsection 3.1.2).

Recall from the introduction that a fusion with the element S of the modular group lead to the ellipsoid partition function S^3_b, while a fusion with the identity element id lead to the sphere index on $S^2\times_q S^1$. We investigate the first case first.

S-Fusion

Gluing with $S\in \mathbb{P}SL(2,\mathbb{Z})$ we have that the complex structure modulus of the torus changes

$$\tau = \beta\epsilon + i\beta\rho^{-1} \longmapsto \tilde{\tau} = -\overline{S\cdot\tau} = \frac{1}{\bar{\tau}} = \frac{\epsilon + i\rho^{-1}}{\beta(\epsilon^2+\rho^{-2})} \overset{\rho\to\infty}{\longrightarrow} \frac{1}{\epsilon\beta},$$

and in fact in the large ρ limit we have individually $\tilde{\beta} = \epsilon^{-1}$ and $\tilde{\epsilon} = \beta^{-1}$. We thus have in this limit

$$q = \exp\left(i\oint_{S^1_\beta|_{r=0}} \omega\right) = e^{2\pi i\beta\epsilon} \equiv e^\hbar \longmapsto \tilde{q} = \exp\left(i\oint_{S^1_{\tilde{\beta}}|_{r=0}} \tilde{\omega}\right) = e^{\frac{2\pi i}{\beta\epsilon}} = e^{-\frac{4\pi^2}{\hbar}}.$$

Now we consider the effect of the gluing on holonomies of a background gauge field. As before, we consider connections $A^{\text{flavor}} = A_0 + \kappa\omega$ with $\kappa \in \mathbb{R}$ and holonomies

$$\frac{1}{2\pi}\oint_{S^1_\beta} A_0 =: \vartheta, \quad \oint_{S^1_\rho} A_0 = 0, \qquad \frac{1}{2\pi}\oint_{S^1_\beta|_{r=0}} \omega = -\beta\epsilon, \quad \oint_{S^1_\beta|_{r\to\infty}} \omega = 0, \quad \frac{1}{2\pi}\oint_{S^1_\rho} \omega = 1.$$

and similarly for the tildes. The S-gluing requires that on the "edge" $(r \to \infty, \tilde{r} \to \infty)$ we need to match the holonomies in "swapped" circles

$$\oint_{S^1_\beta|_{r\to\infty}} A = \oint_{S^1_{\tilde{\rho}}} \tilde{A}, \quad \text{and} \quad \oint_{S^1_{\tilde{\beta}}|_{\tilde{r}\to\infty}} \tilde{A} = \oint_{S^1_\rho} A$$

which implies that we must impose

$$\tilde{\kappa} = \vartheta, \quad \tilde{\vartheta} = \kappa.$$

Choosing anti-topological twist on the left and topological twist on the right we will have holomorphic variables $x = \exp X$ and $\tilde{x} = \exp \tilde{X}$ where (recall the definitions (2.3)):

$$X = 2\pi\beta m^{3d} + (2\pi i\vartheta - \kappa\hbar), \quad \tilde{X} = 2\pi\tilde{\beta} m^{3d} - (2\pi i\tilde{\vartheta} - \tilde{\kappa}\tilde{\hbar}) = \frac{2\pi i}{\hbar} X.$$

This is exactly the relation between $(x; q)$ and $(\tilde{x}; \tilde{q})$ found in the literature [Pas12], which is written in terms of the parameters of the ellipsoid partition function b, μ as

$$\begin{aligned} x &= \exp(2\pi b\mu), \quad q = \exp(2\pi i b^2) \\ \tilde{x} &= \exp(2\pi b^{-1}\mu), \quad \tilde{q} = \exp(2\pi i b^{-2}), \end{aligned} \tag{2.4}$$

where $2\pi i b^2 = \hbar$, $2\pi, ib^{-2} = \tilde{\hbar}$ and one can compute that $\mu = \frac{1}{2} m^{3d}(b^{-1}\beta + b\tilde{\beta}) + i(\vartheta b^{-1} - \kappa b)$.

Identity fusion

Gluing with the identity, we carry out similar steps as in the S-fusion and we have

$$\tau \mapsto \tilde{\tau} = -\bar{\tau} = -\beta\epsilon + i\beta\rho^{-1}$$

thus in the large ρ limit we have simply $\tilde{\beta}\tilde{\epsilon} = -\beta\epsilon$, where again individually

$$\tilde{\beta} = \beta, \quad \tilde{\epsilon} = -\epsilon$$

hold, which in turn implies that

$$q = e^\hbar \mapsto \tilde{q} = e^{-\hbar} = q^{-1}.$$

We move on to match holonomies of background gauge fields, now matching circles without swapping and we obtain

$$\vartheta = \tilde{\vartheta}, \quad \kappa = -\tilde{\kappa} \mod \mathbb{Z}.$$

The last relation implies that there is a *magnetic flux* $F = \mathrm{d}A$ through S^2:

$$-m := \frac{1}{2\pi}\int_{S^2} F = \kappa + \tilde{\kappa}.$$

We can therefore set $\kappa = -\frac{m}{2} + \kappa_0$ and $\tilde{\kappa} = -\frac{m}{2} - \kappa_0$. Again, choosing anti-topological twisting on the left and topological on the right, we can write the relations between left and right parameters in terms of m, ϑ, κ_0 as

$$q = \tilde{q}^{-1}, \quad x = \exp X = q^{\frac{m}{2}}\zeta, \quad \tilde{x} = \exp \tilde{X} = q^{\frac{m}{2}}\zeta^{-1}, \tag{2.5}$$

where $\zeta = \exp(2\pi i\vartheta - \kappa_0\hbar)$ and we have set the masses m^{3d} to zero in (2.3).

2.3 Computing blocks from SUSY quantum mechanics

In this section we want to lay out the detailed methods used to compute the blocks on the "pieces" $D^2 \times_q S^1$. As explained in the outline, this involves reducing the theory on the pieces to supersymmetric quantum mechanics and compute the blocks as partition functions. We compute these as a *localized* finite dimensional contour integral in the (complex) space of gauge scalars, with systematically prescribed integrand and contour. This calculation is still perturbative in $\hbar$, as it is essentially a semi-classical WKB approximation. However, the non-trivial statement from [BDP14] is that one can find the non-perturbative completion, systematically, for both the integrand and the contour. This endeavor involves using the q-difference equations which the blocks satisfy, and we will explain it in chapter 3.

2.3.1 Kaluza-Klein compactification

As we have stated, for the reduction we "work" in the "flat" region of the elongated cigar, in particular in the $\rho \gg \beta$ (and $\hbar = 2\pi i\beta\epsilon$) region of the cigar. The space is locally $T^2 \times \mathbb{R}$, where $T^2 = S^1_\beta \times S^1_\rho$ and we want to reduce the theory to $\mathcal{N} = 4$ quantum mechanics on the $\mathbb{R}$ factor. We do this in two steps: first reduce the S^1_β, and then the S^1_ρ factor (albeit a slightly "off-set" factor in the T^2).

The first reduction yields an effective two dimensional $\mathcal{N} = (2,2)$ theory, whose dynamics are specified by *twisted* F-terms. Wilson lines around the S^1 defined by the background flavor connections A_i complexify real mass parameters $m_i = m_i^{\rm 3d} + \frac{i}{R}\oint_S^1 A_i$ (i runs from 1 to N, where N is the rank of the flavor symmetry which we "gauged") and similarly Wilson lines defined by the gauge connections A_a complexify the real scalars $\sigma_a = \sigma_a^{\rm 3d} + \frac{i}{R}\oint_{S^1} A_a$ of gauge multiplets (a runs from 1 to r, the rank of the dynamical gauge group). Assuming that the Abelian symmetries are compact, invariance under (large) gauge transformation is implemented by periodicity of said scalars:

$$\sigma_a \sim \sigma_a + \frac{2\pi i}{R}, \quad m_i \sim m_i + \frac{2\pi i}{R}.$$

This is a special property unique to two-dimensional $\mathcal{N} = (2,2)$ theories that are descendants of a three-dimensional theory. Any chiral multiplet ϕ in three dimensions yields a KK tower ϕ_n, and if ϕ has real mass $m^{\rm 3d}$ under some $U(1)$ symmetry the mode will have mass

$$m_n = m^{\rm 3d} + \frac{2\pi i n}{R}, \quad n \in \mathbb{Z}$$

The twisted superpotential, which is a function of the dynamical (Σ_a, containing σ_a) and background (M_i, containing m_ϕ) twisted chiral multiplets $\widetilde{W}(\Sigma_a, M_i)$, will receive one-loop corrections from integrating out the massive modes and their contribution can be summed [Wit93; BDP14]

$$\delta\widetilde{W}^{\rm 2d}(M_\phi) = \sum_{n\in\mathbb{Z}} (M_\phi + \frac{2\pi i n}{R})\big(\log(RM_\phi + 2\pi i n) - 1\big) \simeq \frac{r}{4}M_\phi^2 + \frac{1}{R}\,\mathrm{Li}_2(-e^{-RM_\phi}),$$

where M_ϕ is a linear combination of of Σ_a and M_i containing m_ϕ. Any such chiral multiplet ϕ yields such a contribution to the twisted superpotential. Furthermore, the twisted superpotential can include tree-level Chern-Simons terms, as

$$\frac{1}{R}\widetilde{W}^{\rm 2d}_{\rm CS}(\Sigma_a, M_i) = \frac{1}{2}k_{ab}\Sigma^a\Sigma^b + k_{ai}\Sigma^a M_i + \frac{1}{2}k_{ij}M_i M_j,$$

i.e. pure gauge, mixed gauge-flavor and pure background interactions.

There is a caveat to this story: this superpotential is *not invariant* under large gauge transformations, because we have implicitly exchanged the dummy field A_a in the path integral with its field strength, which are auxiliary fields of the twisted chiral multiplets. Invariance is achieved if we require *quantization* of flux of the field strengths, i.e. $\int F_a = 2\pi\mathbb{Z}$ (e.g. by inserting a Dirac comb

$$\sum_{n_a\in\mathbb{Z}} \exp(-2\pi i n_a \int \mathrm{d}^2\theta\Sigma_a) \tag{2.6}$$

in the path integral). Without this factor, the action is multivalued as already signified by the dilogarithm appearing in the asymptotic expansion of the superpotential: it has multiple sheets labeled by integers $(b,c) \in \mathbb{Z}^2$ related to the principle branch by

$$\mathrm{Li}_2(-e^{-x}) \mapsto \mathrm{Li}(-e^{-x}) + 2\pi i b(x + i\pi) + 4\pi^2 c.$$

The addition of the Dirac comb factor means we are summing over all sheets of the covering space $\widetilde{\mathcal{M}}$ of our scalar manifold $\mathcal{M}$ (constant shifts in the dilogarithm vanish in the superspace integration). Thus in the end the integrand of the path integral is *single-valued.*

Reducing one more dimension we obtain at the small $\hbar$ and $\rho \to \infty$ limit where $R = \beta\sqrt{1+\epsilon^2\rho^2} \to \frac{1}{2\pi i}\hbar\rho$ that the superpotential is

$$W^{\rm QM}(\Sigma_a, M_i) \equiv 2\pi\rho\widetilde{W} = \frac{i}{\hbar}\Big[\sum_{\phi_i}(\frac{1}{4}M_\phi^2 + \mathrm{Li}_2(e^{-M_\phi})) + \frac{1}{2}k_{ab}\Sigma^a\Sigma^b + k_{ai}\Sigma^a M_i + \frac{1}{2}k_{ij}M_i M_j\Big], \tag{2.7}$$

where the summation is over the chiral fields ϕ_i of the theory and all the fields involved are "cylinder-valued" i.e. $2\pi i$-periodic. This *dimensionless* superpotential thus defines a one-dimensional (quantum mechanical) $\mathcal{N}=4$ Landau-Ginzburg model with target space $\mathcal{M} := (\mathbb{C}^*)^r$ — the space of gauge scalars. Since the path integral is modified by the Dirac comb term, it is more natural to formulate it on the covering space $\widetilde{\mathcal{M}}$ and then sum over deck transformations, which then descends to a single-valued integrand on $\mathcal{M}$.

2.3.2 Determining the classical vacua and boundary conditions

Vacua

For this one dimensional theory on $\mathbb{R}_+$ we need the boundary conditions at $t=0$ and $t\to\infty$, i.e. a choice of (gapped!) massive supersymmetric vacuum. The equations that determine the vacua are[NW10]

$$\frac{\partial\widetilde{W}}{\partial\sigma_a} = 2\pi i n_a, \tag{2.8}$$

which is written using the single-valued $(\mathbb{C}^*)^r$-variables $s_a = e^{\sigma_a}$ and $x_i = e^{m_i}$ as

$$\exp\left(s_a\frac{\partial\widetilde{W}}{\partial s_a}\right) = 1. \tag{2.9}$$

This equation has a finite number of distinct solutions $s_a^{(\alpha)}$ if and only if all vacua α are massive, and hence we *need* enough flavor symmetry in our theory.

Boundary condition in the flat region

Formulating the sigma model on $\widetilde{\mathcal{M}}$ rather than $\mathcal{M}$ for single-valuedness of the fields we have that for each fixed term $\vec{n} = \{n_\alpha\}$ in the Dirac comb (2.6) the vacuum of choice α will impose the boundary condition that the fields approach the "image of the vacuum α on the appropriate sheet" of $\widetilde{\mathcal{M}}$ where (2.8) can be solved for n_a. This is a boundary condition at infinity (flat region) that is *invariant* under large gauge transformations, implemented here geometrically by deck transformations of the covering space $\widetilde{M}$.

Completion to a Lagrangian submanifold

We make a small digression, which is relevant for our later discussion of line operator identities and q-difference equations. Equation (2.9) can be supplemented [DGG11b; DGG11a; KW06] by

$$\exp\left(x_i\frac{\partial\widetilde{W}}{\partial x_i}\right) = p_i, \quad i=1,\dots,N, \tag{2.10}$$

which define the effective background FI parameters p_i associated to the flavor symmetries $i = 1,\dots,N$. Then, using (2.9) together with (2.10) we can eliminate the s_i, and thus define N polynomial equations:

$$f_i(x,p) = 0$$

that "cut out" a middle-dimensional algebraic variety $\mathcal{L}_{\mathrm{SUSY}} \in (\mathbb{C}^*)^{2N}$ spanned by (x_i,p_i). The subscript is chosen deliberately: The variety is in fact a *Lagrangian* subvariety of $(\mathbb{C}^*)^{2N}$ with respect to the Kähler form (inherited from the canonical Kähler form in the space of logarithms X_i, Y_i)

$$\Omega = \sum_{i=1}^{N}\frac{\mathrm{d}x_i}{x_i}\wedge\frac{\mathrm{d}p_i}{p_i}.$$

These polynomial equations will become difference equations upon quantization, corresponding to the line operator identities.

Note that, if the superpotential is non-degerate, the distinct solutions $s_a^{(\alpha)}$ of (2.9) can also be recovered from the set of polynomial equation defining the Lagrangian subvariety alone: the solutions of (2.9) completely determine the $p_i = p_i^{(\alpha)}$ in (2.10) as functions of the x_j.

Boundary condition at the origin

The boundary condition at $t = 0$ corresponds to the "tip" of the cigar in $D^2 \times_q S^1$. It is constructed as described in [Wit10]: every field s_a is assigned a weight, or equivalently, a supersymmetric *wavefunction* of the sigma model on $\mathcal{M} = (\mathbb{C}^*)^r$ is inserted at the "tip" $t = 0$. Such supersymmetric wavefunctions take values in the exterior bundle $\bigwedge^* \mathcal{M}$ [Wit88], with one contribution (factor) coming from bosons and one contribution (form factor coming from fermion. The correct fermion factor is the holomorphic top-form

$$\bar{\Omega} = \frac{\mathrm{d}s_1}{s_1} \wedge \cdots \wedge \frac{\mathrm{d}s_r}{s_r}$$

while the bosonic part is now determined *perturbatively* in $\hbar$ by a WKB approximation using the twisted superpotential $\widetilde{W}$. This means that in small $\hbar$ perturbation theory, we can determine the wavefunction by summing up contributions to the (two dimensional) twisted superpotential as done previously in subsection 2.3.1 and obtain

$$\Psi_0(s_a, m_i; \hbar) = \bar{\Omega} \exp\left(\frac{1}{\hbar}\widetilde{W}_\hbar(s_a, m_i; \hbar)\right), \tag{2.11}$$

where $\widetilde{W}_\hbar$ is given by

$$\widetilde{W}_\hbar(s_a, m_i; \hbar) = \sum_{\phi_i}[\frac{1}{4}m_\phi^2 + \mathrm{Li}_2(e^{-m_\phi - \frac{\hbar}{2}}; \hbar)] + \frac{1}{2}k_{ab}\sigma^a\sigma^b + k_{ai}\sigma^a m_i + \frac{1}{2}k_{ij}m_i m_j,$$

the m_ϕ are the masses of chiral fields in our theory, $s_a = e^{\sigma_a}$ and B_n are the Bernoulli numbers. The polylogarithm Li with two arguments is defined as

$$\mathrm{Li}_2(x; \hbar) := \sum_{n=0}^{\infty} \frac{B_n \hbar^n}{n!} \mathrm{Li}_{2-n}(x),$$

and it differs from the expression in (2.7) because of twisting, i.e. $m_\phi \to m_\phi + (R_\phi - 1)(i\pi + \hbar/2)$ where R_ϕ is the charge of the chiral ϕ.

2.3.3 Computing the path integral via localization

We finally want to actually evaluate the partition function on $\mathbb{R}_+$ which will yield the holomorphic block on $D^2 \times_q S^1$. The relevant machinery is thoroughly discussed in [Wit10]. Due to the insertion of the Dirac comb (2.6), the bosonic contribution is labeled by integer valued vectors $\vec{n}$ and is given in terms of superfields Σ_a

$$I_{\vec{n}} = \int \mathrm{d}t\, \mathrm{d}^4\theta g_{a\bar{b}} \Sigma^a \bar{\Sigma}^{\bar{b}} + \int \mathrm{d}t \mathrm{d}\theta^2 W_{\vec{n}}^{\mathrm{QM}}(\Sigma_a, M_i) + \mathrm{c.c},$$

where the kinetic (D-)term is Q-exact, hence does not depend on the Kähler metric overall [IS13]. As per usual when the integrand is Q-exact, the path integral can be localized to configurations that are invariant under the action of the supercharges at $t = 0$. These are configurations which satisfy *gradient flow equations*, with respect to $\mathrm{Im}\, Q^{\mathrm{QM}}$ on $\widetilde{M}$, as a function of t, similarly to extremizing potentials in toy models. Explicitly, for each critical point α the path integral localizes to loci (middle-dimensional contours) $\Gamma_\alpha \subset \mathcal{M} = (\mathbb{C}^*)^r$ defined with asymptotics $s_a^{(\alpha)} = \exp(\sigma_a^{(\alpha)})$ at $t \to \infty$ and such that they satisfy the flow equation [Wit10]

$$\frac{\mathrm{d}\sigma_a}{\mathrm{d}t} = g_{a\bar{b}} \frac{\mathrm{d}\,\mathrm{Im}\, W^{\mathrm{QM}}}{\mathrm{d}\bar{\sigma}_{\bar{b}}}.$$

This signifies that the state, meaning the differential form, that "hits" the boundary when $t = 0$, is simply the *Poincaré dual* to the downward gradient-flow cycle associated to α, i.e.

$$\Psi_\alpha(s_a, m_i) \simeq \mathrm{PD}[\Gamma^\alpha].$$

This is precisely the state $|\alpha\rangle$ living in the "flat region" we have mentioned since the outline, and Ψ_0 in (2.11) is precisely $|0_q\rangle$ corresponding to the state on the tip (even though both are states in $\mathcal{H}(T^2)$). The partition function is then the overlap between these two states:

$$B^\alpha(x; q) \equiv \mathcal{Z}_{\mathrm{QM}} \simeq \langle 0_q | \alpha_q \rangle = \int_{\mathcal{M}} \Psi_0 \wedge \star \Psi_\alpha \equiv \int_{\Gamma^\alpha} \underline{\Omega} \exp\left(\frac{1}{\hbar}\widetilde{W}_\hbar(s_a, m_i; \hbar)\right). \tag{2.12}$$

where in the last step we have used the natural pairing of Poincaré duality. This is a valid expression for small $\hbar$. However, the approximation "took place" only in the calculation of the integrand by WKB approximation, and consequently also in the calculation of the contours vie the flow equations. In other words, the calculation holds also non-perturbatively as long as $\underset{\sim}{\Gamma^\alpha}$ and $\widetilde{W}_\hbar$ receive appropriate non-perturbative completion, possibly involving the multiple sheets of $\widetilde{\mathcal{M}}$. As we will see, the form of these contour integrals will be fixed using the line-operator identities in 3.2.2.

3. A non-perturbative completion

In this chapter we want to complete the "recipe" given by the authors of [BDP14], namely make use of the "modern technology" of line operators to determine the non-perturbative completion of the holomorphic blocks, given by (2.12). Motivated by a "trivial", but ultimately important example —the free chiral theory T_Δ—, we review the appearance of line operators and their identities which translate to q-difference constraints for the blocks and hence also for the partition function. Using the q-difference equations which can be determined in two ways, we present the full "recipe" to write down solutions and hence determine the blocks.

3.1 Line operators and q-difference equations

3.1.1 A "trivial" example: the free chiral theory

Before we embark on the full computation of blocks, following [BDP14] we review a basic, but very important example, the *free chiral multiplet*. The results for this case have been worked out in the literature [HHL11; Pas12; DGG11a] for the two backgrounds of interest (S_b^3 and $S^2 \times_q S^1$). The precise field content is specified from the following table

$$T_\Delta : \left\{ \text{chiral fields:}\{\phi\}, \quad \text{charges:} \begin{array}{|cc} F & R \\ \hline 1 & 0 \end{array}, \quad \text{CS matrix:} \begin{array}{c|cc} & F & R \\ \hline F & -\frac{1}{2} & \frac{1}{2} \\ R & \frac{1}{2} & -\frac{1}{2} \end{array} \right\}, \tag{3.1}$$

where the CS matrix denotes the CS levels (couplings) for each pair of background gauge fields coming from flavor (F) or R-symmetry. After weakly gauging the flavor symmetries, the F-F CS terms are pure, the F-R terms are mixed (flavor-R) and the R-R are "contact" terms. The notation T_Δ is from [DGG11b]: the free chiral corresponds to "an ideal tetrahedron". As we will see, it serves as the "building block" for more complicated theories, much like triangulations of manifolds. Turning on masses $m^{3\text{d}}$ changes the CS matrix, and the "bare" CS matrix above cancels the anomaly from fermions in the chiral multiplet ϕ (cf. end of subsection 2.2.1. Usually, one only needs to cancel anomalies for dynamical gauge symmetries, but in this case we need to cancel anomalies coming also from flavor symmetries, as these are weakly gauged. In fact, factorization of the partition function into holomorphic blocks is in fact only possible when all flavor anomalies cancel.

The ellipsoid partition function and sphere index

The ellipsoid S_b^3 partition function is expressed in terms of (μ, b), where b is the real deformation parameter for the ellipsoid geometry and μ is the complexified mass parameter corresponding to the flavor symmetry as determined by (2.4). As we saw before, they relate to our variables as $X = 2\pi b\mu$ and $\hbar = 2\pi i b^2$. The partition function has been computed in [HHL11] and can be written [BDP14] as

$$\mathcal{Z}_\Delta^b(X;\hbar) = \begin{cases} C^2 \prod_{r=0}^{\infty} \frac{1-q^{r+1}x^{-1}}{1-\tilde{q}^{-r}\tilde{x}^{-1}}, & |q|<1 \\ C^2 \prod_{r=0}^{\infty} \frac{1-\tilde{q}^{r+1}\tilde{x}^{-1}}{1-q^{-r}x^{-1}}, & |q|>1 \end{cases}$$

where we recall that $q = e^{\hbar} = e^{2\pi i b^2}$, $\tilde{q} = e^{-\frac{2\pi^2}{\hbar}} = e^{2\pi i b^{-2}}$ and $x = e^X$, $\tilde{x} = e^{\tilde{X}} = e^{\frac{2\pi i}{\hbar}X}$, and the constant is $C = \exp\left(-\frac{1}{24}(\hbar - \frac{4\pi^2}{\hbar})\right) = (q\tilde{q})^{-1/24}$. In more useful (and consistent with our later survey) notation, we write (3.2) in terms of q-Pochhammer symbols, where we can write in uniform notation

$$\mathcal{Z}_\Delta^b(X;\hbar) = (q\tilde{q})^{-1/24}\left(qx^{-1};q\right)_\infty \left(\tilde{q}\tilde{x}^{-1};\tilde{q}\right)_\infty, \quad |q| \gtrless 1. \tag{3.2}$$

We refer the reader to the appendix A.1.1 for some properties of the q-Pochhammer symbols (unique power series expansion for both "chambers" $|q| \gtrless 1$ and the uniform notation etc.).

Similarly, the index for $S^2 \times_q S^1$ was computed in [DGG11a, page 56] to be

$$\mathcal{I}_\Delta(m,\zeta;q) = \prod_{r=0}^{\infty} \frac{1-q^{r+1}x^{-1}}{1-\tilde{q}^{-r}\tilde{x}^{-1}},$$

and in "modern" notation

$$\mathcal{I}_\Delta(m,\zeta;q) = \left(qx^{-1};q\right)_\infty \left(\tilde{q}\tilde{x}^{-1};\tilde{q}\right)_\infty, \tag{3.3}$$

where as before $\tilde{q} = q^{-1}, x = q^{\frac{m}{2}}\zeta, \tilde{x} = q^{\frac{m}{2}}\zeta^{-1}$.

A first block and its properties

We can now "read-off" what the blocks should be using

$$\mathcal{Z}_{M^3} = \langle 0_q | 0_{\tilde{q}} \rangle = \sum_\alpha \langle 0_q|\alpha\rangle \langle \alpha|0_{\tilde{q}}\rangle = \sum_\alpha B^\alpha(x;q) B^\alpha(\tilde{x};\tilde{q}) =: \|B^\alpha(x;q)\|_g^2 \tag{2.1}$$

where the sum disappears since there is a unique vacuum. With some a posteriori wisdom, we define the "tetrahedron block" to be the manifestly holomorphic function

$$B_\Delta(x;q) = \left(qz^{-1};q\right)_\infty \equiv \frac{1}{(x^{-1};q^{-1})_\infty} = \sum_{n=0}^{\infty} \frac{x^{-n}}{(q^{-n},q)_n} = \begin{cases} \prod_{k=0}^{\infty}(1-q^{r+1}x^{-1}) & \text{for } |q|<1, \\ \prod_{k=0}^{\infty}(1-q^{-r}x^{-1})^{-1} & \text{for } |q|>1, \end{cases}$$

where we have used the inversion property (A.1.6). One mismatch is of course the prefactor $C = (q\tilde{q})^{1/24}$ in the case of S_b^3 (3.2), where we "glue" the two blocks coming from $D^2 \times_q S^1$ by $S \in \mathbb{P}SL(2;\mathbb{Z})$, which we will discuss in more detail in subsection 3.2.5 and attribute to *pure, background* CS contact terms. The calculations and investigations should always be modulo any such contact terms, which is ultimately related to working "modulo elliptic factors" for q-difference equations, as we will see in 6.1.2.

Among the properties of this special function, one should note that it is still a *piecewise* defined function; in other words, it converges to two *different* functions for $|q| > 1$ and $|q| < 1$, which both have the same power series expansion for $|z| < 1$, but nevertheless does *not* admit analytic continuation from $|q| < 1$ to $|q| > 1$. This is a prevalent phenomenon throughout this work, in fact it is part of the factorization conjecture: the blocks have an *identical* series expansion for $|q| \gtrless 1$ that however converges to different functions for each "chamber", with no analytic continuation between the two regimes. However, a remarkable feature of the partition function is that once the two blocks, none of which admit an analytic continuation across the unit q-circle, are multiplied (fused), the resulting partition function *does* admit such a continuation at least when $\hbar \in \mathbb{C} \setminus \mathbb{R}_-$. This was established for the case of the free chiral on these backgrounds [Dim+09; DG11], and it is conjectured in [BDP14] that this property will persist for general theories and backgrounds (admitting this description).

The first q-difference equation

Maybe the most important feature of the free chiral theory is the fact that its block satisfies an *operator* constraint that is written as

$$(\hat{p} + \hat{x}^{-1} - 1)B_\Delta(x;q) = 0,$$

where the operators $\hat{p}, \hat{x}$ satisfy the so-called q-Weyl algebra $\hat{p}\hat{x} = q\hat{x}\hat{p}$, or more generally for larger rank, the algebra generated by $\hat{x}_i, \hat{p}_i$ with

$$\hat{x}_i\hat{x}_j = \hat{x}_j\hat{x}_i, \quad \hat{p}_i\hat{p}_j = \hat{p}_j\hat{p}_i, \quad \hat{p}_i\hat{x}_j = q^{\delta_{ij}}\hat{x}_j\hat{p}_i.$$

This operator equation follows from the so-called line operator identities that the block satisfies [DGG11a; DGG11b]. As the name suggests, they come from line operators that can be "inserted" in the partition functions of the Heegaard pieces, with Wilson operators corresponding to "position" operators $\hat{x}_i$ and dual 't Hooft operators corresponding to "momentum" operators $\hat{p}_i$. We will study their appearance in greater detail in the next section.

In the (modern, mathematical) notation of the second part of this work, the constraint is written

$$[\sigma_{q,x} + x^{-1} - 1]B_\Delta(x;q) = 0 \tag{3.4}$$

i.e. $B_\Delta(qx;q) = (1-x^{-1})B_\Delta(x;q)$, which is satisfied in both inside and outside the unit circle as can be checked by (3.1.1). This is our first explicit example of a q-difference equation. A more general study of difference equations is presented in the subsection 3.1.2.

The "classical" limit

One can also consider the "classical" limit $q \to 1 \Leftrightarrow \hbar \to 0$ (from either side of the unit q-circle), where the blocks take the form

$$B_\Delta(x;q) \to \exp \frac{1}{\hbar}\left[\sum_{n=0}^{\infty} \frac{B_n \hbar^n}{n!} \operatorname{Li}_{2-n}(x^{-1})\right],$$

where B_n are Bernoulli numbers and $\operatorname{Li}_n(x)$ are polylogarithms. A proof of this non-trivial expansion is discussed in the appendix in A.1.3.

The term "classical limit" is motivated by the *notation* $q = e^\hbar$ whence $q \to 1$ corresponds to $\hbar \to 0$. However, this limit is not classical in the standard sense, as we are using $\hbar = 2\pi i\beta\epsilon$ (cf. subsection 2.2.2). The limit $q \to 1$ really corresponds to the vanishing of the "twisting" controlled by ϵ (cf. (2.2)). This notation and terminology is a hereditary trait, from the literature, where the parametrization by $\hbar$ — a universal constant, which we now take as a dimensionless *parameter* in $\mathbb{C}$ — is used to parametrize *actual* classical limits. An example is given by [Dim11; DGG11b; GS12] where the $\hbar \to 1$ limit corresponds exactly to classical limits of quantized Chern-Simons gauge theories on three-manifolds with boundary. The classical theories obtained are an intermediate step in the *categorification* of geometric data (three-manifolds, two-manifolds, bordisms etc) which is useful as a preview for the program of Quantization in general. Geometrically, $\hbar = 2\pi i\beta\epsilon$, where β "controlls" the size of the base cycle S^1 and ϵ the "twisting" of the fiber. The classical limit then corresponds to zero twisting, while the size of the cycle survives. In the boundary T^2 quantum mechanics picture, the limit is indeed the "classical" limit, whence the semi-classical WKB approximation becomes dominant.

For us, the limit is of great technical as well as conceptual importance. On the one hand one can use "two-dimensional" constructions, investigations and results and "lift them" to three dimensions by q-generalizing the relevant objects (functions, equations). This is however a *degenerate* process: there are infinite q-generalizations to any object. This is a reflection of the greater mathematical and physical complexity that three-dimensional gauge theories possess. We discuss this procedure in slightly more detail in A.3. On the other hand one can also work in the opposite direction: investigate "three-dimensional" results and constructions by checking against the "two-dimensional" limit $q \to 1$ (e.g. for consistency). Such investigations are of interest in pure mathematics as well (cf. section 6.1). One has to note that the perspective of two versus three dimensions being the $q = 1$ versus $q \neq 1$ is *specific* to this construction, which is motivated by the work in [BDP14]. The perspective of q-generalized versus not-q-generalized is much more broad.

3.1.2 Line operators and q-difference equations

Where do q-difference equations occur?

Similar to the blocks of the free chiral, $\mathcal{N} = 2$ theories on S^3_b, $S^2 \times_q S^1$, but also on other spaces that are realized as fibrations over S^1 have blocks that are constrained by operator equations involving q-difference operators, $\hat{p}_i$ or σ_{q,z_i}. Their appearance in calculations of partition functions and indices is motivated by many seemingly separate known facts. We state some of the perspectives here

- These equations are a consequence of identities in the algebra of Wilson and 't Hooft line operators corresponding to the 'gauged' Abelian flavor symmetry [DGG11a; DGG11b] that "wrap" around the S^1 factor of the base and act on the tip of the cigar. Their insertion can be viewed as the three-dimensional analogue of the chiral operator insertions at the tip of the two-dimensional cigar in the Cecotti-Vafa construction. In particular for each $U(1)_i$ flavor symmetry (in the Cartan subgroup of the global symmetry group), there is [KW06; Kap06] a Wilson line $\hat{x}_i$ measuring the holonomy of the corresponding gauge field, and thus acts by multiplication by the complexified mass x_i on the partition function, i.e.
$$\hat{x}_i B(x;q) = x_i B(x;q),$$
and also a corresponding 't Hooft line $\hat{p}_i$ that shifts the masses as $x_i \mapsto qx_i$, i.e.
$$\hat{p}_i B(x;q) = B(x_1, \ldots, qx_i, \ldots x_N; q).$$

This shift operator can be written as $\hat{p}_i = q^{\frac{\partial}{\partial \log x}} = \exp(\hbar \frac{\partial}{\partial X}) \equiv \sigma_{q,x_i}$, and we have the Weyl q-commutation relations

$$\hat{p}_i \hat{x}_j = q^{\delta_{ij}} \hat{x}_j \hat{p}_i$$

and the $\hat{x}_i, \hat{p}_i$ furnish a representation of the q-Weyl algebra. These line operators can best be thought of as "living" in a four-dimensional bulk theory with underlying manifold $D^2 \times_q S^1 \times \mathbb{R}_+$, with the boundary $\mathbb{R}_+ \ni \sigma = 0$ corresponding to the Heegaard piece. In the bulk they are line operators corresponding to dynamical gauge fields and their duals, whose holonomy is measured on the S^1 factor of the base. Their ordering along R_+ is irrelevant in the bulk, but q-commutes when they act on the boundary partition functions, hence they follow the q-Weyl algebra.

- Another way to interpret these constraints is through the AGT correspondence [AGT10]. As we discussed in the introduction to the first chapter, one can relate the partition functions of four-dimensional theories (which in our case constitute the bulk, and the three-dimensional theory is the boundary) to four-point correlators in some two-dimensional conformal field theory and thus to conformal blocks. Then the q-difference constraints of the three-dimensional partition functions are interpreted as the standard Ward-Takahashi identities satisfied by the conformal blocks.
- A related concept to the Ward-Takahashi identities satisfied by the conformal blocks, but also to the tt^*-equations in the two-dimensional case (or in greater generality, the Picard-Fuchs equations [Cer+93]) is the Knizhnick-Zamolodchikov (KZ) equation related to a Lie algebra $\mathfrak{g}$. As described in [EFK98], the "lift" of the KZ equation to a *quantum* algebra $\hat{g}_q$ is precisely the so-called q-KZ equation which is a q-difference equation. Much of the construction of the (differential) KZ equation, including solutions, formalism, global properties, are still interesting to look at in the case of a q-deformation. We expect the investigation of the relation between these seemingly distinct studies/constructions to be fruitful.

Explicit realization from quantum mechanics

Focusing on our examples where the Heegaard pieces are $D^2 \times_q S^1$, the blocks on such spaces satisfy thus identities of the form

$$f_a(\hat{x}, \hat{p}; q) \cdot B(x; q) = 0, \tag{3.5}$$

where $\hat{f}_a$ are polynomials in $\hat{x}_i, \hat{p}_i$ and q, and a runs up to the number N of total flavor symmetries. We have not chosen the label f by chance; in the "classical" limit $q \to 1$ the equations $f_a(x, p; 1) = 0$ for all a define a Lagrangian submanifold $\mathcal{L}_{\text{SUSY}}$ of $(\mathbb{C}^*)^{2N}$ with respect to the canonical symplectic form $\Omega = \sum_i \frac{\mathrm{d}p_i}{p_i} \wedge \frac{\mathrm{d}x_i}{x_i}$. But this is *precisely* the Lagrangian submanifold we discussed in the digression 2.3.2. In the "quantum" setting ($q \in \mathbb{C} \setminus S^1$) *the blocks* $B^\alpha(x; q)$ *are a complete basis of solutions to the* q*-difference equations* (3.5), in particular they are solutions with certain analytic properties, justifying the term *holomorphic* block. These are the main features of the q-constraints satisfied by the blocks that we take advantage of.

The polynomials $f_a(x_i, p_j)$ *defined by* (2.9) *and* (2.10) *also define the* q*-difference equations for the blocks*

$$f_a(\hat{x}_i, \hat{p}_j) B(x; q) = 0,$$

after promotion of $x_i, p)j$ *to operators in a* q*-difference algebra. The blocks* $B^\alpha(x; q)$ *then form a complete basis of solutions to these* q*-difference equations.*

Fusing the blocks and factorization

In the case of fused geometries like S_b^3 and $S^2 \times_q S^1$, the line operators can act on two places, the two tips of the fused cigars. Since they don't "interfere", the partition function of a fused geometry will in fact satisfy *two* sets of identities

$$\hat{f}_a(\hat{x}, \hat{p}; q) \cdot \mathcal{Z}_{\text{fused}} = 0 = \hat{f}_a(\hat{\tilde{x}}, \hat{\tilde{p}}; \tilde{q}) \cdot \mathcal{Z}_{\text{fused}}$$

where the tilde's commute with the simple operators. This provides some strong evidence for factorization into blocks and was in fact one of the motivations behind the factorization conjecture in [Pas12].

Combining the requirements for the shift operators $\hat{p}, \hat{\tilde{p}}$ with the earlier relations between $(x; q)$ and $(\tilde{x}, \tilde{q})$ in the cases of S_b^3 and $S^2 \times_q S^1$ we find the following relations

$$\begin{aligned} S_b^3: \quad & \hat{x} = e^{2\pi b\mu}, \quad \hat{p} = e^{ib\partial_\mu}, \quad q = e^{2\pi i b^2} \quad \text{and} \\ & \hat{\tilde{x}} = e^{2\pi b^{-2}\mu}, \quad \hat{\tilde{p}} = e^{ib^{-1}\partial_\mu}, \quad \tilde{q} = e^{2\pi i b^{-2}}; \\ S^2 \times_q S^1: \quad & \hat{x} = q^{\frac{m}{2}}\zeta, \quad \hat{p} = e^{\partial_m + \frac{\hbar}{2}\partial_{\log\zeta}}, \quad q = e^{\hbar} \quad \text{and} \\ & \hat{\tilde{x}} = q^{\frac{m}{2}}\zeta^{-1}, \quad \hat{\tilde{p}} = e^{\partial_m - \frac{\hbar}{2}\partial_{\log\zeta}}, \quad \tilde{q} = e^{-\hbar} \end{aligned}$$

and one can check that these satisfy a q-Weyl algebra.

3.2 Non-perturbative construction of blocks

In this section we want to complete the construction from [BDP14], by providing ways of computing the non-perturbative completion of the holomorphic blocks. The main agent used is an integral formula for the blocks which, in complete generality, should provide solutions to the line operator identities.

3.2.1 Uniqueness of blocks and of factorizations in the free chiral

We have established that the holomorphic block of the free chiral theory is given by (3.1.1). It is however an acceptable question to ask, whether this block is unique. In other words, we are asking if it is a unique solution to the q-difference equation such that the factorization formula (2.1) yields the (unique) partition function. It is clear that any solution $B_\Delta(x; q)$ needs to satisfy the following requirements

1. $B_\Delta(x; q)$ is a meromorphic (piecewise) function for $x \in \mathbb{C}$ and $q \in \mathbb{C} \setminus S^1$ with *no* analytic continuation from one chamber to the other $|q| \gtrless 1$.
2. There is a correspondence between the pieces of $B_\Delta(x; q)$ in the chambers $|q| < 1$ and $|q| > 1$, and they both possess the same convergent q-hypergeometric series.
3. $B_\Delta(x; q)$ is a solution to $(1 - \hat{p})f(x; q) = \hat{x}^{-1}f(x; q)$, in both chambers.
4. Of course, $\mathcal{Z}_\Delta^b(X; \hbar) = \|B_\Delta(x; q)\|_S^2$ and $\mathcal{I}_\Delta(m, \zeta; q) = \|B(x; q)\|_{\text{id}}$ with appropriate relations for the arguments dictated by the elements $S, \text{id} \in \mathbb{P}SL(2; \mathbb{Z})$.

As we will discuss in 6.1, solution to q difference equations as in point 3. are only unique up to an *elliptic factor*, i.e.

$$B(x; q) = c(x; q)B_\Delta(x; q)$$

if $(1 - \hat{p})c(x; q) = 0$ or equivalently $c(qx; q) = c(x; q)$. This means that $c(x; q)$ is "invisible" to the q-difference operators. An additional restriction of the possible prefactor $c(x; q)$ is given by the factorization conjecture: In the examples S_b^3 and $S^2 \times_q S^1$ from (3.2) and (3.3) we must have

$$\|c(x; q)\|_g^2 = \begin{cases} (q\tilde{q})^{-1/24} & g = S \\ 1 & g = \text{id} \end{cases}$$

The constraints 1.-4. as well as the one above still do not uniquely determine the possible factors, but these can be nicely parametrized in terms of ratio of Jacobi theta functions *. We refer the reader to the appendix A.2.1 for a short survey of Jacobi theta functions and their properties. In particular we may write

$$c(x; q) = \prod_i \Theta_q \left((-q^{1/2})^{b_i+1} z^{a_i} \right)^{n_i}$$

where $a_i, b_i, n_i \in \mathbb{Z}$ with the conditions

$$\sum_i n_i a_i^2 = 0, \quad \sum_i a_i b_i n_i = 0, \quad \text{and} \sum_i n_i b_i^2 = 0.$$

The first two imply ellipticity (by repeated use of the q-difference equation (A.2.11) that the Jacobi theta function satisfies) and the last one ensures correct behavior when gluing.

*In fact, *all* meromorphic elliptic functions $\mathcal{M}(\mathbb{E}_q)$, i.e. meromorphic functions on $\mathbb{E}_q = \mathbb{C}^*/q^{\mathbb{Z}}$ can be written as such a ratio [HSS16].

3.2.2 Generalization: properties of arbitrary blocks

We have now reached the main part of the "recipe" for computing holomorphic blocks of partition functions. We generalize the requirements for the blocks that were revealed in the case of the free chiral, and discuss further properties.

The main tool discovered by the authors of [BDP14] is a general expression of the blocks as a contour integral. The form of our integral is basically a non-perturbative completion of the quantum mechanical integral we presented in (2.12):

$$\mathcal{B}^\alpha(x;q) = \int_{\Gamma^\alpha} \frac{\mathrm{d}s}{2\pi i s}[\text{CS contribution}] \times [\text{matter contribution}] =: \int_{\Gamma^\alpha} \frac{\mathrm{d}s}{2\pi i s}\Upsilon(s,x;q),$$

where the integral is over a middle-dimensional cycle $\Gamma^\alpha \subset (\mathbb{C}^*)^r$ (where $r = \operatorname{rank} G$ of the gauge group G) determined by the (classical) vacuum α and the integrand $\Upsilon(s,x;q)$ remains to be determined. Recall that physically, $x = x_i = x_1,\dots,x_N$ are the *exponentiated* and *complexified* (by the flavor Wilson loops) massive deformations, $s = s_a = s_1,\dots s_r$ the exponentiated scalar of the gauge multiplet (cf. subsection 2.2.1). The integral should generate solutions to the identities inherited from line operators of the four-dimensional bulk theory. Chiral matter contributions (plus W-bosons of non-Abelian gauge symmetries) will be products of the basic "tetrahedron" block of the free chiral $B_\Delta(x;q)$. These, along with the Chern-Simons contributions are by construction meromorphic functions of x_i and s_a. This integral will formally solve the line operator identities as long as it converges, which implies that $\partial\Gamma$ is empty or at asymptotic infinity and Γ^α stays away from the poles of the integrand. In fact it is conjectured that all blocks $B^\alpha(x;q)$ can be realized by integration over appropriate cycles Γ^α. The cycles are in 1-1 correspondence with the vacua α of the theory and we can determine them perturbatively using gradient flows and correct them afterwards. This discussion is saved for 3.2.7.

The critical points of the integrand α (perturbatively in $\hbar$) are in one-to-one correspondence with the vacua specified by the finite distinct solutions of (2.9). Since the points are the "true supersymmetric vacua" they do not change under small deformations. We will exploit this in practice, and compute the QM superpotential directly from the integrand by expanding $\Upsilon = \exp\left(\frac{\widetilde{W}}{\hbar} + \mathcal{O}(\hbar^0)\right)$ and determining the vacua from $\widetilde{W}$. From these points α we can construct the cycles Γ^α by the downward "gradient flow", while we still need to adjust them to avoid singularities.

We want the blocks $B^\alpha(x;q)$ to have similar properties as the tetrahedron blocks $B_\Delta(x;q)$:

1. The $B^\alpha(x;q)$ are meromorphic functions of $x \in \mathbb{C}$ and $q \in \mathbb{C} \setminus S^1$ with *no analytic continuation* between the regimes $|q| \gtrless 1$.

2. Each $B^\alpha(x;q)$ has a single convergent perturbative expansion in $\hbar$ both inside and outside the unit q-circle for fixed α and x.

3. For each α, the block $B^\alpha(x;q)$ can be written as a single q-hypergeometric series both inside and outside the unit q-circle.

4. The blocks $B^\alpha(x;q)$ form a basis of solutions for the set of q-difference equations $f_i(\hat{x},\hat{p};q)\cdot B^\alpha(x;q) = 0$ obtained from the line operators, where for the tetrahedron block $f(\hat{x},\hat{p};q) = \hat{p} + \hat{x}^{-1} - 1$.

5. Finally, the products (explicit factorizations) $\mathcal{Z}_b(X;\hbar) = \|B^\alpha(x;q)\|_S^2$ and $\mathcal{I}(m,\zeta;q) = \|B^\alpha(x;q)\|_{\mathrm{id}}^2$ reproduce the S_b^3 and $S^2 \times_q S^1$ partition functions, and the S_b^3 partition function can be analytically continued from $\hbar < 0$ to $\hbar > 0$ across the half line $\hbar = 2\pi i b^2 \in i\mathbb{R}_+$.

Properties 1.,2., and 4. follow from the construction of the integral: the integrand satisfies all these conditions and we just need to extend them to the integral. 3. and 5. are conjectures, the last one being the main conjecture of this paper.

3.2.3 Uniqueness of arbitrary blocks, Stokes phenomena and q-monodromy

Since the blocks $B^\alpha(x;q)$ are "built up" from the fundamental tetrahedron block $B_\Delta(x;q)$ they will also inherit the "ambiguity" of elliptic factors $c(x;q)$ which are "invisible" for the difference equations and also vanish for factorizations. This is natural, since the general blocks also satisfy q-difference equations and hence they are determined only modulo elliptic factors.

Furthermore there is another complication/feature of blocks which we have not discussed yet. When there is more than one vacuum, the corresponding solution space of the q-difference equations has dimension greater than 1 and the solution space exhibits interesting global properties: monodromies and Stokes phenomena of the blocks as functions of the parameters x. Hence, when there are multiple massive vacua α which lead to multiple blocks $B^\alpha(x;q)$, we can trace a full set of solutions (known as a "solution vector") $\{B^\alpha(x;q)\}_\alpha$ and follow a path in the parameter space of x. Passing through distinct loci in the parameter space the blocks may be independently rescaled by elliptic prefactors as above and also "mix up" by linear combinations. The total effect is that when passing through the "special loci", also known as Stokes walls, the solutions "jump" according to

$$B^\alpha(x;q) \to \begin{cases} \sum_\beta M^\alpha{}_\beta B^\beta(x;q) & |q|<1 \\ \sum_\beta (M^{-1T})^\alpha{}_\beta B^\beta(x;q) & |q|>1, \end{cases} \tag{3.6}$$

where $M \in GL(|\{\alpha\}|, \mathcal{M}(\mathbb{E}_q))$ is in general an elliptic function-valued matrix, where the "jump" from different regimes is chosen so as to reserve fused products. A specific realization of this phenomenon is the *monodromy of a q-difference equation*, where explicit Stokes walls might not exist, but loops around distinguished points induce a general linear transformation on the solutions. In the case of q-difference equations, this will be an elliptic-valued transformation (constant with respect to q-difference equations, cf. section 6.1).

The two different jumps for the two different chambers $|q| \gtrless$ are not chosen by accident. They come from an interesting input from a very basic physical observation: partition functions *should* not depend in the choice of chamber; thus after fusing two blocks $B(x;q)$ and $B(\tilde{x},\tilde{q})$ the "jumps" induced by the Stokes phenomenon or monodromy should disappear. Given that $|q|<1 \Leftrightarrow |\tilde{q}|>1$ and vice versa, the two blocks will be in opposite chambers and the "jumps" will be complementary according to (3.6) and cancel.

The broader question of monodromies and other global topological of q-difference equations, also known as the Riemann-Hilbert correspondence for q-difference equations [Bir13; RSZ09] is the *main* subject of interest for this work, with "an eye" towards applications in supersymmetric gauge theories. It is a question with much deeper roots and implications, both mathematical and physical, than the author can hope to "tame" within a year. As we will see in the non-trivial example (cf. subsection 3.3.3), the global data of the q-difference equations dictated by line operator identities encode crucial information about realization of 3D mirror symmetry. Motivated also by the physical importance of monodromies of differential equations (tt^*, Picard-Fuchs equations) [Cer+93; Ton00; DT00; Hor03], one is lead to believe investigation of q-monodromies and global behavior of q-difference equations will be of similar physical and mathematical importance.

3.2.4 Explicit line operator identities

It was argued in subsection 3.1.2 that one can obtain the q-difference equations coming from the line operator identities by determining the Lagrangian submanifold from the effective (perturbative) QM superpotential as in subsection 2.3.2. We now want to discuss *another* way to obtain the q-difference equations, as presented in [BDP14]. We assume our superconformal field theory has a UV Lagrangian description as a gauge theory, and we start from some number of free chiral multiplets (whose line operator identities are given by (3.4)) and then we apply a set of elementary moves/modifications to obtain the theory in question. Such modifications include adding Chern-Simons terms, adding superpotential terms and gauging flavor symmetries. This approach is *complementary* to simply "reading-off" the line operator identities from the asymptotic expansion of the resummed twisted superpotential, i.e. the SUSY quantum mechanics approach. Nevertheless, both are necessary to understand the procedure, in particular the discussion about the contribution of Chern-Simons terms in the next subsection.

We start from N copies of the free chiral multiplet theory T_Δ

$$T_\times := T_{\Delta_1} \otimes \ldots \otimes T_{\Delta_N},$$

with $U(1)^N$ Abelian flavor symmetry, where the i-th $U(1)$ factor acts on the ϕ_i-th free chiral and has CS level $-\frac{1}{2}$. The operators $f_i^\times$ of $T^\times$ on $D^2 \times_q S^1$ are simply N copies of the tetrahedron operator

$$f_i^\times = \hat{p}_i + \hat{x}_i^{-1} - 1 \simeq 0,$$

where $\simeq$ implies that annihilation of blocks. Recall that the $\hat{x}_i$ are Wilson loops that act by multiplication, while $\hat{p}_i$ are (dual) 't Hooft loops that act as q-shifts i.e. $\hat{p}_i \hat{x}_j = q^{\delta_{ij}} \hat{x}_j \hat{p}_i$, making these identities q-difference equations.

This theory can now be modified in multiple ways.

1. A redefinition of the flavor symmetry by a rational linear transformation $M \in GL(N, \mathbb{Q})$ (equivalent to a redefinition of a basis for the $U(1)$'s). This induces the transformation

$$\hat{x}_i \mapsto \prod_j (\hat{x}_j)^{M^{-1}_{ij}}, \quad \hat{p}_i \mapsto \prod_j (\hat{p}_j)^{M_{ij}}.$$

Note that this may introduce roots of these operators, which can (need to?) be eliminated by overall multiplication by root factors.

2. A redefinition of the R symmetry by adding it to a multiple of the flavor $U(1)_i$ currents (similarly to the topological twisting), or equivalently, a shift of the $U(1)_i$ gauge field by $A_i \mapsto A_i + \sigma_i A_R$ where σ_i are constants. This induces the mapping

$$\hat{x}_i \mapsto (-q^{\frac{1}{2}})^{\sigma_i} \hat{x}_i,$$

for Wilson line operators.

3. Dual to the above is an introduction of a flavor/R mixed term (background CS interaction) of the form $\sim \sum_i \sigma_i^{(P)} \int A_i \mathrm{d} A_R$, where $\sigma_i^{(P)}$ are again constants, which induces the transformation

$$\hat{p}_i \mapsto (-q^{\frac{1}{2}})^{\sigma_i^{(P)}} \hat{p}_i.$$

4. Addition of Chern-Simons terms for the flavor symmetries, e.g. terms of the form $\sum_{ij} \frac{1}{2} k_{ij} \int A_i \mathrm{d} A_j$ (flavor-flavor) for $k_{ij} \in \mathbb{Z}$ which introduces the transformation

$$\hat{p}_i \mapsto q^{-\frac{1}{2}k_{ij}} \left[\prod_j (\hat{x}_j)^{-k_{ij}} \right] \hat{p}_i. \tag{3.7}$$

Note that this is equivalent to conjugating all line operators with

$$\exp\left(\sum_{ij} \frac{k_{ij}}{2\hbar} \hat{X}_i \hat{X}_j \right), \quad \hat{X}_i = \log \hat{x}_i.$$

5. Gauging of a flavor symmetry $U(1)_i$. This implies that shifts in x_i are trivial since it is now dynamical, thus $\hat{x}_i$ must be eliminated and $\hat{p}_i \mapsto 1$. This operation can be performed formally in a left-ideal of q-difference operators. However we're not done: by gauging a flavor symmetry, we are also enriching the theory with a $U(1)_J$ topological flavor symmetry coupled to the newly gauge $U(1)_i$ gauge field by an FI term. The total effect in the level of line operators is the mapping (interchanging of 't Hooft and Wilson lines)

$$\hat{x}_i \mapsto \hat{p}_J, \quad \hat{p}_i \mapsto \hat{x}_J^{-1}$$

for fixed i. This is equivalent to S-duality in a four-dimensional Abelian gauge theory for which our three-dimensional theory is a "boundary condition".

6. Finally, the theory can be modified by adding a gauge invariant operator $\mathcal{O}_i$ to the superpotential that breaks some $U(1)_i$ flavor symmetry. It's precise form is irrelevant, but it must have R charge equal to two, to preserve $U(1)_R$ symmetry. The effect on line operator identities is that $\hat{x}_i \mapsto 1$ and $\hat{p}_i$ vanishes.

These "moves" are sufficient to construct the Lagrangian for any Abelian $\mathcal{N} = 2$ gauge theory along with its line operator identities, always starting from decoupled copies of the free chiral with CS terms. This procedure can be extended to theories with non-Abelian gauge symmetries, whose details we omit.

3.2.5 Chern-Simons terms and theta function ambiguities

For the case of the free chiral with Chern-Simons level $k = -\frac{1}{2}$ we already saw that the block is given by the q-Pochhammer function (3.1.1) which we restate

$$B_\Delta(x;q) \coloneqq (qx^{-1};q)_\infty = \sum_{n=0}^{\infty} \frac{x^{-n}}{(q^{-n},q)_n} = \begin{cases} \prod_{k=0}^{\infty}(1-q^{r+1}x^{-1}) & \text{for } |q|<1, \\ \prod_{k=0}^{\infty}(1-q^{-r}x^{-1})^{-1} & \text{for } |q|>1. \end{cases}$$

This satisfies the line operator identity $\hat{p} + \hat{x}^{-1} - 1 \simeq 0$. We now consider adding a CS term at level $+1$ for some flavor corresponding to parameter x. In the quantum mechanics approach this would just multiply the integrand by the factor

$$\exp \frac{1}{2\hbar} X^2 = \exp \frac{1}{2\hbar} (\log x)^2,$$

coming from the flavor-flavor CS term of the form $\int A_i \mathrm{d} A_i$. On the other hand, in the other point of view (modifying the line operator identities directly) this factor transforms line operator identities in the correct manner by conjugating

$$\hat{p} \mapsto e^{\frac{X^2}{2\hbar}} \hat{p} e^{-\frac{X^2}{2\hbar}} = q^{-\frac{1}{2}} x^{-1} \hat{p}$$

which reminds one of (3.7). However, these functions are not meromorphic in x or q, and one "trick" is to replace the exponentiated quadratic factor with a Jacobi theta function A.2.1

$$e^{\frac{X^2}{2\hbar}} \rightsquigarrow \frac{1}{\Theta_q(x)}.$$

We can do this because of multiple (physical and mathematical) reasons. First, as one can check the theta functions satisfy

$$\Theta_q(x) \hat{p} \Theta_q(x)^{-1} = q^{-\frac{1}{2}} x^{-1} \hat{p},$$

and in addition they also have the right analytic properties. Furthermore, the asymptotic behavior is matched

$$\Theta_q(x)^{-1} \xrightarrow{\hbar \to 0} C e^{\frac{X^2}{2\hbar}}$$

which after integration is absolute matching as the expansion terminates at $\mathcal{O}(\hbar)$, up to the factor $C = (q\tilde{q})^{-1/24}$. Another motivation for this replacement is that the quadratic exponential actually lives in the covering space defined by X and we need to add up all the equivalent contributions under $X \mapsto X + 2\pi i$ that enforce periodicity, which leads to the theta functions.

From this discussion we can extend our results to a *general prescription* to obtain acceptable Chern-Simons contributions at some level k to the holomorphic blocks. We start with an $N \times N$ Chern-Simons level matrix k_{ij} that couples gauge or flavor symmetries, along with a "level-vector" σ_i for mixed Chern-Simons terms that couple gauge or flavor symmetries and the R-symmetry. The extra term (factor) we obtain for the integrand in the quantum mechanics approach is

$$\exp \left[\frac{1}{2\hbar} \sum_{ij} k_{ij} X_i X_j + \frac{1}{\hbar} \sum_i \sigma_i X_i (i\pi + \frac{\hbar}{2}) \right],$$

and we replace this by the expression

$$\prod_i \Theta_q \left((-q^{\frac{1}{2}})^{(b_i+1)} x^{a_i} \right)^{n_i},$$

where b_i and n_i are integers and a_i are column vectors of N rows (remembering that we write x representing in fact $x = (x_1, \dots x_N)$ parameters), such that

$$\sum_i n_i a_i (a_i)^T = -k, \quad \sum_i n_i b_i a_i = -\sigma,$$

where we have suppressed the i, j indices. These conditions follow from enforcing that both the "quadratic" exponential and the product we replace it with satisfy the same line operator identities and that they have the correct (same) asymptotic expansion for $\hbar \to 0$. Note that there are infinite ways of choosing such a product such that the conditions are satisfied; the physical importance of this for the blocks on $D^2 \times_q S^1$ is unknown. However, we note that, at least in the above example the two choices differ by factors $c(x;q) = \prod_i \Theta_q \left((-q^{\frac{1}{2}})^{(b'_i+1)} x^{a'_i} \right)^{n'_i}$, where $\sum_i n'_i a'_i (a'_i)^T = 0$ and $\sum_i n'_i b'_i a'_i = 0$. This is equivalent to the elliptic factor ambiguity discussed in subsection 3.2.1, meaning that $c(x;q)$ "commutes" through the difference equations dictated by line operators and also is invisible for the fused partition function as $\|c(x;q)\|_S^2 = 1$ modulo powers of C and $\|c(x;q)\|_{\mathrm{id}}^2 = 1$.

3.2.6 The integrand

After discussing the Chern-Simons terms, we can now proceed to the rest of the integrand for the block. We consider *any* $\mathcal{N}=2$ gauge theory with $U(1)_R$ R-symmetry and we choose a maximal torus (Cartan subgroup) $U(1)^N$ for the flavor symmetry group with mass parameters $x_i \in \mathbb{C}^*$ with i running in $1,\ldots,N$. We also choose a maximal torus $U(1)^r$ for the gauge group and denote the *complexified* gauge scalars ("mass parameters") by $s_i \in \mathbb{C}^*$ with i running from $N+1,\ldots N+r$.

Let us now consider the theory $T_\Delta^{R_\phi}$ obtained from the free chiral theory T_Δ by shifting the R-charge of the scalar in the chiral multiplet to R_ϕ by a move "2.". We obtain

$$T_\Delta^{R_\phi}:\left\{\begin{array}{l}\text{chiral fields:}\{\phi\},\\ \text{charges}:\begin{array}{|cc}F & R\\ \hline 1 & R_\phi\end{array},\quad \text{CS matrix: }\begin{array}{c|cc} & F & R\\ \hline F & -\frac{1}{2} & \frac{1}{2}(1-R_\phi)\\ R & \frac{1}{2}(1-R_\phi) & -\frac{1}{2}(1-R_\phi)^2\end{array}\end{array}\right\}. \tag{3.8}$$

The block matrix changes to

$$B_\Delta^{(R_\phi)}(y;q)=\left((-q^{\frac{1}{2}})^{2-R_\phi}y^{-1};q\right)_\infty,$$

where y is the mass parameter of the flavor symmetry. We have a collection of rules for our prescription of obtaining integrand from a gauge theory:

Chiral matter

We group every chiral multiplet ϕ with (scalar) R-charge R_ϕ into a copy of $T_\Delta^{R_\phi}$ as in (3.8), i.e. we attach CS couplings to this chiral as dictated in (3.8) for each such chiral. For every such copy, we add a factor

$$B_\Delta^{(R_\phi)}(y_\phi;q)=\left((-q^{\frac{1}{2}})^{2-R_\phi}y_\phi^{-1};q\right)_\infty$$

in the integrand, where y_ϕ is an appropriate product of x's and s's corresponding to the shifted $U(1)$ under which it transforms. This grouping ensures that there are no anomalous gauge or flavor symmetries.

Chern-Simons terms

We remove the copies of T_Δ and we are left with an $(N+r)\times(N+r)$ integer matrix k_ij matrix of the added CS level couplings and an $(N+r)$-dimensional vector σ of mixed CS couplings between flavor or gauge symmetries and the R symmetry. At this stage we *choose* an appropriate product of theta functions as done in the example(s) of subsection 3.2.5. The general form of the result is as we saw

$$\mathrm{CS}[k,\sigma;x,s,q]=\prod_i \Theta_q\left((-q^{\frac{1}{2}})^{(b_i+1)}x^{a_i}\right)^{n_i},$$

where as before b_i and n_i are integers and a_i are column vectors of $N+r$ integers the appropriate conditions $\sum_i n_i a_i(a_i)^T=-k,\quad \sum_i n_i b_i a_i=-\sigma$. For instance, at a level $k=+1$ coupling for x the CS term becomes just $\Theta_q(x)^{-1}$. A Fayet-Iliopoulos term mixing a gauge symmetry with (scalar) parameter s and a flavor symmetry with parameter x yields the following CS term

$$\mathrm{CS}_{\mathrm{FI}}=\frac{\theta(x;q)\theta(s;q)}{\theta(xs;q)}$$

The recipe

We formally collect all the factors to obtain the general expression for our block

$$\mathcal{B}(x;q)=\int_*\frac{\mathrm{d}s}{2\pi is}\prod_{G\subset\mathcal{G}}\mathrm{gauge}[G;s,q]\times\mathrm{CS}[k,\sigma;x,s,q]\times\prod_\phi B_\Delta^{(R_\phi)}(y_\phi(x,s);q). \tag{3.9}$$

There are a few things to note: first, superpotentials play almost no role: they break flavor symmetries and just restrict the corresponding parameters in the integrand. Secondly, the block defined this way inherits some of the properties 1.-5. outlined in the beginning of this chapter, in particular it is by construction defined for both regimes $|q| \gtrless 1$ with no analytic continuation between them (as the building blocks $B_\Delta(x;q)$ satisfy these properties) — property 1. It is conjectured that the block satisfies in fact all five properties. The integrand $\Upsilon(s,x,k,;q)$ is the non-perturbative completion of the QM superpotential; this means as we have stated that we can expand $\Upsilon = \exp\left(\frac{\widetilde{Q}}{\hbar} + \mathcal{O}(\hbar^0)\right)$ and read off the potential.

The block will satisfy the difference equations dictated by the line operator identities, *almost* by construction: (3.9) can be obtained by the sequence of elementary moves described in subsection 3.2.4, and thus satisfies the difference equations at every step. We say "almost" because of an important subtlety: the integration (gauge) variable s_i in $\int \frac{\mathrm{d}s_i}{s_i}$ in each direction of our cycle needs to have the following effect on the level of q-difference equations: eliminating the corresponding $\hat{s}_i$ operator and setting the corresponding shift operator $\hat{p}_i \mapsto 1$. This holds true if the cycles Γ^α used to evaluate the block are "invariant under q-shifts", i.e. a multiplication of the whole cycle by q can by continuously deformed (homotoped) to its original form. This implies that contours are either closed or end asymptotically at 0 or ∞ in each copy of $\mathbb{C}^*$ in $\mathcal{M}$, and that the contours must remain at distance at least q away from poles of the integrand.

Furthermore, we note that the expression (3.9) is again unique only up to *elliptic* factors. This ambiguity stems in both the *choice* of CS term contribution and also the choice of non-Abelian contribution (whose explanation we skipped; the reader is referred to [BDP14]).

In the case the integral along Γ^α is calculated by summing contributions from residues, the "ambiguity" of the integrand pulls out of the integral: since the integrand is built out of q-Pochhammer symbols $(z;q)_\infty$ the poles are typically countably infinite points spaced out by q: e.g. $(s_0, qs_0, \ldots, q^n s_0, \ldots)$. But points like these are equal under the elliptic factor $c(s_0,x;q) = c(q^n s_0, x; q)$ thus the elliptic factor will be the same (constant) factor for all residues, and thus pulls out of the integral, being still an elliptic ratio of theta functions of the remaining arguments. In explicit examples, we will multiply the results of our computations with some elliptic ratio of theta functions to reach a simpler and more elegant result with as few theta functions as possible.

3.2.7 Defining contours

We now address the problem of finding the integration cycles when the theory has multiple vacua α.

Assuming holomorphicity

As a motivation, let us consider the case where the block is an integral of the form $\int_{\Gamma\subset\mathcal{M}} \mathrm{d}S\, \Upsilon(X,S;\hbar)$ where $\mathcal{M} \cong \mathbb{C}^r$ and the integrand $\Upsilon(X,S,\hbar)$ is a non vanishing *holomorphic* function of $S \in \mathbb{C}^r$, depending on parameters $X, \hbar$. A natural example is the usual expression $\Upsilon = \exp(\hbar^{-1} f(X,S))$ where f is a holomorphic function in S. This would correspond to a finite dimensional path integral with a *holomorphic* action f. We assume (cpw. "Picard-Lefshetz theory") that there is a basis $\{\Gamma^\alpha\}$ of middle-dimensional cycles such that for any contour Γ over which the integral converges, Γ can be written as an integer linear combination of $\{\Gamma^\alpha\}$. The only non-trivial integrals will be over non-compact contours due to the higher-dimensional analogue of Cauchy's theorem and the basis cycles $\{\Gamma^\alpha\}$ must be basis elements of the relative homology group

$$H_r(\mathcal{M}, \mathcal{M}_\Lambda; \mathbb{Z}),$$

where $\mathcal{M}_\Lambda := \{S \in \mathcal{M} | \log|\Upsilon(S,X;\hbar)| \leq \Lambda\}$, for Λ sufficiently large and negative (we let $\Lambda \to -\infty$ later on). $\mathcal{M}_\Lambda$ is the region of $\mathcal{M}$ which the contours reach "asymptotically" and which do not contribute to the integral [Wit10]. It depends on the parameters $X, \hbar$ but the rank of the relative group should not. For fixed, *generic* values of X and $\hbar$ such that Υ has isolated and non-degenerate critical points and such that there is a preferred basis of cycles $\{\Gamma^\alpha\}$ associated to the critical points $\{S^{(\alpha)}\}$: we define Γ^α to be the set of points "reached' by a downward gradient flow from $S^{(\alpha)}$ with respect to $\Upsilon(S,X;\hbar)$ and the Kähler metric on $\mathbb{C}^r$. It is a small exercise to see that the (complex) argument along such flows is constant due to holomorphicity of the Picard-Lefshetz function $\Upsilon(S,X;\hbar)$. Hence, a flow starting from the critical point $S^{(\alpha_1)}$ can hit another critical point $S^{(\alpha_2)}$ if and only if

$$\arg\Upsilon(X,S^{(\alpha_1)};\hbar) = \arg\Upsilon(X,S^{(\alpha_2)};\hbar) \text{ and } \log\left|\Upsilon(X,S^{(\alpha_1)};\hbar)\right| > \log\left|\Upsilon(X,S^{(\alpha_1)};\hbar)\right|. \tag{3.10}$$

These conditions define a real-codimension one *Stokes wall* in $(X,\hbar)$ parameter space. For generic points $(X,\hbar)$ in parameters space, the contour is far away from such walls and thus the flows continue without hitting more critical points and the cycles Γ^α are well defined. However, varying the parameters "through" the Stokes walls in parameter space induces a critical-point basis change according to

$$\begin{pmatrix}\Gamma^{\alpha_1}\\ \Gamma^{\alpha_2}\end{pmatrix} \mapsto \begin{pmatrix}\Gamma^{\alpha_1}{}'\\ \Gamma^{\alpha_2}{}'\end{pmatrix} = \begin{pmatrix}1 & \pm 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Gamma^{\alpha_1}\\ \Gamma^{\alpha_2}\end{pmatrix},$$

where the ± 1 depends on orientation. In fact, the jump is a multiple of the intersection number between the "upward" flow from the "bottom" point $S^{(\alpha_2)}$ and the "downward" flow from the "top" point $S^{(\alpha_1)}$, a statement which also takes care of the sign. This transformation matrix is a *connection matrix*(cf. subsection 6.1.4 and is basically a realization of the so-called Picard-Lefschetz formula. Two Stokes walls intersect (transversally) on a real-codimension two locus in the $(X, \hbar)$ parameters space, and *critical points become degenerate* at these loci. Motion around a closed loop around the loci induces a monodromy that permutes the basis cycles Γ^α, according to the composition of the connection matrices.

Generic integrands

Turning back to our block integrals, we note that our integrals are *not* of the form stated above. First, the domain is $\mathcal{M} = (\mathbb{C}/2\pi i\mathbb{Z})^r$ (or even a Weyl group quotient of this), i.e. in exponentiated variables $s = \exp S$ $\mathcal{M}_s = (\mathbb{C}^*)^r$, so there may exist homology cycles that encircle non-trivial one-cycles in $\mathcal{M}$. Secondly, the integrand is *not* holomorphic, but meromorphic and has infinite lines of poles and zeros. A "good" integration cycle should not cross the lines of poles: the poles "condense" into a branch cut at the limit $\hbar \to 0$, i.e. the separation $\hbar$ between the countable poles vanishes. This is the phenomenon known to mathematicians as *confluence*.

Approximating the contours from quantum mechanics

One thing we learned from the quantum mechanical construction of blocks is that there is an *exact* potential $W_{\text{exact}}(x, s; q)$ whose critical points correspond to the true vacua α of the theory. The potential also generates (downward) flows that are used as cycles Γ^α. In our examples, we have only determined the potential perturbatively in $\hbar$ as shown in (2.7), thus the analysis works for small $\hbar$ (and also $|q| \simeq 1$). Thus, the first approach is to find *approximate* cycles by gradient flows of $\widetilde{W}/\hbar$ keeping track of the full non-perturbative potential $\Upsilon(s, x; q)$ at the same time.

Away from critical points, one can (has to) deform the contours "by hand" to make the block integral consistent. At the $\hbar \to 0$ limit along a ray of constant phase, the (half-line of) zeros and poles of the integrand Υ become *distinguished* branch cuts for the effective potential $\widetilde{W}(X, S)$. The initial cycles obtained from the gradient flow may "hit" the branch cuts coming from poles, but crossing a line of such poles is not allowed on the level of line operator identities, thus the deformation is necessary. On the other hand, contours are allowed to cross or lie on branch cuts coming from zeros, as opposed to poles. In fact, some (deformed) cycles may have to be taken such that they flow "upwards", e.g. if there is an appropriate downward cycle that for some values of the parameters becomes upward-flowing.

We can use these "approximate" and "deformed-by-hand" cycles to study Stokes phenomena for blocks. The analysis is expected to hold as long as the critical points of $\widetilde{W}$ are away from branch cuts, so that gradient flows do not cross the cuts. The approximate of the Stokes walls for a pair of cycles $(\Gamma^{\alpha_1}, \Gamma^{\alpha_2})$ flowing from $S^{(\alpha_1)}, S^{(\alpha_2)}$ respectively is given by

$$\operatorname{Im}\big(\frac{1}{\hbar}\widetilde{W}(X; S^{(\alpha_1)})\big) = \operatorname{Im}\big(\frac{1}{\hbar}\widetilde{W}(X; S^{(\alpha_2)})\big).$$

The cycle with greater value of $\operatorname{Re}\big(\frac{1}{\hbar}\widetilde{W}(X; S^{(\alpha)})\big)$ will be shifted by a copy (or rather, intersection number of copies) of the other cycle as the parameters are varied across the wall. The above equation "lives" on the sheet of $\widetilde{W}(X, s)$ with the distinguished branch cuts, defined by $\hbar \to 0$ in the integrand $\Upsilon(x, s; q)$ along a ray of constant phase.

This analysis also yields some results that help us understand *conjugate* ($|q| \gtrless 1$) Stokes jumps in cycles. We assume that we have fixed two values $\hbar_0, \tilde{\hbar}_0$ of the $\hbar$ parameter with $\hbar_0\tilde{\hbar}_0$ real and negative, and we vary the mass parameter $x = \exp X$. Then the same sheet of $\widetilde{W}$ is the relevant one for analyzing gradient flows at both $\hbar_0$ and $\tilde{\hbar}_0$. Thus the Stokes wall(s) defined above will be at the same X, but $\operatorname{Re}\big(\frac{1}{\hbar_0}\widetilde{W}(X; S)\big)$, $\operatorname{Re}\big(\frac{1}{\tilde{\hbar}_0}\widetilde{W}(X; S)\big)$ will have an opposite sign, thus the jumps across the wall in each case will be different:

$$\begin{pmatrix}\Gamma^{\alpha_1}\\ \Gamma^{\alpha_2}\end{pmatrix} \mapsto \begin{pmatrix}1 & \pm 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Gamma^{\alpha_1}\\ \Gamma^{\alpha_2}\end{pmatrix} \text{ at } \hbar = \hbar_0 \implies \begin{pmatrix}\Gamma^{\alpha_1}\\ \Gamma^{\alpha_2}\end{pmatrix} \mapsto \begin{pmatrix}1 & 0\\ \mp 1 & 1\end{pmatrix}\begin{pmatrix}\Gamma^{\alpha_1}\\ \Gamma^{\alpha_2}\end{pmatrix} \text{ at } \hbar = \tilde{\hbar}_0.$$

We extend this for more general values of parameters: Stokes matrices $M, \widetilde{M}$ that are associated to Stokes phenomena in X-space but at *conjugate* values of $\hbar$ will be constrained by

$$M\widetilde{M}^T = \mathbb{1} \tag{3.11}$$

Non-perturbative completion of the contours: shift-invariant cycles

We now take the full non-perturbative integrand $\Upsilon(x, s; q)$ to be the potential that generated gradient flows. This *cannot* be the exact potential of supersymmetric quantum mechanics, as it has too many (countable) critical points.

We consider the integrand $\Upsilon(x, s; q)$ at finite q, where is it is a meromorphic function in s. Apart from the critical points $s^{(\alpha)}$ (that survive the $q \to 1$ limit) corresponding to vacua, there is a countably infinite set of "quantum" ($q \neq 1$) critical points $s = \hat{s}^{(\beta)}$ occurring between every two consecutive zeros or poles on the (half-)lines. They do not correspond to vacua because the true vacua are uncharged under the rotation that yields the Wilson line that generates q-deformations, thus they cannot "appear spontaneously" by letting $q \neq 1$.

Now we consider the gradient flows from *all* the critical points $s^{(\alpha)}$ and $\hat{s}^{(\beta)}$ with respect to $\log|\Upsilon(s, x; q)|$. The flows yield cycles Γ_q^α and $\hat{\Gamma}_q^\beta$ respectively, on which the block integral is convergent, but they typically end at zeros of the integrand and not at asymptotic infinity. This implies that they are not *invariant under q-shifts* as dictated by line operator identities, thus they are not good contours to compute the blocks. One needs *shift-invariant* cycles that are closed or end at asymptotic infinity, in order for the block integrals to have the desired properties.

This is remedied by taking appropriate linear combinations of Γ_q^α and $\hat{\Gamma}_q^\beta$ that are shift-invariant. Γ_q^α and $\hat{\Gamma}_q^\beta$ form a countable basis of the group $\mathbf{\Gamma}_q$ defined as a direct limit

$$\mathbf{\Gamma}_q \coloneqq \lim_{\Lambda \to -\infty} H_r(\mathcal{M}_q, \mathcal{M}_{q\Lambda}; \mathbb{Z})$$

where $\mathcal{M}_q \coloneqq (\mathbb{C}/2\pi i\mathbb{Z})^r \setminus \{\text{poles of } \Upsilon\}$ and $\mathcal{M}_{q\Lambda} \coloneqq \{S \in (\mathbb{C}/2\pi i\mathbb{Z})^r | \, |\Upsilon(x, s; q)| < e^\Lambda\}$. The notions of convergence and shift-invariance are implemented as follows: A (integral along a) cycle $\Gamma = \sum_\alpha n_\alpha \Gamma_q^\alpha + \sum_\beta \hat{n}_\beta \hat{\Gamma}_q^\beta$ is convergent if

$$\int_\gamma \mathrm{d}S\ \Upsilon \coloneqq \sum_\alpha n_\alpha \int_{\Gamma_q^\alpha} \mathrm{d}S\ \Upsilon + \sum_\beta \hat{n}_\beta \int_{\hat{\Gamma}_q^\beta} \mathrm{d}S\ \Upsilon$$

is finite. Furthermore, such a linear combination is shift-invariant *roughly* if a shift by $\pm\hbar$ in the integration variable S_i (corresponding to a shift $s_i \mapsto q^{\pm 1} s_i$) does not change the integral $\int_\Gamma \mathrm{d}S\ \Upsilon$. To be more precise, a cycle Γ is shift-invariant if for a shift by $\pm\hbar$ in the S_i direction there exist other convergent cycles Γ' and Γ'', such that Γ'' is the shifted Γ' and the integrals agree $\int_\Gamma \mathrm{d}S\ \Upsilon = \int_{\Gamma'} \mathrm{d}S\ \Upsilon = \int_{\Gamma''} \mathrm{d}S\ \Upsilon$.

We can thus define the *subgroup* $\mathbf{\Gamma} < \mathbf{\Gamma}_q$ of convergent, shift-invariant cycles. It is a *finite* rank group whose elements integrate Υ to holomorphic blocks. In "lucky" cases, every element in $\mathbf{\Gamma}$ contains at least one copy of the "true vacuum" cycles Γ_q^α, corresponding to a downward flow from the semi-classical critical points s^α, i.e. there are no convergent shift-invariant cycles that consist only of "quantum" cycles $\hat{\Gamma}_q^\beta$.

Note that as we move in the parameter space of x, q the cycles in $\mathbf{\Gamma}_q$ will shift due to Stokes phenomena, whose walls are dictated by the conditions

$$\arg \Upsilon(x, s^{(\alpha_1)}; q) = \arg \Upsilon(x, s^{(\alpha_2)}; q) \mod 2\pi i \tag{3.12a}$$

$$\arg \Upsilon(x, \hat{s}^{(\beta_1)}; q) = \arg \Upsilon(x, \hat{s}^{(\beta_2)}; q) \mod 2\pi i \tag{3.12b}$$

$$\arg \Upsilon(x, s^{(\alpha)}; q) = \arg \Upsilon(x, \hat{s}^{(\beta)}; q) \mod 2\pi i. \tag{3.12c}$$

The shifts will modify the elements of $\mathbf{\Gamma}$ only by "quantum" cycles $\hat{\Gamma}_q^\beta$ so the basis $\{\Gamma^\alpha\}$ of $\mathbf{\Gamma}$ will not change and thus neither will the vacua associated to them. At some distinguished walls, however, the basis *will* jump. These distinguished walls are related to the *physical* Stokes phenomenon.

3.3 A non-trivial example: the $\mathbb{CP}^1$ sigma-model

We now consider an explicit example detailed in [BDP14] that will show most of the interesting features related to holomorphic blocks. The theory T_{I} we consider has a UV description as a gauged linear sigma model (GLSM) which in the IR becomes a non-linear sigma model with target $\mathbb{CP}^1$ [DT00]. We summarize the contents of the theory

$$T_{\mathrm{I}}:\left\{\begin{array}{l}\text{Dynamical } G=U(1) \text{ gauge theory with chirals } \phi_1,\phi_2;\\ \text{scalar in vector multiplet denoted } \sigma^{3\mathrm{d}}, \text{ complexified to } S, s=\exp S;\\ X\times Y=U(1)_V\times U(1)_J \text{ flavor symmetries with mass parameters } m^{3\mathrm{d}},t^{3\mathrm{d}}\\ \text{complexified to } x,y \text{ respectively;}\\ \text{charges: } \begin{array}{c|cc} & \phi_1 & \phi_2\\ \hline G & 1 & 1\\ X & 1 & -1\\ Y & 0 & 0\\ R & 0 & 0\end{array}\ ,\quad \text{CS matrix: } \begin{array}{c|cccc} & G & X & Y & R\\ \hline G & 0 & 0 & 1 & 0\\ X & 0 & 0 & 0 & 0\\ Y & 1 & 0 & 0 & 0\\ R & 0 & 0 & 0 & *\end{array}\end{array}\right\}.$$

3.3.1 Moduli space of the theory

We now describe the parameter space of T_{I} spanned $(m^{3\mathrm{d}},t^{3\mathrm{d}})$. For $m^{3\mathrm{d}}=0$ and $t^{3\mathrm{d}}>0$ the theory has a $\mathbb{CP}^1$ Higgs branch of vacua, i.e. on upward ray in the $(m^{3\mathrm{d}},t^{3\mathrm{d}})$ plane. On either side of the ray (still for $t^{3\mathrm{d}}>0$ the theory becomes massive and has two Higgs-branch vacua localized at the "poles" of $\mathbb{CP}^1$. At negative $t^{3\mathrm{d}}$ however there are two Coulomb branches of vacua at $m^{3\mathrm{d}}=\pm t^{3\mathrm{d}}$, again rays in the $(m^{3\mathrm{d}},t^{3\mathrm{d}})$ plane, with a massive Coulomb in between the rays. The branches have a $\mathbb{Z}_3$ symmetry under the rotation(s)

$$(m^{3\mathrm{d}},t^{3\mathrm{d}})\overset{\omega/\omega^2}{\longmapsto}\Big(-\frac{m^{3\mathrm{d}}\mp t^{3\mathrm{d}}}{2},\frac{\mp 3m^{3\mathrm{d}}-t^{3\mathrm{d}}}{2}\Big),\tag{3.13}$$

with $\omega^3=\mathrm{id}$ and the transformations ω,ω^2, after promoting them into transformation of the background vector multiplets (a linear redefinition of the flavor symmetries), result in theories that we denote by T_{II} and T_{III} respectively. As we will see, this is an explicit manifestation of mirror symmetry. The rays can be seen as dashed lines in 3.1, where $\mathrm{Re}\,Y\sim t^{3\mathrm{d}}$ and $\mathrm{Re}\,X\sim m^{3\mathrm{d}}$.

Now we compactify T_{I} on a circle of radius β and the mass parameters $m^{3\mathrm{d}},t^{3\mathrm{d}}$ are complexified by Wilson lines $\oint_{S^1}A$ on S^1 and we define the *dimensionless*, single-valued parameters on $\mathbb{C}^*$ and the scalar as

$$\begin{aligned}x&=e^X \text{ with } X:=2\pi\beta m^{3\mathrm{d}}+i\oint_{S^1}A_V\\ y&=e^Y \text{ with } Y:=2\pi\beta t^{3\mathrm{d}}+i\oint_{S^1}A_J\\ s&=e^S \text{ with } S:=2\pi\beta\sigma^{3\mathrm{d}}+i\oint_{S^1}A_R.\end{aligned}$$

where we supplement this also with a Wilson line $i\oint A_R=i\pi$ for the R-symmetry and the effective twisted superpotential in the two uncompactified dimensions is

$$\widetilde{W}_{\mathrm{I}}(S;X,Y)=\frac{1}{2}S^2+\frac{1}{2}X^2+S(Y-i\pi)+\mathrm{Li}_2(e^{-S-X})+\mathrm{Li}_2(e^{-S+X}),$$

which can be interpreted in many ways: as an effective superpotential on a $\mathcal{N}=(2,2)$ theory of vacua on $\mathbb{R}^2$, as an effective $\mathcal{N}=4$ supersymmetric quantum mechanical potential on $\mathbb{R}_+$ from the reduction of T_{I} on $D^2\times_q S^1$, or as describing the perturbative behavior of the integrand of a block integral at $\hbar\to 0$.

The $\mathbb{Z}_3$ transformation (3.13) dubbed "mirror symmetry" should define an equivalence relation among the theories $T_{\mathrm{I}},T_{\mathrm{II}}$ and T_{III}, and it extends to a holomorphic transformation of the complexified parameters

$$(X,Y)\overset{\omega}{\longmapsto}\Big(\frac{Y-X}{2},-\frac{3X+Y}{2}\Big),\tag{3.14}$$

and similarly for ω^2. Supplementing this transformation with Chern-Simons contact terms between $U(1)_R$ and $U(1)_V$ that contribute terms $\pm i\pi X$ to the twisted superpotentials, we find that

$$
\begin{aligned}
T_{\mathrm{II}} : \widetilde{W}_{\mathrm{II}}(S;X,Y) &= \frac{1}{2}S^2 - (2X+i\pi)S + X^2 - X(Y+i\pi) + \mathrm{Li}_2(e^{-S}) + \mathrm{Li}_2(e^{-S+X-Y}) \\
T_{\mathrm{III}} : \widetilde{W}_{\mathrm{III}}(S;X,Y) &= \frac{1}{2}S^2 + (2X-i\pi)S + X^2 + X(Y-i\pi) + \mathrm{Li}_2(e^{-S}) + \mathrm{Li}_2(e^{-S-X-Y}).
\end{aligned}
$$

After some calculation we find that in fact

$$
\widetilde{W}_{\mathrm{II/III}}(S;X,Y) = \widetilde{W}_{\mathrm{I}}\left(S + \frac{Y \mp X}{2}; \frac{\pm Y - X}{2}, \frac{\mp 3X - Y}{2}\right) + \frac{i\pi}{2}(Y - {}^{3}\!/\!{}_{1}X).
$$

Similarly as in (2.10) and right after, we can determine the equations that define the supersymmetric Lagrangian submanifold in the parameter space from the superpotentials, which in fact turn out to be the *same* for all chambers. After some work in solving the equations (2.9) and (2.10) to drop the dependence on s^α we obtain the Lagrangian constraints:

$$
\mathcal{L}_{\mathrm{SUSY}} \coloneqq \{(x,y,p_x,p_y) \in (\mathbb{C}^*)^4 | p_y + \left(\frac{1}{y} - x - \frac{1}{x}\right) + \frac{1}{p_y} = 0,\ p_x p_y - (p_x + p_y)x + 1 = 0\}. \tag{3.15}
$$

Now we turn towards Stokes walls and jumps. The locus in which the theories become massless is the *discriminant* locus $\mathcal{D}$, i.e. the locus determined by requiring the discriminant of the equations defined by (2.9) to be zero. This is the locus that is a *source* for Stokes walls, i.e. where Stokes walls intersect. One must avoid this locus when defining contours for block integrals, as we want our theories to be gapped to avoid IR divergences. For T_{I} we find that the vacuum equations are

$$
\exp \frac{\partial \widetilde{W}}{\partial S} = 1 \implies \frac{1}{y} = (x^{-1} - s)(1 - xs^{-1}),
$$

which is solved by

$$
s^{1,2}(x,y) = -\frac{1}{2}\left[y - x - x^{-1} \pm \sqrt{(y^{-1} - x - x^{-1})^2 - 4}\right] \tag{3.16}
$$

and the discriminant locus is

$$
\mathcal{D} \coloneqq \{(x,y) \in (\mathbb{C}^*)^2 | y^{-1} = x + x^{-1} \pm 2\}.
$$

As expected due to mirror symmetry, while the vacuum equations for T_{I} and T_{II} are different, the respective discriminant loci coincide with $\mathcal{D}$. This is also evident by the $\mathbb{Z}_3$ symmetry that is also present for the loci: $\mathcal{D}$ is invariant under $(x,y) \overset{\omega}{\longmapsto} (x^{-\frac{1}{2}}y^{\frac{1}{2}}, x^{-\frac{3}{2}}y^{-\frac{1}{2}})$.

3.3.2 Determining the q-difference equations and the blocks

The $\mathbb{CP}^1$ model has two massive vacua labeled $\alpha = 1,2$ for generic values of the parameters, and hence there are two holomorphic blocks $B^1(x,y;q), B^2(x,y;q)$ which we compute using the prescription of section 3.2.2, i.e. we write down the integral that will solve the line operator identities and find the contours Γ^α for each distinct vacuum α. As we have seen in subsections 3.1.2 and 3.2.4 there are two ways to determine the q-difference identities. Since we have determined the quantum mechanical potential as well as the Lagrangian submanifold in (3.15) we can read-off the q-difference equations[†]

$$
\left[\hat{p}_y + \left(\frac{1}{\hat{y}} - \hat{x} - \frac{1}{\hat{x}}\right) + \frac{1}{\hat{p}_y}\right] f(x,y) = 0, \tag{3.17a}
$$

$$
\left[q^{-\frac{1}{2}}\hat{p}_x\hat{p}_y - \hat{x}(q^{\frac{1}{2}}\hat{p}_x + \hat{p}_y) + 1\right] f(x,y) = 0. \tag{3.17b}
$$

In the notation of the later chapters $\hat{p}_* = \sigma_{q,*}$. In the second part of this work, we will study the (naive) massless limit $m^{3\mathrm{d}} \to 0$, corresponding to $x \to 1$. The first equation will reduce to a q-deformation of the Bessel equation.

[†]In [BDP14] the authors make a detailed derivation through the process described in subsection 3.2.4 and obtain a more accurate version of the second equation. Since we are later interested in the massless limit of the first chiral, corresponding to $x \to 1$, this is irrelevant.

Following the steps from subsection 3.2.6, we can write down the formal integral that solves these difference equations:

$$\begin{aligned}\mathcal{B}_{\mathrm{I}}(x,y;q) &= \int_* \frac{\mathrm{d}s}{2\pi i s} \frac{\Theta_q(y)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q(sy)} \left(qs^{-1}x^{-1};1\right)_\infty \left(qs^{-1}x;q\right)_\infty \\ &=: \int_* \frac{\mathrm{d}s}{2\pi i s} \Upsilon_{\mathrm{I}}(x,y,s;q),\end{aligned}$$

where the q-Pochhammer functions are contributions of the two chirals, and the theta functions come from the CS levels and the FI term, and are as in subsection 3.2.5 unique up to elliptic factors.

We perform an analysis of the integrand as a function of the cylindrical variable $S = \log s$. Taking $\hbar$ to be real and nonzero, q is real and positive and we can distinguish the two regimes $q \gtrless 1$. For $|q| > 1$ (respectively, $|q| < 1$), the integrand has a line of poles (respectively, zeros) along $\operatorname{Im} S = \operatorname{Im} Y$ coming from the $\Theta_q(sy)$ factor which comes from the FI term, with spacing $|\hbar|$. There are also two parallel half-lines of zeros (respectively, poles) from the chiral contributions starting at $S = \pm X$ extending to $S = -\infty$, with spacing $|\hbar|$. At large $|\operatorname{Re} S|$ the integrand is behaves like $\exp\left(\frac{1}{2\hbar}\operatorname{sign}(\operatorname{Re} S) S^2\right)$.

At the classical limit $\hbar \to 0$ (from either side of the real axis) the integrand behaves like

$$\Upsilon_{\mathrm{I}}(x,y.s;q) \overset{\hbar\to 0}{\sim} \exp\left[\frac{1}{\hbar}\left(\frac{1}{2}(\log x)^2 - \frac{1}{2}(\log(-y))^2 + \frac{1}{2}\log(-sy))^2 + \mathrm{Li}_2(x^{-1}s^{-1}) + \mathrm{Li}_2(xs^{-1})\right)\right],$$

which is the same as $\frac{1}{\hbar}\widetilde{W}_{\mathrm{I}}(S;X,Y)$ with a distinguished (principal) choice of branch cuts, with vacua $s^\alpha(x,y)$ dictated by (3.16).

The evaluation of the integrals is a technical feat, whose details we will not present here. The reader is referred to [BDP14] and we quote the result

$$\begin{aligned} C^{-1}B^1_{\mathrm{I}}(x,y;q) &\coloneqq \int_{\Gamma^1_>} \frac{\mathrm{d}s}{2\pi i s}\Upsilon_{\mathrm{I}} = \frac{\Theta_q(y)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q\left(x^{-1}y\right)}\mathcal{J}(xy^{-1},x^2;q), \\ C^{-1}B^2_{\mathrm{I}}(x,y;q) &\coloneqq \int_{\Gamma^2_>} \frac{\mathrm{d}s}{2\pi i s}\Upsilon_{\mathrm{I}} = \frac{\Theta_q(y)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q(xy)}\mathcal{J}(x^{-1}y^{-1},x^{-2};q), \end{aligned} \quad \text{for } |q|>1 \tag{3.18}$$

where we have introduced a normalization factor $C = \frac{2\pi i}{(q^{-1},q^{-1})_\infty}$ that fixes the "elliptic ambiguity" and the function $\mathcal{J}$ is related to the Hahn-Exton q-Bessel function (cf. subsection 6.3.1) and has a q-hypergeometric series

$$\mathcal{J}(x,y;q) \coloneqq (qy;q)_\infty \sum_{n=0}^\infty \frac{x^n}{(q^{-1},q^{-1})_n (qy;q)_n}, \qquad |q| \gtrless 1,$$

which is convergent for both regimes $|q| \gtrless 1$ and defines a meromorphic function of $x, y \in \mathbb{C}^*$. These blocks define the blocks $B^\alpha_{\mathrm{I}}(x,y;q)$ in the chamber "I".

For the other regime $|q| < 1$ the authors work conjecturally: one of the main properties of blocks that we have stated is that the blocks share a *common* q-hypergeometric series expansion for both regimes $|q| \gtrless 1$. Since the right-hand side of both blocks in (3.18) consists of functions that are defined for *both* regimes, one *conjectures* that in fact

$$\begin{aligned} C^{-1}B^1_{\mathrm{I}}(x,y;q) &\coloneqq \int_{\Gamma^1_>} \frac{\mathrm{d}s}{2\pi i s}\Upsilon_{\mathrm{I}} = \frac{\Theta_q(y)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q\left(x^{-1}y\right)}\mathcal{J}(xy^{-1},x^2;q), \\ C^{-1}B^2_{\mathrm{I}}(x,y;q) &\coloneqq \int_{\Gamma^2_>} \frac{\mathrm{d}s}{2\pi i s}\Upsilon_{\mathrm{I}} = \frac{\Theta_q(y)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q(xy)}\mathcal{J}(x^{-1}y^{-1},x^{-2};q), \end{aligned} \quad \text{for } |q|<1, \tag{3.19}$$

where the constant C is determined by choosing the physically irrelevant Chern-Simons contact terms.

The "q-Bessel" function

We make a digression to discuss the surprising properties of the function $\mathcal{J}(x,y;q)$, some of which are conjectures. Note that the function is defined through the series,

$$\mathcal{J}(x,y;q) \coloneqq (qy;q)_\infty \sum_{n=0}^\infty \frac{x^n}{(q^{-1},q^{-1})_n (qy;q)_n}, \qquad |q| \gtrless 1. \tag{3.20}$$

It is related to the Hahn-Exton q-Bessel function (6.22c) by

$$\mathcal{J}(-z^2, 1; q) = (q)_\infty J^{(3)}_{\nu=0}(z; q).$$

More generally, one can check that

$$\mathcal{J}(-a^2, b; q) = (q)_\infty a^{-\beta} J^{(3)}_\beta(a; q), \tag{3.21}$$

where $b = q^\beta$.

The authors of [BDP14] have determined an interesting list of properties

1. An easy manipulation shows that

$$\mathcal{J}(x, y; q) = \Theta_q\,(qy)\,\mathcal{J}(xy^{-1}, y^{-1}; q^{-1}).$$

2. For $|q| < 1$ we have that

$$\frac{(qy;q)_\infty}{(qy;q)_n} = (q^{n+1}y;q)_\infty = \sum_{r=0}^{\infty} \frac{(q^n y)^r}{(q^{-1},q^{-1})_r}.$$

   Substituting this in the definition of the function we obtain

$$\mathcal{J}(x,y;q) = \sum_{n,r=0}^{\infty} \frac{q^{nr}}{(q^{-1},q^{-1})_n (q^{-1},q^{-1})_r} x^n y^r = \sum_{n,r=0}^{\infty} \frac{(-1)^{n+r} q^{\frac{1}{2}(n+r+1)(n+r)}}{(q,q)_n (q,q)_r} x^n y^r,$$

   where in the last equality we have used the fact that $\frac{1}{(q^{-1},q^{-1})_n} = \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q,q)_n}$. In both expressions we can see that there is a symmetry under interchange of x, y in the right-hand side, hence $\mathcal{J}(x,y;q) = \mathcal{J}(y,x;q)$, thus the function is *symmetric* when $|q| < 1$. Note that this does *not* hold when $|q| > 1$; in particular the expressions do not converge when $|q| > 1$. Conjecturally, it is claimed that

$$\mathcal{J}(x,y;q) - \mathcal{J}(y,x;q) = \frac{\Theta_q\left(qx^{-1}\right);q)\Theta_q\,(qy)}{\Theta_q\,(qx^{-1}y)} \mathcal{J}(x,y;q^{-1}), \quad \text{if } |q| > 1.$$

   This has been verified in [BDP14] numerically to high precision. This expression is necessary for consistent Stoked jumps for I.

3. Combining 1. and 2. we have, for $|q| > 1$

$$\frac{\mathcal{J}(x,y;q)}{\Theta_q\,(qy)} = \mathcal{J}(xy^{-1}, y^{-1}; q^{-1}) \overset{|q|>1}{=} \mathcal{J}(y^{-1}, xy^{-1}; q^{-1}) =: \mathcal{J}(\tilde{x}\tilde{y}^{-1}, \tilde{y}^{-1}; q^{-1}),$$

   where we have made the last definition in order to apply 1. again. We find for consistency that $\tilde{x} = x^{-1}$ and $\tilde{y} = x^{-1}y$ have to hold, and applying 1. again we have indeed that

$$\frac{\mathcal{J}(x,y;q)}{\Theta_q\,(qy)} = \frac{\mathcal{J}(\tilde{x},\tilde{y};q)}{\Theta_q\,(q\tilde{y})} \equiv \frac{\mathcal{J}(x^{-1}, x^{-1}y; q)}{\Theta_q\,(qx^{-1}y)}$$

   which shows that

$$\Theta_q\left(qx^{-1}y\right)\mathcal{J}(x,y;q) = \Theta_q\,(qy)\,\mathcal{J}(x^{-1}, x^{-1}y; q) \quad \text{if } |q| > 1.$$

3.3.3 A Stokes phenomenon, monodromy and mirror symmetry

We now investigate the most interesting feature of blocks: Stokes phenomena. Physically, in the wave-function interpretation of blocks these correspond to regions in parameters space where there can be tunneling between supersymmetric vacuum states $|\alpha\rangle$ of $\mathcal{H}(T^2)$. We concentrate in particular regions of the parameters space where the flows are dictated by the "semi-classical" potential $\mathrm{Re}\left(\frac{1}{\hbar}\widetilde{W}_{\mathrm{I}}(x,y,s;q)\right)$ or the "quantum" potential $\log|\Upsilon_{\mathrm{I}}(x,y,s;q)|$ such that they can connect critical points without passing through branch cuts or lines of poles or zeros. The walls are then located "semi-classically" at

$$\mathrm{Im}\left(\frac{1}{\hbar}\widetilde{W}_{\mathrm{I}}(x,y,s^{(1)};q)\right) = \mathrm{Im}\left(\frac{1}{\hbar}\widetilde{W}_{\mathrm{I}}(x,y,s^{(2)};q)\right)$$

for the critical points $s^{(i)}(x,y)$, $i=1,2$, and "quantum mechanically" at

$$\arg \Upsilon_{\mathrm{I}}(x,y,s^{(2)};q) = \arg \Upsilon_{\mathrm{I}}(x,y,s^{(2)};q) \mod 2\pi.$$

There are *three* codimension-one walls in $\mathbb{C}^2$ meeting at the discriminant locus $\mathcal{D}$ — we say the *emanate* from the locus. The authors analyze the global behavior of the solution at a transverse (to $\mathcal{D}$) 'slice' ($\mathbb{C}$ parametrized by $\operatorname{Re} X$ and $\operatorname{Re} Y$) defined by $\operatorname{Im} Y = 0$ and $\operatorname{Im} X = 4\pi/3$ in $\mathbb{C}^2$, where the discriminant is represented by $0 \in \mathbb{C}$. The discriminant locus $\mathcal{D}$ intersects our plane at the origin and the three Stokes walls separate it into chambers in an anti-parallel fashion compared to the "massless rays" (I Higgs, II+III Coulomb) as discussed in subsection 3.3.1, and thus we label the chambers by I-III. Each of the three chambers represents a theory $T_{\mathrm{I}}, T_{\mathrm{II}}, T_{\mathrm{III}}$ which in the IR is a semi-classical sigma model with target $\mathbb{CP}^1$. The mirror action (3.13) maps these theories to each other (up to some modification of R charges and background θ angles), hence we have three mirror partners: $T_{\mathrm{I}} \cong T_{\mathrm{II}} \cong T_{\mathrm{III}}$.

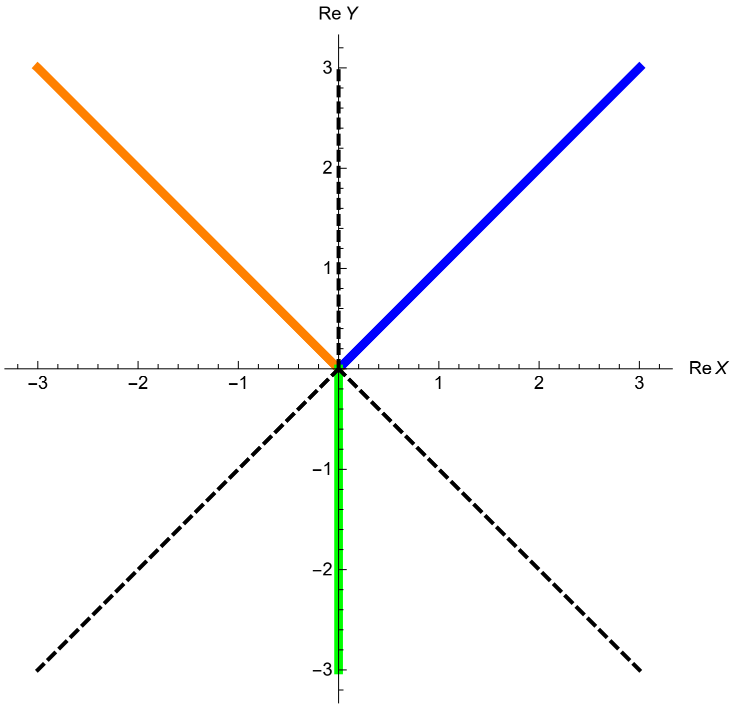


Figure 3.1: The slice $\mathcal{P} = \{\operatorname{Im} Y = 0, \operatorname{Im} X = 4\pi/3\} \subset \mathbb{C}^2$. The colored lines represent the three transverse Stokes walls of emanating from the discriminant locus, which intersects our plane at the origin. The dashed lines enclose the regions I, II and III as described in

$|q| > 1$

We follow the analysis of the authors for $|q|>1$. In the Chamber I, T_{I} is (approximately) a sigma model with target $\mathbb{CP}^1$ with two vacua $\alpha = 1,2$ (one at each pole) and thus two blocks $B^{1,2}_{\mathrm{I}}(x,y;q)$ and two critical points $s^{\alpha}(x,y)$. As we move to other chambers by I $\to$ II $\to$ III $\to$ I the points "circle" around each other in the S plane, while the half-lines of poles (for $|q|>1$) slide relatively to each other in the $\operatorname{Re} S$ direction. Their $\operatorname{Im} S$ direction is fixed and equal to $\operatorname{Im} X$. In each chamber we have two cycles $\Gamma^1_{\mathrm{I,II,III}>}$ and $\Gamma^2_{\mathrm{I,II,III}>}$.

Moving from I to II the cycle $\Gamma^1_{\mathrm{I}>}$ "passes through" the critical point $\alpha=2$ and is shifted by (an intersection number of) a copy of $+\Gamma^2_{\mathrm{I}>}$, the sign fixed by the intersection number, i.e.

$$\begin{pmatrix} \Gamma^1_{\mathrm{II}>} \\ \Gamma^2_{\mathrm{II}>} \end{pmatrix} = M_{>}^{\mathrm{I}\to\mathrm{II}} \begin{pmatrix} \Gamma^1_{\mathrm{I}>} \\ \Gamma^2_{\mathrm{I}>} \end{pmatrix}, \quad \text{with } M_{>}^{\mathrm{I}\to\mathrm{II}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

and similarly for the blocks $B^{1,2}_{\mathrm{II}}$ with respect to $B^{1,2}_{\mathrm{I}}$.

Similarly, moving through II, the first cycle $\Gamma^1_{\mathrm{II}>}$ becomes closed (it wraps around the cylinder) but this is homotopic to the cycle that ends at $\operatorname{Re} S = -\infty$ thus there is no modification in the level of blocks.

Moving from II to III, the second cycle intersects the first critical point $\alpha = 1$ and by skew-symmetry of the intersection number we now have

$$\begin{pmatrix}\Gamma^1_{\mathrm{III}>}\\ \Gamma^2_{\mathrm{III}>}\end{pmatrix} = M^{\mathrm{II}\to\mathrm{III}}_{>}\begin{pmatrix}\Gamma^1_{\mathrm{II}>}\\ \Gamma^2_{\mathrm{II}>}\end{pmatrix}, \quad \text{with } M^{\mathrm{II}\to\mathrm{III}}_{>} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

Finally, passing from III to I, the first cycle intersects the point $\alpha = 2$ again thus,

$$\begin{pmatrix}\tilde\Gamma^1_{\mathrm{I}>}\\ \tilde\Gamma^2_{\mathrm{I}>}\end{pmatrix} = M^{\mathrm{III}\to\mathrm{I}}_{>}\begin{pmatrix}\Gamma^1_{\mathrm{III}>}\\ \Gamma^2_{\mathrm{III}>}\end{pmatrix}, \quad \text{with } M^{\mathrm{III}\to\mathrm{I}}_{>} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

We have suggestively introduced the notation $(\tilde\Gamma^1_{\mathrm{I}>}, \tilde\Gamma^2_{\mathrm{I}>})$ because the product of the matrices is *not* identity

$$M^{\mathrm{III}\to\mathrm{I}}_{>} M^{\mathrm{II}\to\mathrm{III}}_{>} M^{\mathrm{I}\to\mathrm{II}}_{>} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

and there is a *non-trivial monodromy* action when circling around the discriminant locus $\mathcal{D}$ in parameter space, i.e. on a circle in the $\mathrm{Re}\,Y, \mathrm{Re}\,X$ plane that crosses the three Stokes walls as depicted in 3.1. Physically, one expects the supersymmetric ground states $|\alpha\rangle$, $\alpha = 1, 2$ on $\mathcal{H}(T^2)$ to undergo such a monodromy transformation around the discriminant locus.

Explicitly, the blocks are written with respect to the first chamber as

$$\begin{pmatrix}B^1_{\mathrm{II}}\\ B^2_{\mathrm{II}}\end{pmatrix} = \begin{pmatrix}B^1_{\mathrm{I}} + B^2_{\mathrm{I}}\\ B^2_{\mathrm{I}}\end{pmatrix} \text{ and } \begin{pmatrix}B^1_{\mathrm{III}}\\ B^2_{\mathrm{III}}\end{pmatrix} = \begin{pmatrix}B^1_{\mathrm{I}} + B^2_{\mathrm{I}}\\ -B^1_{\mathrm{I}}\end{pmatrix} \quad \text{for } |q| > 1. \tag{3.22}$$

$|q| < 1$

Now we consider blocks at $|q| < 1$: we know what the Stokes matrices should be even without the explicit result for the integrals because of (3.11) since we are in conjugate parameter regime (outside-inside the q circle)

$$M_{<} = (M_{>})^{-1\,T}.$$

In fact using formal integration cycles one is lead to the same result [BDP14]. We thus have for $|q| < 1$

$$\begin{pmatrix}B^1_{\mathrm{II}}\\ B^2_{\mathrm{II}}\end{pmatrix} = (M^{\mathrm{I}\to\mathrm{II}}_{>})^{-1\,T}\begin{pmatrix}B^1_{\mathrm{I}}\\ B^2_{\mathrm{I}}\end{pmatrix} = \begin{pmatrix}B^1_{\mathrm{I}}\\ B^2_{\mathrm{I}} - B^1_{\mathrm{I}},\end{pmatrix} \tag{3.23a}$$

$$\begin{pmatrix}B^1_{\mathrm{III}}\\ B^2_{\mathrm{III}}\end{pmatrix} = (M^{\mathrm{II}\to\mathrm{III}}_{>})^{-1\,T}\begin{pmatrix}B^1_{\mathrm{I}}\\ B^2_{\mathrm{I}}\end{pmatrix} = \begin{pmatrix}B^2_{\mathrm{I}}\\ B^2_{\mathrm{I}} - B^1_{\mathrm{I}},\end{pmatrix} \tag{3.23b}$$

Comparing these expressions to the corresponding ones in $|q| > 1$ from (3.22), we face the following question: how can the different expressions account for the conjecture that the blocks have the *same* q-hypergeometric series expansion for both $|q| \gtrless 1$? There is nothing special about the chamber I we picked, so this conjecture should hold in all chambers.

The resolution is that, as we have stressed previously, having the same q-hypergeometric series expansion for $|q| \gtrless 1$ does *not* imply that the functions are the same analytic functions. In particular they can be different functions with different properties as we have seen in the case of the "q-Bessel" function $\mathcal{J}(x, y; q)$. In fact, precisely the properties of the of the "q-Bessel" function 1.-3. plus the fact that $\Theta_q\left(q^{1/2}z\right) = \Theta_q\left(q^{1/2}z^{-1}\right)$ and the q-difference equation $\Theta_q(qz) = -z^{-1}\Theta_q(z)$ one can verify that for $|q| > 1$

$$B^1_{\mathrm{II}}(x, y; q) \equiv B^1_{\mathrm{I}}(x, y; q) + B^2_{\mathrm{I}}(x, y; q) = \ldots = \frac{\Theta_q(y)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q\left(x^{-1}y\right)}\mathcal{J}(x^2, xy^{-1}; q),$$

which is the same expression as for B^1_{I} in the $|q| < 1$ regime (due to symmetry in x, y when $|q| < 1$). In a similar fashion one can compute that *all* chambers have expressions sharing q-hypergeometric series expansions for both regimes $|q| \gtrless 1$ and the consistent results with a single q-hypergeometric series expression are

$$\begin{aligned} B^1_{\mathrm{II}}(x, y; q) &= \frac{\Theta_q(y)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q\left(x^{-1}y\right)}\mathcal{J}(x^2, xy^{-1}; q), \\ B^2_{\mathrm{II}}(x, y; q) &= \frac{\Theta_q(y)\,\Theta_q\left(qx^2\right)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q\left(x^{-1}y\right)\Theta_q(xy)}\mathcal{J}(xy, x^{-1}y; q) \end{aligned} \quad \text{for } |q| \gtrless 1, \tag{3.24}$$

and

$$\begin{aligned} B^1_{\text{III}}(x,y;q) &= \frac{\Theta_q(y)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q(xy)}\mathcal{J}(x^{-2},x^{-1}y^{-1};q), \\ B^2_{\text{III}}(x,y;q) &= \frac{\Theta_q(y)\,\Theta_q\left(qx^2\right)}{\Theta_q\left(-q^{1/2}x\right)\Theta_q(xy)\,\Theta_q\left(xy^{-1}\right)}\mathcal{J}(x^{-1}y,xy;q) \end{aligned} \qquad \text{for } |q| \gtrless 1. \tag{3.25}$$

Stokes phenomenon as mirror symmetry

We have mentioned that the mirror symmetry transformation (3.13) is an equivalence of the theories $T_{\text{I}}, T_{\text{II}}, T_{\text{III}}$. The transformation leaves the "slice" $\mathbb{C} \subset \mathbb{C}^2$ defined by $\operatorname{Im} X = 4\pi/3$ and $ImY = 0$ invariant, and *permutes* the chambers by I→II→III→I. On the level of exponentiated masses x, y it maps $(x,y) \mapsto \left(\sqrt{y/x}, 1/\sqrt{x^3y}\right)$. The rotation induces a mapping on blocks

$$\begin{pmatrix} B^1_{\text{I}} \\ B^2_{\text{I}} \end{pmatrix} \overset{\omega}{\mapsto} \begin{pmatrix} B^1_{\text{II}} \\ B^2_{\text{II}} \end{pmatrix} \overset{\omega}{\mapsto} \begin{pmatrix} B^1_{\text{III}} \\ B^2_{\text{III}}. \end{pmatrix}$$

It is a small exercise to check from (3.24) and (3.25) that this is *indeed* the case. Needless to say, this is a remarkable agreement; seemingly independent calculations of formal Stokes matrices, non-trivial identities between q-functions and mirror symmetry of the $\mathbb{CP}^1$ sigma model play along perfectly.

In our later discussion in the second part of this work, we will discuss the monodromy in the case the mass deformation associated to the $U(1)$ flavor symmetry is zero, i.e. when $x \to 1$.

4. A lightning-fast review of a GLSM

As the title suggests, we will very briefly go through the study of gauged linear sigma models. We will follow the notation and exposition from [KRS16], using results from [Wit93; HHP08; Joc+; HR13]. The main focus is the calculation of the partition function of the so called gauged linear sigma model on a hemisphere $D^2 \subset S^2$. This is interesting for us because of its similarity with the three-dimensional construction from [BDP14] we reviewed in the previous chapters: the D^2 can be thought of as the two-dimensional analogue of the elongated cigar $D^2 \times_q S^1$ and its partition function an analogue of the three-dimensional holomorphic blocks. This similarity is further affirmed by the *universal* and *integral* formula for the partition function on the hemisphere that the authors in [HR13] derive.

4.1 Generalities

The gauged linear sigma models (GLSM) are two-dimensional gauge theories with $\mathcal{N} = (2, 2)$ supersymmetry. They are especially interesting because in the infrared they flow to a supersymmetric conformal field theory which describes a Calabi-Yau (CY) compactification of string theory. The Fayet-Iliopoylos and theta parameters of the SCFT define parameters on the Kähler moduli space $\mathcal{M}_K$ of the Calabi-Yau. In the SCFT picture they correspond to marginal deformations that are not renormalized under the renormalization group flow. The moduli space is divided into phases (e.g. as we already saw in subsection 3.3.1), each with different low energy descriptions. A phase is called *geometric* if the corresponding low energy theory coming from the GLSM is a non-linear sigma model with a Calabi-Yau target space.

The partition function of the GLSM can be used (by its low energy limit) to compute *exact* Kähler potentials on $\mathcal{M}_K$, which are furthermore used to extract Gromov-Witten invariants. The points of $\mathcal{M}_K$ parameterizing different SCFTs, and there is a vector bundle $\mathcal{H} \to \mathcal{M}_K$ with the fiber being the chiral ring of the SCFT. This vector bundle is equipped with a natural flat connection, the tt^* connection. The (generalized) central charges of D-branes inserted into the SCFTs are (inner products of) flat sections (covariantly constant with respect to tt^*) of this vector bundle. The partition functions of the SCFT are alse expected to be such sections, in particular multivalued, flat, holomorphic sections. One can therefore ask how these sections change when transported between phases, or correspondingly, how are these (local) sections analytically continued beyond their original domain of definition. This question is thus related to the transportation of (central charges of) D-branes between phases of the moduli space.

The authors present a *general*, contour *integral* formula for the calculation of partition functions irrespective of the phase. They also derive a (defining) differential equation for the partition function which is in fact the Picard-Fuchs equation of the periods of the mirror CY. The focus is, as in the 3d $\mathbb{P}^1$ case, Abelian $U(1)$ GLSMs, which admit a geometric phase [Wit93; HHP08] with the target CY being a hypersurface in $\mathbb{P}^N$. The moduli space $\mathcal{M}_K$ can be presented as $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The three singular points correspond to the regular singularities of the Picard-Fuchs equation and represent the so-called large volume ($z = 0$), Landau-Ginsburg ($z = \infty$) and conifold points ($z = 1$).

A gauged linear sigma model in general is mathematically defined* by providing the following data: A (compact) gauge group G, a superpotential $W \in S := \mathrm{Sym}(V^*)$, where V is the m-vector space of chiral fields, a representation $\rho_V : G \to GL(V)$, and a representation of the R-symmetry group $R : U(1) \to GL(V)$. One can decompose the Lie algebra $\mathfrak{g}$ of G into $\mathfrak{g} = \mathfrak{s} + \mathfrak{a}$, where $\mathfrak{s}$ is semi-simple and $\mathfrak{a}$ with $\dim \mathfrak{a} = s$ is Abelian (in particular $\mathfrak{a} \subset \mathfrak{t}$, where $\mathfrak{t}$ is the Cartan subalgebra). Then we define the parameters $t \in \mathfrak{g}^*_{\mathbb{C}}$ such that they factor through the Abelian component $\mathfrak{a}^*_{\mathbb{C}} \hookrightarrow \mathfrak{g}^*_{\mathbb{C}}$. Then we have $t = (t_1, \ldots, t_s$ and $t_i = \zeta_i - i\theta_i$, where ζ are the FI parameters and θ the theta terms. With this data,

* We do not present the full details and instead refer to the literature in [HHP08; HR13; Joc+].

one can then write the D- and F-term equations and compute the classical space of vacua X_ζ, where ζ are the Fayet-Iliopoulos (FI) parameters of the theory.

A particular case that is discussed in [Wit93; HHP08; KRS16] is the GLSM with

$$(G, W, \rho_V, R) = (U(1), W = pG_N(x_1, \dots, x_N), \rho_{\mathbb{C}^{N+1}} : U(1) \to SL(N+1), R) \tag{4.1}$$

where p has weight $-N$ and G_N is a homogeneous polynomial of degree N, with each x_i having weight 1 under $U(1)$.

The functions $e^t \in (\mathbb{C}^*)^s$ on the algebraic torus then provide coordinates for the moduli space $\mathcal{M}_K$. In fact, $\mathcal{M}_K$ is realized as a compactification of $(\mathbb{C}^*)^s$ by removing a closed codimension 1 subset Δ, which is the discriminant locus (cpw. 3.3). The discriminant locus is determined by the superpotential and the $U(1)$ charges of the chirals (weights of the ρ_V representation).

In addition to the field content, for a GLSM on a hemisphere, which is a manifold with boundary, one has to consider the boundary conditions (i.e. the theory on the boundary). Specifying boundary conditions is equivalent to specifying the D-brane data $\mathcal{B}$ for the GLSM (whence the branes are called B-branes. This consists of: a $\mathbb{Z}_2$-graded free S-module $M = M_0 \oplus M_1$, a "factorization" $Q \in \mathrm{End}_S(M)$ of W such that $Q^2 = W\,\mathrm{id}_M$[†] as well as representations $\rho : G \to GL(M)$ and $r_* : \mathfrak{u}(1) \to \mathfrak{gl}(M)$ with some compatibility conditions on Q and M.

4.2 The partition function and grade restriction

The integral formula for the partition function in [HHP08; HR13] depends on the data (G, W, ρ_V, R) of the GLSM as well as the data $\mathcal{B} = (M, Q, \rho, r_*)$ of the D-brane corresponding to the boundary. Furthermore, it also depends on a non-trivial choice of contour $\gamma \subset \mathfrak{t}_\mathbb{C}$ (recall $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{g}$, with coordinates $\sigma \in \mathfrak{t}_\mathbb{C}$), similarly to the integral formula for 3d blocks. The formula reads

$$\mathcal{Z}_{D^2}(\mathcal{B}) = \text{const.} \int_\gamma \mathrm{d}^{\operatorname{rank} G}\sigma \prod_{\mathfrak{g} \text{ roots } \alpha} \alpha(\sigma)\sinh(\pi\alpha(\sigma)) \prod_i^m \Gamma\big(iQ_i(\sigma) + \frac{R_i}{2}\big) e^{it(\sigma)} f_\mathcal{B}(\sigma),$$

where σ are the coordinates on $\mathfrak{t}_\mathbb{C}$, R_i and Q_i are the R- and $U(1)$-charges, α are the positive roots of G and $f_\mathcal{B}(\sigma)$ is the "brane factor". The brane factor is the *only* dependence of $\mathcal{Z}$ on $\mathcal{B}$. The integrand, denoted as $F_\mathcal{B}(\sigma)$, has poles coming from the gamma functions, whose loci $\mathcal{P}$ must be avoided by the contour, hence $\gamma \subset \mathfrak{t}_\mathbb{C} \setminus \mathfrak{P}$.

In fact the choice of integration contour is more subtle with physical implications. It is a Lagrangian submanifold of $\mathfrak{t}_\mathbb{C}$, and not *every* choice of contour is "admissible". Admissible contours are chosen such that the integral exists and converges; Such an admissible contour exists when the FI parameters are *generic*. For non-generic values, there is a condition on the charges q coming from convergence of the integral:

$$-\frac{N}{2} < \frac{\theta}{2\pi} + q < \frac{N}{2}.$$

This is called "charge window" and the procedure for D-branes is called *grade restriction* [HHP08; HR13]. It implies that *not* all D-branes can be transported across phase boundaries, but only the grade restricted ones.

4.3 Differential equation and main focus

The explicit $\mathcal{Z}_{D^2}(\mathcal{B})$ integral is found to satisfy the following differential equation [Joc+]

$$\big[\vartheta_z^{N-1} - z\prod_{j=1}^{N-1}(\vartheta_z + \frac{j}{N})\big] f(z) = 0,$$

where $z = e^t$ are the coordinates on the (unexcised) $\mathcal{M}_K$, N is the number of chiral fields with weight 1 from (4.1), and j are the $U(1)$ weights $q = 1, \dots, N-1$. This equation is of generalized hypergeometric form, and it is the source of our interest in q-deformations of the hypergeometric equation.

[†] This condition implies that $Q = \left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$, with $W = ab$.

In the geometric phase, the IR limit of this GLSM is a degree N hypersurface in $\mathbb{P}^{N-1}$ defined by

$$G_N(x_1, \ldots, x_N) = 0.$$

Our main focus is thus the cubic in $\mathbb{P}^2$. One can check that the defining equation for the partition function is the Gaussian hypergeometric equation with parameters $(\alpha, \beta, \gamma) = (1/3, 2/3, 1)$ (cpw. (5.2)). We are thus interested in studying the q-analogue of the hypergeometric function at the q-analogues of these parameters. As we will see, the most obvious deformation corresponds to the so-called basic hypergeometric equation with parameters $(a, b, c) = (q^{1/3}, q^{2/3}, q)$.

Part II

Classical and “quantum” monodromy

A preliminary discussion: the goal of this work

Results so far

We have spent most of Part I presenting the 3D construction of Beem, Dimofte and Pasquetti [BDP14], and briefly reviewing the 2D GLSM studied in [HHP08; HR13; Joc+; KRS16]. In particular, we saw how q-difference equations appear as *constraints* for holomorphic blocks, which can be thought of as partition functions on "pieces" of three-manifolds. These constraints are in fact *defining* equations for the blocks, as the *space of solutions of the q-difference equations is spanned by the set of blocks*. Determining the holomorphic blocks is equivalent[‡] to determining the partition function on the total manifold: the geometric gluing of the pieces that retrieves the three-manifold extends on the level of gauge theories (modulo deformations) and we can *fuse* the blocks into a factorized equation of the form (2.1). This situation is analogous to two-dimensional gauged linear sigma models [HHP08; HR13; Joc+; Cer+93]: the partition function is again written as a contour integral with a *universal* contour. Restrictions on the choice of contour play a physical role. Furthermore, the partition function satisfies a *defining* differential equation, the Picard-Fuchs equation (corresponding to the periods of the mirror CY). In fact we will take advantage of this feature, and promote the differential constraints into a q-difference constraint, which we conjecture to be (one of many) three-dimensional lifts.

More importantly for our work, the q-difference equation defining the blocks in the non-trivial example of the $\mathbb{P}^1$ sigma model in 3.3 exhibit physically interesting global properties: Stokes phenomena and monodromy. The global data plays an important role in realizing explicit mirror symmetry on the moduli space of this example. More generally, monodromies and analytic continuations of solutions to differential equations (e.g. hypergeometric, Bessel, Painlevé equations) encode physically interesting information: dyon charges [SW94; Ler97], target space duality groups [Cer+93], behavior of D-brane (generalized) central charges between phases [HR13], among others. A ubiquitous feature is that the differential equations that correspond to these monodromies are defining equations for some physically interesting quantity, e.g. partition functions, central charges etc.

Our goal

The main idea behind this work is to

Study the global behavior of q-difference equations appearing in the context of supersymmetric gauge theories.

In particular, we want to study the q-difference equations that are the "lift" of the Picard-Fuchs equations for the cubic in $\mathbb{P}^2$, appearing in [KRS16], as well as the "massless" limit ($x \to 1$) of the q-difference equation of the $\mathbb{P}^1$-sigma model appearing in [BDP14] ((3.17a)). The first set of equations is of basic hypergeometric type, while the second is a deformation of the q-Bessel equation.

Outline

The study of global data in q-difference equations is somewhat lagging behind its counterpart of differential equations. The connection problem for q-difference equations —the precursor to monodromy and Stokes phenomena— was first studied by Birkhoff [Bir13] more than a century ago, and is still a "hot topic" in mathematical research. Some notable references are [Hah49; Aom95; Eti95; PS97; Sau02; Sau03; Sau06; HSS16; Dre17] and of course the detailed work of Ramis, Sauloy and Zhang [RSZ09].

We therefore first devote a chapter on the global properties of *differential* equations, with an aim to develop some methods necessary for the computation of global data (connection matrices, monodromy matrices). This study is done through a classic, yet very instructive and general example: the case of the Gaussian hypergeometric equation. Monodromy and analytic continuation results for these functions are known in the generic cases [Sla09; Bat53], and yet the non-generic, so-called logarithmic cases are the ones which are usually interesting in applications. A systematic treatment of those cases is still active research [Nør55; Nør63; Sch16]. We present general methods of analytic continuation and computing monodromies for the generic cases as presented mainly in [Iwa+12]; the methods provide a picture of

[‡] The equivalence holds conjecturally for general three manifolds, while three-manifolds that are some fibration over an S^1 base are the subject of study of [BDP14].

what we expect to find in the q-difference case as opposed to the differential case. In particular, how we can extract the monodromy matrices from connection matrices and exponents of solutions.

The second chapter is then devoted to the bulk of our work: computing local solutions to q-difference equations finding the analytic continuation formulae that furnish the connection matrix. The difference equations we consider are two main cases: the basic q-hypergeometric equation and the q-Bessel equation and its deformations. We have provided a brief motivation for the study of the q-hypergeometric equation from a lightning fast review of the corresponding two-dimensional gauged linear sigma model corresponding to a degree 3 Calabi-Yau hypersurface in $\mathbb{P}^2$ in section 4.3. The Bessel (difference) equation appeared as one of the two *partial* q-difference equations in the main non-trivial example in our background: the $\mathbb{CP}^1$ sigma model from [BDP14]. We briefly investigate the different deformations of the q-Bessel equations and their physical equivalence, and proceed with the study of a more convenient deformation than the one presented in (3.17a).

5. Monodromy of differential equations: classical methods

5.1 Case study: the hypergeometric equation

We study the *hypergeometric differential equation* defined in one complex variable z as

$$z(1-z)\frac{\mathrm{d}^2u}{\mathrm{d}z^2} + \left[\gamma - (\alpha+\beta+1)z\right]\frac{\mathrm{d}u}{\mathrm{d}z} - \alpha\beta u = 0, \tag{5.1}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are parameters. Equivalently, we can write this as

$$z(\vartheta_z + \alpha)(\vartheta_z + \beta)u - \vartheta_z(\vartheta_z + \gamma - 1)u = 0, \tag{5.2}$$

where $\vartheta_z = z\frac{\mathrm{d}}{\mathrm{d}z}$. In particular we are interested in solutions *around* the singular points of this equation defined on the Riemann sphere (in the generic case these are $0, 1$ and ∞) as dictated by the Frobenius method for solving differential equations, as well as relating the solutions around different points wherever they are both defined. The latter problem is referred to as the "connection problem". In solving the connection problem, one also inherently solves the more interesting for our applications "monodromy problem" — that is the transformation that a set of solutions undergo when analytically continuing the set (through connection *matrices*) along a *non-trivial* loop in the (thrice) punctured Riemann sphere.

Following [Iwa+12], we will present different methods to compute the monodromy groups of the hypergeometric equation, among other things.

5.2 Integral representations

It is in many cases useful if differential equations admit solutions that are of *integral* form i.e. have the form

$$u(z) = \int_{\gamma(z)} K(z,t)\mathrm{d}t,$$

for some *simpler* kernel function $K(x,t)$. In particular, it is helpful when dealing with the *connection problem*, a prime example of the *global* study of solutions:

Connection problem: Let $\mathcal{L}f = 0$ be a linear differential equation of order n and let $\mathcal{F}^{(1)} = \{f_j^{(1)}\}$ and $\mathcal{F}^{(1)} = \{f_j^{(2)}\}$ with $j = 1, \ldots, n$ be solution vectors in the solution space $\mathcal{S} \cong \mathbb{C}^n$ of said differential equation around two points x_1 and x_2 respectively. Furthermore, let C be a path connecting the two points and denote by $C_*\mathcal{F}^{(1)}$ the *analytic continuation* of $\mathcal{F}^{(1)}$ along C. The connection problem amounts to finding a *linear* relation between $C_*\mathcal{F}^{(1)}\}$ and $\mathcal{F}^{(2)}$, i.e.

$$C_*f_j^{(1)} = M_{jk}f_k^{(2)}.$$

For the hypergeometric differential equation we will focus on two types of representations for solutions: the *Euler* integral representation and the *Barnes* integral representation.

5.2.1 Euler integral representation

We denote the hypergeometric series

$$ {}_2F_1(\alpha,\beta;\gamma;z) \equiv F(\alpha,\beta;\gamma;z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} z^n, $$

where $(a)_n := a(a+1)\cdot \ldots \cdot (a+n-1)$ is the Pochhammer symbol. We then have
Theorem 5.2.1. *If* $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$ *then*

$$ F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,\mathrm{d}t, $$

for $|z| < 1$ *where we choose the branch of the (factors in the integrand) according to*

$$ \arg t = 0, \quad \arg(1-t) = 0, \quad |\arg(1-zt)| < \frac{\pi}{2}, \quad \text{for } 0 < t < 1. $$

Equivalently, writing $t = \frac{1}{s}$ *yields*

$$ F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_1^{\infty} s^{\alpha-\gamma}(s-1)^{\gamma-\beta-1}(s-z)^{-\alpha}\,\mathrm{d}s. $$

Proof. The proof relies in the integral representation of the beta function and some manipulations. For details cf. [Iwa+12] page 53. □

Note that both integrals are defined for $z \in \mathbb{C} \setminus [1,\infty)$ and thus define an analytic continuation of ${}_2F_1$ to said domain.

5.2.2 The Euler transform

Given a complex function f of the form $f(t) = (t-a)^{\mu} g(t)$ for $a, \mu \in \mathbb{C}$ (μ is called the *exponent* of f at a), where $g(a) \neq 0$ and holomorphic, we consider the integral transformation, called the *Euler transform* of f

$$ \left(D_a^{-\alpha} f\right)(z) := \frac{1}{\Gamma(\alpha)} \int_a^z (z-t)^{\alpha-1} f(t)\,\mathrm{d}t, \tag{5.3} $$

where the path of integration C has endpoints a and z as noted above, and we have fixed the branches of the multivalued functions $\arg(t-x)$ and $\arg(t-a)$ along C. For $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\mu > -1$, the integral converges, and we want to extend the definition for more values of α and μ. To this end, we consider the shifted *Mellin transform*

$$ H(\nu) = \int_a^b (t-a)^{\nu-1} g(t)\,\mathrm{d}t, $$

over a path γ with endpoints as above, g a holomorphic function in the neighborhood of the path γ and a fixed branch for $\arg(t-a)$.
Lemma 5.2.2. *The function* $H(\nu)$ *can be analytically continued to a meromorphic function in* $\nu \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ *and the poles on* $\mathbb{Z}_{\leq 0}$ *are given by*

$$ \operatorname*{Res}_{\nu=-m} = \frac{1}{m!} g^{(m)}(a). $$

Proof. We pick a point c in the image of γ within the radius of convergence of the Taylor series of g at a. Then, denoting the integrand by $\mathcal{I}$ we have for $\operatorname{Re}\nu > 0$

$$ \begin{aligned} H(\nu) &= \int_a^c \mathcal{I} + \int_c^b \mathcal{I} = \sum_{m=0}^{\infty} \frac{g^{(m)}(a)}{m!} \int_a^c (t-a)^{\nu-1+m}\,\mathrm{d}t + \int_c^b \mathcal{I} \\ &= \sum_{m=0}^{\infty} \frac{g^{(m)}(a)}{m!} \frac{(c-a)^{\nu+m}}{\nu+m} + \text{ holomorphic in } \nu \end{aligned} $$

□

This lemma defines in some sense how to extract *a finite part of a divergent integral*: formally the integral defined by H is divergent for $\mathrm{Re}\,\nu < 0$, but the division of the path into two parts shows that there is a well-defined notion of a finite part. The same procedure can be applied to $D_a^{-\alpha}$ (5.3): The integral is well defined, in terms of its finite part, for any value $\alpha \in \mathbb{C}$, and defines a holomorphic function provided that the exponent $\mu \notin \mathbb{Z}_{\leq 0}$. Some properties of the transform, which show that it can be considered as a generalization of derivatives include

1. If $\mu \neq -1, -2, \ldots$ is the exponent of f at a then the exponent of $D_a^{-\alpha} f$ at a is $\mu + \alpha$ unless $\mu + \alpha$ is e negative integer, in which case the exponent is 0 or a positive integer.
2. There is a composition rule $D_a^{\alpha} \cdot D_a^{\beta} = D_a^{\alpha+\beta}$, as long as the exponent μ of f is not a negative integer.
3. $D_a^m = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m$, if m is a positive integer.
4. It satisfies a “Leibnitz” rule: for a polynomial $p(z)$ of degree n we have that
$$D_a^{\alpha}\big(p(z)f(z)\big) = \sum_{j=0}^{n} \binom{\alpha}{j} p^{(j)}(z)\big(D_a^{\alpha} f\big)(z).$$

The initial point a can also be moved to ∞ with some modification of the above results which we omit.

5.2.3 The hypergeometric Euler transform

We use the results about the Euler transform (5.3) to sind solutions of the hypergeometric equation in integral form.

Consider the set
$$\mathcal{H}_n := \{L = \sum_{j=0}^{n} p_j(z) \frac{\mathrm{d}^j}{\mathrm{d}z^j} \,|\, \deg p_j \leq j, \forall j, \quad \text{and} \quad \deg p_n = n\}$$
of n-th order differential operators with polynomial coefficients as above, called of *hypergeometric type*. We choose $\lambda \in \mathbb{C}$, $f(z)$ a holomorphic function around $a \in \mathbb{C}$ with exponent μ such that $n - \mu \notin \mathbb{N}$. Then, a calculation following from the properties of D_a^{λ} yields (cf. [Iwa+12])
$$D_a^{\lambda}(Lf) = \left(\sum_{k=0}^{n} \sum_{j=k}^{n} \binom{\lambda}{j-k} p_j^{(j-k)}(z) \frac{\mathrm{d}^k}{\mathrm{d}z^k} \right) D_a^{\lambda} f =: (\mathcal{D}^{\lambda} L) D_a^{\lambda} f,$$
where we have defined the map of operators $\mathcal{D}^{\lambda}$ as
$$\mathcal{D}^{\lambda} L := \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{\lambda}{j-k} p_j^{(j-k)}(z) \frac{\mathrm{d}^k}{\mathrm{d}z^k}.$$

We thus have statements

- If the exponent μ of f at a satisfies $n - \mu \notin \mathbb{N}$ then $D_a^{\lambda}(Lf) = (\mathcal{D}^{\lambda} L)(D_a^{\lambda} f)$ for $\lambda \in \mathbb{C}$.
- If the exponent μ of f at $z = \infty$ satisfies $\mu \notin \mathbb{Z}_{<0}$ and $\lambda + \mu \notin \mathbb{Z}_{<0}$ then $D_{\infty}^{\lambda}(Lf) = (\mathcal{D}^{\lambda} L)(D_{\infty}^{\lambda} f)$ for $\lambda \in \mathbb{C}$.

We can thus obtain solutions to $(\mathcal{D}^{\lambda} L) f = 0$ given solutions to $Lf = 0$.

Since the operator $\mathcal{D}^{\lambda} L$ is also of hypergeometric type, we have established that $\mathcal{D}^{\lambda} : \mathcal{H}_n \to \mathcal{H}_n$, i.e. that the map is a transformation of operators of hypergeometric type, called the *hypergeometric Euler transform*. It follows a composition rule $\mathcal{D}^{\lambda} \mathcal{D}^{\nu} = \mathcal{D}^{\lambda+\nu}$.

Through the differential equation known as the *Jordan-Pochhammer equation*, and its solutions integral expression, we may obtain the integral expression of a hypergeometric solution as a special case (we omit the details). We obtain

Theorem 5.2.3. *Let α, β, γ be constants such that $2 - (\alpha - \gamma)$, $2 - (\gamma - \beta - 1)$, $\alpha - \beta - 1$, $-\beta - 1$, $\alpha \notin \mathbb{N}$. Then the hypergeometric equation admits solutions $F_{pq}(z)$ given by*
$$F_{pq}(z) = \int_p^q t^{\alpha-\gamma}(1-t)^{\gamma-\beta-1}(t-z)^{-\alpha}\,\mathrm{d}t,$$
where $p, q = 0, 1, \infty$ or z, and the we consider the finite part of the integral if it is divergent.

This theorem gives us *six* solutions given by the choice of p and q, which in turn are related by Kummer's 24 relations. We can also use *double* loops, cf. Prop 3.3.7 in [Iwa+12].

5.2.4 Barnes integral representation

In addition to the Euler integral representation of solutions to the hypergeometric equation, there is also the Barnes integral representation. There are three ways to derive it, and we discuss two of them here.

Barnes integral from the power series

We consider a function defined by a power series $f(z) = \sum_{m=0}^{\infty} a_m z^m$, as well as a function $g(t)$ that is (i) meromorphic for $|t| < \infty$, (ii) holomorphic at $t = 0, 1, 2, \dots$ and (iii) *interpolates* the sequence $\{a_m\}$ of coefficients, i.e. $g(m) = a_m$ for $m \in \mathbb{N}$. Then it is easy to check that the function

$$h(t) := -g(t)\frac{\pi}{\sin(\pi t)}(-z)^t$$

has simple poles at $t(= m) = 0, 1, 2, \dots$ with corresponding residue $-a_m z^m$. We can thus define a contour integral in t whose path C_N encircles the poles $t = 0, 1 \dots, N$ (with negative orientation) and we obtain

$$\frac{1}{2\pi i}\int_{C_N} h(t)\mathrm{d}t = \sum_{m=0}^{N} a_m z^n \tag{5.4}$$

It becomes obvious what we would like to do: check that h has some suitable asymptotic behavior and let $N \to \infty$ so that we have an integral representation

$$f(z) = \frac{1}{2\pi i}\int_C h(t)\mathrm{d}t,$$

for some path C.

We apply this to the hypergeometric series $F(\alpha, \beta; \gamma; z)$:

$$a_m = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m(1)_m} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)\Gamma(1+m)},$$

assuming that $\alpha, \beta\gamma \neq 0, -1, -2, \dots$. We identify

$$g(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)\Gamma(1+t)},$$

which satisfies the conditions we need. Using the identity $\Gamma(-t)\Gamma(1+t) = -\frac{\pi}{\sin(\pi t)}$ we obtain

$$h(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-z)^t,$$

and the main statement of this subsection is then:

Theorem 5.2.4. *Suppose $\alpha, \beta, \gamma \neq 0, -1, -2 \dots$, $|x| < 1$ and $|\arg(-x)| < \pi$. Then*

$$F(\alpha, \beta; \gamma; z) = \frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_C \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-z)^t\,\mathrm{d}t,$$

where the path C is as in the figure 5.1a, i.e. it lies on the imaginary axis for $|t| \gg 1$, the poles of $\Gamma(\alpha+t)\Gamma(\beta+t)$ lie to the left of C and the poles of $\Gamma(-t)$ to the right of C. This is the Barnes integral representation of the hypergeometric function.

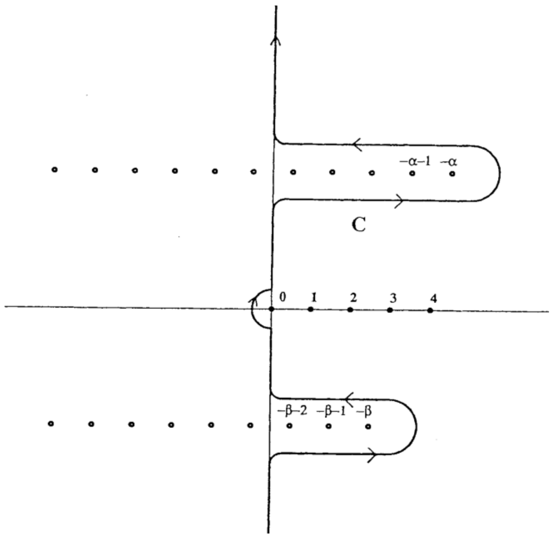
Proof. To prove the theorem, we close up the contour C to a loop $C_N = C + C_N^{(2)}$. By our result (5.4) above we readily have that the integral along C_N yields $\sum_{m=0}^{N} a_m z^m$. We thus only need to prove the "asymptotic property" of h:

$$\int_{C_N^{(2)}} h(t)\mathrm{d}t \text{ converges to 0 as } N \to \infty.$$

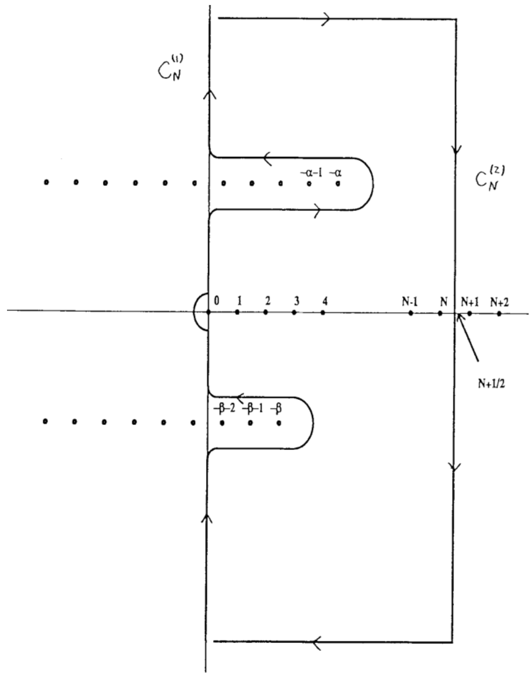
Using the Stirling formula (details in [Iwa+12]), we obtain that when $\delta < |\arg t| \leq \frac{\pi}{2}$ *or* $\mathrm{Re}\, t \in \mathbb{N} + \frac{1}{2}$ then

$$h(t) = \mathcal{O}(t^{\alpha+\beta-\gamma-1}e^{-(\pi|\mathrm{Im}\, t|+\arg(-z)\,\mathrm{Im}\, t)}),$$

for any positive number $\delta < \frac{\pi}{2}$. This formula can be applied to the "three parts" of $C_N^{(2)}$ for all N, and shows that the contribution vanishes for $N \to \infty$. □



(a) The path of integration C for the theorem.



(b) The closed path of integration. The proof of the theorem is to show that the $C_N^{(2)}$ part vanishes.

Figure 5.1

Barnes integral from a difference equation

For the second approach to the Barnes integral, we use the *inverse Mellin transform* of a function $G(t)$:

$$F(z) = \frac{1}{2\pi i}\int G(t)(-z)^t \mathrm{d}t,$$

which is (for now) a formal expression, i.e. no contour is specified. One of the properties of this transform is that it transforms differential equations to difference equations, and when the differential equation is of special form, the difference equation can be solved. We apply this method to the hypergeometric equation. The equation takes also the familiar form

$$[\vartheta_z(\vartheta_z+\gamma-1) - z(\vartheta_z+\alpha)(\vartheta_z+\beta)]f = 0, \quad \vartheta_z := z\frac{\mathrm{d}}{\mathrm{d}z}.$$

We then take the Ansatz that the solution can be written as an inverse Mellin transform

$$f(z) = \frac{1}{2\pi i}\int_C g(t)(-z)^t \mathrm{d}t,$$

and we look for the conditions on g under which this integral converges. We take C to be a vertical path possibly with some deformation to avoid singularities (no coincidence, cf. figure 5.1). Our Ansatz reduces the differential equation to

$$\int_C t(t+\gamma-1)g(t)(-z)^t \mathrm{d}t + \int_C (t+\alpha)(t+\beta)g(t)(-z)^{t+1}\mathrm{d}t = 0.$$

Now note that, *if* the path C can be shifted to the left by 1 without change, then we obtain the condition

$$\int_C \big[(t+1)(t+\gamma)g(t+1) + (t+\alpha)(t+\beta)g(t)\big](-z)^{t+1}\mathrm{d}t = 0$$

which is the difference equation

$$g(t+1) = -\frac{(t+\alpha)(t+\beta)}{(t+\gamma)(t+1)}g(t).$$

This is solved by

$$g(t) = \frac{\Gamma(t+\alpha)\Gamma(t+\beta)\Gamma(-t)}{\Gamma(t+\gamma)},$$

and in fact since $\lim_{\tau\to\infty} t^2 g(t)(-z)^t = 0$ when $t = \sigma + i\tau$ holds for this solution, (i.e. the integrand is finite along an infinite vertical strip), the "formal" integral is exact. We thus obtain the solution

$$f(z) = \frac{1}{2\pi i}\int_C \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-z)^t \,\mathrm{d}t,$$

where C is as in figure 5.1.

The Gauss-Kummer identity

Lastly, we state a result relevant for the calculation of connection matrices.
Theorem 5.2.5. *If* $\mathrm{Re}(\gamma-\alpha-\beta) > 0$ *then the* Gauss-Kummer identity *holds*

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.$$

5.3 Monodromy of the hypergeometric equation

5.3.1 The problems

We formalize some of the notions that we need to actually *define* the monodromy problem. Given a differential equation

$$\left[\frac{\mathrm{d}^n}{\mathrm{d}z^n} + a_1(z)\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} + \ldots + a_2(z)\frac{\mathrm{d}}{\mathrm{d}z} + a_n(z)\right] f(z) = 0$$

defined on some possibly non simply connected domain $D \subset \mathbb{C}$ but without loss of generality, connected (usually just $\mathbb{C}\setminus\{\text{singular locus}\}$), the solutions might be multivalued functions on D, that come from well-defined single valued functions on the universal covering $\tilde{D}$. To describe the "multi-valuedness" of the space of solutions $\mathcal{S} \cong \mathbb{C}^n$ we associate to the equation above a certain subgroup of $GL(\mathcal{S}) \cong GL(n,\mathbb{C})$ called the *monodromy group* associated to the equation, which is in fact the *image* of a *representation* $\rho : \pi_1(D) \to GL(\mathcal{S})$ where π_1 is the fundamental group (due to connectedness we disregard base points).

Given a *local* solution vector $\mathcal{F} = (f_1,\ldots,f_n) \in \mathcal{S}(U)$ in some 1-connected neighborhood U of $b \in D$ and an element $[\alpha] \in \pi_1(D)$, we denote by $\alpha_*\mathcal{F}$ the analytic continuation of $\mathcal{F}$ along the representative loop α. Then, since $\alpha_*\mathcal{F}$ is also a solution vector we have

$$\alpha_*\mathcal{F} = \mathcal{F}\cdot M(\alpha;\mathcal{F})$$

for a matrix $M \in GL(\mathcal{S}(U))$, called the monodromy matrix.

Now, denoting by $\mathcal{G}$ a local solution vector in $\mathcal{S}(V)$ in some 1-connected neighborhood V of $a \in D$, we consider a path γ connecting a and b, as well as the analytic continuation $\gamma_*\mathcal{G}$ of $\mathcal{G}$ along γ. Then there exists a matrix $C \in GL(n,\mathbb{C})$ such that

$$\mathcal{G} = \mathcal{F}C.$$

This matrix is called the *connection matrix*. We can thus compute that for the two different local solution vectors $\mathcal{F},\mathcal{G}$ we have

$$\mathcal{G}M(\alpha;\mathcal{G}) = \mathcal{G}C^{-1}M(\alpha;\mathcal{F})C,$$

i.e. *every two monodromy representations are conjugate*, as we expect since we are disregarding base points for the fundamental groups. We thus have... problems:

Monodromy problem *Given a linear differential equation, find an explicit expression for its monodromy and/or find the presentation of the monodromy group on a fundamental system of solutions.*

More generally, we might analytically continue a solution vector along a path that is not closed. This defines the connection problem, discussed earlier.

Connection problem *Given a path γ from $a \in D$ to $b \in D$ and local solution vectors around a, b find an explicit expression for the connection matrix C.*

5.3.2 Finding the monodromy of the hypergeometric equation

We turn our attention to the case of the hypergeometric equation: the domain of definition is $D = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$, i.e. the thrice punctured Riemann sphere. An elementary consideration in algebraic topology (e.g. $D \simeq S^1 \vee S^1$ by a deformation retraction) shows that $\pi_1(D) = \langle \gamma_0, \gamma_1 \rangle \cong F_2$ i.e. the fundamental group is a *free* group in two generators, generated by the simple loops around 0 and 1, without loss of generality. The loop around ∞ is given by $\gamma_\infty = \gamma_0^{-1}\gamma_1^{-1}$. Since the solution space in the hypergeometric equation is two-dimensional, the monodromy of the hypergeometric equation is a group homomorphism (representation) $\rho : F_2 \to GL(2, \mathbb{C})$.

In this subsection we will describe three ways to compute the monodromy of the hypergeometric equation.

By Euler integrals over paths

We want to solve the connection problem using the Euler integrals introduced earlier, and therefore also the monodromy problem. We introduce some notation: Let

$$\begin{aligned} F_{pq}(z) &= \int_p^q \varphi(t, z; \lambda, \mu, \nu) \, \mathrm{d}t, \text{ where} \\ \varphi(t, z; \lambda, \mu, \nu) &= t^\lambda (1-t)^\mu (z-t)^\nu \text{ and} \\ \lambda = \alpha - \gamma, &\quad \mu = \gamma - \beta - 1, \quad \nu = -\alpha. \end{aligned}$$

and $p, q \in \{0, 1, z, \infty\}$. We also assume that $\operatorname{Im} z > 0$. The integral *makes sense* (at worst through its *finite part*) if the following condition holds

$$\text{none of } \alpha, 1-\beta, \gamma-\alpha \text{ and } \beta-\gamma+1 \text{ is a positive integer} \tag{5.5}$$

and defines a solution of the hypergeometric equation. We obtain 6 integrals

$$F_{01}, F_{1\infty}, F_{\infty 0}, F_{0z}, F_{1z}, \text{ and } F_{z\infty}$$

We need to specify branches for all three factors of the integrand. These are given in the table below

	$\arg t$	$\arg(1-t)$	$\arg(z-t)$	z_0
$\overline{01}$	0	0	$[\xi, \eta]^*$	∞
$\overline{1\infty}$	0	$-\pi$	$[\eta, \pi]$	0
$\overline{\infty 0}$	π	0	$[0, \xi]$	1
$\overline{0z}$	ξ	$[\eta - \pi, 0]$	ξ	0
$\overline{1z}$	$[0, \xi]$	$\eta - \pi$	η	1
$\overline{z\infty}$	ξ	$[\xi - \pi, \eta - \pi]^*$	$\xi + \pi$	∞

where $\overline{pq}$ denotes a path from p to q, and $[m, n]^* := [\min(a, b), \max(a, b)]$, while z_0 denotes the point around which the expression is a fundamental solution (the correspondence is provided later). Then, we have the following result

Theorem 5.3.1. *The six solutions given by $F_{pq}(z)$ with $p, q \in \{0, 1, z, \infty\}$ satisfy*

$$\begin{array}{llllllr} F_{01} & +F_{1\infty} & +F_{\infty 0} & & & & = 0 \\ F_{01} & & & -F_{0z} & +F_{1z} & & = 0 \\ & e^{-2\pi i\mu} F_{1\infty} & & & -F_{1z} & -e^{2\pi i\nu} F_{z\infty} & = 0 \\ & & e^{2\pi i\lambda} F_{\infty 0} & +F_{0z} & & +F_{z\infty} & = 0 \end{array}$$

The proof of the theorem relies in dividing up $\mathbb{C}$ into three contractible domains D_j, $j = 1, 2, 3$ whose boundaries are the six contours above, and we apply Cauchy's theorem

$$\int_{\partial D_j} \varphi(t, z; \lambda, \mu, \nu) \, \mathrm{d}t = 0$$

with appropriate choices of branches.

The general solutions around the singularities are of the form

$$w_{z_0} = A w^{(1)}_{z_0,e_1}(z) + B w^{(2)}_{z_0,e_2}(z) \tag{5.6}$$

with $A, B \in \mathbb{C}$ and e_i the *exponent* introduced earlier i.e. $w_{z_0,e}(z) = (z-z_0)^e g(z-z_0)$ where g is holomorphic around z_0.
We thus have around $z = 0, 1$ and ∞ respectively

$$w_0(z) = A\, F(\alpha,\beta;\gamma;z) + B z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z), \tag{5.7a}$$
$$w_1(z) = A F(\alpha,\beta;1+\alpha+\beta-\gamma;1-z) + B(1-z)^{\gamma-\alpha-\beta} F(\gamma-\beta,\gamma-\alpha;\gamma-\alpha-\beta+1;1-z), \tag{5.7b}$$
$$w_\infty(z) = A z^{-\alpha} F(\alpha,\alpha-\gamma+1;\alpha-\beta+1,z^{-1}) + B z^{-\beta} F(\beta,\beta-\gamma+1;\beta-\alpha+1;z^{-1}). \tag{5.7c}$$

which are defined when α, β, γ and their differences are not integers*. This is what we mean by "generic parameters", in the sense that the set $(\alpha,\beta,\gamma) \in \mathbb{C}^3$ which is "problematic" is of measure 0 in $\mathbb{C}^3$.
Theorem 5.3.2. *The six integrals are in correspondence with the three pairs of fundamental solutions around the three singular points* $0, 1, \infty$ *as*

$$\begin{array}{rll} 0: & F_{1\infty} = c_{1\infty} w^{(1)}_{0,0}, & F_{0z} = c_{0z} w^{(2)}_{0,1-\gamma} \\ 1: & F_{\infty 0} = c_{\infty 0} w^{(1)}_{1,0}, & F_{1z} = c_{1z} w^{(2)}_{1,\gamma-\alpha-\beta} \\ \infty: & F_{01} = c_{01} w^{(1)}_{\infty,\alpha}, & F_{z\infty} = c_{z\infty} w^{(2)}_{\infty,\beta}, \end{array}$$

for some constants c_*.

For details for both theorems cf. [Iwa+12] page 99. Theorem 5.3.1 now provides enough relations to solve the monodromy problem: We can calculate the connection matrix of, say solutions around 0 and 1:

$$(F_{1\infty}, F_{0z}) = (F_{\infty 0}, F_{1z}) \frac{P}{e^{2\pi i\nu} - e^{-2\pi i\mu}}, \text{ where } P = \begin{pmatrix} e^{2\pi i(\nu+\lambda)} - e^{2\pi i\nu} & e^{-2\pi i\mu} - e^{2\pi i(\lambda+\nu)} \\ e^{2\pi i\nu} - 1 & 1 - e^{-2\pi i\mu} \end{pmatrix}$$

and we can calculate the monodromy with respect to the fundamental system around 0. We have
Theorem 5.3.3. *Let* γ_0, γ_1 *be loops based at* $z = \frac{1}{2}$ *circling the singular points 0,1 respectively in the positive direction once and suppose* α, β, γ *satisfy* (5.5). *Then the analytic continuation of* $\mathcal{F} = (F_{1\infty}, F_{0z})$ *along the paths, i.e. the monodromy, is given by*

$$\gamma_{0*}\mathcal{F} = F A_0, \quad \gamma_{1*}\mathcal{F} = F A_1$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i\gamma} \end{pmatrix}, \quad A_1 = P^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(\gamma-\alpha-\beta)} \end{pmatrix} P.$$

with P *as above. The monodromy group with respect to the solution vector* $\mathcal{F}$ *is generated by* A_0 *and* A_1.

We skip the derivation using double loops.

5.3.3 By Barnes integrals

We saw earlier that we have the Barnes integral representation for solutions to the hypergeometric equation

$$F(\alpha,\beta;\gamma;z) = \frac{1}{2\pi i} \int_C h(t,z)\, \mathrm{d}t,$$

where the integrand is

$$h(t,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)} (-z)^t.$$

The poles form the sets

$$\begin{aligned} \mathcal{P}_+ &= \{0, 1, 2, \ldots\} \\ \mathcal{P}_- &= \{-\alpha, -\alpha-1, \ldots\} \cup \{-\beta, -\beta-1, \ldots\}, \end{aligned}$$

while the contour C is a vertical line, mostly on the imaginary axis, possibly deformed so the $\mathcal{P}_+$ lies to the right of it and $\mathcal{P}_-$ lies to the left of C, cpw. figure 5.1a. We now *change* the path from C to C_N as shown in figure 5.2.

* Specific solutions *can* exist even when some of the parameters or their differences are integers

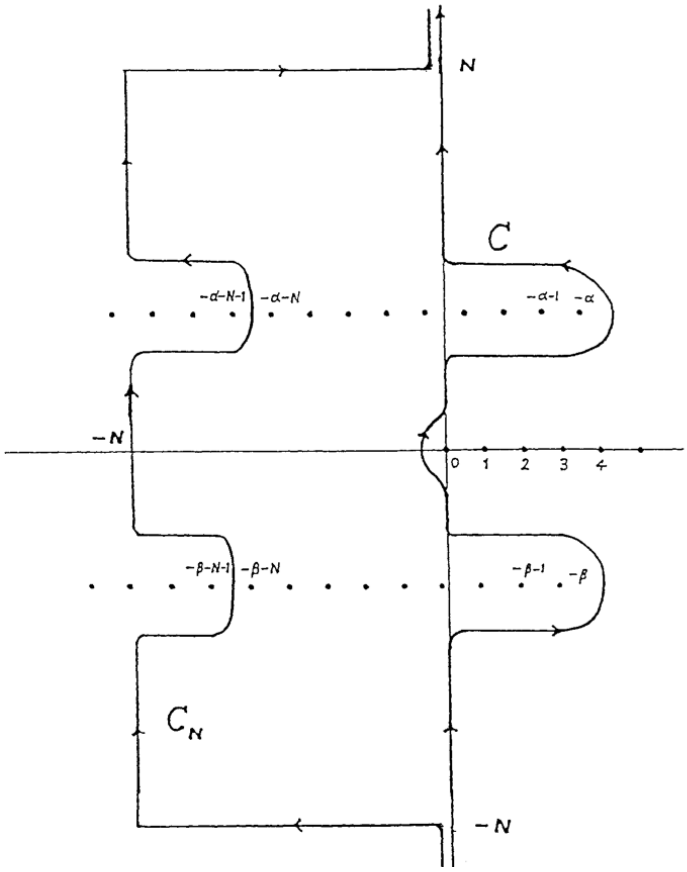


Figure 5.2: The paths C and C_N.

Theorem 5.3.4. *Suppose that none of $\alpha, \beta, \gamma, \gamma-\alpha$ and $\gamma-\beta$ is an integer. Then, in the notation of (5.6), we have that*

$$(w_{0,0}^{(1)}(z), w_{0,1-\gamma}^{(2)}(z)) = (w_{\infty,\alpha}^{(1)}(z), w_{\infty,\beta}^{(2)}(z)) \cdot P,$$

with connection matrix

$$P = \begin{pmatrix} c(\alpha,\beta,\gamma) & c(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma) \\ c(\beta,\alpha,\gamma) & c(\beta-\gamma+1,\alpha-\gamma+1,2-\gamma) \end{pmatrix}, \quad c(\lambda,\mu,\nu) := e^{-\pi i\lambda}\frac{\Gamma(\nu)\Gamma(\mu-\lambda)}{\Gamma(\mu)\Gamma(\nu-\lambda)}.$$

Proof. The proof relies on calculating the residues on the poles $\mathcal{P}_-$ that are included in the closed contour $C - C_N$. We obtain formally due to Cauchy's theorem

$$I_C - I_{C_N} \equiv w_{0,0}^{(1)}(z) - I_{C_N} = \sum \big(\text{poles in } C - C_N\big),$$

which explicitly reads

$$w_{0,0}^{(1)}(z) = c(\alpha,\beta,\gamma)w_{\infty,\alpha}(z,N) + c(\beta,\alpha,\gamma)w_{\infty,\beta}(z,N) + I_{C_N},$$

where $w_{\infty,\nu}(z,N)$ is a truncated series, coming from the pole contributions, converging to $w_{\infty,\alpha}(z)$ for $N\to\infty$. It remains to show that the I_{C_N} contribution vanishes in the $N\to\infty$ limit. For the second fundamental solution $w_{0,1-\gamma}^{(2)}(z)$ we replace the parameters α,β,γ by $\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma$ and multiply by $z^{1-\gamma}$ to obtain in total

$$\begin{aligned} w_{0,0}^{(1)}(z) &= c(\alpha,\beta,\gamma)w_{\infty,\alpha}^{(1)}(z) + c(\beta,\alpha,\gamma)w_{\infty,\beta}^{(2)}(z) \\ w_{0,1-\gamma}^{(1)}(z) &= c(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma)w_{\infty,\alpha}^{(1)}(z) + c(\beta-\gamma+1,\alpha-\gamma+1,2-\gamma)w_{\infty,\beta}^{(2)}(z). \end{aligned}$$

For details cf. [Iwa+12, page 111] . ◻

The monodromy is then easy to calculate:

Theorem 5.3.5. *Let γ_0, γ_1 be loops based at $z = \frac{1}{2}$ only circling the singular points 0,∞ respectively in the positive direction once, and suppose none of $\alpha, \beta, \gamma, \gamma - \alpha, \gamma - \beta$ is an integer. Then the monodromies around these loops, i.e. the analytic continuations of $\mathcal{F} = (w^{(1)}_{0,0}(z), w^{(2)}_{0,1-\gamma}(z))$ along γ_0, γ_∞ are given by*

$$\gamma_{i*}\mathcal{F} = FA_i, \quad i = 0, \infty,$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i\gamma}, \end{pmatrix} \quad A_\infty = P^{-1}\begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix} P,$$

and the monodromy group is generated by these matrices.

6. q-monodromy

In this chapter we dwell into the core of our work: we want to study the global behavior of the solutions to q-difference equations.

The cases that interest us as we stated in the discussion II the 'lift' of the "differential" case stemming from the gauged linear sigma model in [KRS16], as well as a variation of (the massless limit of) the three-dimensional $\mathbb{P}^1$-sigma model from [BDP14] (cf. 3.3, e.g. (3.17a)). We lay out case studies of these difference equations, which have been investigated to some extent in the mathematics literature [Sau02; Zha03; Mor11]. In particular, we are interested in the analytic continuation of their solutions, i.e. the connection matrices, with an eye towards applications in physics. In addition we investigate the possibility of computing monodromy matrices.

6.1 Some generalities on q-difference equations

Difference equations and in particular q-difference equations are *functional* equations involving "shifts" in the argument of the function $f(z) \to f(qz)$, where the function is the unknown , and q is a parameter. The parameter can be taken to be in $\mathbb{C}$, but we restrict ourselves to $q \in \mathbb{C} \setminus U(1)$ as the unit q-circle presents with complications*. The q-difference equation can be thought of as some operator $\mathcal{L}_q$, built out of the 'position' operator acting by $\hat{z}f = zf$ and the q-shift operator $\sigma_{q,z} \equiv \sigma_q := q^{z\frac{\mathrm{d}}{\mathrm{d}z}}$ which induces q-shifts in the arguments: $\sigma_q f(z) = f(qz)$. They act on some space of functions, but we are not interested in the formal aspect of these spaces or the structures therein (Galois theory, Picard-Vessiot extensions of difference rings etc).

q-difference equations can also be thought of as a generalization of differential equations: A q-difference equation can be written as so-called q-*differential* equation involving the q-derivative operator $\mathcal{D}_q$ by

$$\sigma_q = (q-1)z\mathcal{D}_q + \mathrm{id}\,.$$

and one can now take the limit $q \to 1$ where $\mathcal{D}_q \overset{q\to 1}{\to} \frac{\mathrm{d}}{\mathrm{d}z}$ (or conveniently $z\mathcal{D}_q \to \vartheta_z = z\frac{\mathrm{d}}{\mathrm{d}z}$), and in generic cases the q-difference equation becomes a differential one (up to some rescaling of the dependent or independent variable). It is an interesting question to ask whether solutions of q-difference equations have well defined $q \to 1$ limits and if they do, whether they correspond to solutions of differential equations. In other words we have the "diagram"

$$\begin{array}{ccc} \mathcal{L}(\sigma_q, \hat{z})f = 0 & \text{is solved by} & f(z;q) \\ {\scriptstyle q\to 1}\big\downarrow & & \vdots\,{\scriptstyle q\to 1} \\ \mathcal{L}(\vartheta_z, z)f = 0 & \text{is solved by} & f(z) \end{array}$$

If the diagram 'commutes' in the sense that $\lim_{q\to 1} f(z;q) = f(z)$ then we have what is called *confluence* of solutions [HSS16]. Confluence is a phenomenon that is not well-understood yet and is a subject of current research in mathematics [Dre15; Dre17; DZ09]. In many of our examples (e.g. the basic hypergeometric equation (6.12)) the non-trivial aspect appears as a 'condensation of discrete poles into a branch cut' of the limiting function, but we do not discuss it further in this work. It should be noted that along with solutions to q-difference equations, also connection matrices and other global data (Stokes matrices) undergo a confluence phenomenon.

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* As we saw in Part I, we write the parameter q as $q = e^{\hbar} = e^{2\pi i\beta\epsilon}$ [BDP14], with the $q \to 1$ limit corresponding to $\epsilon \to 0$. This corresponds to vanishing of the twisting in the fibration $D^2 \times_q S^1$.

We want to introduce q-difference equations in a more structured manner, albeit not by a detailed mathematical treatment. For the more formal aspect of q-difference equations one can turn to [HSS16; Ern12; KC01; PS97]. We will mostly follow the work of Sauloy in [HSS16; Sau02; Sau03].

6.1.1 Some simple examples

The free chiral block

Let us consider a simple and ubiquitous example of a difference equation

$$\sigma_q f(z) = (1-z)f(z) \tag{6.1}$$

with 'initial condition' $f(0) = 1$. Similarly to differential equations, we can look for solutions that are holomorphic around some point. Let us consider a holomorphic solution around 0: $f(z) = \sum_{n\geq 0} c_n z^n$. Substituting in the equation we obtain

$$\sum_{n=0}^{\infty} c_n(q^n - 1)z^n + \sum_{n=1} c_{n-1}z^n = 0$$

, and the first term of the first sum vanishes, so we may rearrange the sums and obtain the recursion formula

$$c_{n-1} = (1-q^n)c_n, \quad c_0 = 1$$

which is solved by $c_n = \frac{1}{(q;q)_n}$ and we recover the well-documented [Sau02; HSS16] solution (modulo elliptic factors)

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}.$$

This is precisely the equation and solution of the line operator identity of the free chiral block (3.4) up to the transformations $(z,q) \to (z^{-1}, q^{-1})$.

In a completely similar fashion we can consider

$$\sigma_q f(z) = \frac{1}{1-z} f(z),$$

with initial condition $f(0) = 1$ and find the solution

$$f(z) = (z;q)_\infty\,.$$

Monodromy of the q-Pochhammer symbol

We can use the asymptotic expansion of the q-Pochhammer symbol (A.1.10) to obtain a first glimpse of monodromies of q-functions. We essentially rely on the known monodromy properties of the polylogarithms Li_n which are given in [Bro09]. Small loops M_0 around $0 \in \mathbb{C}$ and small loops M_1 around $1 \in C$ induce the following monodromy transformations

$$M_0^* \operatorname{Li}_n(z) = \operatorname{Li}_n(z), \quad M_1^* \operatorname{Li}_n(z) = \operatorname{Li}_n(z) + \frac{2\pi i}{(n-1)!} \log^{(n-1)}(z).$$

The monodromy around 0 looks trivial, but in fact is not (more precisely, the Riemann surface is still branched): if one transforms by a loop around 1, the induced logarithms will make subsequent loops around 0 non-trivial. Using these, we find that the loop around 1 induces the transformation on the asymptotic expansion (A.1.10)

$$M_1^* \frac{1}{(z;q)_\infty} = \frac{1}{(z;q)_\infty} \exp[-2\pi i \frac{\log z}{\log q}].$$

This implies that we have

$$\begin{aligned}
(M_1^*)^k \frac{1}{(z;q)_\infty} &= e^{-2\pi i k \frac{\log z}{\log q}} \frac{1}{(z;q)_\infty} \\
M_0^* \frac{1}{(z;q)_\infty} &= \frac{1}{(z;q)_\infty} \\
(M_0^*)^n (M_1^*)^k \frac{1}{(z;q)_\infty} &= e^{nk \frac{4\pi}{\log q}} e^{-2\pi i k \frac{\log z}{\log q}} \frac{1}{(z;q)_\infty}.
\end{aligned}$$

It is clear that the transformed functions are *still* solutions to the simple q-difference equation $[\sigma_q - (1-z)]f = 0$: the exponential factors that are z-dependent are an archetypal example (cf. [Jac10; Hah49]) of elliptic functions, i.e. q-shift invariant functions. We discuss this in the next subsection 6.1.2. Of course a big caveat is actually the use of the expansion formula: it is not guaranteed to reflect the actual global properties of $\frac{1}{(z;q)_\infty}$.

A generalization

We can consider the generalization of the free chiral q-difference equation

$$[(1-\sigma_q)^n - z]f(z) = 0,$$

where the free chiral corresponds to the case $a = 1$. We can solve these equations with a "trick":

Claim: The above equation is solved up to order ϵ^n by the following Ansatz

$$f_n(z, \epsilon; q) = \sum_{k=0}^{\infty} \frac{x^{k-\epsilon}}{(q^{1-\epsilon}; q)_k^n}, \tag{6.2}$$

and thus the n linearly independent solutions are given by the first n coefficients $f_{n,k}$ $(k = 0, 1, \ldots, n-1)$ of the expansion

$$f_n(z, \epsilon; q) = \sum_{m=0}^{\infty} f_{n,k}(z; q)\epsilon^k.$$

Using the fact that on monomials $F(\sigma_q)x^m = F(q^m)x^m$ we can easily find after some calculation that

$$(1-\sigma_q)^n f_n(z, \epsilon; q) = (1-q^{-\epsilon})^n z^{-\epsilon} + z f_n(z, \epsilon; q),$$

where the first term comes from the $k = 0$ term in the infinite sum. It is easy to see that this term is of order $\mathcal{O}(\epsilon^n)$, verifying our claim.

We will study the case $n = 2$ in detail later.

6.1.2 The field of constants

One immediate question one might ask is, to what extent are the solutions to difference equations unique? It is clear that the solution space of a finite order n q-difference equation forms an n-dimensional vector space, but... over which field? Clearly $\mathbb{C}$ is part of it: $\alpha f(z) + \beta g(z)$ solves a linear q-difference equation of which f, g are solutions. But this can be extended, as we have seen in subsection 3.2.1 one can multiply a solution by *any* σ_q-invariant function $c(z) = c(qz)$ and the result is still a solution. Thus the field of constants is precisely the field of such functions. Through the identification (isomorphism)

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\exp"] \arrow[d, "\pi^1_{\text{can.}}"'] & \mathbb{C}^* \arrow[d, "\pi^2_{\text{can.}}"] \\
\mathbb{C}/\Gamma_\tau \arrow[r, "\cong"] & \mathbb{C}^*/q^{\mathbb{Z}}
\end{tikzcd}$$

where $q = e^{2\pi i \tau}$, we find that functions that are q-periodic and $X + 2\pi i$ periodic where $x = e^X$ are precisely the *elliptic functions* $\mathcal{M}(\mathbb{E}_q)$ on the complex torus with complex modulus τ. Thus, any solution of a q-difference equation is determined *modulo an elliptic factor*.

6.1.3 General difference equations and systems

Let us consider a linear q-difference equation of order n with meromorphic coefficients. It is written in most general form as

$$[a_n(z)\sigma_q^n + \ldots + a_1(z)\sigma_q + a_0(z)]f(z) = h(z). \tag{6.3}$$

Since a_n cannot be identically zero, we can divide by it and without loss of generality consider instead

$$[\sigma_q^n + a_{n-1}(z)\sigma_q^{n-1} \ldots + a_1(z)\sigma_q + a_0(z)]f(z) = h(z)$$

As in the case of differential equations, the equation is called homogeneous when $h = 0$. We are only concerned with homogeneous equations in this work. Similarly to ordinary linear differential, the solution space of linear degree n q-difference equations forms a vector space of dimension n [Car12] (at least locally around 0 and ∞).

We can write the linear q-difference equation of order n can be written in terms of a linear system of first-order q-difference equations of dimension n. We introduce the solution vector

$$\bar{\phi} = \begin{pmatrix} f \\ \sigma_q f \\ \vdots \\ \sigma_q^{n-1} f \end{pmatrix}$$

with the help of which it is an easy exercise to verify that (6.1.3) is equivalent to the vector equation

$$\sigma_q \bar{\phi} = A\bar{\phi},$$

where

$$A = A(z) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{n-1} & -a_{n-2} & -a_{n-3} & \dots & -a_0 \end{pmatrix}. \tag{6.4}$$

This is called the coefficient matrix and it plays a significant role in solving and classifying q-difference equations. It becomes clear that linear differential equations of order n are special cases of n-dimensional linear systems. However, in practice we will only work with the form (6.4). We can furthermore assemble the n linearly independent (over $\mathcal{M}(\mathbb{E}_q)$) solutions $\phi_1(z), \dots, \phi_n(z)$ into the so-called fundamental solution matrix

$$\Phi(z) = \begin{pmatrix} \phi_1 & \dots & \phi_n \\ \sigma_q \phi_1 & \dots & \sigma_q \phi_n \\ \vdots & \ddots & \vdots \\ \sigma_q^{n-1}\phi_1 & \dots & \sigma_q^{n-1}\phi_n \end{pmatrix}(z) \tag{6.5}$$

which satisfies the matrix equation equivalent to (6.1.3)

$$\sigma_q \Phi(z) = A(z)\Phi(z). \tag{6.6}$$

This is the form we will be using to investigate the connection problem. Note that the presentation of the system is not unique: there are *gauge transformations* $G[A]$ of the coefficient matrix $A(z)$. The systems defined by $A(z)$ and $B(z)$ are gauge-equivalent if there exists a matrix $G \in GL_n(\mathbb{C}(\{z\}))$ such that

$$B = G[A] \coloneqq (\sigma_q G)AG^{-1}.$$

It is easy to see that a gauge transformation is a redefinition of the proposed solutions $\bar{\phi} = (\phi_1, \dots, \phi_n)$, i.e. a new Ansatz $G\bar{\phi}$ in the original equation, transforming the coefficients appropriately.

There are some general definitions that characterize our difference system based on the coefficient matrix. We present these here. Given a system $(\sigma_q F)(z) = A(z)F(z)$ with $A \in GL_n(\mathbb{C}(\{z\}))^\dagger$ we have that the system is called

Fuchsian at 0 if $A(z)$ is holomorphic near 0 and $A(0) \in GL_n(\mathbb{C})$. A Fuchsian system is furthermore called **non-resonant** at 0 if $\sigma(A(0)) \cap q^{\mathbb{Z}^*}\sigma(A(0)) = \varnothing$, where $\sigma(A)$ denotes the spectrum of the operator A.

regular singular at 0 if there is a matrix $R^{(0)}(z) \in GL_n(\mathbb{C}(\{z\}))$ such that the system with coefficient matrix $(R^0(qz))^{-1}A(z)R^0(z)$ is Fuchsian. Clearly, every Fuchsian system is trivially regular singular.

irregular singular when such a matrix $R^{(0)}$ does not exist.

non-logarithmic if $A(0)$ is semi-simple (over $\mathbb{C}$ this is equivalent to diagonalizability).

One can extend these definitions to the point infinity by investigating the matrix $A(z^{-1})$. The equations that we will consider are Fuchsian for the section 6.2, while we will have both Fuchsian and irregular equations in 6.3.

$^\dagger\mathbb{C}(\{z\})$ denotes the meromorphic functions in $\mathbb{C}$.

6.1.4 Local solutions and connection matrix

In this subsection we use the methods developed by Sauloy [Sau02] to solve Fuchsian $q$difference equations, also used by [Roq08]. They are the q-difference analogue of the classical Frobenius method to solve Fuchsian differential equations. We do not give a full exposition and instead present only the methods that are relevant. The methods give a systematic way to find local solutions of Fuchsian q-difference equations around the points $0, \infty$. Note that these points are always mathematically interesting, since they are fixed points of the shift operator σ_q. The solutions of q-difference equations around finite points is, as far as we know, an open problem both of technical and conceptual difficulty. For a short discussion, refer to subsection 6.2.4.

We will utilize some basic linear algebra facts to simplify the search for solutions to difference systems. Supposing that our coefficient matrix $A(z)$ in is non-resonant (which will always be the case) and Fuchsian (to which we will reduce the equations – even for irregular singular equations) we denote by J_{z^*} the *Jordan normal form* of $A(z^*)$ where $z^* = 0, \infty$, i.e. $J_{z^*} = M(z^*)A(z^*)M(z^*)^{-1}$ for some similarity matrix $M(z^*)$. Sauloy shows in [Sau03] that there is a solution $F_{z^*} \in GL_n(\mathbb{C}(\{z\}))$ to the modified system

$$(\sigma_q F_{z^*})(z)J_{z^*} = A(z)F_{z^*}(z). \tag{6.7}$$

This result is useful since we can now solve the simpler, constant-coefficient system

$$(\sigma_q X_{J_{z^*}})(z) = J_{z^*}X_{J_{z^*}}(z) \tag{6.8}$$

where the matrix $X_{J_{z^*}}$ is called the *character matrix* and then the *fundamental solution matrices* to (6.6) around $z = z^*$ are given by

$$\Phi_{z^*}(z) = F_{z^*}(z)X_{J_{z^*}}(z) \tag{6.9}$$

as can be easily checked. The "decomposition" of the solution to Fuchsian equations in to two parts is in complete analogy with the Frobenius method for solving Fuchsian differential equations: the local solutions to such differential equations are of the form $z^\lambda \sum_{n\geq 0} c_n(\lambda)z^n$. The sum-factor is the holomorphic part of the solutions while the factor z^λ, which is called "exponent" or "character", is determined by the indicial equation. As we saw in chapter 5 it is relevant for the global behavior of the solution, namely because for non-integer exponents γ it is a multivalued-function. Analogously, the matrix $F_{z^*}(z)$ from above encodes the "holomorphic part" of the solution, while the character matrix $X_{z^*}(z)$ encodes all the information about the q-analogues of exponents (characters).

The system (6.8) can in fact be solved for general *constant* matrices J_{z^*}. This relies on the fact that every invertible matrix J admits a Jordan-Chevalley(-Dunford) decomposition $J = J_DJ_U = J_UJ_D$ where J_D is a semi-simple (over $\mathbb{C}$, diagonalizable) matrix, while J_U is a unipotent matrix which implies that $(\mathbb{1} - J_U) = 0$ for some positive n. Let us first assume that the matrix F_{z^*} is purely diagonalizable. We want to define a q-analogue of simple monomials ('q-exponents' or 'q-characters') i.e. a function that satisfies $\sigma_q f_\lambda(z) = \lambda f_\lambda(z)$ for $\lambda \in \mathbb{C}^*$. There are many (infinite, infact) choices one can make and $z\ell$ with $q^\ell =: \lambda$ is one them. However it is recommended to pick different functions as characters[‡].:

$$e_\lambda(z) := \frac{\Theta_q(z)}{\Theta_q(\lambda z)}, \quad \lambda \in \mathbb{C}^* \tag{6.10}$$

where $\Theta_q(z) = (q;q)_\infty(z;q)_\infty(qz^{-1};q)_\infty = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}n(n-1)}z^n$ is the Jacobi theta function. In view of the difference equation $(\sigma_q\theta_q)(z) = -z^{-1}\Theta_q(z)$ satisfied by the theta function, our q-character has the desired property. Note that this choice is not (mathematically) canonical in any way, but it could have physical significance. Why do we choose these functions over the monomials z^ℓ? For a discussion of this we refer to the Appendix A.2.2 as well as discussions in [Sau03; Sau02].

Now, for a diagonalizable $n \times n$ matrix $D = P\,\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}P^{-1}$ where $P \in GL_n(\mathbb{C})$ we can define

$$e_D(z) := P\,\mathrm{diag}\{e_{\lambda_1}(z), \ldots, e_{\lambda_n}(z)\}P^{-1}.$$

For the unipotent part we define so called q-logarithms

$$\ell_q(z) := -z\frac{\frac{\mathrm{d}}{\mathrm{d}z}\Theta_q(z)}{\Theta_q(z)}.$$

[‡]Our choice coincides with the one from Roques [Roq08], while Sauloy [HSS16; Sau02] chooses $e_a(z) = \frac{\Theta_q(c/z)}{\Theta_q(1/z)}$

Differentiating the q-difference equation of the theta function we find that the q-logarithms satisfy the q-difference equation

$$(\sigma_q \ell_q)(z) = \ell_q(z) + 1.$$

Then, given a unipotent matrix U we define

$$e_U(z) := \sum_{k=0} \binom{\ell_U(z)}{k} (U - \mathbb{1})^k,$$

where $\binom{\ell_U(z)}{k} = \frac{1}{k!} \prod_{i=0}^{k-1} (\ell_q(z) - i)$. Note that the sum truncates since by definition $(U - \mathbb{1})^n = 0$ for some n. We can now claim to have solved (6.8) by setting the character matrix

$$X_{J_0}(z) = e_{J_D}(z)\, e_{J_U}(z). \tag{6.11}$$

Note that, in practice, the Jordan normal form of *our* coefficient matrices will be either purely diagonal(-izable) for generic values of parameters or purely unipotent for non-generic values. We will denote the similarity matrix that brings $A(z^*, \text{parameters})$ to its Jordan normal form by

$$M(z^*, \text{parameters}) = \begin{cases} S(z^*, \text{generic parameters}) \\ U(z^*, \text{non-generic parameters}) \end{cases}$$

depending on the parameters, where S stands for "semi-simple" and U stands for "unipotent".

To summarize, for Fuchsian, non-resonant systems we have that

1. the coefficient matrix A at $z = 0, \infty$ has Jordan normal form $J_{z^*} = M(z^*)A(z^*)M(z^*)^{-1}$ (where $z^* = 0, \infty$), which in turn has a decomposition into a semi-simple (diagonalizable) and unipotent factor $J_{z^*} = J_D J_U$. When the unipotent part is non-trivial, the system is logarithmic.
2. These factors determine the contribution $X_{J_{z^*}}$ of the q-characters to the solution matrix by (6.11).
3. The 'holomorphic part' F_{z^*} of the solution is then determined by (6.7) and thus finally,
4. the solution matrix of (6.6) is given by (6.9).

Once we have local fundamental solution matrices Φ_0 and Φ_∞ respectively, then the *connection matrix* (also called the Birkhoff matrix) from ∞ to 0 is given by $P = \Phi_\infty^{-1}\Phi_0$. It takes values in the field of constants with respect to the shift operator σ_q — the field of elliptic functions and it is the goal of our calculations. As we will see, the main technical difficulty in its calculation for the cases of interest is the following: most of the data (solutions to (6.7) and (6.8) etc) exists in the literature but *for generic values of the parameters* appearing in the equations. In particular, for parameters for which the system is non-logarithmic. However, the situations that interest us are in fact the ones with the non-generic parameters and one has to rely on tricks to compute the solutions and thus the connection matrices.

6.1.5 Differential vs. difference monodromy

We recall the toy case of the differential hypergeometric equation of chapter 5 in particular the theorems 5.3.3 and 5.3.5. We observe the following: the monodromy group associated to a differential equation $\mathcal{D}f = 0$, is the image of a representation $\rho_{\mathcal{D}} : \pi_1(X, p) \to GL(\mathcal{S})$, where X is the space where the differential equation is defined (e.g. $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ for us) and $\mathcal{S}$ is the vector space of (local) solutions. In the case of a regular singular differential equation like the hypergeometric equation, the fundamental group of X is generated by the two loops around two of the singular points, with the third given by the relation $\gamma_0\gamma_1\gamma_\infty = 1$. If we are given solutions around a point, say 0, then the monodromy of the equation on a small loop around 0 is *dictated by the exponents (characters)* of the given solutions. In particular for two solutions of the form $(z - z_0)^{\epsilon_i} \times \big(\text{analytic in } (z - z_0)\big)$, the corresponding monodromy about the small loop around z_0 is dictated by the exponents by

$$\begin{pmatrix} e^{2\pi i \epsilon_1} & 0 \\ 0 & e^{2\pi i \epsilon_2}, \end{pmatrix}$$

and similarly for the other singular points. It is straightforward to see that the monodromy is a consequence of the multi-valuedness of the characters: for $\epsilon_i \neq \mathbb{Z}$ we have $z^{\epsilon_i} = \exp(\epsilon_i \log z)$. The monodromy around another point, say ∞, but with respect to the same basis is found as shown in the above theorems: one conjugates the connection matrix P with the small loop around infinity, in the basis of solutions around infinity. We therefore see that two generators of the monodromy group (in the case of the hypergeometric equation, the two generators span the group) can be computed if one knows the exponents of the solutions around two points, as well as the connection matrix connecting the two solutions

In the q-difference case, things are a bit more complicated: as we saw, it is *recommended* to choose different functions as q-characters than z^α, the q-characters $e_a(z)$ we introduced in (6.10). As mentioned in the appendix A.2.2, these functions are *single-valued*, meromorphic functions on the complex plane. One might then ask, *what does monodromy mean for q-difference equations*?

The answer to this question is mathematically obvious, from the simple examples considered in this thesis. However, arguments from Galois and Picard-Vessiot theory have been made by many authors (cf. [Eti95; Sau02; Sau03; RSZ09; HSS16]) that

the q-analogue of monodromies for ordinary differential equations is the Birkhoff matrix.

As a last note, one *can* still choose the functions z^α as exponents, in which case one can "read-off" the monodromy as in the differential case.

6.2 The lift of the cubic: the basic hypergeometric equation

In this section we want to investigate the three-dimensional lift of the Picard-Fuchs equation of the cubic in $\mathbb{P}^2$ as described in subsection 4.3. The differential equation satisfied by the partition function of the GLSM is the classical hypergeometric equation group for the parameters $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}, \gamma = 1$. It is therefore interesting to q-generalize this equation into a q-difference equation and ask what the corresponding q-monodromy is. The value of $\gamma = 1$ brings additional complications, as we have mentioned: the generic solutions around $z = 0$ in (5.7a) are not both valid, because they coincide. Instead, the second solution involves logarithms. This is all detailed in the references [KRS16; Sch16].

We extrapolate the three-dimensional lift from these calculations by replacing the differential (operator $\frac{\mathrm{d}}{\mathrm{d}z}$ Picard-Fuchs equations that the hemisphere partition function satisfies by the *corresponding* (operator $\mathcal{D}_q$) q-difference equation. In other words, we want to ultimately study the so-called *basic hypergeometric equation* with parameters $a = q^\alpha, b = q^\beta$ and $c = q^\gamma$. As in the case of the classical hypergeometric equation, the value of the parameter $c = q$ will lead to problems relating to the logarithmic nature of the solutions.

6.2.1 The basic hypergeometric q-difference equation

The basic hypergeometric q-difference equation and series was first studied by Heine [Hei47], and subsequently by a large number of authors (notably Rev. F.H. Jackson [Jac10], Smith[§], Hahn [Hah49] to name a few) as a (q-)generalization of Gauss' hypergeometric series ${}_rF_s$ and equation.

The basic hypergeometric difference equation is

$$\left[\sigma_q^2 - \frac{(a+b)z - (1+c/q)}{abz - c/q}\sigma_q + \frac{z-1}{abz - c/q}\right]\phi(z) = 0, \tag{6.12}$$

where $\sigma_q \equiv q^{z\frac{\mathrm{d}}{\mathrm{d}z}}$ is the q-shift operator and a, b, c are parameters which are generic if they are in $\mathbb{C} \setminus q^{\mathbb{Z}}$. We are to think of them as $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, and in the $q \to 1$ limit these reduce to the classical hypergeometric parameters. One solution around $z = 0$ is of course the *basic hypergeometric series* (a.k.a. Heine's series)

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q; z\right) := \sum_{n=0}^{\infty} \frac{(a,b;q)_n}{(q,c;q)_n} z^n, \qquad |z| < 1.$$

The second solution is obtained by

$$z^{1-\gamma}\,{}_2\phi_1\left(\begin{matrix} aq/c, bq/c \\ q^2/c \end{matrix} \,\middle|\, q; z\right),$$

[§] Smith, Edwin Raymond. *Zur Theorie der Heineschen Reihe und ihrer Verallgemeinerung.* München: Straub, 1911. <http://eudml.org/doc/203472>.

where the first factor is the non-trivial exponent. As discussed in 6.1.4, we will replace this branched factor with the globally defined meromorphic q-character $e_{q/c}(z)$ and the second solution is

$$e_{q/c}(z)\ {}_2\phi_1\left(\begin{matrix} aq/c, bq/c \\ q^2/c \end{matrix}\,\middle|\, q; z\right).$$

Similarly, a transformation $z \mapsto z^{-1}$ leaved the form of the difference equation (6.12) intact, and we can find the solutions around infinity

$$e_a(z)^{-1}\ {}_2\phi_1\left(\begin{matrix} a, aq/c \\ aq/b \end{matrix}\,\middle|\, q; \frac{cq}{abz}\right)$$
$$e_b(z)^{-1}\ {}_2\phi_1\left(\begin{matrix} b, bq/c \\ bq/a \end{matrix}\,\middle|\, q; \frac{cq}{abz}\right).$$

The solutions around 0 and infinity can easily be seen to reduce the the solutions around 0 and infinity for the classical hypergeometric equation (5.1) given in (5.7a), (5.7c). What about solutions which converge to the solutions around 1 (5.7b)? As it turns out this is a difficult open problem. We discuss this in short in subsection 6.2.4.

We want to apply the procedure outlined in the previous section to the case of the basic hypergeometric equation and in particular, to the case of interest: $a = q^{\frac{1}{3}}, b = q^{\frac{2}{3}}$ and $c = q$. As we will see, because $c = q$ our system is logarithmic, which is a source of complication in the basic (q-generalized) case, as much as it is in the classical case [Sch16; Nør63].

As discussed in the in the subsection 6.1.3, we can rewrite this equation as a matrix equation using the fundamental solution matrix (6.5)

$$(\sigma_q\Phi)(z) = A(a,b;c;z)\Phi(z), \text{ with } A(a,b;c;z) = \begin{pmatrix} 0 & 1 \\ -\mu(z) & \lambda(z) \end{pmatrix}, \tag{6.13}$$

where $\mu(z) \equiv \mu(a,b;c;z) = \frac{z-1}{abz-c/q}$ and $\lambda(z) \equiv \lambda(a,b;c;z) = \frac{(a+b)z-(1+c/q)}{abz-c/q}$ are the coefficients in our equation. The matrix A takes in fact values in the rational functions, i.e. $A \in GL_n(\mathbb{C}(z))$. According to our characterization of q-difference systems, since clearly $A(0), A(\infty) \in GL_n(\mathbb{C})$, we find that the system (6.13) is *Fuchsian*, thus also *regular singular* at both points $0, \infty$. The eigenvalues of $A(0)$ for this system are easily computed to be $\sigma(A(0)) = \{\frac{1}{2}(1 + \frac{q}{c} \pm \sqrt{(1-\frac{q}{c})^2})\}$. The case that interests us is $c = q$ and the two eigenvalues coincide. According to our characterization the system is *non-resonant* for all values of c. Concerning diagonalizability, we have a differentiation at $c = q$: the system becomes logarithmic. We thus treat the two cases $c \neq q$, $c = q$ separately. Following [Roq08], we obtain the logarithmic case from the non-logarithmic case as a limit $c \to q$ and we extract meaningful result.

6.2.2 Analytic continuation around infinity (Case $c \neq q$)

We therefore start by providing fundamental solution matrices around 0 and ∞. We study the matrix

$$A(z) = \begin{pmatrix} 0 & 1 \\ -\mu(z) & \lambda(z) \end{pmatrix} \text{ with } \lambda(a,b;c;z) = \frac{(a+b)z-(1+c/q)}{abz-c/q} \text{ and } \mu(a,b;c;z) = \frac{z-1}{abz-c/q}.$$

We start with the solutions around $z = 0$. For $c \neq q$ the matrix $A(0) = \begin{pmatrix} 0 & 1 \\ -\frac{q}{c} & \frac{c+q}{c} \end{pmatrix}$ is in fact diagonalizable with similarity matrix $M_0(c) \equiv S_0(c) := \begin{pmatrix} 1 & 1 \\ 1 & \frac{q}{c} \end{pmatrix}$, i.e.

$$A(0) = \begin{pmatrix} 1 & 1 \\ 1 & \frac{q}{c} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{q}{c} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \frac{q}{c} \end{pmatrix}^{-1} = \frac{1}{c-q}\begin{pmatrix} 1 & 1 \\ 1 & \frac{q}{c} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{q}{c} \end{pmatrix}\begin{pmatrix} -q & c \\ c & -c \end{pmatrix}. \tag{6.14}$$

Hence the system is non-resonant and non-logarithmic in the case $c \neq q$, while for $c = q$ the obstruction to diagonalizability is clear from the above factor $\frac{1}{c-q}$. We recognize the trivial Jordan normal form $J_0(c) = \operatorname{diag}\{1, q/c\}$. Applying the procedure, we write the solution around $z = 0$ as

$$\Phi_0(z;c) = F_0(z;c)X_{J_0(c)}(z),$$

where the character matrix is

$$X_{J_0(c)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & e_{q/c}(z) \end{pmatrix},$$

while the solution to the modified system (6.7) is can be 'read-off' as the analytic part of the known solutions

$$F_0(z;c) = \begin{pmatrix} {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right) & {}_2\phi_1\left(\begin{matrix}aq/c,bq/c\\q^2/c\end{matrix}\middle| q;z\right) \\ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;qz\right) & q/c\;{}_2\phi_1\left(\begin{matrix}aq/c,bq/c\\q^2/c\end{matrix}\middle| q;qz\right) \end{pmatrix}$$

In this case the "exponent" in the q-case is the q/c eigenvalue of $A(0)$ which corresponds to the exponent $1-\gamma$ in the classical case.

We repeat the procedure at $z=\infty$. The matrix $A(\infty) = \begin{pmatrix} 0 & 1 \\ -(ab)^{-1} & a^{-1}+b^{-1}\end{pmatrix}$ is also diagonalizable

$$A(\infty) = \begin{pmatrix} 1 & 1 \\ a^{-1} & b^{-1}\end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1}\end{pmatrix}\begin{pmatrix} 1 & 1 \\ a^{-1} & b^{-1}\end{pmatrix}^{-1} = \frac{1}{a-b}\begin{pmatrix} 1 & 1 \\ a^{-1} & b^{-1}\end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1}\end{pmatrix}\begin{pmatrix} a & -ab \\ -b & ab\end{pmatrix}.$$

The solution around $z=\infty$ is then

$$\Phi_\infty(z) = F_\infty(z) X_{J_\infty}(z),$$

where for $J_\infty(a,b) \equiv J_\infty = \mathrm{diag}\{a^{-1}, b^{-1}\}$ we have the character matrix

$$X_{J_\infty}(z) = \begin{pmatrix} e_{a^{-1}}(z) & 0 \\ 0 & e_{b^{-1}}(z)\end{pmatrix}$$

and the solutions to the modified system (6.7) are

$$F_\infty(z) = \begin{pmatrix} {}_2\phi_1\left(\begin{matrix}a,aq/c\\aq/b\end{matrix}\middle| q;\frac{cq}{abz}\right) & {}_2\phi_1\left(\begin{matrix}b,bq/c\\bq/a\end{matrix}\middle| q;\frac{cq}{abz}\right) \\ \frac{1}{a}\;{}_2\phi_1\left(\begin{matrix}a,aq/c\\aq/b\end{matrix}\middle| q;\frac{c}{abz}\right) & \frac{1}{b}\;{}_2\phi_1\left(\begin{matrix}b,bq/c\\bq/a\end{matrix}\middle| q;\frac{c}{abz}\right) \end{pmatrix}.$$

The connection matrix

We can now use the analytic continuation of ${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right)$ by the celebrated (Mellin-Barnes-)Watson formula [GR04] which states that, if $|\arg(-z)|<\pi$ and $c, a/b \notin q^{\mathbb{Z}}$, we have

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right) &= \frac{(b,c/a;q)_\infty \Theta_q(az)}{(c,b/a;q)_\infty \Theta_q(z)}\;{}_2\phi_1\left(\begin{matrix}a,aq/c\\aq/b\end{matrix}\middle| q;\frac{cq}{abz}\right) \\ &\quad + \frac{(a,c/b;q)_\infty \Theta_q(bz)}{(c,a/b;q)_\infty \Theta_q(z)}\;{}_2\phi_1\left(\begin{matrix}b,bq/c\\bq/a\end{matrix}\middle| q;\frac{cq}{abz}\right) \\ &= \frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)} e_a(z)^{-1}\;{}_2\phi_1\left(\begin{matrix}a,aq/c\\aq/b\end{matrix}\middle| q;qc/abz\right) \\ &\quad + \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)} e_b(z)^{-1}\;{}_2\phi_1\left(\begin{matrix}b,bq/c\\bq/a\end{matrix}\middle| q;qc/abz\right), \end{aligned} \tag{6.15}$$

where $\Gamma_q(z)$ are the q-gamma functions (cf. appendix A.2.3) and we are using the compact notation $f(a)f(b) =: f(a,b)$ for the Γ_q factors. We have also 'picked logarithms' of q writing $a=q^\alpha$ as usual. Note that the z-dependent factors in each summand are solutions to the q-difference equation (6.12), albeit with *different* choice of q-characters (recall $e_{a^{-1}}(z) \neq e_q(z)^{-1}$ but (A.2.13a) gives $e_{a^{-1}}(z) = e_a(z/a)^{-1}$). Replacing $a \mapsto aq/c,\ b \mapsto bq/c,\ c \mapsto q^2/c$, we obtain the analytic continuation of the second set of solutions:

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix}aq/c,bq/c\\q^2/c\end{matrix}\middle| q;z\right) &= \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\Theta_q(aqz/c)}{\Theta_q(z)}\;{}_2\phi_1\left(\begin{matrix}a,aq/c\\aq/b\end{matrix}\middle| q;\frac{cq}{abz}\right) \\ &\quad + \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\Theta_q(bqz/c)}{\Theta_q(z)}\;{}_2\phi_1\left(\begin{matrix}b,bq/c\\bq/a\end{matrix}\middle| q;\frac{cq}{abz}\right) \\ &= \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)} e_{aq/c}(z)^{-1}\;{}_2\phi_1\left(\begin{matrix}a,aq/c\\aq/b\end{matrix}\middle| q;\frac{cq}{abz}\right) \\ &\quad + \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)} e_{bq/c}(z)^{-1}\;{}_2\phi_1\left(\begin{matrix}b,bq/c\\bq/a\end{matrix}\middle| q;\frac{cq}{abz}\right). \end{aligned}$$

These two relations essentially are the evaluation of $F_\infty^{-1}F_0$, given by the (elliptic) coefficients of the (Mellin-Barnes-)Watson formula. We collect these results together with the characters in the connection matrix $P = \Phi_\infty^{-1}\Phi_0 = X_{J_\infty}^{-1}F_\infty^{-1}F_0X_{J_0}$

$$\begin{aligned}P(z) &= X_{J_\infty}^{-1}(z)\begin{pmatrix}\frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)}e_a(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)}e_{aq/c}(z)^{-1}\\ \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)}e_b(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)}e_{bq/c}(z)^{-1}\end{pmatrix}X_{J_0}(z)\\ &= \begin{pmatrix}\frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)}e_{a^{-1}}(z)^{-1}e_a(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)}e_{q/c}(z)e_{a^{-1}}(z)^{-1}e_{aq/c}(z)^{-1}\\ \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)}e_{b^{-1}}(z)^{-1}e_b(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)}e_{q/c}(z)e_{b^{-1}}(z)^{-1}e_{bq/c}(z)^{-1}\end{pmatrix}\end{aligned}$$

The elliptic functions and constants (with respect to z, but not the parameters) are non-zero under genericity assumptions on the parameters a, b, c and their ratios (cf. [Roq08]). One obtains many cases according to the (non-)genericity of the parameters, but here we focus solely on the case of interest, namely $c = q, a = q^{1/3}, b = q^{2/3}$.

A last note, one can replace the q-character matrix by the classical monomials, yielding the so-called *twisted* connection matrix

$$\begin{aligned}\check{P}(z) &= \begin{pmatrix}z^{-\alpha} & 0\\ 0 & z^{-\beta}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)}e_a(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)}e_{aq/c}(z)^{-1}\\ \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)}e_b(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)}e_{bq/c}(z)^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^{1-\gamma}\end{pmatrix}\\ &= \begin{pmatrix}\frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)}z^\alpha e_a(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)}z^{\alpha-\gamma+1}e_{aq/c}(z)^{-1}\\ \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)}z^\beta e_b(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)}z^{\beta-\gamma+1}e_{bq/c}(z)^{-1}\end{pmatrix}.\end{aligned}$$

6.2.3 Analytic continuation around infinity (Case $c = q$, and $a/b \notin q^{\mathbb{Z}}$)

We start again around $z = 0$. It is already obvious from the calculations in the previous case that $A(0) = \left(\begin{smallmatrix}0 & 1\\ -1 & 2\end{smallmatrix}\right)$ corresponding to $c = q$ is *not* diagonalizable (i.e. the matrix is not semi-simple and the system is logarithmic). Its Jordan normal form $J_0(c = q)$ is found by the similarity matrix $M_0(c = q) \equiv U_0 := \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$

$$A(0) = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

We can now find solutions to the modified system (6.7) with $J_0 = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ i.e.

$$(\sigma_q\tilde{F})(z)J_0 = A(z)\tilde{F}(z)$$

from the case $c \neq q$ by taking the limit. We introduce some notation to distinguish the two cases: We add the dependence on c in the coefficient matrix $A(z; c)$ and the solution $F_{z^*}(z; c)$ to the modified system (6.7). Note that the normal form of the matrix at $z = 0$ when $c \neq q$ (i.e. diagonal form) is $A(0, c) = M_0(c)J_0(c)M_0(c)^{-1} \equiv S_0(c)J_0(c)S_0(c)^{-1}$ as given in (6.14), where $J_0(c)$ for $c \neq 0$ is diagonal (recall, $J_0(\cdot)$ is a *discontinuous* map of c). The right-hand side does *not* have a (factor-by-factor) limit $c \to q$ and needs to be replaced by the above $A(0, q) = U_0J_0(q)U_0^{-1}$. From the case $c \neq q$ we know that $F_0(z; c)$ for $c \neq q$ satisfies (6.7) with the normal matrix $J_0(c)$. We calculate

$$\begin{aligned}&(\sigma_qF_0)(z; c)\ J_0(c) = A(z, c)\ F_0(z; c)\\ \Longrightarrow\ &(\sigma_qF_0)(z; c)\ S_0(c)^{-1}A(0, c) = A(z, c)\ F_0(z; c)S_0(c)^{-1}\\ \Longrightarrow\ &(\sigma_qF_0)(z; c)\ S_0(c)^{-1}A(0, c)\ U_0 = A(z, c)\ F_0(z; c)S_0(c)^{-1}U_0.\end{aligned} \tag{6.16}$$

Since $A(0, c), S_0(c)$ and U_0 are constants with respect to σ_q the last line is equivalent to

$$(\sigma_qF_0S_0^{-1}U_0)(z; c)\underbrace{U_0^{-1}A(0, c)\ U_0}_{=J_0(0)} = A(z, c)\ (F_0S_0^{-1}U_0)(z; c).$$

We readily see that if $(F_0S_0^{-1}U_0)(z, c)$ has a well-defined limit at $c \to q$, then it is a solution to (6.7) for $J_0(c = q)$, thus yielding a solution to the system. This is indeed the case:

$$(F_0S_0^{-1}U_0)(z;c) = \begin{pmatrix} {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right) & {}_2\phi_1\left(\begin{matrix}aq/c,bq/c\\q^2/c\end{matrix}\middle| q;z\right) \\ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;qz\right) & q/c\,{}_2\phi_1\left(\begin{matrix}aq/c,bq/c\\q^2/c\end{matrix}\middle| q;qz\right)\end{pmatrix} \begin{pmatrix}1 & 1\\ 1 & q/c\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix} {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right) & \frac{c}{c-q}\left({}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right) - {}_2\phi_1\left(\begin{matrix}aq/c,bq/c\\q^2/c\end{matrix}\middle| q;z\right)\right) \\ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;qz\right) & \frac{c}{c-q}\left({}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;qz\right) - q/c\,{}_2\phi_1\left(\begin{matrix}aq/c,bq/c\\q^2/c\end{matrix}\middle| q;qz\right)\right)\end{pmatrix},$$

and the limit yields derivatives $\frac{\mathrm{d}}{\mathrm{d}c}\big|_{c=q}$ of the functions in the second column. Denoting

$$\frac{\mathrm{d}}{\mathrm{d}c}\bigg|_{c=q} {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right) =: \zeta(a,b;z) \text{ and } \frac{\mathrm{d}}{\mathrm{d}c}\bigg|_{c=q} {}_2\phi_1\left(\begin{matrix}aq/c,bq/c\\q^2/c\end{matrix}\middle| q;z\right) =: \xi(a,b;z) \tag{6.17}$$

we find the solution to (6.7) for $J_0(c=q)$:

$$\tilde{F}_0(z) := F_0(z;c=q) = \begin{pmatrix} {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right) & q\big(\zeta(a,b;z)-\xi(a,b;z)\big) \\ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;qz\right) & {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;qz\right) + q\big(\zeta(a,b;qz)-\xi(a,b;qz)\big).\end{pmatrix}$$

It remains to find the characters $X_{J_0}(z)$. The Jordan(-Chevalley-Dunford) decomposition of our $J_0(q)$ is trivial, i.e. $J_0(q) = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is just a unipotent matrix. Our formula for the unipotent part then yields

$$X_{J_0}(z) = \sum_{k=0} \binom{\ell_q(z)}{k}(J_0-\mathbb{1})^k = \begin{pmatrix}1 & \ell_q(z)\\ 0 & 1\end{pmatrix}.$$

We can now finally write the fundamental solution matrix around $z=0$ for the resonant case $c=q$:

$$\Phi_0(z) = \tilde{F}_0(z)X_{J_0}(z),$$

where

$$\tilde{F}_0(z) = \begin{pmatrix} {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;z\right) & q\big(\zeta(a,b;z)-\xi(a,b;z)\big) \\ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;qz\right) & {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| q;qz\right) + q\big(\zeta(a,b;qz)-\xi(a,b;qz)\big).\end{pmatrix},$$

and

$$X_{J_0}(z) = \begin{pmatrix}1 & \ell_q(z)\\ 0 & 1\end{pmatrix}.$$

We proceed to the solutions around $z=\infty$: in this case the $c=q$ condition does *not* change our results. We recall:

$$A(\infty) = \begin{pmatrix}0 & 1\\ -(ab)^{-1} & a^{-1}+b^{-1}\end{pmatrix} = \frac{1}{a-b}\begin{pmatrix}1 & 1\\ a^{-1} & b^{-1}\end{pmatrix}\begin{pmatrix}a^{-1} & 0\\ 0 & b^{-1}\end{pmatrix}\begin{pmatrix}a & -ab\\ -b & ab\end{pmatrix}.$$

and thus solution around $z=\infty$ is

$$\Phi_\infty(z) = F_\infty(z)X_{J_\infty}(z),$$

where for $J_\infty = \mathrm{diag}\{a^{-1},b^{-1}\}$ we have

$$X_{J_\infty}(z) = \begin{pmatrix}e_{a^{-1}}(z) & 0\\ 0 & e_{b^{-1}}(z)\end{pmatrix}$$

and the solutions to the modified system (6.7) are

$$F_\infty(z) = \begin{pmatrix} {}_2\phi_1\left(\begin{matrix}a,a\\aq/b\end{matrix}\middle| q;\frac{q^2}{abz}\right) & {}_2\phi_1\left(\begin{matrix}b,b\\bq/a\end{matrix}\middle| q;\frac{q^2}{abz}\right) \\ \frac{1}{a}\,{}_2\phi_1\left(\begin{matrix}a,a\\aq/b\end{matrix}\middle| q;\frac{q}{abz}\right) & \frac{1}{b}\,{}_2\phi_1\left(\begin{matrix}b,b\\bq/a\end{matrix}\middle| q;\frac{q}{abz}\right)\end{pmatrix}.$$

The connection matrix

For the calculation of the connection matrix when $c = q$, we extract an analytic continuation formula essentially from Watson's formula (6.15) by taking the limit $c \to q$. The procedure is completely analogous to that of finding the solutions to the modified system (6.7) for the $c = q$ case. In the $c \neq q$ case, we had $P(c) = X_{J_\infty}^{-1} F_\infty^{-1} F_0 X_{J_0(c)}$, where here $J_0(c)$ is a diagonal matrix. Similarly, in the $c = q$ case we expect that $\tilde{P} = P(c = q) = X_{J_\infty}^{-1} F_\infty^{-1} \tilde{F}_0 X_{J_0(q)}$, where now $J_0(q) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ as above. But we can write this as

$$\begin{aligned} \tilde{P} &= X_{J_\infty}^{-1} F_\infty^{-1} \tilde{F}_0 X_{J_0(q)} \\ &= X_{J_\infty}^{-1} F_\infty^{-1} \lim_{c \to q} \left(F_0 S_0^{-1} U_0 \right) X_{J_0(q)} \\ &= X_{J_\infty}^{-1} \lim_{c \to q} \left(F_\infty^{-1} F_0 S_0^{-1} U_0 \right) X_{J_0(q)}. \end{aligned} \tag{6.18}$$

Thus if the limit L of the term in the brackets exists, we have computed the connection matrix for the logarithmic case. Explicitly, we check

$$\begin{aligned} L &= \lim_{c \to q} \left[\begin{pmatrix} \frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)} e_a(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)} e_{aq/c}(z)^{-1} \\ \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)} e_b(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)} e_{bq/c}(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & q/c \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right] \\ &= \lim_{c \to q} \left[\frac{1}{c-q} \begin{pmatrix} \frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)} e_a(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)} e_{aq/c}(z)^{-1} \\ \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)} e_b(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)} e_{bq/c}(z)^{-1} \end{pmatrix} \begin{pmatrix} c-q & c \\ 0 & -c \end{pmatrix} \right] \\ &= \begin{pmatrix} \frac{\Gamma_q(\beta-\alpha)}{\Gamma_q(\beta,1-\alpha)} e_a(z)^{-1} & q \frac{\mathrm{d}}{\mathrm{d}c}\Big|_{c=q} \left[\frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)} e_a(z)^{-1} - \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)} e_{aq/c}(z)^{-1} \right] \\ \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha,1-\beta)} e_b(z)^{-1} & q \frac{\mathrm{d}}{\mathrm{d}c}\Big|_{c=q} \left[\frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)} e_b(z)^{-1} - \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)} e_{bq/c}(z)^{-1} \right]. \end{pmatrix} \end{aligned}$$

We use the easily verifiable property

$$\frac{\mathrm{d}}{\mathrm{d}c}[e_{f(c)}(z)^{-1}] = -\frac{f'(c)}{f(c)} \ell_q(f(c)z)\, e_{f(c)}(z)^{-1}$$

which implies that $\frac{\mathrm{d}}{\mathrm{d}c}\big|_{c=q} e_{aq/c}(z)^{-1} = \frac{1}{q} \ell_q(az)\, e_a(z)^{-1}$, then also $\frac{\mathrm{d}}{\mathrm{d}c} = \frac{1}{\log qc} \frac{\mathrm{d}}{\mathrm{d}\gamma}$ which yields (recall $\Gamma_q(1) \equiv 1$)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\gamma}\Big|_{\gamma=1} \frac{\Gamma_q(\gamma)}{\Gamma_q(\gamma-\alpha)} &= \frac{\Psi_q(1) - \Psi_q(1-\alpha)}{\Gamma_q(1-\alpha)} \\ \frac{\mathrm{d}}{\mathrm{d}\gamma}\Big|_{\gamma=1} \frac{\Gamma_q(2-\gamma)}{\Gamma_q(\beta-\gamma+1)} &= \frac{-\Psi_q(1) + \Psi_q(\beta)}{\Gamma_q(\beta)}, \end{aligned}$$

where

$$\Psi_q(z) \coloneqq \frac{1}{\Gamma_q(z)} \frac{\mathrm{d}\Gamma_q(z)}{\mathrm{d}z}$$

is the q-digamma function. we obtain after some calculation

$$L = \begin{pmatrix} \frac{\Gamma_q(\beta-\alpha)}{\Gamma_q(\beta,1-\alpha)} e_a(z)^{-1} & (\log q)^{-1} e_a(z)^{-1} \frac{\Gamma_q(\beta-\alpha)}{\Gamma_q(\beta,1-\alpha)} [2\Psi_q(1) - \Psi_q(1-\alpha) - \Psi_q(\beta) + \log q\, \ell_q(az)] \\ \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha,1-\beta)} e_b(z)^{-1} & (\log q)^{-1} e_b(z)^{-1} \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha,1-\beta)} [2\Psi_q(1) - \Psi_q(1-\beta) - \Psi_q(\alpha) + \log q\, \ell_q(bz)] \end{pmatrix}.$$

We can further simplify this by expressing the q-logarithm in terms of q-gamma and q-digamma functions. We have

$$\ell_q(z) = -z \frac{\frac{\mathrm{d}}{\mathrm{d}z} \Theta_q(z)}{\Theta_q(z)} \equiv -z \frac{\frac{\mathrm{d}}{\mathrm{d}x}\big|_{x=z} \Theta_q(x)}{\Theta_q(z)},$$

but picking logarithms $z = q^\zeta, x = q^\chi$ we can easily express (for $|q| < 1$, but the final result is valid also for $|q| > 1$)

$$\Theta_q(z) = (q;q)_\infty^3 \frac{1-q}{\Gamma_q(\zeta, 1-\zeta)},$$

and thus

$$\ell_q(z) = -\frac{1}{\log q} \frac{1}{\Theta_q(z)} \frac{\mathrm{d}}{\mathrm{d}\chi}\Big|_{\chi=\zeta} \frac{1}{\Gamma_q(\chi, 1-\chi)} = \frac{1}{\log q} [\Psi_q(\zeta) - \Psi_q(1-\zeta)].$$

We thus find that the connection matrix for the logarithmic case $c = q$ is given by the beautiful formula

$$\begin{aligned}\tilde{P} = &\begin{pmatrix} e_{a^{-1}}(z)^{-1} & 0 \\ 0 & e_{b^{-1}}(z)^{-1} \end{pmatrix} \\ &\times \begin{pmatrix} \frac{\Gamma_q(\beta-\alpha)}{\Gamma_q(\beta,1-\alpha)} e_a(z)^{-1} & (\log q)^{-1} e_a(z)^{-1} \frac{\Gamma_q(\beta-\alpha)}{\Gamma_q(\beta,1-\alpha)} \Upsilon_q(\zeta;\alpha,\beta) \\ \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha,1-\beta)} e_b(z)^{-1} & (\log q)^{-1} e_b(z)^{-1} \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha,1-\beta)} \Upsilon_q(\zeta;\beta,\alpha) \end{pmatrix} \\ &\times \begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1 \end{pmatrix},\end{aligned}$$

where we have defined for brevity

$$\Upsilon_q(\zeta;\alpha,\beta) \coloneqq 2\Psi_q(1) - \Psi_q(1-\alpha) - \Psi_q(\beta) + \Psi_q(\zeta+\alpha) - \Psi_q(1-\zeta-\alpha).$$

The *twisted* connection matrix is again obtained by replacing the q-characters by the "classical" exponent functions, i.e.

$$\begin{aligned}\check{P} = &\begin{pmatrix} z^{\alpha} & 0 \\ 0 & z^{\beta} \end{pmatrix} \\ &\times \begin{pmatrix} \frac{\Gamma_q(\beta-\alpha)}{\Gamma_q(\beta,1-\alpha)} e_a(z)^{-1} & (\log q)^{-1} e_a(z)^{-1} \frac{\Gamma_q(\beta-\alpha)}{\Gamma_q(\beta,1-\alpha)} \Upsilon_q(\zeta;\alpha,\beta) \\ \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha,1-\beta)} e_b(z)^{-1} & (\log q)^{-1} e_b(z)^{-1} \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha,1-\beta)} \Upsilon_q(\zeta;\beta,\alpha) \end{pmatrix} \\ &\times \begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Note that our choice of q-characters and q-logarithms, whether it is $e_a(z)$, z^{α} or any other elliptic multiple of these, spuriously enters into the entries of the connection matrix. The importance of choices of characters and logarithms will be discussed in the end of our work.

6.2.4 Solutions around the "third" singular point – a discussion

We saw in the beginning of this section that the basic hypergeometric q-difference equation

$$\left[\sigma_q^2 - \frac{(a+b)z - (1+c/q)}{abz - c/q}\sigma_q + \frac{z-1}{abz-c/q}\right]\phi(z) = 0 \tag{6.12}$$

has solutions around 0 given by

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| q; z\right), \quad e_{q/c}(z)\ {}_2\phi_1\left(\begin{matrix} aq/c, bq, c \\ q^2/c \end{matrix} \middle| q; z\right)$$

as well as solutions around infinity given by

$$e_a(z)^{-1}\ {}_2\phi_1\left(\begin{matrix} a, aq/c \\ aq/b \end{matrix} \middle| q; \frac{cq}{abz}\right), \quad e_b(z)^{-1}\ {}_2\phi_1\left(\begin{matrix} b, bq/c \\ bq/a \end{matrix} \middle| q; \frac{cq}{abz}\right).$$

The equation can be rewritten in terms of the q-derivative operator $\mathcal{D}_q$ as

$$\left[z(c-abz)\mathcal{D}_q^2 + \left(\frac{1-c}{1-q} + \frac{(1-a)(1-b)-(1-abq)}{1-q}z\right)\mathcal{D}_q - \frac{(1-a)(1-b)}{1-q}\right] f(z) = 0,$$

which clearly reduces to the hypergeometric differential equation (5.1) when $q \to 1$ and $q^{\alpha} = a$ etc. The solutions to the q-difference equation in turn, also converge to [HSS16] the corresponding solutions of the differential equation, i.e.

$$\begin{aligned}{}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| q; z\right) &\to {}_2F_1(\alpha,\beta;\gamma;z) \\ e_{q/c}(z)\ {}_2\phi_1\left(\begin{matrix} aq/c, bq, c \\ q^2/c \end{matrix} \middle| q; z\right) &\to z^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z) \\ e_a(z)^{-1}\ {}_2\phi_1\left(\begin{matrix} a, aq/c \\ aq/b \end{matrix} \middle| q; \frac{cq}{abz}\right) &\to z^{-\alpha}{}_2F_1(\alpha,\alpha-\gamma+1;\alpha-\beta+1,z^{-1}) \\ e_b(z)^{-1}\ {}_2\phi_1\left(\begin{matrix} b, bq/c \\ bq/a \end{matrix} \middle| q; \frac{cq}{abz}\right) &\to z^{-\beta}{}_2F_1(\beta,\beta-\gamma+1;\beta-\alpha+1;z^{-1}).\end{aligned}$$

What about the third type of solutions of the classical hypergeometric differential equation? The aim of this subsection is to discuss the existence (or not) of q-analogues of the classical solutions around the point $z = 1$ (for generic parameters). The classical solutions for the equation (5.1) are:

$${}_2F_1(\alpha, \beta, \alpha+\beta+1-\gamma; 1-z) \quad \text{and } (1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\beta, \gamma-\alpha, \gamma-\alpha-\beta+1; 1-z).$$

This means we have to find solutions of (6.12) whose $q \to 1$ limit reduces to the above classical solutions. We will not present a thorough discussion of the subject, instead present some (somewhat independent) remarks on the topic, and the uninterested reader may skip them.

1. **There is no *unique* "third" singular point.** The third singular point is not simply $z = 1$. The form of the basic hypergeometric equation seems to suggest that a singularity appears at $z = \frac{c}{abq}$ instead of $z = 1$, which is also evident by the coefficient matrix A in (6.13). This is in agreement with the $q \to 1$ limit: recall that $a = q^\alpha, b = q^\beta$ and q^γ, thus $\frac{abq}{c}$ does degenerate into 1 in case the parameters α, β, γ we choose are real. So is $z = \frac{c}{abq}$ the "third" singular point? The answer seems to be yes; however, the problem looks to be slightly more complicated. This leads to the second remark.

2. **The transformation $z \mapsto 1-z$ is "ill-behaved".** In the classical case (differential equations) when a regular singular point z_0 appears in the equation, the Frobenius method dictates that (i) we look for solutions of the form $F = \sum_n c_n(\lambda)(z-z_0)^{\lambda+n}$ or (ii) in case the solutions around $z = 0$ are known, that we transform $z \mapsto z - z_0$ ($z \mapsto 1/z$ for the point ∞) and approaches (i) and (ii) are in fact equivalent. In the case of the hypergeometric differential equation, the transformations $z \mapsto z - z_0$ for $z_0 = 0, 1, \infty$ transform the initial equation into an equation that is again of hypergeometric type.

   However, the same will *not* hold for difference equations and certainly not for the basic hypergeometric equation. The reason is that the difference operator $\sigma_q = q^{z\frac{\mathrm{d}}{\mathrm{d}z}}$ does not "behave well" under translations $z \mapsto z - z_0$, unlike the differential operator $\frac{\mathrm{d}}{\mathrm{d}z} \mapsto \frac{\mathrm{d}}{\mathrm{d}(z-z_0)} = \frac{\mathrm{d}}{\mathrm{d}z}$. In fact, translation operators appear: $\sigma_{q,z} = q^{z\frac{\mathrm{d}}{\mathrm{d}z}} \mapsto q^{(z-z_0)\frac{\mathrm{d}}{\mathrm{d}(z-z_0)}} = T_{z_0(q-1)} \circ \sigma_{q,z}$, making the new, (now functional) equation more difficult to solve. What about looking for power series solutions around $z = z_0$ as in (i)? In difference equations, it is *unwise* to look directly for such solutions, even though a solution might admit such an expansion. The reason is tied to the previous point about translations: the (translated) power series Ansatz $F = \sum_n c_n(\lambda)(z-z_0)^{\lambda+n}$ is *not* a smart Ansatz for difference equations.

3. **A different Ansatz – q-shifted powers** One *can* however, use any of the q-generalizations of translated powers: one can replace $(z-z_0)^\alpha$ by any of the "q-shifted powers" $(z;q)_\alpha$, $(z;q)^{-1}_{-\alpha}$ or related objects with limits $(1-z)^\alpha$. These functions are "better-behaved" compared to the simple translated powers. This leads to Ansätze of the form

$$(z;q)_\lambda \sum_{n=0}^{\infty} c_n(\lambda)\,(z;q)_n\,,$$

   or similar ones. This idea has been introduced and investigated by Ryde [Ryd21], and subsequently by Hahn [Hah49; Hah52] and Exton [Ext83].

4. **Identities with q-analogues hinting that "third" solutions exist.** There are a number of identities of q-hypergeometric functions that seem yield evidence *for* the existence of solutions that are q-expansions around 1. To name two, first consider the Gauss-Kummer identity from theorem 5.2.5:

$${}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma, \gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha, \gamma-\beta)}.$$

   This is precisely the *finite* value of the (holomorphic) solution around $z_0 = 1$ for $z = 0$. This identity *does* have a q-analogue, the q-Gauss sum [GR04]

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle|\, q; \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty} = \frac{\Gamma_q(\gamma, \gamma-\alpha-\beta)}{\Gamma_q(\gamma-\alpha, \gamma-\beta)}.$$

   This yields evidence that there should be a solution-expansion around $z_0 = 1$ (or more generally, $z_0 = f(q)$ with $f(1) = 1$) with the above value for $z = 0$. Secondly, there is one of the so-called Kummer's formulas

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha}{}_2F_1\left(\alpha, \gamma-\beta; \gamma; \frac{z}{z-1}\right).$$

This identity also admits a q-generalization, proved by Heine [Hei47]

$$ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\;\middle|\;q;z\right)=\frac{(az;q)_\infty}{(z;q)_\infty}\frac{(b;q)_\infty}{(c;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\frac{c}{b},z\\az\end{matrix}\;\middle|\;q;b\right). $$

The last line *resembles* the Ansätze by Ryde, which again gives some evidence for the existence of such solutions.

5. **A claim by Hahn.** Both Exton [Ext83] and Hahn in fact wrote down a solution of the above form [Hah49]: Hahn proposes that

$$ \varphi(z)=:\;{}_3\phi_2\left(\begin{matrix}a,b,z\\\frac{abq}{c},0\end{matrix}\;\middle|\;q;q\right)=\sum_{n=0}^{\infty}c_n\frac{q^n}{(q;q)_n}(z;q)_n $$

where $c_n=\frac{(a,b;q)_n}{\left(\frac{abq}{c};q\right)_n}$, is a solution to the basic hypergeometric equation "around $z=1$". However, we argue that this is *not* a solution. We set $y=1-z$ and obtain

$$ \begin{aligned}\varphi(z)&=\sum_{n=0}^{\infty}\frac{q^nc_n}{(q;q)_n}(z;q)_n=1+y\sum_{n=1}^{\infty}\frac{q^nc_n}{(q;q)_n}\prod_{k=1}^{n-1}(1-q^k+q^ky)\\&=1+y\sum_{n=1}^{\infty}\frac{q^n(q;q)_{n-1}c_n}{(q;q)_n}\prod_{k=1}^{n-1}(1+\frac{q^k}{q^k-1}y)\end{aligned} $$

We note that for $n\geq 2$ and $a_k=:\frac{q^k}{q^k-1}$ we have that

$$ \prod_{k=1}^{n-1}(1+a_ky)=\sum_{k=0}^{n-1}\sigma_k(a_{\{n-1\}})y^k\equiv 1+(a_1+\dots+a_{n-1})y+\ldots+a_1\cdots a_{n-1}y^{n-1}, $$

where $\sigma_k(a_{\{n\}})$ denotes the k-th elementary symmetric polynomial in $\{a_1,\ldots,a_n\}$. We obtain

$$ \begin{aligned}\varphi(z)&=1+\frac{qc_1}{1-q}y+\sum_{n=2}^{\infty}\frac{q^nc_n}{1-q^n}\sum_{k=0}^{n-1}\sigma_k(a_{\{n-1\}})y^{k+1}\\&=1+\left(\sum_{n=1}^{\infty}\frac{q^nc_n}{1-q^n}\right)y+\sum_{n=2}^{\infty}\frac{q^nc_n}{1-q^n}\sum_{k=1}^{n-1}\sigma_k(a_{\{n-1\}})y^{k+1}\end{aligned} $$

Now using elementary manipulations it is easy to check that for $y=1-kz$ with $k\in\mathbb{C}^*$ we have that $\sigma_{q,z}\equiv\sigma_{q,y}\circ T_{1-q}\equiv T_{q^{-1}-1}\circ\sigma_{q,y}$, where T_a denotes the translation operator (in y). We can then transform the basic hypergeometric difference equation (6.12) to a (slightly more complicated) functional difference equation, using that $\sigma_q^2\equiv\sigma_{q^2}$:

$$ \left[(kaby+c/q-kab)\sigma_{q^2,y}\circ T_{1-q^2}+\big(-k(a+b)y+(a+b)k-c/q-1\big)\sigma_{q,y}\circ T_{1-q}+ky+1-k\right]f(z)=0. $$

Choosing $k=1$ as Hahn suggests in his claim in page 360 of [Hah49] (but not according to (5.6) in the same paper; (5.6) suggests to take $k=ab/c$ but we disregard this for now) we obtain

$$ \left[(aby+c/q-ab)\sigma_{q,z}q^2y\circ T_{1-q^2}+\big(-(a+b)y+(a+b)-c/q-1\big)\sigma_{q,y}\circ T_{1-q}+y\right]f(z)=0 $$

We can substitute $\varphi(z)$ in this expression; if it is a solution, it should vanish at all orders of y, as well as all orders of q. Plugging in the Ansatz we obtain

$$ \begin{aligned}0\stackrel{!}{=}&(aby+c/q-ab)\Bigg[1+\left(\sum_{n=1}^{\infty}\frac{q^nc_n}{1-q^n}\right)(q^2y+1-q^2)\\&\qquad+\sum_{n=2}^{\infty}\frac{q^nc_n}{1-q^n}\sum_{k=1}^{n-1}\sigma_k(a_{\{n-1\}})(q^2y+1-q^2)^{k+1}\Bigg]\\&+\big(-(a+b)y+(a+b)-c/q-1\big)\Bigg[1+\left(\sum_{n=1}^{\infty}\frac{q^nc_n}{1-q^n}\right)(qy+1-q)\\&\qquad+\sum_{n=2}^{\infty}\frac{q^nc_n}{1-q^n}\sum_{k=1}^{n-1}\sigma_k(a_{\{n-1\}})(qy+1-q)^{k+1}\Bigg]\\&+y\Bigg[1+\left(\sum_{n=1}^{\infty}\frac{q^nc_n}{1-q^n}\right)y+\sum_{n=2}^{\infty}\frac{q^nc_n}{1-q^n}\sum_{k=1}^{n-1}\sigma_k(a_{\{n-1\}})y^{k+1}\Bigg].\end{aligned} $$

Now, collect the 0-th order terms of the above 'plugged-in' expression in y. If $\varphi(z)$ is a solution, they should vanish identically, for generic values of the remaining parameters. The shifted powers $(ay+b)^k$ only contribute b^k in the 0-th order, and we have that the following expression needs to vanish identically

$$0 \stackrel{!}{=}(c/q-ab)\left[1+\left(\sum_{n=1}^{\infty}\frac{q^n c_n}{1-q^n}\right)(1-q^2)+\sum_{n=2}^{\infty}\frac{q^n c_n}{1-q^n}\sum_{k=1}^{n-1}\sigma_k(a_{\{n-1\}})(1-q^2)^{k+1}\right]$$
$$+(a+b-c/q-1)\left[1+\left(\sum_{n=1}^{\infty}\frac{q^n c_n}{1-q^n}\right)(1-q)+\sum_{n=2}^{\infty}\frac{q^n c_n}{1-q^n}\sum_{k=1}^{n-1}\sigma_k(a_{\{n-1\}})(1-q)^{k+1}\right]$$

It is clear that when $|q|<1$ we can expand $a_k=\frac{q^k}{q^k-1}$ in positive powers of q. When $|q|>1$ one has to expand $a_k=\frac{q^k}{q^k-1}$ in terms of *negative* powers of q, and an obvious obstruction to cancellation is harder to see. We expand the terms in the brackets up to order q^2:

$$0 \stackrel{!}{=} (c/q-ab)\left[1+qc_1+(c_1+c_2)q^2+\mathcal{O}(q^3)\right]+(a+b-c/q-1)\left[1+qc_1+(2c_1+c_2)q^2+\mathcal{O}(q^3)\right],$$

which yields order by order

$$0 \stackrel{!}{=} 0\cdot q^{-1}+(a+b-ab-1)q^0-c_1cq^1+\mathcal{O}(q^2),$$

which cannot vanish for generic a,b,c. Hence the proposed solution, cannot solve the basic hypergeometric q-difference equation.

6.3 The case of $\mathbb{P}^1$: a q-Bessel equation

In this section we move on to the next problem: studying the difference equations from [BDP14] in the zero mass limit, corresponding to $x\to 1$ in (3.17a). The difference equation in question after changing to $z=y^{-1}$ becomes, in our notation, $[\sigma_q+(z-2)+\sigma_q^{-1}]f(z)=0$ which is equivalent to

$$[\sigma_q^2-(2-qz)\sigma_q+1]f(z)=0. \tag{6.19}$$

One can argue that solving this function is *physically* equivalent to solving the modified equation

$$[\sigma_q^2-2\sigma_q+(1-z)]f(z)=0, \tag{6.20}$$

up to a choice of Chern-Simons terms. We make a small digression to show this equivalence later. We would like to study the global properties of this equation, in particular find its local solutions and connection matrix if possible, as in the previous section. A careful reader will notice that (6.20) is of the same form as the basic hypergeometric equation (6.12) of the previous section for the parameters $(a,b,c)=(0,0,q)$. However, we cannot carry over the discussion to this case: the Mellin-Barnes-Watson formula (and therefore the whole derivation of the connection matrix) does not apply at $a=b=0$. This failure boils down to the fact that for $a=b=0$ the basic hypergeometric equation has an irregular singularity at infinity, as we will see. Is this enough to give up? No: the literature comes to our rescue.

The above equation is the limit one of *three* different q-generalizations of the Bessel equation which are studied in [Hah49; Ext78; Zha03; Mor11] among others. We will explore the first two generalizations and their relation, and compute the connection matrix using the results of these papers as well as the methods used in the previous section.

6.3.1 The classical Bessel equation and its q-generalizations

One of the most important classical ordinary differential equations of second order is the Bessel equation, defined as

$$(\vartheta_z-\nu)(\vartheta_z+\nu)u+z^2u=0 \Leftrightarrow \frac{\mathrm{d}^2u}{\mathrm{d}z^2}+\frac{1}{z}\frac{\mathrm{d}u}{\mathrm{d}z}+(1-\nu^2z^{-2})u=0$$

For generic values of ν ($\nu \notin \mathbb{Z}$) there are two linearly independent solutions $J_\nu(z), J_{-\nu}(z)$ around $z = 0$ given by the Bessel functions

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(-;\nu+1;-z^2/4\right),$$

where ${}_0F_1\left(-;a;z\right) = \sum_{n=0}^\infty \frac{z^n}{(a)_n n!}$. As in the case of the hypergeometric equation, if $\nu \in \mathbb{Z}\setminus\{0\}$ the solutions involve logarithms and require going through the Frobenius method machinery.

One can now ask 'what are the q-generalizations of this equation and their corresponding solutions?' Reverend Jackson already introduced *two* generalizations of these functions [GR04]

$$\begin{aligned} J_\nu^{(1)}(x;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu\ {}_2\phi_1\left(\begin{matrix}0,0\\ q^{\nu+1}\end{matrix}\;\middle|\; q;-x^2/4\right), \quad |x|<2 \\ J_\nu^{(2)}(x;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu\ {}_0\phi_1\left(\begin{matrix}—\\ q^{\nu+1}\end{matrix}\;\middle|\; q;-\frac{x^2q^{\nu+1}}{4}\right), \quad x\in\mathbb{C}, \end{aligned}$$

with limits [Mor11]

$$\lim_{q\to1^-} J_\nu^{(1)}\big((1-q)x;q\big) = J_\nu(x) = \lim_{q\to1^-} J_\nu^{(2)}\big((1-q)x;q\big).$$

Hahn [Hah49] showed that they are in fact related by[¶]

$$J_\nu^{(2)}(x;q) = \left(-x^2/4;q\right)_\infty J_\nu^{(1)}(x;q).$$

Harald Exton[‖] [Ext78] along with Hahn independently introduced a third q-generalization of the Bessel function

$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}x^\nu\ {}_1\phi_1\left(\begin{matrix}0\\ q^{\nu+1}\end{matrix}\;\middle|\; q;qx^2\right), \quad x\in\mathbb{C},$$

with limit

$$\lim_{q\to1^-} J_\nu^{(3)}\big((1-q)x;q\big) = J_\nu(2x).$$

These three generalizations satisfy three different q-difference equations [Mor11]

$$J_\nu^{(1)}(x;q): \quad \left[\sigma_q - \left(q^{\nu/2}+q^{-\nu/2}\right)\sigma_{q^{1/2}} + \left(1+\frac{x^2}{4}\right)\right]f(x) = 0 \tag{6.22a}$$

$$J_\nu^{(2)}(x;q): \quad \left[\left(1+\frac{x^2}{4}\right)\sigma_q - \left(q^{\nu/2}+q^{-\nu/2}\right)\sigma_{q^{1/2}} + 1\right]f(x) = 0 \tag{6.22b}$$

$$J_\nu^{(3)}(x;q): \quad \left[\sigma_q - \left[\left(q^{\nu/2}+q^{-\nu/2}\right) - q^{-\nu/2+1}x^2\right]\sigma_{q^{1/2}} + 1\right]f(x) = 0, \tag{6.22c}$$

all of which are q-generalizations of the Bessel equation. When $\nu\notin\mathbb{Z}$, the $\{J_\nu^{(k)}, J_{-\nu}^{(k)}\}$ are linearly independent solutions to the above equations for $k=1,2,3$. When $\nu\in\mathbb{Z}$ one has to go again through the machinery of Sauloy [Sau02], the analogue of the Frobenius method.

6.3.2 Relation between the deformations and the line operator identities

Note that the third deformation $J_\nu^{(3)}(x;q)$ fully describes the q-difference equation (3.17a): Setting $q^{-\nu/2}x^2 =: y^{-1}$ and also $f_\nu(x) = f_\nu\left(\sqrt{q^{\nu/2}y^{-1}}\right) =: g_\nu(y)$ in (6.22c) we have that $\sigma_{q^{1/2},x} = \sigma_{q,y}^{-1}$ and that the x from (3.17a) is related to (6.22c) by $x = q^{\nu/2}$[a]. We obtain from (6.22c)

$$\left[\sigma_{q,y}^{-1} - \left[(x+x^{-1}) - y^{-1}\right] + \sigma_{q,y}\right]g_\nu(y) = 0,$$

[¶] This is a special case of the more general relation [GR04, page 241]

$$\frac{(z;q)_\infty}{(az;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}a,0\\ c\end{matrix}\;\middle|\; q;z\right) = {}_1\phi_2\left(\begin{matrix}a\\ c,az\end{matrix}\;\middle|\; q;cz\right) = \frac{1}{(c;q)_\infty}\ {}_1\phi_1\left(\begin{matrix}z\\ az\end{matrix}\;\middle|\; q;c\right). \tag{6.21}$$

[‖] Interesting note: Exton was a musician by profession, practicing mathematics on the side. F.H. Jackson was himself a reverend and a military man, also practicing mathematics on the side. (see [Ext83], and the note by Chaundy doi:10.1112/jlms/s1-37.1.126).

which is exactly (3.17a), with $x = q^{\nu/2}$. Thus, the solutions $J_\nu^{(3)}(z;q)$ of (6.22c) are related to the solutions (blocks) $B(x,y;q)$ of (3.17a) and (3.17b) by

$$B(x,y;q) = F(x)E(x,y;q)J_\nu^{(3)}\left(\sqrt{-x/y};q\right), \quad \text{with } x = q^{\nu/2},$$

where E is a possible elliptic factor, and $F(x)$ is there to "fix" the second equation (3.17b). In particular, we can rewrite the relation (3.21)

$$\mathcal{J}(-a^2,b;q) = (q)_\infty e_b\,(a)^{-1}\,J_\beta^{(3)}(a;q),$$

where we have replaced the factor $a^{-\beta}$ with a corresponding q-character (recall $b = q^\beta$). Replacing the q-characters by any function satisfying the same q-difference equation allows us to match the proposed solutions (3.18) and (3.19).

$$B_I^1(x,y;q) = E(x,y;q)\frac{1}{\Theta_q\left(-q^{1/2}x\right)}J_\nu^{(3)}\left(\sqrt{-x/y};q\right),$$
$$B_I^2(x,y;q) = E(x^{-1},y;q)\frac{1}{\Theta_q\left(-q^{1/2}x\right)}J_{-\nu}^{(3)}\left(\sqrt{-(xy)^{-1}};q\right),$$

where $x = q^{\nu/2}$ and the factors E are elliptic factors that contain (at least) the products

$$E(x,y;q) \sim e_{x^{-1}}\,(y)\,e_{x^2}\left(\sqrt{-\frac{x}{y}}\right)^{-1}.$$

The first factor comes from the formulas (3.18) and (3.19) from [BDP14], while the second is the modified q-character $e_b\,(a)^{-1}$ we *chose* above.

The monodromy matrix described in [BDP14] (cf. subsection 3.3.3) is defined about a loop in the $\operatorname{Re}X, \operatorname{Re}Y$ plane. We now want to compute the monodromy in the massless ($x \to 1$) case, in the variable $z = y^{-1}$.

[a] The notation is degenerate: we replaced the x in (6.22c) with $\sim y^{-1}$, which is the now the y in (3.17a).

6.3.3 Physical equivalence of $J_\nu^{(1)}$ and $J_\nu^{(3)}$ in the massless limit

As we saw above, the massless limit corresponds to $x \to 1$ (equivalently, $\nu \to 0$). In that limit the two solutions from [BDP14] collapse to a single solution, hinting at the existence of logarithmic solutions. The solution they collapse to is of form

$$B_I^1(y;q) \sim \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q;q)_n^2}y^{-n}.$$

Now recall our Ansatz (6.2) for solving the equations of the form $[(1-\sigma_q)^n - z]f(z) = 0$ and let us modify it by inserting a $q^{P(n)}$ factor as follows

$$f_n(z,\epsilon;q) = \sum_{k=0}^{\infty}\frac{q^{P(n)}x^{k-\epsilon}}{(q^{1-\epsilon};q)_k^n},$$

where $P(n)$ is a polynomial in n. The solution to the modified q-Bessel equation (6.20) corresponding to $J_\nu^{(1)}$ which we want to study corresponds to $P(n) = \text{const.}$ (or rather the ϵ-expansion of the Ansatz). The equations and solutions (6.19) corresponds to the choice $P(n) = \frac{n(n-1)}{2}$. These differing contributions can be tracked down to dynamical Chern-Simons terms $\sim \int AdA$ in the block integrals 3.2.2, which contribute such factors due to their poles. It would be an interesting extension to see if different choices of Chern-Simons terms would give other meaningful q-deformations of the Bessel equation.

Not to our surprise, the physically interesting case lies in the non-generic value $\nu = 0$: the equation for $J_\nu^{(3)}$ after setting $z := q^{-\nu/2}Nox^2$ reduces to (6.19) when $\nu \to 0$, while the equation for $J_\nu^{(1)}$ reduces to the modified but physically equivalent equation (6.20) when we set $z := -\frac{x^2}{4}$. This is quite similar the three-dimensional lift of the previous section: the basic hypergeometric equation with parameters $(a, b, c) = \left(q^{\frac{1}{3}}, q^{\frac{2}{3}}, q\right)$ is logarithmic.

Much like in the previous section, we will lay out the derivation of solutions and connection matrices for these equations in the generic case ($\nu \notin \mathbb{Z}$) and then compute solutions and connection matrices of the $\nu = 0$ case following the techniques of Roques [Roq08].

6.3.4 The two deformations $J_\nu^{(1)}$, $J_\nu^{(2)}$ (Case $\nu \notin \mathbb{Z}$)

Note, throughout this subsection we will assume that $|q| < 1$ as is done in [Zha03]. We will remark on the $|q| > 1$ chamber later. Furthermore, we will not discuss $J_\nu^{(2)}$ as all the results for $J_\nu^{(1)}$ carry over to $J_\nu^{(2)}$ by Hahn's formula (6.21).

Solutions around $x = 0$ We first focus on the deformations $J_\nu^{(1)}$, $J_\nu^{(2)}$ of the Bessel function, and in particular on $J_\nu^{(1)}$ since $J_\nu^{(2)}$ is easily related to it. As we already stated when ν is not an integer we have two solutions $\{J_\nu^{(1)}, J_{-\nu}^{(1)}\}$ around $x = 0$ given by

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \, {}_2\phi_1\left(\begin{matrix}0,0\\ q^{\nu+1}\end{matrix}\,\middle|\, q; -x^2/4\right), \quad |x| < 2,$$

where we can clearly recognize the holomorphic part and the character $\left(\frac{x}{2}\right)^\nu$. As in the previous section, it is recommended we replace the monomial by a q-character (see A.2.2) of the form $e_{q^\nu}(x/\kappa) = \frac{\Theta_q(x/\kappa)}{\Theta_q(q^\nu x/\kappa)}$ with $\kappa \in \mathbb{C}^*$. As we have discussed (cf. discussion in 6.1.4), this choice is not canonical; in fact in the literature [Zha03] it turns out to be more convenient to choose**

$$J_{\nu,\lambda}^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\frac{\Theta_{q^{1/2}}\left(\lambda q^{\nu/2}/x\right)}{\Theta_{q^{1/2}}\left(\lambda/x\right)} \, {}_2\phi_1\left(\begin{matrix}0,0\\ q^{\nu+1}\end{matrix}\,\middle|\, q; -x^2/4\right), \quad |x| < 2, \tag{6.23}$$

where $\lambda \in \mathbb{C}^*$ is arbitrary (for now) and similarly for $J_{-\nu,\lambda}^{(1)}(x;q)$. Note that the q-difference properties of $J_{\nu,\lambda}^{(1)}(x;q)$ do not depend on λ. However, λ is essentially a *choice* of exponent up to elliptic factors. In our notation, the character we have chosen is

$$\frac{\Theta_{q^{1/2}}\left(\lambda q^{\nu/2}/x\right)}{\Theta_{q^{1/2}}\left(\lambda/x;\right)} \equiv e_{q^{\nu/2}}\left((x/\lambda)^{-1};q^{1/2}\right)^{-1} = q^{\nu/2} e_{q^{-\nu/2}}\left(x/\lambda;q^{1/2}\right)^{-1} = q^{\nu/2} e_{q^{\nu/2}}\left(q^{\nu/2}x/\lambda;q^{1/2}\right)$$

where we have used (A.2.13a)-(A.2.13c). So up to scaling (or picking $\lambda = q^{\nu/2}$) this is the 'standard' choice (the only 'legal' choice, according to Sauloy [Sau03] — this is not a mathematical statement!). We might hope that λ will encode physical information. The set $\{J_{\nu,\lambda}, J_{-\nu,\lambda}\}$ forms our basis of solutions at $x = 0$ for $\nu \notin \mathbb{Z}$.

Solutions around $x = \infty$ For the solutions around infinity, the situation is slightly more complicated: much like the classical Bessel function, the q-generalizations (6.22a) to (6.22c) have an irregular singularity at infinity. Irregular q-difference equations (with analytic coefficients) are of course harder to handle: their classification results are more complicated and involve the so-called Newton polygons. We will not present the fully developed theory here (which one can find in [RSZ09; HSS16] among others). Instead we will focus on a case-specific plan for the equations in which we are interested in.

One can still look for formal solutions around irregular singularities (in the form of divergent power series), and then hope to find a way to make them into convergent solutions. This is achieved through the so-called q-Borel-Laplace transformations that are used to handle q-difference equations with irregular singular points. We take a moment to explain these transformations.

**Note that there is a difference in our convention of theta functions with respect to the cited ones. This leads to a redefinition $\lambda \mapsto -\lambda$ compared to [Zha03].

Digression: q-Borel and q-Laplace transformations

Given a (formal) power series $f(t) = \sum_{n\geq 0} f_n t^n$ with $f_0 = 1$ we define the *q-Borel transformation of order one* [Zha00][a] by

$$\big(\mathcal{B}_{q;1}\big) f(\tau) \coloneqq \sum_{n\geq 0} f_n q^{-\binom{n}{2}} \tau^n \equiv g(\tau).$$

Note that definitions of this transformation vary across the literature, even amongst the same authors. Also, $g(\tau)$ could be divergent, when $|q| < 1$. The operator $\mathcal{B}_{q;1} \equiv \mathcal{B}_q$ satisfies the following useful operator identity

$$\mathcal{B}_q(t^m \sigma_q^n) = q^{-\binom{m}{2}} \tau^m \sigma_q^{n-m} \mathcal{B}_q, \quad \forall m, n \in \mathbb{N} \tag{6.24}$$

which implies that we can 'lower' the degree of some shift-operators σ_q in a q-difference equation.

One can now justifiably ask whether this operation has an inverse and the answer is yes. Recall that given two power series $\alpha(z) = \sum_{n\geq 0} a_n z^n$ and $\beta(z) = \sum_{n\geq 0} b_n z^n$ we can form their *Hadamard product*

$$\big(\alpha \odot \beta\big)(z) = \sum_{n\geq 0} a_n b_n z^n.$$

It is a small exercise in series manipulation to verify by Cauchy's formula that

$$\big(\alpha \odot \beta\big)(z) \equiv \frac{1}{2\pi i} \oint_{|s|=r} \frac{\mathrm{d}s}{s} \alpha(s)\beta(zs^{-1})$$

Coming back to the q-Borel transformation, we immediately see that $f(t)$ is the Hadamard product of its Borel transformation $(\mathcal{B}_q f)(\tau) = g(\tau)$ with the power series $\theta(\tau) = \sum_{n\geq 0} q^{\binom{n}{2}} \tau^n$. Of course we have not chosen to call this function θ by mistake: the integral expression of the Hadamard product does not "see" the negative-power terms of a bilateral Laurent series $\beta(\tau)$ and thus we may extend $\theta(\tau)$ to our beloved $\Theta_q(\tau)$. We thus have the formal inverse $\mathcal{L}_q$ to $(\mathcal{B}_q)$ given by

$$\big(\mathcal{L}_q g\big)(t) = \frac{1}{2\pi i} \oint_{|\tau|=R} \frac{\mathrm{d}\tau}{\tau} g(\tau) \Theta_q\left(t\tau^{-1}\right).$$

where R is a large enough radius. $\mathcal{L}_q$ satisfies

$$\big(\mathcal{L}_q \circ B_q f\big)(t) = f(t)$$

for convergent power series $f(t)$ and thus it also satisfies the inverted operator equation

$$\mathcal{L}_q(\tau^m \sigma_q^n) = q^{\binom{m}{2}} t^m \sigma_q^{n+m} \mathcal{L}_q.$$

$\mathcal{L}_q$ is called a *q-Laplace transform*. The name is motivated by viewing the theta function as a q-analogue of the exponential function, making the defining expression of $\mathcal{L}_q$ remind us of the (differential) Laplace transform. Note again, that there are different definitions of the q-Laplace transform across the literature.

[a] We will not consider higher orders in this work.

We are now ready to find solutions of (6.22a) at infinty. First we transform (6.22a) via $t = x^{-1}$ (whence $\sigma_{q,x} = \sigma_{q^{-1},t} = \sigma_t^{-1}$) and $f(t^{-1}) = h(t)$ to get

$$\left[\left(1 + \frac{1}{4q^2t^2}\right)\sigma_q - (q^{\nu/2} + q^{-\nu/2})\sigma_{q^{1/2}} + 1\right] h(t) = 0. \tag{6.25}$$

This can be rewritten in matrix form $\sigma_{q^{1/2}}\Phi = A(t)\Phi$ as in the previous section 6.1.3 where

$$A(t) = \begin{pmatrix} 0 & 1 \\ \frac{q^{\nu/2}+q^{-\nu/2}}{1+\frac{1}{4q^2t^2}} & -\frac{1}{1+\frac{1}{4q^2t^2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{4q^2t^2(q^{\nu/2}+q^{-\nu/2})}{4q^2t^2+1} & -\frac{4q^2t^2}{4q^2t^2+1}. \end{pmatrix}$$

Clearly, $A(0)$ is holomorphic around $t = 0$ but it is *not* in $GL_n(\mathbb{C})$, thus the singular point $t = 0$ is irregular. It is therefore recommended[††] [Zha03] to look for solutions of the form $h(t) = E(t)p(t)$, where

[††] This 'recommendation' comes from the study of differential equations with irregular singularities.

$E(t)$ is a q-exponential function and $p(t)$ denotes a holomorphic function. Zhang chooses

$$E(t) = E_\alpha(t) := \frac{1}{\Theta_{q^{1/2}}(\alpha t)}, \quad t \notin \frac{q^{\frac{\mathbb{Z}}{2}}}{\alpha}$$

satisfying $\sigma_{q^{1/2}} E_\alpha(t) = -\alpha t E_\alpha(t)$, $\sigma_q E_\alpha(t) = \alpha^2 q^{1/2} t^2 E_\alpha(t)$ and $\alpha \in \mathbb{C}^*$. With some a posteriori wisdom he also chooses α such that $\alpha^2 = -4q^{3/2}$. The two solutions of this equation will yield the two linearly independent solutions at infinity. We can substitute this Ansatz $h(t) = h_\alpha(t)$ in the equation (6.25), use the difference properties of $E_\alpha(t)$ and α and obtain the following equation for $p(t) = p_\alpha(t)$ which is analytic by assumption

$$\left[-(1 + 4q^2t^2)\sigma_q + \alpha(q^{\nu/2} + q^{-\nu/2})t\sigma_{q^{1/2}} + 1\right]p_\alpha(t) = 0 \tag{6.26}$$

We will *now* use the q-Borel-Laplace transform: Instead of trying to solve the above equation, we can solve the one for $r_\alpha(\tau) := (B_{q^{1/2}} p_\alpha)(\tau)$. We apply $\mathcal{B}_{q^{1/2}}$ on the left of the equation and use (6.24) to obtain

$$\left[\sigma_q - \left(1 + \alpha(q^{\nu/2} + q^{-\nu/2}) - 4q^{3/2}\tau^2\right)\right] r_\alpha(\tau) = 0.$$

We now see the a posteriori wisdom in the choice of α: we can write this equation as

$$\left[\sigma_q - (1 + \alpha t q^{\nu/2})(1 + \alpha q^{-\nu/2}\tau)\right] r_\alpha(\tau) = 0.$$

which we can solve immediately, up to elliptic functions

$$r_\alpha(\tau) = \frac{1}{\left(-\alpha q^{\nu/2}\tau; q\right)_\infty \left(-\alpha q^{-\nu/2}\tau; q\right)_\infty}.$$

We note here that this result is derived in [Zha03] on the premise that $q \in (0,1)$ for convergence. On the one hand his is unnecessary, up to this point: the function on the right-hand side is defined for all q such that $|q| \neq 1$ and satisfies the same q-difference equation regardless of the "chamber" of q. On the other hand, the pole structure of of $r_\alpha(\tau)$ is vastly different when passing from the unit q-circle: the function has simple poles on the set $\left\{-\alpha^{-1}q^{\pm\nu/2+\mathbb{Z}_{\geq 0}}\right\}$ when $|q| < 1$, but *no* poles when $|q| > 1$. This will lead to a big caveat (see later).

We can now compute $p_\alpha(t) = \left(\mathcal{L}_q r_\alpha\right)(t)$ and obtain the full solution to (6.22a) around infinity by $h_\alpha(t) = \frac{p_\alpha(t)}{\Theta_{q^{1/2}}(\alpha t)}$. The calculation is a technical one and it is done in [Zha03] for $|q| < 1$. We will simply quote the results here. We first introduce the notation‡‡

$$j^{(1)}_{\nu,\alpha}(t;q) = \left(q^{1/2}, q^{1/2}; q^{1/2}\right)_\infty h_\alpha(t;q) = \left(q^{1/2}; q^{1/2}\right)_\infty \frac{p_\alpha(t)}{\Theta_{q^{1/2}}(\alpha t)},$$

and note that due to the symmetry of (6.26) we have that

$$j^{(1)}_{\nu,\alpha}(t;q) = j^{(1)}_{-\nu,\alpha}(t;q) = j^{(1)}_{\nu,-\alpha}(-t;q).$$

Thus the solutions around infinity will be spanned by $\{j^{(1)}_{\nu,\alpha}, j^{(1)}_{\nu,-\alpha}(t;q)\}$. Finally we state the result of the contour integral calculation of $p_\alpha(t)$:
Theorem ([Zha03]). *When $|q| < 1$, $\alpha = \pm 2iq^{3/4}$, $xt = 1$, $|x| < 2$ and $\nu \notin \mathbb{Z}$ we have that*

$$\begin{aligned} j^{(1)}_{\nu,\alpha}(t;q) = &\frac{\left(q^{1/2}, q^{1/2}; q\right)_\infty}{\left(q^{1+\nu}, q^{-\nu}; q\right)_\infty} \frac{\Theta_{q^{1/2}}\left(\alpha q^{\nu/2} t\right)}{\Theta_{q^{1/2}}(\alpha t)} \frac{\Theta_{q^{1/2}}(\lambda t)}{\Theta_{q^{1/2}}\left(\lambda q^{\nu/2} t\right)} J^{(1)}_{\nu,\lambda}(x;q) \\ &+ \frac{\left(q^{1/2}, q^{1/2}; q\right)_\infty}{\left(q^{1-\nu}, q^{+\nu}; q\right)_\infty} \frac{\Theta_{q^{1/2}}\left(\alpha q^{-\nu/2} t\right)}{\Theta_{q^{1/2}}(\alpha t)} \frac{\Theta_{q^{1/2}}(\lambda t)}{\Theta_{q^{1/2}}\left(\lambda q^{-\nu/2} t\right)} J^{(1)}_{-\nu,\lambda}(x;q). \end{aligned}$$

We have to state the caveat here: this calculation depends on the "chamber" $|q| \gtrless 1$ that we choose, since the calculation depends on the pole structure of the integrand of $\left(\mathcal{L}_q r_\alpha\right)(t)$. When $|q| < 1$ the poles come from $r_\alpha(\tau)$ and they are all contained in a disc of radius $R < R_0 =:= \max\{q^{\pm\nu/2}/|\alpha|\}$, while for

‡‡ The reason for this rescaling is that $j^{(1)}$ has a 'nice' limit at $q \to 1$, namely

$$\lim_{q \to 1^-} j^{(1)}_{\nu,\alpha}\left(\frac{t}{1-q}; q\right) = ie^{-\nu\pi i/2}\left(\frac{J_{-\nu}(x) - e^{\nu\pi i}J_\nu(x)}{-i\sin\nu\pi}\right)$$

(something similar for $j^{(2)}$, where the term in the bracket is related to the Hankel functions.

$|q| > 1$ the poles come from the theta function $\frac{1}{\Theta_{q^{1/2}}(\frac{\alpha t}{\tau})}$. But the poles of the theta function are of the form $\{tq^{\mathbb{Z}/2}\}$ and hence are not all contained in a disc, and hence the integration depends on the contour. Thus the results cannot be obviously carried over to $|q| > 1$.

To conclude this paragraph, we state two more results from [Zha03] that allow us to express the solutions $\{j^{(1)}_{\nu,\alpha}, j^{(1)}_{\nu,-\alpha}(t;q)\}$ in terms of known functions.

Lemma 6.3.1 ([Zha03]). *We have for $p_{\alpha,\nu}(t)$ as in (6.26) that:*

(a) $p_{\alpha,\nu}(t) = \left(\alpha q^{-1/4}t; q^{1/2}\right)_\infty \; {}_2\phi_1\left(\begin{matrix}(q^{1/2})^{\nu+1/2}, (q^{1/2})^{-\nu+1/2} \\ -q^{1/2}\end{matrix} \,\middle|\, q^{1/2}; \alpha q^{-1/4}t\right),$

as well as the identity

(b) $\left(x; q^{1/2}\right)_\infty \; {}_2\phi_1\left(\begin{matrix}0, 0 \\ q^{\nu+1}\end{matrix} \,\middle|\, q; -x^2\right) = {}_2\phi_1\left(\begin{matrix}(q^{1/2})^{\nu+1/2}, -(q^{1/2})^{\nu+1/2} \\ (q^{1/2})^{2\nu+1}\end{matrix} \,\middle|\, q^{1/2}; -x\right).$

Part b) will be useful for the case $\nu \to 0$.

With (part a) of) this lemma we can express $j^{(1)}_{\nu,\alpha}$ as

$$j^{(1)}_{\nu,\alpha}(t) = \frac{\left(q^{1/2}, q^{1/2}; q\right)_\infty}{\left(q^{3/4}\alpha^{-1}t^{-1}; q^{1/2}\right)_\infty} \frac{\Theta_{q^{1/2}}\left(\alpha q^{-1/4}t\right)}{\Theta_{q^{1/2}}\left(\alpha t\right)} \; {}_2\phi_1\left(\begin{matrix}(q^{1/2})^{\nu+1/2}, (q^{1/2})^{-\nu+1/2} \\ -q^{1/2}\end{matrix} \,\middle|\, q^{1/2}; \alpha q^{-1/4}t\right) \tag{6.27}$$

The connection matrix for $J^{(1)}$ Introducing the notation for the ($q^{1/2}$-)elliptic coefficients

$$C_{\nu,\alpha}(\lambda, t; q) := \frac{\left(q^{1/2}, q^{1/2}; q\right)_\infty}{\left(q^{1+\nu}, q^{-\nu}; q\right)_\infty} \frac{\Theta_{q^{1/2}}\left(\alpha q^{\nu/2}t\right)}{\Theta_{q^{1/2}}\left(\alpha t\right)} \frac{\Theta_{q^{1/2}}\left(\lambda t\right)}{\Theta_{q^{1/2}}\left(\lambda q^{\nu/2}t\right)},$$

we can rewrite the contents of the theorem into the connection matrix

$$\begin{pmatrix} j^{(1)}_{\nu,\alpha}(t;q) \\ j^{(1)}_{\nu,-\alpha}(t;q) \end{pmatrix} = \begin{pmatrix} C_{\nu,\alpha}(\lambda,t;q) & C_{-\nu,\alpha}(\lambda,t;q) \\ C_{\nu,-\alpha}(\lambda,t;q) & C_{-\nu,-\alpha}(\lambda,t;q) \end{pmatrix} \begin{pmatrix} J^{(1)}_{\nu,\lambda}(x;q) \\ J^{(1)}_{-\nu,\lambda}(x;q) \end{pmatrix}, \quad \lambda \in \mathbb{C}^*, xt = 1, |x| < 2. \tag{6.28}$$

One can see that the connection matrix puts a constraint on the choice of λ: one *cannot* choose $\lambda = \alpha = \pm 2iq^{3/4}$ otherwise the connection matrix will not be invertible and the two solutions $j^{(1)}_{\nu,\pm\alpha}(t;q)$ will coincide.

6.3.5 The two deformations $J^{(1)}_\nu$, $J^{(2)}_\nu$ (Case $\nu = 0$)

We now turn to our case of interest: we want to solve the connection problem for the equation (6.20), which is the same equation as (6.22a) in the limit $\nu \to 1$ sunder the transformation $z = -\frac{x^2}{4}$. However, the limit $\nu \to 0$ yields complications as was the case in the previous section, namely the solutions around 0 become logarithmic. For this reason we must rewrite the results from the case $\nu \notin \mathbb{Z}$ in terms of first order difference systems and take the limit appropriately as prescribed by Roques [Roq08]. We recall the outline of the procedure: we write the equation for generic parameters as a first-order system. Using the Jordan(-Chevalley-Dunford) decomposition of the coefficient matrix (which in the generic case is just the diagonal form of the matrix), write down solutions around 0 and infinity . For the non-generic parameters we need to use the full Jordan decomposition into semi-simple and unipotent factors and if we're lucky enough, we can compute the results for the non-generic parameters as a limit of the generic parameters.

After transforming via $z = -\frac{x^2}{4}$ we obtain the equation for generic ν

$$[\sigma_q^2 - (q^{\nu/2} + q^{-\nu/2})\sigma_q + (1 - z)]g(z) = 0 \tag{6.29}$$

which we can rewrite as the system $\sigma_q \Phi = A\Phi$ where the coefficient matrix is

$$A(z;\nu) = \begin{pmatrix} 0 & 1 \\ z - 1 & q^{\nu/2} + q^{-\nu/2} \end{pmatrix}.$$

First off the bat: we do not study *this* coefficient matrix around ∞ as it is clearly not holomorphic around $z^{-1} = 0$. Instead we will later "read-off" the relevant information (solution matrix Φ_∞) from the discussion in the subsection 6.3.4. We thus focus now on the study around $z = 0$: $A(z;\nu)$ is clearly a holomorphic matrix around $z = 0$ and this the system is *Fuchsian* at 0 6.1.3, hence also regular singular. As to resonance, we find its eigenvalues:

$$\sigma(A) = \{\frac{1}{2}(q^{\nu/2} + q^{-\nu/2}) \pm \sqrt{(1-q^\nu)^2 - 4q^\nu z}\} =: \{\lambda_1(z), \lambda_2(z)\}.$$

In particular, $\sigma(A(0;\nu) = \{q^{\nu/2}, q^{-\nu/2}\}$ and thus the system is non-resonant at 0 for all values of $\nu \notin \mathbb{Z}_*$: $\sigma(A(0;\nu)) \cap q^{\mathbb{Z}_*}\sigma(A(0;\nu)) = \varnothing$ when $\nu \notin \mathbb{Z}_*$, even in the degenerate case $\nu = 0$ where $\sigma(A(0;0)) = \{1,1\}$ (recall, degenerate eigenvalues are still Fuchsian). Whether the system is logarithmic or not is determined by the semi-simplicity (equivalent to diagonalizability when working over $\mathbb{C}$) of $A(0;\nu)$. For generic ν (equivalent to $\lambda_1 \neq \lambda_2$), the Jordan decomposition is given by

$$A(z;\nu) = \begin{pmatrix} \frac{\lambda_1(z)}{1-z} & \frac{\lambda_2(z)}{1-z} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1(z) & 0 \\ 0 & \lambda_2(z) \end{pmatrix} \begin{pmatrix} \frac{\lambda_1(z)}{1-z} & \frac{\lambda_2(z)}{1-z} \\ 1 & 1 \end{pmatrix}^{-1} =: M_z(\nu)J_z(\nu)M_z(\nu)^{-1}. \tag{6.30}$$

Note that here the similarity matrix $M_z(\nu)$ is *not* a continuous mapping of ν: for $\nu \neq 0$, $J_0(\nu)$ only has a semi-simple (in fact, diagonal) component (to which the similarity matrix $M_z(\nu) = S_z(\nu)$ brings $A(0;\nu)$, where S stands for semi-simple). In particular, at $z = 0$ and generic ν we have

$$A(0;\nu) = \begin{pmatrix} q^{\nu/2} & q^{-\nu/2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q^{\nu/2} & 0 \\ 0 & q^{-\nu/2} \end{pmatrix} \begin{pmatrix} q^{\nu/2} & q^{-\nu/2} \\ 1 & 1 \end{pmatrix}^{-1} \equiv S_0(\nu)J_0(\nu)S_0(\nu)^{-1}, \quad \nu \neq 0$$

whence the system is non-logarithmic. However, when $\nu = 0$ we have the decomposition

$$A(0;0) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} =: M_0(0)J_0(0)M_0(0)^{-1} \equiv U_0(0)J_0(0)U_0(0)^{-1}.$$

and now $J_0(0)$ is unipotent and hence the system is logarithmic for $\nu = 0$. We can use the results of the previous section on the analytic continuation of the hypergeometric function: For generic ν we solve the auxiliary system

$$\big(\sigma_q F_0\big)(z;\nu)J_0(\nu) = A(z;\nu)F_0(z;\nu) \tag{6.7}$$

which gives the holomorphic contribution and the constant-coefficient system of the character matrix

$$\big(\sigma_q X_{J_0(\nu)}\big)(z) = J_0(\nu)X_{J_0(\nu)}. \tag{6.8}$$

and the fundamental solution matrix is then given by

$$\Phi_0(z;\nu) = F_0(z;\nu)X_{J_0(\nu)}$$

and similarly around infinity (with some complication in the case of the Bessel functions). The connection matrix is then computed

$$P(z;\nu) = \big(\Phi_\infty^{-1}\Phi_0\big)(z;\nu).$$

When $\nu = 0$ we can use the same reasoning as in (6.16): the solution $F_0(z;\nu=0) \equiv \tilde{F}_0(z)$ to (6.7) for $J_0(\nu = 0)$ is given by the limit of

$$\big(\sigma_q F_0 S_0^{-1}U_0\big)(z;\nu)\underbrace{U_0^{-1}A(0;\nu)U_0}_{=J_0(\nu=0)} = A(z;\nu)\big(F_0S_0^{-1}U_0\big)(z;\nu)$$

as $\nu \to 0$. Similarly, for the connection matrix in the case $\nu \to 0$ we can use the reasoning from (6.18): The connection matrix at $\nu = 0$ is given by

$$\begin{aligned} P(\nu=0) = \tilde{P} &= \Phi_\infty^{-1}\tilde{F}_0 X_{J_0(\nu=0)} \\ &= \Phi_\infty^{-1}\lim_{\nu\to 0}\big[(F_0S_0^{-1}U_0)(\nu)\big]X_{J_0(\nu=0)} \\ &= \lim_{\nu\to 0}\big[(\Phi_\infty^{-1}F_0S_0^{-1}U_0)(\nu)\big]X_{J_0(\nu=0)} \end{aligned} \tag{6.31}$$

We collect all the data for $\nu \neq \mathbb{Z}$: The equation is

$$[\sigma_q^2 - (q^{\nu/2} + q^{-\nu/2})\sigma_q + (1-z)]g(z) = 0 \tag{6.29}$$

or as a system $\sigma_q\Phi = A\Phi$ with A as in (6.30). Around 0, the holomorphic part $F_0(z;\nu)$ of the solution matrix for generic ν can be read-off from (6.23)

$$F_0(z;\nu) = \begin{pmatrix} \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} 0,0 \\ q^{\nu+1}\end{matrix}\,\middle|\, q;z\right) & \frac{(q^{-\nu+1};q)_\infty}{(q;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} 0,0 \\ q^{-\nu+1}\end{matrix}\,\middle|\, q;z\right) \\ q^{\nu/2}\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} 0,0 \\ q^{\nu+1}\end{matrix}\,\middle|\, q;qz\right) & q^{-\nu/2}\frac{(q^{-\nu+1};q)_\infty}{(q;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} 0,0 \\ q^{-\nu+1}\end{matrix}\,\middle|\, q;qz\right) \end{pmatrix},$$

while the character matrix is dictated by $J_0(\nu) = \begin{pmatrix} q^{\nu/2} & 0 \\ 0 & q^{-\nu/2} \end{pmatrix}$. Instead of *choosing* the characters $e_{q^{\pm\nu/2}}(z;q)$, we choose as in (6.23):

$$X_{J_0(\nu),\lambda} = \begin{pmatrix} \frac{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\sqrt{\frac{q^\nu}{z}}\right)}{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\frac{1}{\sqrt{z}}\right)} & 0 \\ 0 & \frac{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\sqrt{\frac{q^{-\nu}}{z}}\right)}{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\frac{1}{\sqrt{z}}\right)} \end{pmatrix}.$$

With this choice, the fundamental solution matrix at $z=0$ is written $(x = 2i\sqrt{z})$

$$\Phi_0(z;\nu,\lambda) = F_0(z;\nu)X_{J_0(\nu),\lambda} = \begin{pmatrix} J^{(1)}_{\nu,\lambda}(2i\sqrt{z};q) & J^{(1)}_{-\nu,\lambda}(2i\sqrt{z};q) \\ \sigma_q J^{(1)}_{\nu,\lambda}(2i\sqrt{z};q) & \sigma_q J^{(1)}_{-\nu,\lambda}(2i\sqrt{z};q) \end{pmatrix}.$$

The situation around $z=\infty$ cannot be obviously decomposed into a holomorphic and character contribution as the equation has an irregular singularity at infinity. We can still read-off the fundamental solution matrix from (6.27) $(t = \frac{1}{x} = -\frac{i}{2\sqrt{z}})$

$$\Phi_\infty(z;\nu,\alpha) = \begin{pmatrix} j^{(1)}_{\nu,\alpha}\left(-\frac{i}{2\sqrt{z}}\right) & j^{(1)}_{\nu,-\alpha}\left(-\frac{i}{2\sqrt{z}}\right) \\ \sigma_q j^{(1)}_{\nu,\alpha}\left(-\frac{i}{2\sqrt{z}}\right) & \sigma_q j^{(1)}_{\nu,-\alpha}\left(-\frac{i}{2\sqrt{z}}\right) \end{pmatrix},$$

and we can certainly re-write the contents of the connection matrix from (6.28). Denoting the matrix in that equation by $P^{-1\,T}(z;\nu,\lambda,\alpha)$, one can check that

$$\left(\Phi_\infty^{-1}\Phi_0\right)(z;\nu,\lambda,\alpha) \equiv P(z;\nu,\lambda,\alpha), \tag{6.32}$$

with

$$P(z;\nu,\lambda,\alpha) = \begin{pmatrix} C_{\nu,\alpha}(\lambda,-\frac{i}{2\sqrt{z}};q) & C_{\nu,-\alpha}(\lambda,-\frac{i}{2\sqrt{z}};q) \\ C_{-\nu,\alpha}(\lambda,-\frac{i}{2\sqrt{z}};q) & C_{-\nu,-\alpha}(\lambda,-\frac{i}{2\sqrt{z}};q) \end{pmatrix}^{-1}$$

where

$$C_{\nu,\alpha}(\lambda,t;q) = \frac{\left(q^{1/2};q^{1/2}\right)_\infty}{\left(q^{1+\nu},q^{-\nu};q\right)_\infty}\frac{\Theta_{q^{1/2}}\left(\alpha q^{\nu/2}t\right)}{\Theta_{q^{1/2}}\left(\alpha t\right)}\frac{\Theta_{q^{1/2}}\left(\lambda t\right)}{\Theta_{q^{1/2}}\left(\lambda q^{\nu/2}t\right)}.$$

Let us now collect the data for $\nu=0$. Around 0, the 'holomorphic part' of the solution $F_0(z;\nu=0) \equiv \tilde{F}_0(z)$ is given by

$$\tilde{F}_0(z) = \lim_{\nu\to 0} F_0 S_0^{-1}U_0 = \lim_{\nu\to 0}\left[\begin{pmatrix} \omega(z;\nu) & \omega(z;-\nu) \\ q^{\nu/2}\omega(qz;\nu) & q^{-\nu/2}\omega(qz;-\nu) \end{pmatrix}\begin{pmatrix} q^{\nu/2} & q^{-\nu/2} \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\right]$$
$$= \begin{pmatrix} \omega(z,0) & \lim\limits_{\nu\to 0} q^{\nu/2}\frac{\omega(z;-\nu)-\omega(z;\nu)}{q^\nu-1} \\ \omega(qz,0) & \lim\limits_{\nu\to 0}\frac{\omega(qz;-\nu)-q^\nu\omega(qz;\nu)}{q^\nu-1} \end{pmatrix}$$

where we have set

$$\begin{aligned} \omega(z;\nu) &:= \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}0,0\\q^{\nu+1}\end{matrix}\,\middle|\,q;z\right) \\ &= \frac{(1-q)^{-\nu}}{\Gamma_q(\nu+1)}\,{}_2\phi_1\left(\begin{matrix}0,0\\q^{\nu+1}\end{matrix}\,\middle|\,q;z\right) \\ &\equiv \frac{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\frac{1}{\sqrt{z}}\right)}{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\sqrt{\frac{q^\nu}{z}}\right)} J^{(1)}_{\nu,\lambda}(z;q). \end{aligned}$$

The limits become derivatives and we have

$$\begin{aligned}
\lim_{\nu\to 0} q^{\nu/2}\frac{\omega(z;-\nu)-\omega(z;\nu)}{q^\nu-1} &= \frac{1}{\log q}\frac{\mathrm{d}}{\mathrm{d}\nu}\bigg|_{\nu=0}\Big[\omega(z;-\nu)-\omega(z;\nu)\Big]\\
&= -\frac{1}{\log q}\left[\frac{\mathrm{d}}{\mathrm{d}(-\nu)}\bigg|_{-\nu=0}\omega(z;-\nu)+\frac{\mathrm{d}}{\mathrm{d}\nu}\bigg|_{\nu=0}\omega(z;\nu)\right]\\
&= -\frac{2}{\log q}\frac{\mathrm{d}}{\mathrm{d}\nu}\bigg|_{\nu=0}\omega(z;\nu)\\
&= -\frac{2}{\log q}\frac{\mathrm{d}}{\mathrm{d}\nu}\bigg|_{\nu=0}\left[\frac{(1-q)^{-\nu}}{\Gamma_q(\nu+1)}\,{}_2\phi_1\left(\begin{matrix}0,0\\q^{\nu+1}\end{matrix}\,\bigg|\,q;z\right)\right]\\
&= 2\frac{\Psi_q(1)+\log(1-q)}{\log q}\,{}_2\phi_1\left(\begin{matrix}0,0\\q\end{matrix}\,\bigg|\,q;z\right)-2q\,\zeta(0,0;z)
\end{aligned}$$

where $\Psi_q(z)$ is the q-digamma function and ζ denotes the derivative of the basic hypergeometric function with respect to parameters as defined in (6.17). The second limit is

$$\begin{aligned}
\lim_{\nu\to 0}\frac{\omega(qz;-\nu)-q^\nu\omega(qz;\nu)}{q^\nu-1} &= \frac{1}{\log q}\frac{\mathrm{d}}{\mathrm{d}\nu}\bigg|_{\nu=0}\big[\omega(qz;-\nu)-q^\nu\omega(qz;\nu)\big]\\
&= -\frac{1}{\log q}\frac{\mathrm{d}}{\mathrm{d}\nu}\bigg|_{\nu=0}\big[(1+q^\nu)\omega(qz;\nu)\big]\\
&= \left[2\frac{\Psi_q(1)+\log(1-q)}{\log q}-1\right]{}_2\phi_1\left(\begin{matrix}0,0\\q\end{matrix}\,\bigg|\,q;qz\right)-2q\,\zeta(0,0;qz).
\end{aligned}$$

Collecting the results we have that the logarithmic solutions around 0 are given by

$$\tilde F_0(z)=\begin{pmatrix}{}_2\phi_1\left(\begin{matrix}0,0\\q\end{matrix}\,\bigg|\,q;z\right) & 2\frac{\Psi_q(1)+\log(1-q)}{\log q}\,{}_2\phi_1\left(\begin{matrix}0,0\\q\end{matrix}\,\bigg|\,q;z\right)-2q\,\zeta(0,0;z)\\ {}_2\phi_1\left(\begin{matrix}0,0\\q\end{matrix}\,\bigg|\,q;qz\right) & \left[2\frac{\Psi_q(1)+\log(1-q)}{\log q}-1\right]{}_2\phi_1\left(\begin{matrix}0,0\\q\end{matrix}\,\bigg|\,q;qz\right)-2q\,\zeta(0,0;qz),\end{pmatrix}$$

where $\Psi_q(z)=\frac{1}{\Gamma_q(z)}\frac{\mathrm{d}\Gamma_q(z)}{\mathrm{d}z}$ is the q-digamma function and $\zeta(\alpha,\beta;z)\equiv\frac{\mathrm{d}}{\mathrm{d}c}\big|_{c=q}\,{}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\,\bigg|\,q;z\right)$. Note that $\tilde F_0$ does *not* depend on λ, as λ only appears in our choice of q-character and $\tilde F_0$ is just the holomorphic part of the solutions.

Similarly, we want to compute the connection matrix at $\nu=0$ by (6.31), i.e. we want to evaluate $(\Phi_\infty^{-1}F_0S_0^{-1}U_0)(\nu)\times X_{J_0(\nu=0)}$ and take the limit $\nu\to 0$. We find that $\big(\Phi_\infty^{-1}F_0\big)(z;\nu,\alpha,\lambda)$ can be "read-off" from the connection matrix in (6.28) or equivalently (6.32). We have

$$\begin{aligned}
\big(\Phi_\infty^{-1}F_0\big)(z;\nu,\lambda,\alpha) &= \big(\Phi_\infty^{-1}\Phi_0\big)(z;\nu,\lambda,\alpha)X^{-1}_{J_0(\nu),\lambda}\\
&\equiv P(z;\nu,\lambda,\alpha)X^{-1}_{J_0(\nu),\lambda}\\
&= \left[\begin{pmatrix}\frac{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\sqrt{\frac{q^\nu}{z}}\right)}{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\frac{1}{\sqrt z}\right)} & 0\\ 0 & \frac{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\sqrt{\frac{q^{-\nu}}{z}}\right)}{\Theta_{q^{1/2}}\left(\frac{\lambda}{2i}\frac{1}{\sqrt z}\right)}\end{pmatrix}\begin{pmatrix}C_{\nu,\alpha}(\lambda,-\frac{i}{2\sqrt z};q) & C_{\nu,-\alpha}(\lambda,-\frac{i}{2\sqrt z};q)\\ C_{-\nu,\alpha}(\lambda,-\frac{i}{2\sqrt z};q) & C_{-\nu,-\alpha}(\lambda,-\frac{i}{2\sqrt z};q)\end{pmatrix}\right]^{-1}\\
&= \begin{pmatrix}\tilde C_{\nu,\alpha}(-\frac{i}{2\sqrt z};q) & \tilde C_{\nu,-\alpha}(-\frac{i}{2\sqrt z};q)\\ \tilde C_{-\nu,\alpha}(-\frac{i}{2\sqrt z};q) & \tilde C_{-\nu,-\alpha}(-\frac{i}{2\sqrt z};q)\end{pmatrix}^{-1}
\end{aligned}$$

where

$$\begin{aligned}
\tilde C_{\nu,\alpha}(t;q) &= \frac{\left(q^{1/2},q^{1/2};q\right)_\infty}{\left(q^{1+\nu},q^{-\nu};q\right)_\infty}\frac{\Theta_{q^{1/2}}\left(\alpha t q^{\nu/2}\right)}{\Theta_{q^{1/2}}\left(\alpha t\right)}\\
&= \frac{\left(q^{1/2},q^{1/2};q\right)_\infty}{\left(q^{1+\nu},q^{-\nu};q\right)_\infty}e_{q^{\nu/2}}\left(\alpha t;q^{1/2}\right)^{-1}.
\end{aligned}$$

Again, the dependence on λ drops out, since none of the matrices Φ_∞ or F_0 depend on it. We set $e_{q^{\nu/2}}\left(\alpha t;q^{1/2}\right)^{-1}=:\chi_{\nu,\alpha}(t)$, noting that these functions tend to 1 as $\nu\to 0$. We compute the inverse of the matrix explicitly, suppressing the argument $t = -\frac{i}{2\sqrt{z}}$

$$\left(\Phi_\infty^{-1}F_0\right)(z;\nu,\lambda,\alpha) = \frac{1}{\left(q^{1/2},q^{1/2};q\right)_\infty}\frac{1}{\chi_{\nu,\alpha}\chi_{-\nu,-\alpha}-\chi_{-\nu,\alpha}\chi_{\nu,-\alpha}}\times$$
$$\times\begin{pmatrix}\left(q^{1+\nu},q^{-\nu};q\right)_\infty\chi_{-\nu,-\alpha} & -\left(q^{1-\nu},q^{\nu};q\right)_\infty\chi_{\nu,-\alpha}\\ -\left(q^{1+\nu},q^{-\nu};q\right)_\infty\chi_{-\nu,\alpha} & \left(q^{1-\nu},q^{\nu};q\right)_\infty\chi_{\nu,\alpha}\end{pmatrix}.$$

The denominator coming from the determinant is zero at $\nu=0$, but We also have that

$$S_0^{-1}(\nu)U_0 = \begin{pmatrix}q^{\nu/2} & q^{-\nu/2}\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\frac{1}{1+q^{\nu/2}} & \frac{q^{\nu/2}}{1-q^{\nu}}\\ \frac{1}{1+q^{-\nu/2}} & \frac{-q^{\nu/2}}{1-q^{\nu}}\end{pmatrix},$$

hence we obtain

$$(\Phi_\infty^{-1}F_0S_0^{-1}U_0)(\nu) = \frac{1}{\left(q^{1/2},q^{1/2};q\right)_\infty}\frac{1}{\chi_{\nu,\alpha}\chi_{-\nu,-\alpha}-\chi_{-\nu,\alpha}\chi_{\nu,-\alpha}}\times$$
$$\times\begin{pmatrix}\frac{\left(q^{1+\nu},q^{-\nu};q\right)_\infty\chi_{-\nu,-\alpha}}{1+q^{\nu/2}}-\mathrm{idem}(\nu,-\nu) & \frac{q^{\nu/2}}{1-q^{\nu}}\left[\left(q^{1+\nu},q^{-\nu};q\right)_\infty\chi_{-\nu,-\alpha}+\mathrm{idem}(\nu,-\nu)\right]\\ \frac{-\left(q^{1+\nu},q^{-\nu};q\right)_\infty\chi_{-\nu,\alpha}}{1+q^{\nu/2}}-\mathrm{idem}(\nu,-\nu) & -\frac{q^{\nu/2}}{1-q^{\nu}}\left[\left(q^{1+\nu},q^{-\nu};q\right)_\infty\chi_{-\nu,\alpha}+\mathrm{idem}(\nu,-\nu)\right]\end{pmatrix}.$$

Taking the limit $\nu\to 0$. Care is needed to take the limit; as $\nu\to 0$, the global determinant factor is singular and each $(q^{\pm\nu};q)_\infty$ factor has a (simple) zero. Furthermore, non-trivial singularity/zero cancellations may occur due to the addition/subtraction of identical terms with the sign of ν switched. It is a small exercise to see that

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\chi_{\nu,\alpha}(t) = -\frac{q^{-\nu/2}\log q}{2}\ell_{q^{1/2}}\left(\alpha t q^{\nu/2}\right)\chi_{\nu,\alpha}(t).$$

where as before

$$\ell_q\left(z\right) = -z\frac{\frac{\mathrm{d}}{\mathrm{d}z}\Theta_q\left(z\right)}{\Theta_q\left(z\right)}$$

From this we find

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\chi_{\nu,\alpha}\chi_{-\nu,-\alpha}-\chi_{-\nu,\alpha}\chi_{\nu,-\alpha}\right] = -\frac{\log q}{2}\Bigg(\left[q^{-\nu/2}\ell_{q^{1/2}}\left(\alpha t q^{\nu/2}\right)-q^{\nu/2}\ell_{q^{1/2}}\left(-\alpha t q^{-\nu/2}\right)\right]\chi_{\nu,\alpha}\chi_{-\nu,-\alpha}$$
$$+\left[q^{\nu/2}\ell_{q^{1/2}}\left(\alpha t q^{-\nu/2}\right)-q^{-\nu/2}\ell_{q^{1/2}}\left(-\alpha t q^{\nu/2}\right)\right]\chi_{-\nu,\alpha}\chi_{\nu,-\alpha}\Bigg),$$

in particular,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\nu}\right|_{\nu=0}\det X_{\nu,\alpha} \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}\nu}\right|_{\nu=0}\left[\chi_{\nu,\alpha}\chi_{-\nu,-\alpha}-\chi_{-\nu,\alpha}\chi_{\nu,-\alpha}\right] = \log q\left[\ell_{q^{1/2}}\left(-\alpha t\right)-\ell_{q^{1/2}}\left(\alpha t\right)\right]$$

Using this result we may take the limit of the first column. We need to consider both terms in the first matrix element and we find

$$\frac{1}{\left(q^{1/2},q^{1/2};q\right)_\infty}\lim_{\nu\to 0}\left[\frac{\frac{\left(q^{1+\nu},q^{1-\nu};q\right)_\infty\chi_{-\nu,-\alpha}}{1+q^{\nu/2}}(1-q^{-\nu})-\frac{\left(q^{1+\nu},q^{1-\nu};q\right)_\infty\chi_{\nu,-\alpha}}{1+q^{-\nu/2}}(1-q^{\nu})}{\chi_{\nu,\alpha}\chi_{-\nu,-\alpha}-\chi_{-\nu,\alpha}\chi_{\nu,-\alpha}}\right]$$
$$=\frac{1}{\left(q^{1/2},q^{1/2};q\right)_\infty}\lim_{\nu\to 0}\Bigg[\frac{\left(q^{1+\nu},q^{1-\nu};q\right)_\infty\chi_{-\nu,-\alpha}}{1+q^{\nu/2}}\frac{1-q^{-\nu}}{\nu}\left(\frac{\mathrm{d}X_{\nu,\alpha}}{\mathrm{d}\nu}\right)^{-1}$$
$$-\frac{\left(q^{1+\nu},q^{1-\nu};q\right)_\infty\chi_{\nu,-\alpha}}{1+q^{-\nu/2}}\frac{1-q^{\nu}}{\nu}\left(\frac{\mathrm{d}X_{\nu,\alpha}}{\mathrm{d}\nu}\right)^{-1}\Bigg].$$

Using the fact that $\lim_{\nu\to 0}\frac{1-q^{\pm\nu}}{\nu} = \mp\log q$ we deduce that the limit of the first matrix element exists and is equal to

$$\frac{\left(q,q;q\right)_\infty}{\left(q^{1/2},q^{1/2};q\right)_\infty}\frac{1}{\ell_{q^{1/2}}\left(-\alpha t\right)-\ell_{q^{1/2}}\left(\alpha t\right)}.$$

Similarly, the lower-left matrix element is obtained by a global sign and an interchange $\alpha\mapsto -\alpha$, which changes nothing, and the lower-left matrix element is also given by the above expression.

The right column of the matrix may also be obtained; at first glance it looks like it diverges, but a careful analysis shows otherwise. We write the limit of the top-right matrix element

$$\frac{1}{\left(q^{1/2},q^{1/2};q\right)_\infty}\lim_{\nu\to 0}\left[\frac{\frac{q^{\nu/2}}{1-q^\nu}\left[\left(q^{1+\nu},q^{-\nu};q\right)_\infty\chi_{-\nu,-\alpha}+\left(q^{1-\nu},q^{\nu};q\right)_\infty\chi_{\nu,-\alpha}\right]}{\chi_{\nu,\alpha}\chi_{-\nu,-\alpha}-\chi_{-\nu,\alpha}\chi_{\nu,-\alpha}}\right]$$
$$\frac{1}{\left(q^{1/2},q^{1/2};q\right)_\infty}\lim_{\nu\to 0}\left(q^{\nu/2}\left(q^{1-\nu},q^{1+\nu};q\right)_\infty\right)\left[\frac{\frac{1-q^{-\nu}}{1-q^\nu}\chi_{-\nu,-\alpha}+\chi_{\nu,-\alpha}}{\nu}\frac{\nu}{X_{\nu,\alpha}}\right]$$

We can now take the limit as all the factors have well-defined limits. In particular, we find after some calculation

$$\lim_{\nu\to 0}q^{\nu/2}\left(q^{1-\nu},q^{1+\nu};q\right)_\infty=(q,q;q)_\infty$$
$$\begin{aligned}\lim_{\nu\to 0}\left[\frac{\frac{1-q^{-\nu}}{1-q^\nu}\chi_{-\nu,-\alpha}+\chi_{\nu,-\alpha}}{\nu}\right]&=\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\frac{1-q^{-\nu}}{1-q^\nu}\chi_{-\nu,-\alpha}+\chi_{\nu,-\alpha}\right]_{\nu=0}\\&=\log q\left[1-\ell_{q^{1/2}}\left(-\alpha t\right)\right]\end{aligned}$$

and of course

$$\lim_{\nu\to 0}\frac{\nu}{X_{\nu,\alpha}}=\left[\frac{\mathrm{d}X_{\nu,\alpha}}{\mathrm{d}\nu}\right]^{-1}_{\nu=0}=\frac{1}{\log q\left[\ell_{q^{1/2}}\left(-\alpha t\right)-\ell_{q^{1/2}}\left(\alpha t\right)\right]}.$$

We collect the factors to obtain the top-right matrix element

$$\frac{(q,q;q)_\infty}{\left(q^{1/2},q^{1/2};q\right)_\infty}\frac{1-\ell_{q^{1/2}}\left(-\alpha t\right)}{\ell_{q^{1/2}}\left(-\alpha t\right)-\ell_{q^{1/2}}\left(\alpha t\right)},$$

and the lower-right matrix element is obtained again by a global sign and a transformation $\alpha\mapsto-\alpha$ (which now is not the identity). Assembling the elements we find that

$$\lim_{\nu\to 0}(\Phi_\infty^{-1}F_0S_0^{-1}U_0)(\nu)=\frac{(q,q;q)_\infty}{\left(q^{1/2},q^{1/2};q\right)_\infty}\frac{1}{\ell_{q^{1/2}}\left(-\alpha t\right)-\ell_{q^{1/2}}\left(\alpha t\right)}\begin{pmatrix}1 & 1-\ell_{q^{1/2}}\left(-\alpha t\right)\\ 1 & -[1-\ell_{q^{1/2}}\left(\alpha t\right)]\end{pmatrix}.$$

The (logarithmic) character matrix $X_{J_0(\nu=0)}(z)$ is dictated (up to q-difference equivalence) by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and can be *chosen* to be

$$X_{J_0(\nu=0)}(z)=\begin{pmatrix}1 & -\ell_{q^{1/2}}\left(\lambda t\right)\\ 0 & 1\end{pmatrix}\tag{6.33}$$

since by $\sigma_{q,z}\equiv q^{\frac{\mathrm{d}}{\mathrm{d}(\log z)}}=q^{\frac{\mathrm{d}}{\mathrm{d}(\log t^{-2})}}=\sigma^{-1}_{q^{1/2},t}$ we deduce that

$$\sigma_{q,z}(-\ell_{q^{1/2}}\left(\lambda t\right))=\sigma^{-1}_{q^{1/2},t}(-\ell_{q^{1/2}}\left(\lambda t\right))=-\ell_{q^{1/2}}\left(\lambda t\right)+1$$

hence $-\ell_{q^{1/2}}\left(\lambda t\right)$ satisfies the same q-difference equation as $\ell_q\left(z\right)$, with $z\sim t^{-2}$. We finally have *the connection matrix in the logarithmic case* $\nu=0$

$$\tilde{P}(z)=\frac{(q,q;q)_\infty}{\left(q^{1/2},q^{1/2};q\right)_\infty}\frac{1}{\ell_{q^{1/2}}\left(-\alpha t\right)-\ell_{q^{1/2}}\left(\alpha t\right)}\begin{pmatrix}1 & 1-\ell_{q^{1/2}}\left(-\alpha t\right)-\ell_{q^{1/2}}\left(\lambda t\right)\\ 1 & -[1-\ell_{q^{1/2}}\left(\alpha t\right)+\ell_{q^{1/2}}\left(\lambda t\right)],\end{pmatrix}$$

where $z=\frac{1}{4t^2}$ and $\alpha=2iq^{3/4}$. It is of crucial importance to note that the dependence of $\tilde{P}$ on λ is an artifact of our choice in (6.33).

6.4 A recap and outlook

6.4.1 Results

We recap our results: We have computed the connection matrices

- The connection matrix of the basic hypergeometric equation (6.12) for generic ($\notin q^{\mathbb{Z}}$) values of the parameters (a,b,c) is given by

$$P(z)=X^{-1}_{J_\infty}(z)\begin{pmatrix}\frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)}e_a\left(z\right)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)}e_{aq/c}\left(z\right)^{-1}\\ \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)}e_b\left(z\right)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)}e_{bq/c}\left(z\right)^{-1}\end{pmatrix}X_{J_0}(z),$$

where the character matrices are dictated by the Jordan matrices

$$J_0 = \begin{pmatrix} 1 & 0 \\ 0 & q/c \end{pmatrix}, \quad \text{and } J_\infty = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

Choosing the q-characters to be the "standard" ones ($e_a(z)$, etc.) we obtain

$$P(z) = \begin{pmatrix} \frac{\Gamma_q(\gamma,\beta-\alpha)}{\Gamma_q(\beta,\gamma-\alpha)} e_{a^{-1}}(z)^{-1} e_a(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\beta-\gamma+1,1-\alpha)} e_{q/c}(z) e_{a^{-1}}(z)^{-1} e_{aq/c}(z)^{-1} \\ \frac{\Gamma_q(\gamma,\alpha-\beta)}{\Gamma_q(\alpha,\gamma-\beta)} e_{b^{-1}}(z)^{-1} e_b(z)^{-1} & \frac{\Gamma_q(2-\gamma,\beta-\alpha)}{\Gamma_q(\alpha-\gamma+1,1-\beta)} e_{q/c}(z) e_{b^{-1}}(z)^{-1} e_{bq/c}(z)^{-1} \end{pmatrix}.$$

- The connection matrix of the basic hypergeometric equation for the specific values $a = q^{1/3}, b = q^{2/3}$ and $c = q$ coming from the GLSM which corresponds to the cubic in $\mathbb{P}^2$ [Joc+; KRS16] is

$$\tilde{P}(z) = X_{J_\infty}^{-1}(z) \begin{pmatrix} \frac{\Gamma_q(1/3)}{\Gamma_q(2/3,2/3)} e_{q^{1/3}}(z)^{-1} & (\log q)^{-1} e_{q^{1/3}}(z)^{-1} \frac{\Gamma_q(1/3)}{\Gamma_q(2/3,2/3)} \Upsilon_q(\zeta; 1/3, 2/3) \\ \frac{\Gamma_q(-1/3)}{\Gamma_q(1/3,1/3)} e_{q^{2/3}}(z)^{-1} & (\log q)^{-1} e_{q^{2/3}}(z)^{-1} \frac{\Gamma_q(-1/3)}{\Gamma_q(1/3,1/3)} \Upsilon_q(\zeta; 2/3, 1/3) \end{pmatrix} X_{J_0}(z),$$

where $\Upsilon_q(\zeta;\alpha,\beta) = 2\Psi_q(1) - \Psi_q(1-\alpha) - \Psi_q(\beta) + \Psi_q(\zeta+\alpha) - \Psi_q(1-\zeta-\alpha)$ and the q-character matrices are dictated by

$$J_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and } J_\infty = \begin{pmatrix} q^{-1/3} & 0 \\ 0 & q^{-2/3} \end{pmatrix}.$$

The "standard" choices are

$$X_{J_0}(z) = \begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1 \end{pmatrix}, \quad X_{J_\infty}(z) \begin{pmatrix} e_{q^{-1/3}}(z) & 0 \\ 0 & e_{q^{-2/3}}(z) \end{pmatrix}.$$

- The connection matrix for the first q-deformation (6.29) of the Bessel equation for generic $\nu \notin \mathbb{Z}$ (corresponding to one equation from the $\mathbb{CP}^1$ sigma model from [BDP14]) is

$$P^{-1}(z;\nu,\lambda,\alpha) = \begin{pmatrix} C_{\nu,\alpha}(-\frac{i}{2\sqrt{z}};q) & C_{\nu,-\alpha}(-\frac{i}{2\sqrt{z}};q) \\ C_{-\nu,\alpha}(-\frac{i}{2\sqrt{z}};q) & C_{-\nu,-\alpha}(-\frac{i}{2\sqrt{z}};q) \end{pmatrix}^{-1} X_{J_0(\nu),\lambda}$$

where

$$C_{\nu,\alpha}(t;q) = \frac{(q^{1/2};q^{1/2})_\infty}{(q^{1+\nu},q^{-\nu};q)_\infty} \frac{\Theta_{q^{1/2}}(\alpha q^{\nu/2} t)}{\Theta_{q^{1/2}}(\alpha t)},$$

$\alpha = 2iq^{3/4}$, and the character matrix is dictated by

$$J_0(\nu) = \begin{pmatrix} q^\nu & 0 \\ 0 & q^{-\nu} \end{pmatrix}.$$

- The connection matrix for the non-generic case $\nu = 0$ (corresponding to the massless limit in (3.17a)) is given by

$$\tilde{P}(z) = \frac{(q,q;q)_\infty}{(q^{1/2},q^{1/2};q)_\infty} \frac{1}{\ell_{q^{1/2}}(-\alpha t) - \ell_{q^{1/2}}(\alpha t)} \begin{pmatrix} 1 & 1-\ell_{q^{1/2}}(-\alpha t) \\ 1 & -[1-\ell_{q^{1/2}}(\alpha t)] \end{pmatrix} X_{J_0(\nu=0)}(z)$$

where $z = \frac{1}{4t^2}$, $\alpha = 2iq^{3/4}$ and the the character matrix is dictated by

$$J_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

6.4.2 Discussion

What is clear from all 4 cases (2 + 2 "degenerate" ones) is that there is a piece of datum for each case that is *independent* of choices, coming from an analytic continuation formula (e.g. Watson's formula (6.15) for the hypergeometric equation, and Zhang's theorem (6.28) for the Bessel equation), as well as the character matrices for each specific case, which one has to *choose*. The specific choices one makes then *enter into the connection matrix*, which is supposed to hold physically relevant information. We make some hypotheses about what this could imply in the following discussion.

In line with the work from Beem, Dimofte and Pasquetti, the analytic continuation from $|z| < 1$ to $|z| > 1$ is given by the connection matrix of the Bessel equation. The continuation corresponds to transporting the blocks from $|y| > 1$ to $|y| < 1$, i.e. from positive to negative real part of the complexified mass deformation Y associated to the topological $U(1)_J$ symmetry (cf. 3.3).

As we mention above, the choice of character matrices enters in the connection matrix; in particular, the choice of q-logarithm for the (degenerate) solutions around $z = 0$ will directly enter in the matrix. In the main text above, we chose the q-logarithm $\ell_{q^{1/2}}(\lambda t)$ which is admissible since it satisfies the correct q-difference equation. Any other admissible q-logarithm will be an elliptic multiple of this choice.

According to [BDP14], the elliptic factors $c(z;q)$ should be constrained by the factorization conjecture, meaning that $\|c(z;q)\|_g^2 = 1$ modulo some pure q-dependent terms related to R-R Chern-Simons terms, called contact terms. However purely from the point-of-view of the q-difference equations any choice is admissible. One possible explanation

1. *The (contour) integral solutions to q-difference equations "hold" more information than the respective equations.*

   An integral expression for the solution to a q-difference equation will also specify the elliptic factor ambiguity, if one has a scheme for choosing a contour. One can be further convinced that integral solutions are special by the fact that solutions to q-difference (or differential) equations admitting an (non-trivial) integral representation is a *rare* phenomenon.

   This means that one *cannot* analyze the physical interpretation of such global data of q-difference equations, without further input from the physics (e.g. explicit integrand and contour) even when the equations stem from physical theories. This is reflected in the "infinity" of choices for arbitrary parameters that can enter in the character matrices, e.g. by rescalings $\frac{\Theta_q(z)}{\Theta_q(az)} \mapsto \frac{\Theta_q(\lambda z)}{\Theta_q(a\lambda z)}$.

Similarly, one can take hints from the two-dimensional GLSM (cubic in $\mathbb{P}^2$): The analytic continuation of the blocks, which can be viewed as three-dimensional lifts of the hemisphere partition function, corresponds to transportation of "brane data", in the form of boundary condition, across phases of the theory. The partition function $\mathcal{Z}$ of the three-dimensional theory on the *total* manifold M^3 should be invariant of the choices in the elliptic factors.

Another possible resolution of the ambiguity would be to *relax the factorization condition*: similarly to the two dimensional case [Hor03], one would construct the partition function as a quantum mechanical amplitude

$$\mathcal{Z}(M^3) = \sum B^\alpha(x)\eta_{\alpha,\bar{\beta}}(x,\bar{x})B^\beta(\bar{x}) \equiv \|B\|_g^2\,, \tag{6.34}$$

where we have labeled the parameters of the theory by x and the "metric" $\eta_{\alpha,\bar{\beta}}$ can be interpreted as a vacuum amplitude on a(n infinitely long) cylinder times S^1 (actually a two-point function of the corresponding operators). The gluing is geometrically determined by the boundary map g and should fix the relation between $x, \bar{x}$. In the examples $S_b^3, S^2 \times_q S^1$ from [BDP14], the gluing map was an element in the automorphism group of the torus. This generalization of factorization is applicable also when one does not have the *same theory* on the two pieces. Now, transporting the blocks from negative to positive real mass deformations in our two examples induces the transformation described by the connection matrix, i.e. informally $B \mapsto PB$, where P is elliptic and can contain arbitrary parameters. Since the total partition function should not depend on the choice of these parameters, we con hypothesize that

2. *The arbitrary choice of elliptic factors in the connection matrix appear as "covariant" transformations of the "metric" η in (6.34).*

   Explicitly, across the phases the blocks change by elliptic connection matrices $B \mapsto P_1 B$. Making another choice of q-characters is always related to the original one by elliptic factors, thus we can say that $B \mapsto P_1 B$ where $P_2 = EP_1$, with all matrices being elliptic valued. To achieve invariance of $\mathcal{Z} \sim B^T \eta B$ we must have

   $$\mathcal{Z}(x) \stackrel{!}{=} \mathcal{Z}'(x) = B'^T(x)\eta'(x,\bar{x})B'(\bar{x})$$

   independently of the choices. This would imply that

   $$\eta \stackrel{!}{=} P_1^T(x)\eta_1'(x,\bar{x})P_1(\bar{x}) \stackrel{!}{=} P_2^T(x)\eta_1'(x,\bar{x})P_2(\bar{x}) = P_1^T(x)E^T(x)\eta_2'(x,\bar{x})E(\bar{x})P_1(\bar{x})$$

   is sufficient for invariance of the partition function.

6.4.3 Outlook and open questions

We briefly remark on some open question and possible future directions:

From 2D to 3D: One of the main goals of this large program is to study 3D (conformal) field theories through gauge theories. It would be interesting to see further results established for 2D theories find their 3D counterparts, e.g. grade restriction of D-branes [HHP08; HR13].

From 3D to 2D: Similarly, it is also interesting to look at the inverse procedure: study if and how three-dimensional results have a limit in two dimensions. In particular, it is an interesting project, both physically and mathematically, to determine even the existence the $q \to 1$ limit q-functions and q-difference equations systematically.

Elliptic factors: We suspect that the choice of elliptic factors will encode more (physical) information other than the choice of (background) Chern-Simons terms. It would be interesting to further understand ind interpret the dependence on the choice of elliptic factors in physical terms. It would also be interesting to ask if there is geometric information encoded in the elliptic factors, in analogy to the two-dimensional case where the partition function on the sphere is supposed to compute the quantum Kähler potential, related to genus zero Gromov-Witten invariants.

Further directions: Other points of further study would be to apply the prescription given by [BDP14] to other backgrounds, i.e. a topologically of smoothly different choice of three-manifold (but always fibered over S^1) and/or different theories defined on it. This would provide further evidence for the validity of the dictionary in terms of blocks, and also further illuminate the factorization conjecture.

Lastly, it would certainly be engaging to fully understand the relationship between line operator identities, the q-difference equations stemming from them and the emergence of quantum spectral curves encoding non-perturbative information [DGG11b; Dim11].

Appendix

A. Fun with q-functions

A.1 The q-Pochhammer symbols and generalizations

A.1.1 The q-Pochhammer symbol and its properties

Inside the unit q-circle

Motivated by the *Pochhammer symbol* also called "shifted (rising) factorial"

$$(z)_n \coloneqq z(z+1)\cdots(z+n-1)$$

with $(z)_0 \equiv 1$ we define the following symbol, called the q-Pochhammer symbol:

$$(z;q)_n \coloneqq \begin{cases} 1 & \text{if } n=0 \\ (1-z)(1-zq)\cdots(1-zq^{n-1}) & \text{n=1,2,\dots} \end{cases} \quad \text{for } z,q \in \mathbb{C}.$$

The motivation comes from the simple but essential fact that the q-analogue of a number $z \in \mathbb{C}$ is $[z]_q = \frac{1-q^z}{1-q}$

$$\lim_{q\to 1} \frac{1-q^z}{1-q} = z,$$

for all $z \in \mathbb{C}$. This is essentially the starting point of *all* q-analogues, which are objects — functions, operators, etc — that reduce to a known object for $q = 1$. Note that we will often work in the "multi-valued plane", i.e. work with $z \in \mathbb{C}$ instead of ζ, where $z = q^\zeta$. This choice also involves "picking a branch" to properly define such exponentials, but we disregard such details in this work, and assume we have made a choice for the logarithm of $\log q =: \hbar$, with $\hbar \in \mathbb{C}$.

We want to extend these definitions for more values of $n \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, in particular $n \to \infty$, for as many values of z, q as possible. A simple convergence argument* convinces one that when $|q| < 1$ we can define the (absolutely) convergent product

$$(z;q)_\infty \coloneqq \prod_{k=0}^{\infty} (1-zq^k), \qquad \text{for } |q|<1. \tag{A.1.1}$$

One can then easily verify the identity $(z;q)_n = \frac{(z;q)_\infty}{(zq^n;q)_\infty}$. An identity that is *not* as easy to verify is the so called *q-binomial theorem* that asserts that

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \qquad \text{for } |z|<1, |q|<1. \tag{A.1.2}$$

Note that the left-hand side makes sense even when $|z| > 1$, and has simple poles at $z = q^{-k}$ for $k \in \mathbb{N}_0$, but the right-hand side has radius of convergence $|z| < 1$† when $|q| < 1$, since $z = 1$ is the first pole that a disc of increasing radius centered at 0 "encounters". Also note that the left-hand side has some obvious zeros ($z = a^{-1}q^{-k}$), which are not obvious at all on the right-hand side.

*Recall, an infinite product $\prod_{k=1}^{\infty} c_n$ converges if and only if the sum $\sum_{k=1}^{\infty} \ln c_n$ converges. In addition, products of the form $\prod_{k=1}^{\infty}(1-c_n)$ for $c_n \in \mathbb{C}$ and for $\sum_{k=1}^{\infty} |c_n|^2 < \infty$ converge if and only if the sum $\sum_{k=0}^{\infty} c_n$ converges.

†Recall, the radius of convergence R for $\sum_{n=0}^{\infty} c_n(z-a)^n$ is given by $R \coloneqq \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$.

Proof. We are following [GR04]: We set

$$h_a(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \text{ for } |z| < 1, |q| < 1$$

and compute the difference[‡]

$$\begin{aligned} h_a(z) - h_{aq}(z) &= \sum_{n=1}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n \\ &= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} \big((1-a) - (1-aq^n)\big) z^n \\ &= -a \sum_{n=1}^{\infty} \frac{(1-q^n)(aq;q)_{n-1}}{(q;q)_n} z^n \\ &= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_{n-1}} z^n \\ &= -az h_{aq}(z). \end{aligned}$$

Next, we compute the difference

$$\begin{aligned} h_a(z) - h_a(qz) &= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n} (z^n - q^n z^n) \\ &= \sum_{n=1}^{\infty} \frac{(1-q^n)(a;q)_n}{(q;q)_n} z^n \\ &= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_{n-1}} z^n \\ &= (1-a)z \sum_{n'=0}^{\infty} \frac{(aq;q)_{n'}}{(q;q)_{n'}} z^{n'} \\ &= (1-a)z h_{aq}(z). \end{aligned}$$

Combining the two results to eliminate $h_{aq}(z)$ we obtain $h_a(z) = \frac{1-az}{1-z} h_a(qz)$. Iterating we obtain

$$h_a(z) = \frac{1-az}{1-z} h_a(qz) = \frac{1-az}{1-z} \frac{1-aqz}{1-qz} h_a(q^2 z) = \ldots = \frac{(az;q)_k}{(z;q)_k} h_a(q^k z).$$

We can now take the limit $k \to \infty$, using $q^k \to 0$ for $|q| < 1$ and $h_a(0) = 1$, to obtain indeed that

$$h_a(z) \equiv \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \quad \text{ for } |q| < 1, |z| < 1.$$

□

Using this result, we can prove more interesting identities, using the “freedom” that there is no restriction on a. Setting $a = 0$ we readily arrive at

$$\frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} \quad \text{ for } |q| < 1, |z| < 1. \tag{A.1.3}$$

Note again that the left-hand side makes sense for $|z| > 1$, and has simple poles at $z = q^{-k}$ for $k \in \mathbb{N}_0$ as before, while the right-hand side converges only for $|z| < 1$, when $|q| < 1$. Another important result is the q-series expansion of the q-Pochhammer symbol:

$$(z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)} z^n}{(q;q)_n} \quad \text{ for } |q| < 1.$$

[‡]The motivation for these calculations comes from the commuting case $q = 1$, where the analogous result holds for $f_a(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$.

To see this identity, set $z \mapsto \frac{z}{a}$ in the q-binomial theorem to obtain

$$\frac{(z;q)_\infty}{(za^{-1};q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n a^{-n}}{(q;q)_n} z^n,$$

and we note that the numerator in the sum is written as

$$(a;q)_n a^{-n} = (a^{-1}-1)(a^{-1}-q)\cdots(a^{-1}-q^{n-1}).$$

We now take the limit $a \to \infty$ (which lifts the constraint $|z|<1$), where the numerator reduces to $(-1)\cdots(-q^{n-1}) = (-1)^n q^{\frac{1}{2}n(n-1)}$ and we have finally our desired result

$$(z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q;q)_n} z^n, \qquad \text{for } |q|<1. \tag{A.1.4}$$

Outside the unit q-circle

We started our discussion saying we want to extend the definition of the q-Pochhammer symbol $(z;q)_n$ to $n \to \infty$ for as many values of z, q as possible; in particular, the *finite* q-Pochhammer symbol is defined also "outside the unit q-circle" $|q|>1$. To implement this we simply take (A.1.4) as the definition of our symbol: it reduces to (A.1.1) for $|q|<1$ as we would wish. What about $|q|>1$? A ratio test shows that the right-hand side of (A.1.4) has a radius of convergence $R=|q|$ when $|q|>1$. We claim in fact that

$$(z;q)_\infty := \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q;q)_n} z^n = \begin{cases} \prod_{k=0}^{\infty}(1-zq^k) & \text{if } |q|<1, \\ \prod_{k=1}^{\infty}(1-zq^{-k})^{-1} & \text{if } |q|>1. \end{cases} \tag{A.1.5}$$

The sum converges for $|z|<|q|$, when $|q|>1$ while the very right-hand side is defined for all z when $|q|<1$ and for all $z \in \mathbb{C}\setminus\{q^k\}_{k\in\mathbb{N}}$ (where the function has simple poles) when $|q|>1$. To see this claim, we can use the definition of $(z;\ell)_\infty$ for $|\ell|<1$ and the results we have so far: Starting from the $|q|>1$ case of the right-hand side of (A.1.5) we have

$$\prod_{k=1}^{\infty}(1-zq^{-k})^{-1} = \prod_{k=0}^{\infty}(1-zp^{k+1})^{-1} = \frac{1}{(pz;p)_\infty},$$

for $p=q^{-1}$. We are thus in the first regime $|p|<1$ and we can use the property (A.1.3) when $|z|<|q|$ (avoiding even the first pole at $z=q$) to obtain

$$\frac{1}{(pz;p)_\infty} = \sum_{n=0}^{\infty} \frac{p^n z^n}{(p;p)_n},$$

and we investigate the coefficients

$$\frac{p^n}{(p;p)_n} = \frac{q^{-n}}{(1-q^{-1})\cdots(1-q^{-n})} = \frac{(-1)^n q^{-n} q^{\sum_{k=1}^n k}}{(1-q)\cdots(1-q^n)} = \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q,q)_n},$$

which shows our claim in (A.1.5).

We thus have one of the most useful identities in the study of q-functions with focus on both $|q| \gtrless 1$: the *inversion formula*

$$(z;q)_\infty = \frac{1}{(pz;p)_\infty}, \qquad \text{with } (|q|-1)(|p|-1)<0. \tag{A.1.6}$$

At this point we need to make an important remark: even though we have defined $(z;q)_\infty$ by a power series in z — called a q-hypergeometric series — convergent for both $|q| \gtrless 1$, it is crucial to remember that it represents in fact two *different* functions, that happen to have the same q-series expansion. This is already seen by the inversion formula: it would make little sense if the symbol $(\cdot\,;\cdot)_\infty$ represented the same function for both regimes. In addition the function has no poles in one regime (inside the q-circle) while the it has countable poles in the other (outside the q-circle), which is also evident in the differing radii of convergence. The same holds for the zeros of each function.

The inversion formula can also be derived for the "finite" index symbol

$$(z;q)_\alpha = (pz;p)_{-\alpha}^{-1}, \quad p=q^{-1}. \tag{A.1.7}$$

Next we want to ask, does the q-binomial theorem (A.1.2) hold for our *newly* defined function $(z;q)_\infty$ (A.1.5)? In particular, does it hold when $|q|>1$? The answer is *yes*: We simply need to repeatedly use that $(z;q)_\infty = (pz;p)_\infty^{-1}$ for $p=q^{-1}$ and invoke the q-binomial theorem for inside the unit p-circle:

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \frac{(pz;p)_\infty}{(paz;p)_\infty} = \sum_{n=0}^{\infty} \frac{(\tilde{a};p)_n}{(p;p)_n} \tilde{z}^n \quad \text{for } |z| < \left|\frac{q}{a}\right|,$$

where $\tilde{a}=a^{-1}$ and $\tilde{z}=paz$, and we investigate the coefficient of z^n:

$$\frac{p^n a^n (\tilde{a};p)_n}{(p;p)_n} = \frac{q^{-n}a^n(a^{-1};q^{-1})_n}{(q^{-1};q^{-1})_n} = \frac{q^{-n}a^n(1-a^{-1})\cdots(1-a^{-1}q^{-n+1})}{(1-q^{-1})\cdots(1-q^{-n})} = \frac{(a;q)_n}{(q;q)_n},$$

which recovers (A.1.2) with a radius of convergence $R=\left|\frac{q}{a}\right|$.

The q-Pochhammer symbol for *complex* index

We can now define the q-Pochhammer symbol for *any* index $\alpha\in\mathbb{C}$:

$$(z;q)_\alpha \coloneqq \frac{(z;q)_\infty}{(q^\alpha z;q)_\infty}, \quad \text{with } z\in\mathbb{C},\ q\in\mathbb{C}\setminus U(1),\ \alpha\mathbb{CP}^1.$$

It is easy to verify that the ratio of these a-priori infinite products truncates in the case $n\in\mathbb{Z}$ and we obtain the q-Pochhammer symbols for positive and negative integer n:

$$(z;q)_n = \begin{cases} (1-z)(1-qz)\dots(1-q^{n-1}z) & n=1,2,\dots \\ 1 & n=0 \\ [(1-zq^{-1})(1-zq^{-2})\dots(1-zq^{n})]^{-1} & n=-1,-2,\dots. \end{cases}$$

A.1.2 Summary

We sum up the relevant definition and properties:

- We define the (infinite) q-Pochhammer symbol as

$$(z;q)_\infty \coloneqq \begin{cases} \prod_{k=0}^{\infty}(1-zq^k) & \text{if } |q|<1, \\ \prod_{k=1}^{\infty}(1-zq^{-k})^{-1} & \text{if } |q|>1. \end{cases}$$

  For $|q|<1$, the function is defined for all $z\in\mathbb{C}$ and has simple zeros at $z=q^{-k}$ for $k\in\mathbb{N}_0$. For $|q|>1$ the function has simple poles at $z=q^k$ for $k\in\mathbb{N}$ and has no non-trivial zeros (except at $z=\infty$).

- The symbol has a single q-hypergeometric series expansion for both regimes $|q|\gtrless 1$, justifying the single symbol

$$(z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q;q)_n} z^n, \text{ convergent for } \begin{cases} |z|<\infty \text{ if } |q|<1, \\ |z|<|q| \text{ if } |q|>1. \end{cases}$$

  Despite this, the symbol denotes two *different* functions for each regime $|q|\gtrless 1$, and one can "connect" the two regimes by the formula

$$(z;q)_\infty = \frac{1}{(pz;p)_\infty}, \quad p=q^{-1} \text{ for both } |q|\gtrless 1.$$

- The symbol satisfies the identity

$$\frac{(z;q)_\infty}{(q^\alpha z;q)_\infty} = (z;q)_\alpha$$

  for all $\alpha\in\mathbb{C}$ and in both chambers $|q|\gtrless 1$.

- The symbol satisfies the q-binomial theorem in both regimes, which asserts that

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \text{ valid for } \begin{cases} |z| < 1 \text{ if } |q| < 1, \\ |z| < \left|\frac{q}{a}\right| \text{ if } |q| > 1. \end{cases}$$

  This evidently also shows that we can write

$$\frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n, \text{ valid for } \begin{cases} |z| < 1 \text{ if } |q| < 1, \\ |z| < \infty \text{ if } |q| > 1. \end{cases}$$

- Maybe the most important property for our purposes is that it satisfies the q-difference equation (cf. section 6.1)

$$\left[\sigma_q - (1-z)^{-1}\right] f(z) = 0$$

  for *both* chambers $|q| \gtrless 1$.

For many more properties of the (finite or infinite symbols) we refer the reader to the rich literature: the "bible" [GR04], as well as the books by Harald Exton and Joan Slater [Ext83; Sla09]. Notable mentions in no particular order include [KC01; Ern12; Ern03; Jac10; Tho69; Car12].

A.1.3 The q-Pochhammer symbol as an exponential series

There is one more interesting property of the q-Pochhammer symbol, namely an asymptotic series. We calculate, for $|q| < 1, |z| < 1$

$$\log(z;q)_\infty = \log\left(\prod_{k=0}^{\infty}(1-zq^k)\right) = \sum_{k=0}^{\infty} \log(1-zq^k) = -\sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{z^n q^{nk}}{n} = -\sum_{n=1}^{\infty} \frac{z^n}{n(1-q^n)}.$$

where we have used the Taylor expansion $\log(1-a) = -\sum_{n=1}^{\infty} \frac{a^n}{n}$ for $|a| < 1$. This shows that

$$(z;q)_\infty = \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n(1-q^n)}\right].$$

Setting $z = q^x$ we recognize the exponent as the Lambert series

$$\mathcal{L}_q(s,x) \coloneqq \sum_{n=1}^{\infty} \frac{n^s q^{nx}}{1-q^n}, \qquad s \in \mathbb{C}, |q| < 1$$

for $s = -1$, i.e. $(q^x;q)_\infty = \exp[-\mathcal{L}_q(-1,x)]$. We can relate the Lambert series with the polylogarithm functions $\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$ for $|x| < 1$ by

$$\sum_{n=0}^{\infty} \mathrm{Li}_s(q^{n+x}) = \mathcal{L}_q(-s,x),$$

which is easily verified. In addition one can verify that

$$\mathcal{L}_q(s,x) = -\frac{\frac{\mathrm{d}}{\mathrm{d}x}}{e^{\frac{\mathrm{d}}{\mathrm{d}x}} - 1} \frac{\mathrm{Li}_{1-s}(q^x)}{\log q}. \tag{A.1.8}$$

which, combined with the above yields

$$(q^x;q)_\infty = \exp\left[\frac{\frac{\mathrm{d}}{\mathrm{d}x}}{e^{\frac{\mathrm{d}}{\mathrm{d}x}} - 1} \frac{\mathrm{Li}_2(q^x)}{\log q}\right],$$

which after setting back $z = q^x$ can be rewritten as

$$(z;q)_\infty = \exp\left[\frac{\theta_z}{q^{\theta_z} - 1} \mathrm{Li}_2(z)\right] = \exp\left[(\log q)^{-1} \frac{\theta_z \log q}{e^{\theta_z \log q} - 1} \mathrm{Li}_2(z)\right], \tag{A.1.9}$$

where $\theta_z \coloneqq z\frac{\mathrm{d}}{\mathrm{d}z}$ and we also recognize the q-shift operator $\hat{p} = q^{\theta_z}$. To verify (A.1.8) one needs to use the expansion [BW16]

$$\frac{te^{at}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(a)t^n}{n!}.$$

where $B_n(x)$, $(B_n := B_n(0))$ are the Bernoulli polynomials (numbers[§]). We can use this expansion again in (A.1.9) to obtain

$$(z;q)_\infty = \exp\left[\sum_{n=0}^{\infty} \frac{B_n(\log q)^{n-1}}{n!}\theta_z^n \operatorname{Li}_2(z)\right] = \exp\left[\sum_{n=0}^{\infty} \frac{B_n(\log q)^{n-1}}{n!} \operatorname{Li}_{2-n}(z)\right], \tag{A.1.10}$$

where we have used that $\theta_z \operatorname{Li}_s(z) = \operatorname{Li}_{s-1}(z)$. Setting now $q = e^\hbar$ with $\operatorname{Re}\hbar < 0$ (to be in the $|q| < 1$ regime) we obtain the result

$$(z;e^\hbar)_\infty = \exp\left[\frac{1}{\hbar}\sum_{n=0}^{\infty} \frac{B_n \hbar^n}{n!} \operatorname{Li}_{2-n}(z)\right], \qquad \text{for } \left|e^\hbar\right| < 1.$$

For $\operatorname{Re}\hbar > 0$ we have to calculate using our result for inside the unit q-circle

$$\begin{aligned}(z;e^\hbar)_\infty \equiv \frac{1}{(e^{-\hbar}z;e^{-\hbar})_\infty} &= \exp\left[\frac{1}{\hbar}\sum_{n=0}^{\infty} \frac{B_n(-1)^n\hbar^n}{n!} \operatorname{Li}_{2-n}(e^{-\hbar}z)\right] \\ &= \exp\left[\frac{1}{\hbar}\sum_{n,m=0}^{\infty} \frac{B_n(-\hbar)^{n+m}}{n!m!} \operatorname{Li}_{2-n-m}(z)\right], nn\end{aligned}$$

where we have used $\operatorname{Li}_n(e^\alpha z) = \sum_{m=0}^\infty \frac{\alpha^m}{m!}\operatorname{Li}_{n-m}(z)$. The double sum can be re-summed as a Cauchy product

$$\sum_{n,m=0}^{\infty} \frac{B_n(-\hbar)^{n+m}}{n!m!} \operatorname{Li}_{2-n-m}(z) = \sum_{k=0}^{\infty}(-\hbar)^k \operatorname{Li}_{2-k}(z)\sum_{\ell=0}^{k}\frac{B_\ell}{\ell!(k-\ell)!},$$

and one can check that $\sum_{\ell=0}^k \frac{B_\ell}{\ell!(k-\ell)!} = \frac{(-1)^k B_k}{k!}$. We thus have that

$$(z;e^\hbar)_\infty = \exp\left[\frac{1}{\hbar}\sum_{n=0}^{\infty} \frac{B_n \hbar^n}{n!} \operatorname{Li}_{2-n}(z)\right], \qquad \text{for both } \left|e^\hbar\right| \gtrless 1.$$

Looking towards our applications, we have also verified in the previous that the "tetrahedron block" satisfies

$$B_\Delta(x;q) := (qx^{-1};q)_\infty = \exp\left[\frac{1}{\hbar}\sum_{n=0}^{\infty} \frac{\tilde{B}_n \hbar^n}{n!} \operatorname{Li}_{2-n}(x^{-1})\right], \qquad \text{for } q = e^\hbar, |q| \gtrless 1,$$

where $\tilde{B}_n$ are the *second* Bernoulli numbers $\tilde{B}_n = (-1)^n B_n$ (which some authors call "Bernoulli numbers"), and we have used the identity $(qz;q)_\infty = (z;q^{-1})_\infty^{-1}$.

A.2 Other special q-functions

We review some of the q-functions that we are going to use throughout this work and we investigate some of their useful properties.

A.2.1 The Jacobi theta function

One of the most useful functions to define is the Jacobi theta function which we[¶] define as

$$\Theta_q(z) := (q)_\infty (z;q)_\infty \left(\frac{q}{z};q\right)_\infty, \qquad \text{with } \begin{cases} z\in\mathbb{C} & \text{if } |q|<1, \\ z\in\mathbb{C}\setminus q^{\mathbb{Z}} & \text{if } |q|>1\end{cases},$$

where we have set $(q)_\infty = \begin{cases} (q;q)_\infty & \text{if } |q|<1 \\ \left(q^{-1};q^{-1}\right)_\infty^{-1} & \text{if } |q|>1.\end{cases}$

[§] B_n are also called *first* Bernoulli numbers. Note that for some authors (e.g. for [BDP14]), the symbol B_n is reserved for the *second* Bernoulli numbers, here denoted by $\tilde{B}_n := B_n(1) \equiv (-1)^n B_n$.

[¶] One has to be careful when comparing expressions in the literature. The definition of the Jacobi theta function often varies. We use the definition from [Roq08].

It satisfies the Jacobi triple product identity [GR04]

$$\Theta_q(z) = \begin{cases} \sum\limits_{n\in\mathbb{Z}} (-1)^n q^{\binom{n}{2}} z^n & \text{if } |q|<1, \\ \left(\sum\limits_{n\in\mathbb{Z}} (-1)^n q^{-\binom{n+1}{2}} z^n \right)^{-1} & \text{if } |q|>1 \end{cases}$$

and more importantly, it satisfies the q-difference equation

$$\sigma_q \Theta_q(z) = -\frac{1}{z}\Theta_q(z)\,, \tag{A.2.11}$$

and more generally

$$\sigma_q^n \Theta_q(z) = (-z)^{-n} q^{-\binom{n}{2}} \Theta_q(z)\,.$$

More properties are easy to verify

$$\Theta_p(z) = \Theta_q(qz)^{-1} \equiv -z\Theta_q(z)^{-1}\,, \text{ where } p = q^{-1} \tag{A.2.12a}$$

$$\Theta_q\left(z^{-1}\right) = \Theta_q(qz) \equiv -\frac{1}{z}\Theta_q(z) \tag{A.2.12b}$$

thus also $\Theta_q\left(z^{-1}\right) = \Theta_p(z)^{-1}$. We also have
Lemma ([Mor11])**.** *For $x \in \mathbb{C}\setminus(-\infty, 0]$ and a constant $K\in\mathbb{C}$ we have*

$$\Theta_q\left(-q^{1/2}\right)\Theta_q\left(-\frac{K}{x}\right) = \Theta_{q^{1/2}}\left(\sqrt{\frac{K}{x}}\right)\Theta_{q^{1/2}}\left(-\sqrt{\frac{K}{x}}\right).$$

Modular properties

We want to compute the monodromy of $\Theta_q(z)$, i.e. see how it behaves when z is transported around a loop around 0. This means we must write it as a function of ζ, τ where $z = e^{2\pi i\zeta}$ and $q = e^{\pi i\tau}$ and transform $\zeta \mapsto \zeta + 1$. We can write this function in terms of

$$\vartheta(\zeta,\tau) := \sum_{n\in\mathbb{Z}} \exp(\pi i\tau n^2 + 2\pi i\zeta) = \sum_{n\in\mathbb{Z}} q^{n^2} z^n,$$

where we have set $q = e^{\pi i\tau}$ and $z = e^{2\pi i\zeta}$. We see immediately that we have

$$\vartheta(\zeta,\tau) = \Theta_{q^2}(-qz) \quad \text{and thus also} \quad \Theta_q(z) = \vartheta(\zeta - \frac{\tau}{4} + \frac{1}{2}, \frac{\tau}{2}).$$

The known modular properties of ϑ are $\vartheta(\zeta+\alpha+\beta\tau) = \exp(-\pi i\beta^2\tau - 2\pi i\beta\zeta)\vartheta(\zeta,\tau)$. In particular, $\vartheta(\zeta+1,\tau) = \vartheta(\zeta,\tau)$, therefore also

$$\Theta_q\left(ze^{2\pi i}\right) = \vartheta(\zeta + 1 - \frac{\tau}{4} + \frac{1}{2}, \frac{\tau}{2}) = \vartheta(\zeta - \frac{\tau}{4} + \frac{1}{2}, \frac{\tau}{2}) \equiv \Theta_q(z)\,,$$

where we have sloppily written $e^{2\pi i}z$ to indicate the shift $\zeta+1$ in the covering space. Thus the monodromy properties of $\Theta_q(z)$ are *trivial*. The full transformation of ϑ recovers the behavior of $\Theta_q(z)$ as in the difference equation (A.2.11)

A.2.2 The q-characters

In this subsection we explore some of the peculiar properties of the q-characters $e_a(z;q) \equiv e_a(z)$ introduced to replace the "traditional" characters z^α in 6.1.4. Recall we defined

$$e_a(z) := \frac{\Theta_q(z)}{\Theta_q(az)} = \frac{(z;q)_\infty \left(\frac{q}{z};q\right)_\infty}{(az;q)_\infty \left(\frac{q}{az};q\right)_\infty}, \quad z\in\mathbb{C}, a\in\mathbb{C}^*, |q| \gtrless 1.$$

Setting $q^\alpha = a$, this can also be neatly written as

$$e_{q^\alpha}(z) = (z;q)_\alpha \left(\frac{q}{z};q\right)_{-\alpha}.$$

It satisfies the q-difference equation

$$\sigma_q e_a(z) = a e_a(z).$$

So why do we choose these functions? The main advantage of using these functions as q-characters instead of the monomials z^α is of course the fact that they are globally defined (up to the poles on $q^{\mathbb{Z}}$): the monomial z^α is only well defined up to a branch cut. Furthermore, the theta functions are used to describe the complete field of elliptic functions [HSS16, Remark 3.3.3]. Lastly, a circumstantial advantage is that these functions "popped up" in calculations in [Hah49; GR04; Mor11; Mor13] replacing the role of monomials in an almost natural fashion.

This choice is however not "a free meal": the functions $e_a(z)$ have *bad* multiplicative properties, meaning that $e_a(z)\, e_b(z) \neq e_{ab}(z)$. Furthermore, it is far from canonical: there are many choices of functions that satisfy the same q-difference equation, e.g. take $\frac{\Theta_q(az^{-1})}{\Theta_q(z^{-1})}$ or the function one obtains by rescaling z by any non-zero complex number. However, there are still some properties of z^α which extend to this function, and we would like to list some of them.

Consider $f(z,\alpha) := z^\alpha$ as a function of both arguments. It clearly satisfies

$$f(z,-\alpha) \overset{(i)}{=} f(z,\alpha)^{-1} \overset{(ii)}{=} f\left(z^{-1},\alpha\right) \overset{(iii)}{=} f(z,-\alpha)$$

We investigate what the corresponding relations are in the case of $g(z,\alpha) := e_{q^\alpha}(z)$. We find after some easy manipulations involving the properties (A.2.12a),(A.2.12b) that $g(z,-\alpha) = g(qz^{-1},\alpha)$ which implies that

$$g(z^{-1},\alpha) = g(qz,-\alpha) = q^{-\alpha} g(z,-\alpha) \qquad \text{[replaces (iii)]}.$$

For the total inverse we find using (A.1.7)

$$\begin{aligned} g(z,\alpha)^{-1} &= (z;q)_\alpha^{-1} \left(qz^{-1};q\right)_{-\alpha}^{-1} = (zq^\alpha;q)_{-\alpha} \left(qz^{-1}q^{-\alpha};q\right)_\alpha \\ &= g(zq^\alpha,-\alpha) && \text{[replaces (i)]} \\ &= g(qz^{-1}q^{-\alpha},\alpha) && \text{[replaces (ii)]} \end{aligned}$$

We can rewrite these equations as

$$e_a(z)^{-1} = e_{a^{-1}}(az) \tag{A.2.13a}$$
$$e_a(z)^{-1} = e_a\left(qa^{-1}z^{-1}\right) \tag{A.2.13b}$$
$$e_a\left(z^{-1}\right) = a^{-1} e_{a^{-1}}(z). \tag{A.2.13c}$$

A.2.3 The q-gamma function

The q-gamma function $\Gamma_q(\zeta)$ is the q-generalization of the classical Gamma function $\Gamma(z)$ satisfying

$$\Gamma(z+1) = z\Gamma(z),$$

with $\Gamma(1) = 1$. It has simple poles at $z = \mathbb{Z}_{\leq 0}$ with residue $\operatorname{Res}(\Gamma,-k) = \frac{(-1)^k}{k!}$ and it also satisfies the useful identity

$$\frac{1}{\Gamma(z,1-z)} = \frac{\pi}{\sin \pi z}.$$

Similarly, the q-gamma function satisfies the q-difference equation

$$\Gamma_q(\zeta+1) = \frac{1-q^\zeta}{1-q}\Gamma_q(\zeta),$$

with $\Gamma_q(1) = 1$. It is defined as follows

$$\Gamma_q(\zeta) = \begin{cases} \frac{(q;q)_\infty}{(q^\zeta;q)_\infty}(1-q)^{1-\zeta} \equiv \frac{(q;q)_{\zeta-1}}{(1-q)^{\zeta-1}} & \text{if } |q|<1, \\ q^{\binom{\zeta}{2}} \frac{\left(q^{-1};q^{-1}\right)_\infty}{\left(q^{-\zeta};q^{-1}\right)_\infty}(q-1)^{1-\zeta} \equiv q^{\binom{\zeta}{2}} \left(q^{-1};q^{-1}\right)_{\zeta-1}(q-1)^{1-\zeta} & \text{if } |q|>1. \end{cases}$$

An interesting feature is the analogue of the interesting relation of $\Gamma(z)$. We have

$$\Gamma_q(\zeta,1-\zeta) = \begin{cases} (1-q)\,(q;q)_\infty^3 \frac{1}{\Theta_q(z)} & \text{if } |q|<1 \\ \frac{q^{\zeta(\zeta-1)}\left(q^{-1};q^{-1}\right)_\infty^3}{q-1}\Theta_q(z) & \text{if } |q|>1, \end{cases}$$

where we have picked a logarithm $q^\zeta = z$. It also has simple poles at $\zeta \in \mathbb{Z}_{\leq 0}$ with residues

$$\operatorname{Res}(\Gamma_q,-k) = -\frac{(1-q)^{k+1}}{\log q\,(q^{-k};q)_k}.$$

A.3 q-generalizations

It is obvious that the quest for a *canonical* definition of a q-generalization of some object, e.g. a function $f(z)$ is pointless. There are an infinite number of ways one can q-generalize such a function: in principle, any function $g(z,q)$ with $\lim_{q\to 1} g(z,q) = f(z)$ will do. We try to impose further restrictions on this deformation, mainly motivated by empirical examples. This is by no means a rigorous list of requirements; a q-generalization may follow all or some of these rules.

Rule: the q-generalization $f_q(z)$ of a function $f(z)$ that satisfies a differential equation $\mathcal{L}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right) f = 0$ must satisfy

$$\mathcal{L}\left(\mathcal{D}_q\right)\phi_q = 0.$$

This restriction is of course relevant up to multiplication by elliptic factors, as discussed in section 6.1. Note also that the substitution $\frac{\mathrm{d}}{\mathrm{d}z} \mapsto \mathcal{D}_q$ is degenerate: the coefficients of the differential equation may be deformed by arbitrary functions of q with limit equal to 1.

Rule: the q-generalization $f_q(z)$ of an analytic function $f(z) = \sum_{n\geq 0} f_n z^n$ is given by a function

$$f_q(z) = \sum_{n=0}^{\infty} f_n(q) z^n,$$

such that the series converges and $\lim_{q\to 1} f_n(q) = f_n$ up to a redefinition of z. (e.g. $\lim_{q\to 1} f_q\big(g(q)z\big) = f(z)$). Furthermore one might require that the series converges for *both regimes* $|q| \gtrless 1$.

Rule: The q-generalization of a function that solves a q-difference equation, has the same form for $|q| \gtrless 1$ either for generic or special (integer) values of the arguments.

By same form, we mean something stronger than just "the same symbol", e.g. same convergent power series in z for both chambers. An example is of course the q-Pochhammer symbol $(z;q)_\infty$ and more generally $(z;q)_\alpha$ both of which satisfy the same q-difference equation for $|q| \gtrless 1$. In addition, in *both* regimes, choosing α in $\mathbb{Z}$ reduces the infinite products to finite ones.

An example of a q-function that does not have the same form in the two chambers is the q-gamma function. However, one can certainly motivate a connection between the two forms: picking $\zeta = n \in \mathbb{Z}$ we have for $|q| > 1$:

$$\Gamma_q(n) = q^{\binom{n}{2}} \frac{(p;p)_\infty}{(p^n;p)_\infty} (q-1)^{1-n} = q^{\binom{n}{2}} (p;p)_{n-1} (q-1)^{1-n},$$

where $p = q^{-1}$. But for integer n we have that $q^{\binom{n}{2}} (p;p)_{n-1} = (-1)^{n-1} (q;q)_{n-1}$ and thus

$$\Gamma_q(n) = (q;q)_{n-1}(1-q)^{1-n},$$

which is exactly the same form as for $|q| < 1$.

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Statement of authorship

I hereby declare that I am the sole author of this master thesis and that I have not used any sources other than those listed in the bibliography and identified as references.

Munich, September 11, 2017 ....