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Master's thesis

The moduli space of higher rank Higgs bundles

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0 Introduction

For a complex vector bundle E on a compact Riemann surface M, the theorem of Narasimhan and Seshadri [NS64] gives a correspondence of stable holomorphic structures to irreducible connections with constant central curvature or, equivalently, unitary representations of the fundamental group. A generalization of constant central curvature connections, which can e.g. be used to establish a similar correspondence for representations in $\mathrm{GL}(n, \mathbb{C})$, is the notion of a *Higgs bundle*, introduced by Hitchin [Hit87b]. A Higgs bundle is a solution (∇, Φ) of the equations

$$\begin{aligned} F^\nabla + [\Phi \wedge \Phi^*] &= c \,\mathrm{id}_E \otimes \omega, \\ \mathrm{d}^\nabla \Phi &= 0, \end{aligned} \tag{1}$$

where ∇ is a unitary connection on E, $\Phi \in \Omega^{1,0}(\mathrm{End}\, E)$ an $\mathrm{End}\, E$–valued $(1,0)$–form and ω the volume form on M. Because M is a surface, the second equation is equivalent to $\overline{\partial}^\nabla \Phi = 0$, so it expresses that Φ is a holomorphic form.

These equations are invariant under unitary vector bundle automorphisms, also called *gauge transformations*, and the equivalence classes of solutions form a *moduli space* $\mathscr{M}$. When we restrict our study to the moduli space $\mathscr{M}^*$ of irreducible solutions, that is solutions which cannot be decomposed into a product of solutions on lower–rank subbundles, it becomes a finite–dimensional smooth manifold. Somewhat surprisingly, this moduli space has an interesting geometric structure, and is therefore often studied in its own right (e.g. [Hit87a], [Sch13], [MSWW14]). Most notably, it admits a *Hyperkähler* structure, consisting of a Riemannian metric and three compatible complex structures satisfying the quaternion relations. Each of these complex structures defines, together with the metric, an independent symplectic structure on $\mathscr{M}^*$.

While defining this Hyperkähler structure (for the case $\mathrm{rk}\, E = 2$), Hitchin [Hit87b] already noted that it can be formally interpreted as arising from a Hyperkähler structure on the configuration space by a certain quotient construction, called the *Hyperkähler quotient*. This is an analogue of the Marsden–Weinstein quotient for symplectic manifolds. Consider a free group action on a Hyperkähler manifold, which is Hamiltonian with respect to all three symplectic structures. Then the combined zero set of the associated moment maps is invariant, and its quotient by the group action is again a smooth Hyperkähler manifold.

A goal of this thesis is to make this Hyperkähler quotient construction of the moduli space precise. We show that, under some technical conditions, the Hyperkähler quotient generalizes to proper actions of infinite–dimensional Lie groups on infinite–dimensional Hyperkähler manifolds, both modeled on Hilbert spaces. Then we prove that the configuration space of Hitchin's equations, completed to a suitable Sobolev space, is in fact such a Hyperkähler manifold and that the unitary gauge transformations form a proper Hamiltonian action of a Hilbert Lie group. The moment maps are then exactly the equations (1). Furthermore, we compute the stabilizer groups of this action and show that the irreducible solutions are exactly the solutions where the stabilizer is minimal. After quotienting out this subgroup, we get a free action and thus a smooth Hyperkähler quotient.

Of course we are finally interested in smooth solutions, not only Sobolev–regular ones. By using a suitable gauge fixing condition, Hitchin's equations become a quasilinear elliptic system. With this property, we show that every gauge orbit of solutions contains a smooth solution.

One way to better understand the moduli space $\mathscr{M}$ is by studying the map

$$\mathscr{M} \to \bigoplus_{i=1}^{m} H^0(M, K^{\otimes i}), \quad [\nabla, \Phi] \mapsto (s_1(\Phi), \dots, s_m(\Phi)),$$

where $H^0(M, K^{\otimes i})$ is the space of holomorphic sections of powers of the canonical line bundle and s_i are the coefficients of the characteristic polynomial. This map is called the *Hitchin fibration.* One of the interesting features about it is that (when restricted to $\mathscr{M}^*$) it is a *completely integrable system* with respect to a complex symplectic structure on $\mathscr{M}^*$. Furthermore, it is a proper map, so its fibers are compact. The combination of these two facts implies that every regular fiber containing only irreducible solutions is biholomorphic to a complex torus. This analysis was already carried out for the rank 2 case in Hitchin's original paper [Hit87b] and later extended to higher ranks (see e.g. [Wen14]). However, we will fill in some analytical details here, in particular giving a slightly more explicit description of the Hamiltonian vector fields than [Hit87b].

The first section reviews some basic definitions and fixes notation. It is concerned with complex manifolds, especially Riemann surfaces, complex vector bundles and their classification on compact Riemann surfaces as well as different ways to describe holomorphic structures on such bundles. Finally, we will state Hitchin's equations and the class of gauge transformations which leave it invariant.

The second section then discusses smooth group actions of infinite–dimensional Lie groups on infinite–dimensional manifolds, both modeled on Hilbert spaces. The main part is a proof of the slice theorem. As a consequence of this, when restricted to an open subset of points with minimal stabilizer, the group action yields a smooth quotient. Building on this, rest of the section exhibits the construction of the Hyperkähler quotient in the infinite–dimensional setting.

These results are applied to the solutions of Hitchin's equations in the third section. First we complete the configuration space and the gauge group with respect to a Sobolev norm. In a next step, we describe the Hyperkähler structure on the configuration space and show that the gauge group action is Hamiltonian with respect to all three symplectic forms, with the moment maps given by Hitchin's equations. We then show that the action is proper and compute its stabilizers. Next, we show that infinitesimally, Hitchin's equations and the gauge transformations form an elliptic complex. In particular, this allows to compute the dimension of the moduli space using the Atiyah–Singer index theorem. Finally, we discuss gauge fixing and the regularity theory of Hitchin's equations.

The fourth and last section is concerned with the Hitchin fibration. We first discuss symmetric polynomials in general and how to apply them to endomorphism–valued forms to define the map. Then we prove that it is proper, using Uhlenbeck's weak compactness theorem and the regularity theory obtained above. Finally, we prove that regular fibers have a basis consisting of certain Hamiltonian vector fields, making the Hitchin fibration a completely integrable system over its regular values.

1 Preliminaries

In this chapter we will review some basic definitions, particularly concerning complex and holomorphic vector bundles, and fix notation. Then we will introduce Hitchin's equations, the solutions of which will be the main object of interest throughout this thesis.

1.1 Complex manifolds

Let M be a smooth manifold of even dimension, so that it is locally diffeomorphic to $\mathbb{R}^{2n} \cong \mathbb{C}^n$. A complex structure on M is given by a maximal subatlas whose coordinate changes are holomorphic. This allows to define holomorphic functions between two complex manifolds. The tangent spaces T_xM become $\mathbb{C}$–vector spaces by requiring that the differentials of the holomorphic charts be $\mathbb{C}$–linear. This is independent of the chart since holomorphic coordinate changes have $\mathbb{C}$–linear differentials. Scalar multiplication by $i \in \mathbb{C}$ on the tangent spaces yields a bundle automorphism $j \colon TM \to TM$ which is anti–involutive, i.e. $j^2 = -\,\mathrm{id}$. A smooth map $f \colon M \to N$ between complex manifolds is holomorphic if and only if its differential intertwines these complex structures, $Df \circ j_M = j_N \circ Df$, at any point. For $M = \mathbb{C}^m$ and $N = \mathbb{C}^n$ this identity is just the Cauchy–Riemann equations.

This provides a description of complex manifolds in more differential geometric terms. Let M again be a smooth manifold. An *almost complex structure* on M is an anti–involutive bundle homomorphism $j \colon TM \to TM$. If the Nijenhuis tensor

$$N_j(X,Y) = -j^2[X,Y] + j[jX,Y] + j[X,jY] - [jX,jY]$$

vanishes, the almost complex structure is called *integrable* and by the Newlander–Nirenberg theorem [NN57] there is a holomorphic subatlas so that j is given by multiplication by i as above, so M is a complex manifold. Conversely, the almost complex structure induced by a complex manifold is always integrable. When we speak of complex structures in this thesis, we will usually mean an integrable almost complex structure.

1.2 Riemann surfaces

We now consider the special case that M has real dimension two. Let j be an almost complex structure on M. Then there is always a Riemannian metric g on M which is compatible to j in the sense that $g(jv,jw) = g(v,w)$ for all $v,w \in T_xM$. Locally one can find such a metric using a non–vanishing local vector field $u \in \Gamma(TM|_U)$. Then u, ju cannot be $\mathbb{R}$–linearly dependent as $ju = \lambda u$ for $\lambda \in \mathbb{R}$ contradicts $j^2u = -u$. So we can define g on the basis $u, ju \in T_xM$ by $g(u,u) = g(ju,ju) = 1$ and $g(u,ju) = g(ju,u) = 0$. The existence of a global metric follows using partitions of unity. It is unique up to multiplication by a smooth positive function: If g_1 and g_2 are two j–compatible Riemannian metrics and $0 \neq u \in T_xM$ then

$$g_i(ju,u) = g_i(u,ju) = g_i(ju,j^2u) = -g_i(ju,u) = 0 \quad \text{and} \quad g_i(ju,ju) = g_i(u,u)$$

for $i = 1,2$, so we can check on the basis u, ju that

$$g_2(v,w) = \frac{g_2(u,u)}{g_1(u,u)} g_1(v,w) \quad \forall v,w \in T_xM.$$

The 2–form $\omega(v,w) = g(jv,w)$ is trivially closed and non–degenerate since j is an isomorphism, so it is a symplectic form on M. It is also a Riemannian volume form for the metric g which endows M with a canonical orientation. If $e \in T_xM$ is a unit vector, then (e, je) is a positively oriented orthonormal basis. The dual basis of T_x^*M is $(e^\flat, -je^\flat)$, so the Hodge star operator $\star^g\colon \Omega^1 \to \Omega^1$ of any j–compatible metric g equals $-j$.

Extending e to a local vector field we can easily check that $N_j(e, je) = 0$. Since N_j is C^∞–linear and antisymmetric this implies $N_j = 0$, so every almost complex structure on a surface is integrable.

1.3 Complex vector bundles

Let M be a complex manifold with complex structure $j\colon TM \to TM$ and $E \to M$ a complex vector bundle. To simplify notation when dealing with vector bundle–valued forms, we will often write Λ^k for the bundle $\Lambda^k T^*M$ and $\Lambda^k \otimes E$ for $\Lambda^k T^*M \otimes E$. The complex structure j on M induces a complex linear anti–involution $j\colon \Lambda^1 \otimes E \to \Lambda^1 \otimes E$ by mapping $\alpha \otimes e$ to $(\alpha \circ j) \otimes e$. It is easy to see that $\Lambda^1 \otimes E = T^*M \otimes_{\mathbb{R}} E$ then splits into two complex subbundles called $\Lambda^{1,0} \otimes E$ and $\Lambda^{0,1} \otimes E$, both of equal dimension, on which j acts by multiplication with i and $-i$, respectively. We call the sections in $\Omega^{1,0}(E) := \Gamma(\Lambda^{1,0} \otimes E)$ and $\Omega^{0,1}(E) := \Gamma(\Lambda^{0,1} \otimes E)$ the *E–valued $(1,0)$–forms* and *E–valued $(0,1)$–forms* on M and denote the projections onto these spaces by $\mathrm{pr}^{1,0}$ and $\mathrm{pr}^{0,1}$. If we view the cotangent spaces of M as complex vector spaces with the imaginary unit given by the endomorphism $j\colon T^*M \to T^*M$, then $\Lambda^{1,0} \otimes E$ is the complex tensor product of T^*M and E while $\Lambda^1 \otimes E$ is the real tensor product of the same bundles endowed with the complex structure of E. In particular, if $E = \underline{\mathbb{C}}$ is the trivial complex line bundle, then $\Lambda^{1,0} \otimes \underline{\mathbb{C}}$ is just T^*M with the fiberwise complex structure given by j. It is called the *holomorphic cotangent bundle* or, if M is a surface, the *canonical bundle* K of M. When we write $\Lambda^{1,0}$ or $\Lambda^{0,1}$ on their own, we mean the the complex bundles $\Lambda^{1,0} \otimes \underline{\mathbb{C}}$ and $\Lambda^{0,1} \otimes \underline{\mathbb{C}}$, respectively. This is compatible with the notation $\Lambda^{1,0} \otimes E$ and $\Lambda^{0,1} \otimes E$ for the eigenspaces of $\Lambda^1 \otimes_{\mathbb{R}} E$.

Starting with complex vector bundles $E \to M$ and $F \to M$, their direct sum $E \oplus F$, tensor product $E \otimes F$, dual E^* and homomorphism bundle $\mathrm{Hom}(E,F)$ are also complex vector bundles. If E and F carry a Hermitian bundle metric (which by convention will always be conjugate linear in the first argument), there is also an induced metric on $E \oplus F$, $E \otimes F$, E^* and $\mathrm{Hom}(E,F)$. A special case which will occur frequently is the endomorphism bundle $\mathrm{End}\ E = \mathrm{Hom}(E,E)$. The Hermitian metric defines a conjugation on this bundle which assigns to every $\varphi \in \mathrm{End}\ E$ an element φ^* in the same fiber which satisfies $\langle v, \varphi^* w\rangle = \langle \varphi v, w\rangle$ for $v, w \in E_{\pi(\varphi)}$. The important subset

$$\mathfrak{u}\, E = \{\varphi \in \mathrm{End}\ E \mid \varphi^* = -\varphi\}$$

of *skew–Hermitian endomorphisms* is a real subbundle of $\mathrm{End}\ E$ and the restriction of the Hermitian metric forms a real bundle metric on $\mathfrak{u}\, E$. The subsets $\mathrm{GL}(E)$ of invertible endomorphisms and $\mathrm{U}\, E$ of unitary endomorphisms are also fiber subbundles, but not vector bundles.

There are some important (multi–)linear bundle maps on $\mathrm{End}\ E$, namely the commutator $[\varphi, \psi] = \varphi\psi - \psi\varphi$ and the trace $\mathrm{tr}\colon \mathrm{End}\ E \to \underline{\mathbb{C}}$ mapping into the trivial line bundle $\underline{\mathbb{C}}$, which is defined using the identification $\mathrm{End}\ E = E^* \otimes E$ by $\mathrm{tr}(\alpha \otimes v) = \alpha(v)$. The induced bundle metric on $\mathrm{End}\ E$ satisfies $\langle \varphi, \psi\rangle = \mathrm{tr}(\varphi^*\psi)$ for all $\varphi, \psi \in \mathrm{End}\ E$. For $\psi, \varphi \in \mathrm{End}\ E$,

$\alpha, \beta \in \Lambda^1$, $\xi, \zeta \in \Lambda^2$ and $\gamma \in \Lambda^k$ all in a fiber over the same point $x \in M$, we further define the following maps, which extend to multilinear bundle maps on various tensor product bundles:

$$\begin{aligned}
(\gamma \otimes \psi)^* &= \gamma \otimes \psi^* \\
[(\alpha \otimes \psi), \varphi] &= \alpha \otimes [\psi, \varphi] \\
[\psi, (\alpha \otimes \varphi)] &= \alpha \otimes [\psi, \varphi] \\
[(\alpha \otimes \psi) \wedge (\beta \otimes \varphi)] &= (\alpha \wedge \beta) \otimes [\psi, \varphi] \\
\langle [(\alpha \otimes \psi), (\beta \otimes \varphi)] \rangle &= \langle \alpha, \beta \rangle [\psi, \varphi] \\
[(\alpha \otimes \psi) \lrcorner (\xi \otimes \varphi)] &= (\alpha^\sharp \lrcorner \xi) \otimes [\psi, \varphi] \\
\star(\gamma \otimes \psi) &= (\star \gamma) \otimes \psi.
\end{aligned}$$

Here $\star$ is the Hodge star and $\lrcorner$ is the inner product $v \lrcorner \xi = \xi(v, -)$. While $[-,-]$ and $\langle [-,-] \rangle$ are antisymmetric, $[- \wedge -]$ is symmetric due to the antisymmetry of both parts.

Let $\nabla \colon \Gamma(E) \to \Omega^1(E) = \Gamma(T^*M \otimes E)$ be a unitary connection on E, that is it satisfies

$$X\langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle.$$

Then its curvature $F^\nabla \in \Omega^2(\mathrm{End}\, E)$ is skew–Hermitian, i.e. $F^\nabla \in \Omega^2(\mathfrak{u}\, E)$. If ∇_1, ∇_2 are both unitary connections on E, their difference $\alpha = \nabla_1 - \nabla_2$ is a skew–Hermitian endomorphism–valued 1–form, $\alpha \in \Omega^1(\mathfrak{u}\, E)$. Let $\mathscr{A}(E)$ be the space of all unitary connections on the complex bundle E. When $\nabla^E \in \mathscr{A}(E)$ and $\nabla^F \in \mathscr{A}(F)$ are given, there is a unique induced unitary connection $\nabla^{E \otimes F} \in \mathscr{A}(E \otimes F)$ which satisfies

$$\nabla_X^{E \otimes F}(v \otimes w) = \nabla_X^E v \otimes w + v \otimes \nabla_X^E w \qquad \forall v \in \Gamma(E), w \in \Gamma(F).$$

This also works if one of E, F is a real bundle with a symmetric connection. In particular, using the Levi–Civita connection on T^*M, we get induced connections on $T^*M^{\otimes k} \otimes E$. There is also an induced differential operator $\mathrm{d}^\nabla \colon \Omega^k(E) \to \Omega^{k+1}(E)$, the *covariant exterior derivative*, which is uniquely defined by

$$\mathrm{d}^\nabla(\alpha \otimes v) = \mathrm{d}\alpha \otimes v + (-1)^k \alpha \wedge \nabla v \qquad \forall \alpha \in \Omega^k, v \in \Gamma(E).$$

The covariant exterior derivative can be viewed as an anti–symmetrization of $\nabla^{T^*M^{\otimes k} \otimes E}$ and generally contains fewer partial derivatives.

1.4 Holomorphic vector bundles

To define holomorphic sections, we additionally need a holomorphic structure on E. As with complex manifolds, there are again different ways to describe this structure. First, we can equip the total space E with a complex structure J and choose a compatible system of trivializations $E|_U \cong U \times \mathbb{C}^m$ which are holomorphic with respect to J on the left side and the complex structures on $U \subset M$ and $\mathbb{C}^m$ on the right side. The projection $E \to M$ is then automatically holomorphic and holomorphic sections of E are smooth sections which are holomorphic with respect to J and j. Alternatively to specifying J explicitly, we could also restrict to a trivialization with holomorphic coordinate changes, which then defines a complex structure on E.

Now let M be a Riemann surface. A *Dolbeault operator* is a $\mathbb{C}$–linear operator $\overline{\partial}\colon \Gamma(E) \to \Omega^{0,1}(E)$ which satisfies the product rule $\overline{\partial}(fs) = \mathrm{pr}^{0,1}\, \mathrm{d}f \otimes s + f\overline{\partial}s$ for every section $s \in \Gamma(E)$ and complex function f on M. For a given holomorphic structure on E there is a unique Dolbeault operator $\overline{\partial}$ such that $\overline{\partial}s|_U = 0$ whenever a section $s \in \Gamma(E)$ is holomorphic on an open set $U \subset M$.

Let us briefly proof its existence and uniqueness. If two such operators exist, their difference is an End E–valued (0,1)–form which vanishes on all vectors $v \in E_x$ for which there exists a local holomorphic section $s\colon U \to E|_U$ such that $s(x) = v$. But these local sections always exist, so the two Dolbeault operators are equal. To show that $\overline{\partial}$ exists locally, let $s\colon U \to E|_U$ now be a smooth section on an open set $U \subset M$ where $E|_U$ is trivial. A holomorphic trivialization induces a bundle map $c\colon TE|_{E|_U} \to TM|_U \oplus E|_U$ covering $\pi\colon E|_U \to U$ which is an isomorphism on the fibers and holomorphic in the sense that $c \circ J = (j \oplus i) \circ c$. The composition $\mathrm{d}s = p_2 \circ c \circ Ds\colon TM|_U \to E|_U$ thus satisfies $\mathrm{d}s(jX) = i\,\mathrm{d}s(X)$ if and only if s is holomorphic. If we regard $\mathrm{d}s$ as an E–valued 1–form, its (0,1)–part therefore vanishes if and only if s is holomorphic. So we can set $\overline{\partial}s = \mathrm{pr}^{0,1}(\mathrm{d}s)$ on U. In particular the Dolbeault operator exists locally. By uniqueness these local realizations coincide on their overlaps and can be combined to a global Dolbeault operator.

Conversely, for every Dolbeault operator $\overline{\partial}$ on a complex bundle $E \to M$ there is a unique holomorphic structure such that the kernel of $\overline{\partial}$ are exactly the holomorphic sections [DK90, Section 2.2]. We can therefore specify holomorphic structures by giving their associated Dolbeault operators. This equivalence still holds for higher–dimensional M but one has to require an additional integrability condition for Dolbeault operators.

If we fix a Hermitian bundle metric $\langle -, - \rangle$ on E, there is a third way to describe holomorphic structures. For every Dolbeault operator $\overline{\partial}$ there is a unique unitary connection $\nabla\colon \Gamma(E) \to \Omega^1(E)$ such that $\overline{\partial} = \mathrm{pr}^{0,1} \circ \nabla$. [Kob14, Proposition I.4.9]. Conversely, the (0,1)–part of a unitary connection is a Dolbeault operator. So holomorphic structures on a Hermitian vector bundle are in a one–to–one correspondence to unitary connections.

1.5 Tensor products and holomorphic forms

Let $E \to M$ and $F \to M$ be holomorphic vector bundles on a Riemann surface M. Then $E \otimes F \to M$ carries a natural induced holomorphic structure which is described by restricting to local trivializations constructed from a pair of holomorphic trivializations of E and F. If $\{f_i\}_{i=1,\dots,r}$ is a local holomorphic frame of F over some open set $U \in M$, then $\sum_i e_i \otimes f_i \in \Gamma(E \otimes F|_U)$ is holomorphic in this structure if and only if all $e_i \in \Gamma(E)$ are holomorphic sections of E.

If $\overline{\partial}^E$ and $\overline{\partial}^F$ are Dolbeault operators describing the holomorphic structures on E and F respectively, then the Dolbeault operator $\overline{\partial}^{E \otimes F}$ corresponding to the induced holomorphic structure on the tensor product is given by

$$\overline{\partial}^{E \otimes F}(e \otimes f) = \overline{\partial}^E e \otimes f + e \otimes \overline{\partial}^F f \qquad \forall e \in \Gamma(E), f \in \Gamma(F).$$

This is clear by the above description using a holomorphic frame of F. Consequently, when identifying unitary connections with holomorphic structures by projecting to the (0,1)–part, the induced holomorphic structure on $E \otimes F$ corresponds to the induced unitary connection on $E \otimes F$.

The canonical bundle $K = \Lambda^{1,0}$ unsurprisingly also carries a natural holomorphic structure with the holomorphic trivializations given by the differential of a holomorphic coordinate chart for M. If $\mathrm{d}z$ is a local holomorphic section of K arising as differential of a holomorphic coordinate. Then $f\mathrm{d}z \in \Gamma(K)$ is holomorphic if and only if $\overline{\partial} f = 0$. But this is equivalent to $\mathrm{d}f \in \Omega^{1,0}\underline{\mathbb{C}}$ which in turn hods if and only if $\mathrm{d}(f\mathrm{d}z) = 0$. So a complex valued $(1,0)$–form is holomorphic if and only if it is closed. Given a Hermitian bundle E with unitary connection ∇, the tensor product bundle $\Lambda^{1,0} \otimes E$ therefore has a holomorphic structure induced by ∇ and the standard structure on $\Lambda^{1,0}$. Then the holomorphic sections of it are exactly the kernel of the exterior covariant derivative $\mathrm{d}^\nabla \colon \Omega^{1,0}(E) \to \Omega^2(E)$.

1.6 Degree of a vector bundle

Let $E \to M$ be a complex vector bundle on a compact surface M and ∇ an arbitrary connection on E with curvature $F^\nabla \in \Omega^2(\mathrm{End}\, E)$. Its trace $\operatorname{tr} F^\nabla$ is then a complex–valued 2–form on M, which can be integrated to get the *degree*

$$\deg(E) = \frac{i}{2\pi} \int \operatorname{tr} F^\nabla. \tag{2}$$

The degree is an integer which does not depend on the choice of connection ∇ [Bau14, pp. 224-225; Bau14, Satz 6.1]. The degree of direct sums, tensor products and dual bundles satisfies the identities

$$\begin{aligned}
\deg(E \oplus F) &= \deg(E) + \deg(F) \\
\deg(E \otimes F) &= \operatorname{rk}(F)\deg(E) + \operatorname{rk}(E)\deg(F) \\
\deg(E^*) &= -\deg(E)
\end{aligned}$$

for all complex vector bundles E, F. This can be seen by taking connections on E and F and calculating the curvature of the induced connections on $E \oplus F$, $E \otimes F$ and E^*. If a bundle admits a flat connection, in particular if it is trivial, it has degree zero. The converse is also true. In fact, if two complex vector bundles E and F on a compact surface M have the same rank and degree, they are isomorphic by the following proposition.

Proposition 1.1. *Every complex vector bundle $E \to M$ on a compact surface M decomposes as a direct sum $E = T \oplus L$ of a trivial bundle T and a complex line bundle L.*

Proof. If E has a nowhere vanishing section s, the image of s defines a trivial subbundle and we can find a complement using an arbitrary bundle metric. Iterating this gives the proposition. So we have to show that a nonvanishing section exists if $r = \operatorname{rk} E \geq 2$.

Let $\varphi_i \colon E|_{U_i} \to U_i \times \mathbb{C}^r$ for $i = 1, \ldots, k$ be a family of trivializations and $\psi_i \colon M \to \mathbb{R}$ a partition of unity with $\operatorname{supp} \psi_i \subset U_i$. Then define

$$s \colon \mathbb{C}^{rk} \times M \to E, \qquad s(v_1, \ldots, v_k, x) = \sum_{i=1}^{k} \psi_i(x)\varphi_i^{-1}(x, v_i).$$

This map is clearly a smooth submersion. The parametric transversality theorem [Mro04, Theorem 18.3] then states that for almost every $v \in \mathbb{C}^{rk}$ the section $s_v = s(v, -) \colon M \to E$ is transversal to the zero section $Z \subset E$, that is $T_{s(v,x)}E = T_{s(v,x)}Z + Ds_v(T_xM)$ for every

$x \in s_v^{-1}(Z)$. But $\dim T_{s(v,x)}E = 2 + 2r$, $\dim T_{s(v,x)}Z = 2$ and $\dim Ds_v(T_xM) = 2$, so this is impossible for $r \geq 2$. So for almost every $v \in \mathbb{C}^{rk}$, the image $s_v(M)$ does not intersect with Z, i.e. s_v is a nonvanishing section. □

Corollary 1.2. *All complex vector bundles on a compact connected surface M of same rank and degree are isomorphic.*

Proof. Let $E_1 \to M$ and $E_2 \to M$ be complex vector bundles with $\mathrm{rk}(E_1) = \mathrm{rk}(E_2)$ and $\deg(E_1) = \deg(E_2)$. By Proposition 1.1 there are splittings $E_1 = T_1 \oplus L_1$ and $E_2 = T_2 \oplus L_2$ with T_1, T_2 trivial and L_1, L_2 one–dimensional. The bundles T_1 and T_2 are clearly isomorphic. To show that L_1 and L_2 are also isomorphic, note that they have equal degree. By [Hat09, Proposition 3.10] complex line bundles are classified by their first integral Chern class, of which [KN09, Theorem 3.1] shows that it equals $i(2\pi)^{-1}F^\nabla$ as a class in real cohomology. But since $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ for the oriented surface M, the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$ is injective, so line bundles of equal degree have the same integral Chern class and are thus isomorphic. □

1.7 Hitchin's equations

Let ∇ be a unitary connection on a complex vector bundle $E \to M$ over a compact connected Riemann surface M and $\Phi \in \Omega^{1,0}(\mathrm{End}\, E)$ an End E–valued $(1,0)$–form on a compact Riemann surface M. We say that the pair (∇, Φ) is a solution of *Hitchin's equations* if (see [MSWW14])

$$F^\nabla + [\Phi \wedge \Phi^*] = c\,\omega \otimes \mathrm{id}_E \tag{3}$$

$$\mathrm{d}^\nabla \Phi = 0 \tag{4}$$

The form Φ is called a *Higgs field* and the pair (∇, Φ) is a *Higgs pair* or a *Higgs bundle* when ∇ is viewed as defining a holomorphic structure on the bundle E.

Chern–Weil theory shows that c is not a dynamical variable of the equation, but a predefined constant, which can be calculated from properties of E and M alone. More precisely, as $\mathrm{tr}\,[\Phi \wedge \Phi^*] = 0$, taking the trace of (3) and then integrating yields

$$-2\pi i \deg(E) = \int_M \mathrm{tr}\, F^\nabla = c\, \mathrm{rk}(E)\, \mathrm{vol}(M).$$

So solutions for (3) can only exist if

$$c = -2\pi i \, \frac{\mu(E)}{\mathrm{vol}(M)},$$

where $\mu(E) = \deg(E)/\,\mathrm{rk}(E)$ is the *slope* of the bundle E.

Since M has one complex dimension, (4) just states that Φ is a holomorphic End E–valued 1-form with respect to the holomorphic structure defined by ∇. Conversely, the Higgs field Φ restricts ∇ by specifying its curvature.

A special class of solutions are those (∇, Φ) for which $\Phi = 0$. The connection ∇ then satisfies the equation $F^\nabla = c\,\omega \otimes \mathrm{id}_E$, i.e. it is a *constant central curvature connection.*

1.8 Gauge transformations

Consider the bundle

$$\mathrm{U}\,E = \{g \in \mathrm{End}\ E \mid g^*g = gg^* = \mathrm{id}\}$$

of unitary endomorphisms of E. Its sections $\Gamma(\mathrm{U}\,E)$ are the precisely the automorphisms of E leaving its bundle metric $\langle -, - \rangle$ invariant and are called *unitary gauge transformations* of E. Such a gauge transformation $g \in \Gamma(\mathrm{U}\,E)$ acts naturally on a unitary connection $\nabla \in \mathscr{A}(E)$ by conjugation, i.e. the transformed connection is given by the composition

$$\Gamma(E) \xrightarrow{g} \Gamma(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\mathrm{id}\otimes g^{-1}} \Omega^1(E).$$

To keep the notation simple, we will just write $g^*\nabla = g^{-1} \circ \nabla \circ g$ for this right action. The unitary gauge transformations also act by conjugation on $\Gamma(\mathrm{End}\ E)$ and accordingly on $\Omega^1(\mathrm{End}\ E)$ and the configuration space $\mathscr{A}(E) \times \Omega^1(\mathrm{End}\ E)$ of Hitchin's equations.

One can easily see that Hitchin's equations are invariant under these gauge transformations, i.e. if (∇, Φ) is a solution of Hitchin's equations, then so is $(g^*\nabla, g^{-1}\Phi g)$. So a single solution already implies the existence of a whole family of solutions parametrized by $\Gamma(\mathrm{U}\,E)$, although they are not all different. In particular any gauge transformation of the form $g = \lambda\,\mathrm{id} \in \Gamma(\mathrm{U}\,E)$ with $\lambda \in S^1$ maps (∇, Φ) to itself.

2 Hilbert manifolds and Lie groups

A very conceptual way of establishing a manifold structure on the moduli space of solutions is by interpreting it as a quotient of an infinite dimensional manifold of all solutions by an action of the infinite dimensional group of unitary automorphisms. The purpose of this section is to find a general set of conditions for such a quotient to admit a manifold structure.

As we want to use the inverse function theorem and avoid the technical difficulties of Fréchet manifolds, we will extend the spaces of sections involved in the Hitchin equations to appropriate Sobolev spaces (details of this will follow in Section 3). This section will be concerned with Hilbert manifolds. The basic theory of Hilbert (and Banach) manifolds used here can be found in [Lan99].

As a convention, M will always denote an arbitrary smooth manifold and G a smooth Lie group. Throughout this section, all manifolds, Lie groups, principal bundles, etc. are considered to be modeled on possibly *infinite–dimensional* separable Hilbert spaces. They are always assumed Hausdorff and second countable and are therefore metrizable using the Riemannian distance with respect to a Riemannian metric constructed with partitions of unity [Lan99, Sections II.3,VII.6].

For $g \in G$ we denote by $\ell_g, r_g, c_g \colon G \to G$ the left and right multiplication and conjugation maps defined by $\ell_g(h) = gh$, $r_g(h) = hg$ and $c_g(h) = ghg^{-1}$. An action $G \times M \to M$ can also be considered a form of 'multiplication' in this sense, so for $g \in G$ and $x \in M$ there are also the maps $\ell_g \colon M \to M, x \mapsto gx$ and the evaluation map $r_x \colon G \to M, g \mapsto gx$. The unit element of a Lie group will often be called e and the Lie algebra corresponding to a Lie group will carry the lowercase Fraktur version of the group's name, e.g. $\mathfrak{g}, \mathfrak{h}, \mathfrak{g}_x, \mathfrak{gl}(n), \mathfrak{u}(n)$ are the Lie algebras of the Lie groups $G, H, G_x, \mathrm{GL}(n), \mathrm{U}(n)$.

2.1 Proper group actions

Of course, not every quotient of a manifold M by a Lie group G is itself a manifold, not even in finite dimensions. A typical example is the irrational winding of the torus, where the quotient space carries the indiscrete topology. However, if G is compact and acts freely, it is well–known that M/G is indeed a manifold. Unfortunately, neither of these conditions is true for the case at hand. That the action is not free is less of a problem, as it turns out that the stabilizers behave very well (in the irreducible locus) and can be quotiented out of the Lie group to give a free action. The issue of G not being compact can not be lifted so easily. In fact, the infinite dimensional Lie group of unitary automorphisms is not even locally compact. But its action on the configuration space is *proper*, which is a very useful notion capturing many of the nice properties of compact group actions. A good treatment of proper group actions in finite dimensions can be found in [GGK02, Appendix B]. This section will generalize some of these results to infinite dimensions. In doing so, it is a bit more general than necessary for our problem.

Definition 2.1. A *Lie group* G is a group which is also a manifold such that the group multiplication $G \times G \to G$ is a smooth map. An (embedded) Lie subgroup $H \subset G$ is an embedded submanifold with is also a subgroup.

Remark 2.2. Smoothness of the inversion map $g \mapsto g^{-1}$ is sometimes stated as another axiom. However, this follows automatically using the inverse function theorem: Consider for a Lie group G the map $\Phi\colon G \times G \to G \times G$ given by $\Phi(g,h) = (g, gh)$. Its derivative at a point is given by $D_{(g,h)}\Phi(X,Y) = (X, D_g r_h(X) + D_h \ell_g(Y))$, which is inverted by $(X,Z) \mapsto (X, D_{gh}\ell_{g^{-1}}(Z - D_g r_h(X)))$. So Φ is a local diffeomorphism by the inverse function theorem and, since it is bijective, a diffeomorphism. So the inversion map $p \circ \Phi^{-1} \circ i$ is smooth, where p is the projection to the second factor and $i(g) = (g, e)$.

Lemma 2.3. *Let G be a Lie group and $H \subset G$ an embedded Lie subgroup. Then H is closed as a subset of G.*

Proof. Let $\overline{H}$ be the closure of H in G. If $m\colon G \times G \to G$ is the group multiplication, then $m^{-1}(\overline{H})$ is a closed subset of $G \times G$. Since $H \times H \subset m^{-1}(H) \subset m^{-1}(\overline{H})$ this implies that $\overline{H} \times \overline{H} = \overline{H \times H} \subset m^{-1}(\overline{H})$ and $\overline{H}$ is thus a closed subgroup of G. In particular, left multiplication by some $g \in \overline{H}$ is a homeomorphism of $\overline{H}$ onto itself. Since H is locally closed, there is an open neighbourhood U of the identity in G such that $H \cap U = \overline{H} \cap U$. Hence $gH \cap gU = \overline{H} \cap gU$ is an open subset of $\overline{H}$ for every $g \in \overline{H}$. But $gH = \bigcup_{g' \in gH} g'H \cap g'U$, so gH is open in $\overline{H}$. As $\overline{H}$ is a disjoint union of such open orbits, each of them is also closed in $\overline{H}$. In particular $H = \overline{H}$. □

Though many of the following statements are only stated for left group actions $G \times M \to M$, their obvious counterparts also hold for right actions $M \times G \to M$ and will sometimes be used in this form (most prominently, the gauge group acts on the solutions of Hitchin's equation from the right). This can usually be seen by considering the associated left action $(g,x) \mapsto xg^{-1}$.

Definition 2.4. An action $G \times M \to M$ is *proper* if the map $G \times M \to M \times M, (g,x) \mapsto (gx, x)$ is proper, i.e. the preimage of every compact subset of $M \times M$ is compact.

Since G and M are metrizable, there is another characterization of proper group actions:

Lemma 2.5. *The action $G \times M \to M$ is proper if and only if for all sequences $x_i \in M$ and $g_i \in G$ such that x_i and $g_i x_i$ both converge in M, g_i has a convergent subsequence.*

Proof. Assume that $x_i \to x$ and $g_i x_i \to x'$ and $f\colon G \times M \to M \times M, (g,x) \mapsto (gx, x)$ is a proper map. Then the set $K = \{(g_0 x_0, x_0), \ldots, (x', x)\} \subset M \times M$, consisting of all elements of the sequence and its limit, is compact, so $f^{-1}(K)$ is compact. Therefore, the sequence $(g_i, x_i) \in f^{-1}(K)$ has a convergent subsequence and in particular g_i has a convergent subsequence.

For the converse, let $K \subset M \times M$ be compact and let (g_i, x_i) be any sequence in $f^{-1}(K)$. Then $(g_i x_i, x_i) \in K$ has a convergent subsequence $(g_{i_j} x_{i_j}, x_{i_j})$. If the criterion in the lemma is satisfied, this implies that a subsequence $g_{i_{j_k}}$ of g_{i_j} converges. So $(g_{i_{j_k}}, x_{i_{j_k}})$ is a convergent subsequence of (g_i, x_i) and therefore $f^{-1}(K)$ is compact. □

The most direct consequence of a proper action is that the quotient topology is Hausdorff. This is of course very important if we want to have any hope that the quotient becomes a manifold.

Proposition 2.6 (Infinite dimensional version of [GGK02, Proposition B.8])**.** *If G acts properly on M, every orbit of G is closed and M/G is Hausdorff.*

Proof. For $x \in M$ let $g_i x$ be any sequence in the orbit Gx converging to $y \in M$. Since the action is proper, a subsequence of g_i then converges to some $g \in G$, which satisfies $gx = y$ by continuity. So $y \in Gx$ and therefore Gx is closed.

Next assume M/G is not Hausdorff and let $x, y \in M$ be such that their orbits cannot be separated. In an arbitrary metric on M, let U_n and V_n be balls of radius $1/n$ around x and y. Then for every $n \in \mathbb{N}$ there exist $g_n \in G$, $x_n \in U_n$ and $y_n \in V_n$ such that $y_n = g_n x_n$, as otherwise $G \cdot U_n$ and $G \cdot V_n$ were disjoint and their projections to M/G would therefore separate x and y. Since $x_n \to x$ and $y_n \to y$ and the action is proper, a subsequence of g_n converges to some $g \in G$ with $y = gx$, so x and y are in the same orbit. □

There is another technical condition that a group action should satisfy to have a smooth quotient. If G or M are finite dimensional, this is automatically satisfied.

Definition 2.7. A smooth map between manifolds *splits* if its differential has a closed image at every point. A group action $G \times M \to M$ *splits* if for every $x \in M$ the map $r_x \colon G \to M$ splits.

Remark 2.8. Let $f \colon N \to M$ be a splitting map and $x \in \operatorname{im} f$. As we assume M to be modeled on a Hilbert space, there is a scalar product on $T_x M$ compatible to its topology. This induces a direct sum decomposition of $T_x M$ into $\operatorname{im} D_x f$ and a complement, which is however generally not unique.

The purpose of the following lemma is to clarify the technical details of the smooth structure on a quotient. Its proof is mostly straightforward.

Lemma 2.9. *Consider a smooth action $G \times M \to M$ such that M/G is Hausdorff. Assume that M is covered by open subsets $\{U_i\}_{i \in I}$ which are G–equivariantly diffeomorphic to products $P_i \times Q_i$, where G acts transitively on P_i and trivially on Q_i. Then there is a unique smooth structure on M/G such that the projection $\pi \colon M \to M/G$ is a smooth submersion. With this structure, M/G is locally diffeomorphic to Q_i.*

Proof. If $\{V_i\}_{i \in I}$ is a countable topological basis of M, then $\{\pi(V_i)\}_{i \in I}$ is a basis of G/M where $\pi \colon M \to M/G$ is the canonical projection. This follows from the surjectivity of π together with the fact that $\pi(V_i)$ is open in M/G since $\pi^{-1}(\pi(V_i)) = \bigcup_{g \in G} gV_i$.

To prove the existence of a smooth structure, let $U \subset M$ and (omitting indices from now on) $\psi \colon P \times Q \to U$ be a G–equivariant diffeomorphism as in the lemma. Then for an arbitrary $p \in P$ define $\widetilde{\psi} \colon Q \to U/G$ by $\widetilde{\psi}(q) = \pi(\psi(p,q))$. By transitivity and equivariance this does not depend on the choice of p. Consequently, $\widetilde{\psi}(\operatorname{pr}_Q(\psi^{-1}(x))) = [x]$ for every $[x] \in U/G$, so $\widetilde{\psi}$ is surjective. It is also injective, as for $q_1, q_2 \in Q$ with $\widetilde{\psi}(q_1) = \widetilde{\psi}(q_2)$, we have $\psi(p, q_1) = g\psi(p, q_2) = \psi(gp, q_2)$ for some $g \in G$, so $q_1 = q_2$ since ψ is bijective. To show that $\widetilde{\psi}$ is an open map, let $O \subset Q$ be open. Then $\widetilde{\psi}(O) = \pi(\psi(P \times O))$, which is clearly open in U/G. So $\widetilde{\psi}$ is a homeomorphism.

To show that these maps $\widetilde{\psi}$ are smoothly compatible and define a smooth structure, consider two such diffeomorphisms, $\psi_1 \colon P_1 \times Q_1 \to U$ and $\psi_2 \colon P_2 \times Q_2 \to U$, restricted to a subdomain small enough that their images coincide. We have to show that $\widetilde{\psi}_2^{-1} \circ \widetilde{\psi}_1 \colon Q_1 \to U/G \to Q_2$ is smooth. But $\widetilde{\psi}_1(q_1) = \widetilde{\psi}_2(q_2)$ is equivalent to $\psi_1(p_1, q_1) = g\psi_2(p_2, q_2) = \psi_2(gp_2, q_2)$, so $\widetilde{\psi}_2^{-1}(\widetilde{\psi}_1(q)) = \operatorname{pr}_{Q_2}(\psi_2^{-1}(\psi_1(p_1, q)))$, which is smooth. So there is a smooth structure on M/G such that the maps $\widetilde{\psi}$ are diffeomorphisms.

The canonical projection π can be locally described as $\pi|_U = \widetilde{\psi} \circ \mathrm{pr}_Q \circ \psi^{-1}$ and is therefore a smooth submersion. For the uniqueness part, assume we have two smooth structures on M/G such that $\pi\colon M \to M/G$ is a submersion. By the inverse function theorem, there are local charts in M and M/G in which π is just a linear projection onto a closed subspace. So in particular there exists a smooth local section. Composing this with the projection onto M/G equipped with the other smooth structure, we get a smooth identity function, which shows that the smooth structures are equivalent. □

Proposition 2.10 (Infinite dimensional version of [GGK02, Proposition B.18])**.** *Let $H \subset G$ be an embedded Lie subgroup. Then the set G/H of left cosets is a smooth manifold modeled on any complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ in $\mathfrak{g}$ and $G \to G/H$ is a smooth principal H–bundle.*

The inverse of a chart of G/H around the identity is given by

$$U \to V/H, \quad v \mapsto [\exp(v)]$$

where $0 \in U \subset \mathfrak{h}^\perp$ and $e \in V \subset G$ are open subsets and V is H–invariant. A trivialization is in this chart given by

$$U \times H \to V/H \times H \to V, \quad (v, g) \mapsto ([\exp(v)], g) \mapsto \exp(v)g.$$

Proof. The right action of H on G is clearly proper since H is closed, so G/H is Hausdorff by Proposition 2.6. The result then follows with Lemma 2.9 if for every $g \in G$ we can construct an H–equivariant diffeomorphism $\Psi\colon U \times H \to V$ with U an open subset of a complement of $\mathfrak{h}$ in $\mathfrak{g}$ and V an H–invariant subset of G. It suffices to assume $g = e$ since we can always append left multiplication by g to get a diffeomorphism around g.

The assumption that H is a submanifold implies a splitting of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ (where $\mathfrak{h}^\perp$ can be any complement, orthogonality is not required). Consider the map

$$\Psi\colon \mathfrak{h}^\perp \times H \to G, \quad (v, g) \mapsto \exp(v) \cdot g$$

where $\exp\colon \mathfrak{g} \to G$ is the exponential map in G. The differential $D_{(0,e)}\Psi\colon \mathfrak{h}^\perp \times \mathfrak{h} \to \mathfrak{g}$ is clearly the identity (it is the sum of $D_0 \exp = \mathrm{id}_{\mathfrak{h}^\perp}$ and $D_e \mathrm{id} = \mathrm{id}_{\mathfrak{h}}$). Continuity then implies the existence of an open neighborhood U of the origin in $\mathfrak{h}^\perp$ such that $D_{(v,e)}\Psi$ is an isomorphism for all $v \in U$. If we denote right multiplication by $r_g(h) = hg$, then $r_{g^{-1}} \circ \Psi \circ (\mathrm{id} \times r_g) = \Psi$ and therefore $D_{(v,g)}\Psi$ is also an isomorphism for every $g \in H$. So $\Psi|_{U\times H}$ is a local diffeomorphism.

In fact, by making U smaller one can also ensure that $\Psi|_{U\times H}$ is injective, and therefore a diffeomorphism onto its (open) image. To see this, assume the contrary. This yields two sequences $(v_n, g_n) \neq (u_n, h_n) \in \mathfrak{h}^\perp \times H$ with u_n, v_n both converging to 0, such that $\exp(v_n) \cdot g_n = \exp(u_n) \cdot h_n$. Then $\exp(v_n) \cdot g_n h_n^{-1} = \exp(u_n) \cdot e$ and $g_n h_n^{-1}$ converges to $e \in G$. But Ψ is a diffeomorphism in a neighborhood of $(0, e)$, so for large enough n, it must hold that $g_n h_n^{-1} = e$ and $v_n = u_n$, which contradicts the assumption $(v_n, g_n) \neq (u_n, h_n)$. So $\Psi|_{U\times H}$ is a diffeomorphism.

We have already proved that the trivializing map in the statement of the lemma is a H–equivariant diffeomorphism. This makes G/H a principal H–bundle. □

Lemma 2.11. *Let $G \times M \to M$ be a smooth action and $x \in M$. Then the stabilizer $G_x = \{g \in G \mid gx = x\} \subset G$ is an (embedded) Lie subgroup with Lie algebra $\mathfrak{g}_x = \ker D_e r_x$.*

Proof. Around any $g \in G_x$ we want to find a chart for G which maps G_x into a linear subspace. It is sufficient to assume $g = e$ as we can transport this chart to every other g by left multiplication. Consider the exponential map $\exp\colon \mathfrak{g} \supset U_1 \to V_1 \subset G$ on open sets U_1, V_1 small enough such that it is a diffeomorphism and U_1 is star–shaped around 0. The identity component of $G_x \cap V_1$ is an open subset, so it equals $G_x \cap V_1 \cap V_2$ for some open subset $V_2 \subset G$. Writing $V = V_1 \cap V_2$ and $U = \exp^{-1}(V)$ the restriction $\exp\colon U \to V$ is still a diffeomorphism and $G_x \cap V$ is connected and contains the identity.

Let $\mathfrak{g}_x = \ker D_e r_x$ be the kernel of the evaluation map. We assert that $\exp(\mathfrak{g}_x \cap U) = G_x \cap V$. This will be sufficient to prove the lemma. First, if $v \in \mathfrak{g}_x \cap U$, then the curve in M defined by $\gamma(t) = \exp(tv)x$ is constant since $\dot\gamma(t) = D_x \ell_{\exp(tv)}(D_e r_x(v)) = 0$ for all t. So $\exp(v)x = \gamma(1) = x$ and $\exp(v) \in G_x$.

Conversely, let $g \in G_x \cap V$. Since $G_x \cap V$ is connected, there is a curve $\alpha\colon I \to G_x \cap V$ from e to g. Let $\beta_t(s) = \exp(t \exp^{-1}(\alpha(s)))x$ for all $s, t \in I$. It satisfies

$$\beta_0(s) = x, \qquad \beta_t(0) = x, \qquad \beta_1(s) = \alpha(s)x = x \quad \text{and} \quad \dot\beta_t(s) = t\dot\beta_1(s) = 0.$$

So $\beta_t(s) = \beta_t(0) = x$ for all $s, t \in I$ and therefore $D_e r_x(\exp^{-1}(g)) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\beta_t(1) = 0$, i.e. $\exp^{-1}(g) \in \mathfrak{g}_x$. □

Definition 2.12. Let G be a Lie group, M a manifold, F a Hilbert space, $\pi_P\colon P \to M$ a principal G–bundle and $\rho\colon G \to \mathrm{End}\,(F)$ a representation of G in the bounded linear operators on F. Then define

$$P \times_\rho F = (P \times F)/\sim, \qquad (pg^{-1}, \rho(g)f) \sim (p, f) \quad \forall g \in G.$$

This defines $P \times_\rho F$ as a vector bundle over M with fiber F. Every local trivialization $\pi_P^{-1}(U) \cong U \times G$ of P induces a local trivialization of $P \times_\rho F$ using the identification $F \to G \times_\rho F, v \mapsto [e, v]$.

$$\begin{array}{ccccc} \pi_P^{-1}(U) \times_\rho F & \xrightarrow{\sim} & U \times G \times_\rho F & \xleftarrow{\sim} & U \times F \\ \downarrow{\scriptstyle \pi_{P \times_\rho F}} & & & & \\ U & & & & \end{array}$$

$$U \times G \times_\rho F \to U, \qquad U \times F \to U$$

If the choice of representation ρ is clear from the context, we will also write $P \times_H F = P \times_\rho F$. Any $\rho(G)$–invariant subset $U \subset F$ defines a subset $P \times_\rho U \subset P \times_\rho F$.

The map $F \to G \times_\rho F$ is a homeomorphism since it is inverted by $[g, v] \mapsto \rho(g)v$. If we view local trivializations of P as smooth maps $\psi_i\colon U \to G$ then the corresponding transition maps $\tau_{ij}\colon U \to \mathrm{End}\,(F)$ are given by $\tau_{ij}(x) = \rho(\psi_i(x)\psi_j(x)^{-1})$, so $P \times_\rho F$ indeed satisfies the axioms of an infinite–dimensional vector bundle (as stated in [Lan99, Section III.1]).

A significant example arises from the action of the stabilizer G_x on the tangent space T_xM by

$$G_x \times T_xM \to T_xM, \quad (g, X) \mapsto D_x\ell_g(X)$$

This defines a smooth representation $\rho\colon G_x \to \mathrm{End}\,(T_xM)$ of G_x in the bounded linear operators on the Hilbert space T_xM. It will be used together with the principal G_x–bundle $G \to G/G_x$ to construct vector bundles $G \times_{G_x} F \to G/G_x$ for invariant subspaces $F \subset T_xM$.

Theorem 2.13 (Slice theorem). *Let $G \times M \to M$ be a proper splitting action and $x \in M$.*

1. *There is a decomposition*
$$T_xM = T_x(Gx) \oplus F$$
where $T_x(Gx) = \operatorname{im} D_e r_x$ and F are both G_x–invariant closed subspaces of T_xM.

2. *There are G_x–invariant open neighbourhoods $U \subset T_xM$ of 0 and $V \subset M$ of x and a G_x–equivariant diffeomorphism*
$$Q \colon U \to V$$
which maps 0 to x.

3. *For any F, U, V, Q satisfying the above, there are open subsets $0 \in \widetilde{U} \subset U \cap F$ and $x \in \widetilde{V} \subset V$, $\widetilde{U}$ being G_x–invariant and $\widetilde{V}$ being G–invariant, such that the map*
$$\Psi \colon G \times_{G_x} \widetilde{U} \to \widetilde{V}, \quad [g, u] \mapsto g \cdot Q(u)$$
is a G–equivariant diffeomorphism.

Proof. For the first part, let $\pi \colon T_xM \to T_xM$ be a linear projection to the tangent space of the orbit, i.e. a bounded operator with $\operatorname{im} \pi = \operatorname{im} D_e r_x$ and $\pi^2 = \pi$. Its image $\operatorname{im} \pi$ is G_x–invariant, since $D_x \ell_g \circ D_e r_x = D_e r_x \circ D_e c_g$ for all $g \in G_x$. As $\pi|_{\operatorname{im} \pi}$ is the identity, this implies $\pi g_* \pi = g_* \pi$ for any $g \in G_x$ (where $g_* = D_x \ell_g \in \operatorname{End}(T_xM)$). Now define an operator $\overline{\pi}$ by
$$\overline{\pi} = \int_{G_x} g_* \pi g_*^{-1} \, \mathrm{d}g.$$
Since the G–action is proper, G_x is compact and this integral is thus well–defined as a Bochner integral with respect to the Haar measure. For all $g \in G_x$, the identities
$$\overline{\pi}^2 = \overline{\pi} \qquad \overline{\pi}\pi = \pi \qquad \pi\overline{\pi} = \overline{\pi} \qquad g_*\overline{\pi} = \overline{\pi} g_*$$
easily follow from $\pi g_* \pi = g_* \pi$ and basic properties of the integral. These identities show that $\overline{\pi}$ is a projector with $\operatorname{im} \overline{\pi} = \operatorname{im} \pi = \operatorname{im} D_e r_x$ and that its kernel $F = \ker \overline{\pi}$ is G_x–invariant. This gives the desired splitting and proves the first part.

For the second part, let $\widetilde{q} \colon M \to T_xM$ be a smooth map with $\widetilde{q}(x) = 0$ and $D_x \widetilde{q} = \mathrm{id}_{T_xM}$ (such a map can e.g. be constructed in local coordinates and extended by a smooth bump function). Define the map $\widetilde{Q} \colon M \to T_xM$ by
$$\widetilde{Q}(y) = \int_{G_x} g_*^{-1}(\widetilde{q}(gy)) \, \mathrm{d}g.$$
As above, this is to be understood as a Bochner integral on the Haar measure. The map $\widetilde{Q}$ is smooth, satisfies $\widetilde{Q}(x) = 0$ as well as $D_x \widetilde{Q} = \mathrm{id}$, and is G_x–equivariant in the sense that $\widetilde{Q}(gy) = g_* \widetilde{Q}(y)$ for all $y \in M$ and $g \in G_x$. By the inverse function theorem $\widetilde{Q}$ is a diffeomorphism in an open neighborhood of x. Its inverse Q is also G_x–equivariant, i.e. $Q(g_* y) = gQ(y)$ for all $g \in G_x$ and therefore satisfies the conditions for the second part of the theorem.

For the third part, choose any norm $\|\_\|$ on F compatible with the topology and let $U_n = \bigcup_{g \in G_x} g_* B_{1/n} \subset F$ where $B_{1/n}$ is the ball of radius $1/n$ around the origin in F. The set U_n is open, G_x–invariant, and contained in a ball of radius of $1/n \cdot \sup_{g\in G_x} \|g_*\|$ around the origin, which is finite since G_x is compact and the function $g \mapsto \|g_*\|$ is continuous. For n large enough that U_n is in the domain of Q, let $V_n = \bigcup_{g\in G} gQ(U_n)$ and consider the function

$$\Psi\colon G \times_{G_x} U_n \to V_n, \quad [g, u] \mapsto g \cdot Q(u).$$

This function is clearly G–equivariant and well–defined since $\Psi([gh^{-1}, hu]) = gh^{-1}Q(h_*u) = gQ(u) = \Psi([g, u])$ for all $h \in G_x$. To see that it is smooth, note that there are neighbourhoods $0 \in A \subset \mathfrak{g}_x^\perp \subset \mathfrak{g}$ ($\mathfrak{g}_x^\perp$ being a complement of the finite dimensional subspace $\mathfrak{g}_x$) and $e \in B \subset G$ such that the following is a chart for $G \times_{G_x} U_n$ around $[e, 0]$:

$$\begin{aligned} \varphi\colon A \times U_n &\longrightarrow B/G_x \times U_n \longrightarrow B \times_{G_x} U_n \\ (v, u) &\longmapsto ([\exp v], u) \longmapsto [\exp(v), u]. \end{aligned}$$

In this chart $\Psi(\varphi(v, u)) = \exp(v)\widetilde{Q}(u)$, so Ψ is smooth around $[e, 0]$. Also $D_0(\Psi \circ \varphi) = D_e r_x \times \mathrm{id}\colon \mathfrak{g}_x^\perp \times F \to T_x M = (\mathrm{im}\, D_e r_x) \times F$, which is an isomorphism since $\mathfrak{g}_x = \ker D_e r_x$. Therefore, $D_{[e,0]}\Psi$ is an isomorphism and by continuity so is $D_{[e,u]}\Psi$ for all $u \in B_{1/n}$ for large enough n. Then by G–equivariance and the inverse function theorem, Ψ is a local diffeomorphism.

In a last step, we show that Ψ is bijective for some $n \in \mathbb{N}$. Surjectivity is clear from the definition of V_n. Assume that Ψ is not injective for any $n \in \mathbb{N}$, then we get sequences $[g_i, u_i] \neq [h_i, v_i] \in G \times_H F$ such that $u_i, v_i \to 0$ and $\Psi([h_i^{-1}g_i, u_i]) = \Psi([e, v_i])$. Since Ψ is a diffeomorphism in a neighborhood of $[e, 0]$, this is only possible if $[h_i^{-1}g_i, u_i] = [e, v_i]$ for large i, which is a contradiction. So Ψ is a diffeomorphism for some $n \in \mathbb{N}$. $\square$

Corollary 2.14. *Consider a proper and splitting action $G \times M \to M$ as in Theorem 2.13 and let $H \subset G$ be an embedded Lie subgroup. Then the G–invariant set*

$$M_{(H)} = \{x \in M \mid \exists g \in G\colon G_x = gHg^{-1}\}$$

of points with stabilizers conjugate to H is a (possibly empty) embedded submanifold of M. If every stabilizer G_x for $x \in M$ contains a subgroup conjugate to H, then $M_{(H)} \subset M$ is open.

If $G_x = H$, $F \subset T_x M$ is an H–invariant finite–dimensional complement of the G–orbit as in Theorem 2.13, and F^H is the subspace of fixed points of this action, then the codimension of $M_{(H)}$ in M equals $\dim F - \dim F^H$.

Proof. Let $x \in M_{(H)}$, so $G_x = gHg^{-1}$ for some $g \in G$. Then $G_{g^{-1}x} = H$, so by Theorem 2.13 there is a G–invariant open neighborhood $V \subset M$ of $g^{-1}x$ (and thus also of x) with $V \cong G \times_H U$ equivariantly, so that $M_{(H)} \cap V$ is mapped to $(G \times_H U)_{(H)}$. Here U is an open subset of some Hilbert space F on which H acts linearly. The fixed point set $F^H \subset F$ of this action is a closed H–invariant subspace and therefore $G \times_H F^H \subset G \times_H F$ is a subbundle over G/H and in particular $G \times_H U^H \subset G \times_H U$ is a submanifold. If F is finite–dimensional, $G \times_H F^H$ and $G \times_H F$ are vector bundles of rank $\dim F^H$ and $\dim F$, so $G \times_H U^H \subset G \times_H U$ has codimension $\dim F - \dim F^H$.

We assert that $(G \times_H U)_{(H)} = G \times_H U^H$. As these sets are both G–invariant, it suffices to check their identity at points of the form $[e, u]$ with $u \in U$. Clearly, the stabilizer of

$G \times_H U$ at such a point is contained in H. So if $[e, u] \in (G \times_H U)_{(H)}$, i.e. the stabilizer is conjugate to H, then it must be equal to H, so $[e, hu] = [h, u] = [e, u]$ for all $h \in H$ and thus $[e, u] \in G \times_H U^H$. Conversely, if $[e, u] \in G \times_H U^H$ then the stabilizer at this point is equal to H, so $[e, u] \in (G \times_H U)_{(H)}$. This shows that around every $x \in M_{(H)}$ there is an open set V such that $M_{(H)} \cap V$ is a submanifold of V, which implies the first statement of the corollary.

Now let in the above setting $[g, u] \in G \times_H U$ and $g' \in G_{[g,u]}$. Then $g'[g, u] = [g, u]$, which means there exists $h \in H$ such that $g'gh^{-1} = g$ and $hu = u$. In particular $g' = ghg^{-1} \in gHg^{-1}$, so $G_{[g,u]} \subset gHg^{-1}$. But if the stabilizer at every point in M contains a subgroup conjugate to H, then the same is true for $G_{[g,u]}$, i.e. there is $\hat{g}$ with $\hat{g}H\hat{g}^{-1} \subset G_{[g,u]} \subset gHg^{-1}$. Since both $\hat{g}H\hat{g}^{-1}$ and gHg^{-1} are closed embedded subgroups of G, they carry the subspace topology and $\hat{g}H\hat{g}^{-1}$ is also a closed subset of gHg^{-1}. But these groups are also diffeomorphic, so the inclusion is open by invariance of domain and, since they are compact, both have the same finite number of connected components. So $\hat{g}H\hat{g}^{-1} = gHg^{-1}$ and in particular $G_{[g,u]}$ is conjugate to H, so the image of $[g, u]$ and therefore all of V, which is an open neighborhood of x in M, is contained in $M_{(H)}$. □

Corollary 2.15. *Consider a proper and splitting action $G \times M \to M$ as in Theorem 2.13. Then there exist only countably many orbit types, i.e. there are countably many subgroups $H_1, H_2, \dots \subset G$ such that $M = \bigcup_{i \in \mathbb{N}} M_{(H_i)}$.*

Proof. By our countability assumptions on manifolds, M is covered by countably many open subsets equivariantly diffeomorphic to $G \times_H U$ as in Theorem 2.13. All stabilizer groups occurring at points inside $G \times_H U$ must be closed subgroups of the compact Lie group H, of which there are only countably many conjugation classes [Pal99, Corollary 1.7.27]. This is still true when considering them as conjugation classes in G, as two groups conjugated in H are still conjugated in G. □

Corollary 2.16. *Consider a proper and splitting action $G \times M \to M$ as in Corollary 2.14 and let $H \subset G$ be a Lie subgroup. Then $M_{(H)}/G$ has a unique manifold structure such that $M_{(H)} \to M_{(H)}/G$ is a smooth submersion.*

Proof. The proof of Corollary 2.14 already shows that locally at a point $x \in M_{(H)}$, the manifold $M_{(H)}$ is G–equivariantly diffeomorphic to $G \times_H F^H$ for some closed subspace $F \subset T_xM$. But $G \times_H F^H \cong G/H \times F^H$ G–equivariantly, so by Lemma 2.9 there is a smooth structure on $M_{(H)}/G$ as claimed. □

2.2 Hyperkähler quotients

Now that we have established conditions for a quotient of a manifold to admit a manifold structure, we will apply this to Hamiltonian actions on Hyperkähler manifolds to obtain an infinite–dimensional version of the Hyperkähler quotient construction.

Definition 2.17. A 2–form $\omega \in \Omega^2(M)$ is a *symplectic form* if it is closed and the map

$$T_xM \to T_x^*M, \quad v \mapsto v \lrcorner \omega_x = \omega_x(v, -)$$

is an isomorphism for every $x \in M$.

Definition 2.18. Let M be a manifold and ω a symplectic form on M. Let $G \times M \to M$ be a group action preserving ω, i.e. $\ell_g^* \omega = \omega$ for all $g \in G$. A *moment map* $\mu\colon M \to \mathfrak{g}^*$ for this action is a map satisfying

$$\mu(gx) = c_{g^{-1}}^* \mu(x) \qquad \mathrm{d}\langle \mu, \xi \rangle = \xi^M \lrcorner\, \omega \qquad \forall x \in M, g \in G, \xi \in \mathfrak{g}$$

where $\langle \mu, \xi \rangle \colon M \to \mathbb{R}$ is given by $\langle \mu, \xi \rangle(x) = \mu(x)(\xi)$, $\xi^M \in \Gamma(TM)$ is the infinitesimal action defined by $(\xi^M)_x = D_e r_x(\xi)$ and $c_{g^{-1}}^* \mu(x) = \mu(x) \circ D_e c_{g^{-1}}$ is the coadjoint action. If a moment map exists, the action is called *Hamiltonian.*

Lemma 2.19. *Using the notation* $A^\perp = \{f \in \mathfrak{g}^* \mid f(\xi) = 0 \ \forall \xi \in A\} \subset \mathfrak{g}^*$ *for* $A \subset \mathfrak{g}$, *moment maps have the following properties:*

1. *The moment map for a given action is unique up to addition of some constant* $f \in \mathfrak{g}^*$ *with* $c_g^* f = f$ *for all* $g \in G$ *on every connected component of* M.

2. *If the moment map* μ *splits, the image of its differential is* $\operatorname{im} D_x \mu = (\ker D_e r_x)^\perp$.

Proof. The first part is trivial. For the second part, since $\langle D_x \mu(v), \xi \rangle = \mathrm{d}\langle \mu, \xi \rangle(v) = \omega(\xi^M, v)$ for every $v \in T_x M$, $\xi \in (\operatorname{im} D_x \mu)^\perp$ is equivalent to $\omega(\xi^M, v) = 0$ for every $v \in T_x M$, which in turn is equivalent to $(\xi^M)_x = D_e r_x(\xi) = 0$ by non–degeneracy. So $(\operatorname{im} D_x \mu)^\perp = \ker D_e r_x$. Reflexivity of the Hilbert space $\mathfrak{g}$ and closedness of $\operatorname{im} D_x \mu$ then imply the statement. □

Definition 2.20. A *Hyperkähler manifold* is a manifold M together with a Riemannian metric g and almost complex structures J_1, J_2, J_3 which

- are compatible with g, i.e. $g(J_i X, J_i Y) = g(X, Y)$ for $i = 1, 2, 3$,
- satisfy the quaternion relations $J_1^2 = J_2^2 = J_3^2 = -1$ and

$$J_1 J_2 = -J_2 J_1 = J_3 \qquad J_2 J_3 = -J_3 J_2 = J_1 \qquad J_3 J_1 = -J_1 J_3 = J_2,$$

- and induce differential forms $\omega_i = g(J_i \_, \_)$ which are closed for $i = 1, 2, 3$ and are thus symplectic forms.

Lemma 2.21. *If two Hyperkähler structures on a manifold* M *have the same symplectic forms* $\omega_1, \omega_2, \omega_3$, *then they are equal.*

Proof. We will write all parts of the Hyperkähler structure in terms of its symplectic forms. Let $K_i \colon T_x M \to T_x^* M$ be the isomorphism defined by $K_i(X) = X \lrcorner\, \omega_i$. Then

$$K_1(X) = X \lrcorner\, \omega_1 = J_1 X \lrcorner\, g = -J_2 J_1 X \lrcorner\, \omega_2 = J_3 X \lrcorner\, \omega_2 = K_2(J_3 X),$$

for any $X \in T_x M$, so $J_3 = K_2^{-1} K_1$. The other complex structures J_1, J_2 can similarly be expressed by K_1, K_2, K_3 and the metric is given by $g(X, Y) = -\omega_i(J_i X, Y)$ for any $i \in \{1, 2, 3\}$. □

Lemma 2.22 (from [Hit87b, Lemma 6.8])**.** *Let* (M, g, J_1, J_2, J_3) *be a finite–dimensional Hyperkähler manifold. Then* J_1, J_2, J_3 *are integrable.*

Proof. Applying a Lie derivative $\mathcal{L}_X$ on the identity $\omega_i(Y,Z) = (Y \lrcorner \omega_i)(Z)$ yields

$$(\mathcal{L}_X\omega_i)(Y,Z) + \omega_i([X,Y],Z) + \omega_i(Y,[X,Z]) = \mathcal{L}_X(Y \lrcorner \omega_i)(Z) + \omega_i(Y,[X,Z]).$$

Now, after eliminating and rearranging, we can apply Cartan's formula to get

$$[X,Y] \lrcorner \omega_i = \mathcal{L}_X(Y \lrcorner \omega_i) - Y \lrcorner \mathcal{L}_X\omega_i = \mathcal{L}_X(Y \lrcorner \omega_i) - Y \lrcorner \mathrm{d}(X \lrcorner \omega_i). \tag{5}$$

Inserting into this (e.g. for $i = 2$) the identity

$$Z \lrcorner \omega_2 = J_2 Z \lrcorner g = J_3 J_1 Z \lrcorner g = J_1 Z \lrcorner \omega_3$$

on both sides gives

$$J_1[X,Y] \lrcorner \omega_3 = \mathcal{L}_X(J_1 Y \lrcorner \omega_3) - Y \lrcorner \mathrm{d}(J_1 X \lrcorner \omega_3). \tag{6}$$

We want to show that the Nijenhuis tensor

$$N_{J_1}(X,Y) = [X,Y] + J_1[J_1X,Y] + J_1[X,J_1Y] - [J_1X,J_1Y]$$

vanishes. Using (5) with $i = 3$ on the first and fourth term of the Nijenhuis tensor and (6) on the second an third term, we get

$$\begin{aligned} N_{J_1}(X,Y) \lrcorner \omega_3 &= \mathcal{L}_X(Y \lrcorner \omega_3) - Y \lrcorner \mathrm{d}(X \lrcorner \omega_3) + \mathcal{L}_{J_1X}(J_1Y \lrcorner \omega_3) - Y \lrcorner \mathrm{d}(J_1^2 X \lrcorner \omega_3) \\ &\quad + \mathcal{L}_X(J_1^2 Y \lrcorner \omega_3) - J_1Y \lrcorner \mathrm{d}(J_1X \lrcorner \omega_3) - \mathcal{L}_{J_1X}(J_1Y \lrcorner \omega_3) + J_1Y \lrcorner \mathrm{d}(J_1X \lrcorner \omega_3) \\ &= 0. \end{aligned}$$

So $N_{J_1} = 0$ by the non–degeneracy of ω_3. The almost complex structure J_1 is therefore integrable by the Newlander–Nirenberg Theorem [NN57]. An analogous argument shows the integrability of J_2 and J_3. □

Remark 2.23. It is unknown if the Newlander–Nirenberg Theorem generalizes to infinite–dimensional Hilbert manifolds [Pat00] (it is false for Banach manifolds). So for most purposes it would be more sensible to require integrability explicitly in the definition of infinite–dimensional Hyperkähler manifolds. But since we are only interested in finite dimensional quotients where Lemma 2.22 holds, the above definition is sufficient.

Theorem 2.24 (Hyperkähler quotients, generalization of [HKLR87, Theorem 3.1])**.** *Let M be a Hyperkähler manifold with metric g, almost complex structures J_1, J_2, J_3 and associated symplectic forms $\omega_1, \omega_2, \omega_3$. Let $G \times M \to M$ be a proper splitting action preserving g, J_1, J_2, J_3 which is Hamiltonian with respect to ω_1, ω_2 and ω_3 with the splitting moment maps μ_1, μ_2, μ_3. Let $H \subset G$ be a Lie subgroup such that $M_{(H)}$ is open and non–empty. Then*

$$M_{(H)}/\!\!/\!\!/G = N/G = \big(M_{(H)} \cap \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)\big)/G$$

is a smooth manifold and admits a unique Hyperkähler structure $\widehat{g}, \widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3, \widehat{J}_1, \widehat{J}_2, \widehat{J}_3$ such that

$$p^*\widehat{\omega}_i = \iota^*\omega_i, \qquad i = 1,2,3, \tag{7}$$

where $p\colon N \to N/G$ and $\iota\colon N \to M$ are the projection and injection maps.

Proof. As all relevant structures seamlessly restrict to open subsets, we can assume $M_{(H)} = M$. The map

$$\widetilde{\mu}\colon M \to (\mathfrak{g}^*)^3, \quad x \mapsto (\mu_1(x), \mu_2(x), \mu_3(x))$$

can be viewed as a composition of the diagonal $\Delta^3\colon M \to M^3$ and $\widehat{\mu} = \mu_1 \times \mu_2 \times \mu_3$. For any triple $(x, y, z) \in M^3$, Lemma 2.19 shows that

$$(\operatorname{im} D_{(x,y,z)}\widehat{\mu})^\perp = (\ker D_e r_x) \oplus (\ker D_e r_y) \oplus (\ker D_e r_z).$$

As G_x, G_y, G_z are conjugate to H, these summands are all isomorphic to $\mathfrak{h} \subset \mathfrak{g}$, which has finite dimension since H is compact. So $\widehat{\mu}$ has constant corank $3 \dim \mathfrak{h}$ and $W = \widehat{\mu}^{-1}(0) \subset M^3$ is a submanifold. The map Δ^3 is transversal over W: Let $x \in M$ with $(x, x, x) \in W$, then the subspace of $(T_xM)^3$ tangent to W is given by

$$\ker D_{(x,x,x)}\widehat{\mu} = (\ker D_x\mu_1) \oplus (\ker D_x\mu_2) \oplus (\ker D_x\mu_3).$$

But $v \in \ker D_x\mu_i$ is equivalent to $0 = \langle D_x\mu_i(v), \xi\rangle = \mathrm{d}\langle \mu_i(x), \xi\rangle(v) = \omega_i(\xi^M, v) = g(J_i\xi^M, v)$ for all $\xi \in \mathfrak{g}$, so $\ker D_x\mu_i = \operatorname{im}(J_i \circ D_e r_x)^\perp$. So transversality is the assertion that $\{(v, v, v) \mid v \in T_xM\}$ and

$$\operatorname{im}(J_1 \circ D_e r_x)^\perp \oplus \operatorname{im}(J_2 \circ D_e r_x)^\perp \oplus \operatorname{im}(J_3 \circ D_e r_x)^\perp$$

together generate $(T_xM)^3$. This is clearly true if the spaces $\operatorname{im}(J_i \circ D_e r_x)$ are pairwise orthogonal, which follows from e.g. $\mu_3(x) = 0$ (since $(x, x, x) \in W$) using

$$g_x(J_1\xi^M, J_2\zeta^M) = -g_x(J_2J_1\xi^M, \zeta^M) = (\omega_3)_x(\xi^M, \zeta^M) = \langle \mu_3(x), [\xi, \zeta]\rangle = 0.$$

So $N = \widetilde{\mu}^{-1}(0) = (\Delta^3)^{-1}(W) \subset M$ is a submanifold.

Since $\mu_i(gx) = c^*_{g^{-1}}\mu_i(x) = 0$ for all $g \in G$ and $x \in N$, the G–action restricts to N. The restricted action is also proper and splitting and has the same stabilizers as the original action. So by Corollary 2.16 the space N/G has a smooth structure and $p\colon N \to N/G$ is a submersion.

To find the induced Hyperkähler structure on $M/\!\!/\!\!/G$, first observe that the condition (7) determines it uniquely: Since p is a submersion, (7) completely defines the symplectic structures $\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3$. By Lemma 2.21 these in turn define the metric and complex structures.

It is useful for constructing the Hyperkähler structure to examine the 'horizontal subspace' of T_xN, which is the $g|_{T_xN}$–orthogonal complement $H_x = (\ker D_xp)^\perp = (\operatorname{im} D_e r_x)^\perp \subset T_xN$. It is isomorphic to $T_{[x]}(N/G)$ via D_xp and for any $v \in H_x$ and $\xi \in \mathfrak{g}$ we have

$$\begin{aligned}
\langle D_x\mu_1(J_1v), \xi\rangle &= \mathrm{d}\langle \mu_1, \xi\rangle(J_1v) = \omega_1(\xi^M, J_1v) = g(\xi^M, v) = 0,\\
\langle D_x\mu_2(J_1v), \xi\rangle &= \omega_2(\xi^M, J_1v) = \omega_3(\xi^M, v) = \langle D_x\mu_3(v), \xi\rangle = 0,\\
\langle D_x\mu_3(J_1v), \xi\rangle &= \omega_3(\xi^M, J_1v) = -\omega_2(\xi^M, v) = -\langle D_x\mu_2(v), \xi\rangle = 0,\\
g(\xi^M, J_1v) &= -\omega_1(\xi^M, v) = -\langle D_x\mu_1(v), \xi\rangle = 0,
\end{aligned}$$

where $(\xi^M)_x = D_e r_x(\xi)$ is the infinitesimal action of ξ on M. This shows that $J_1v \in \ker D_x\widetilde{\mu} = T_xN$ and $J_1v \in (\operatorname{im} D_e r_x)^\perp = H_x$, so H_x is J_1–invariant. Analogously H_x is invariant under J_2 and J_3. Thus we can define the induced almost complex structures $\widehat{J}_1, \widehat{J}_2, \widehat{J}_3$ on N/G by restricting J_1, J_2, J_3 to H_x and identifying H_x with the tangent space $T_{[x]}(N/G)$. The independence of the choice of $x \in [x]$ follows from G–invariance of J_i.

The metric g and symplectic forms $\omega_1, \omega_2, \omega_3$ carry over to $\widehat{g}, \widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3$ on N/G in the same way, i.e. by restricting to H_x and identifying this with $T_{[x]}(N/G)$ via $D_x p$. It is easy to see that $\widehat{\omega}_i = \widehat{g}(\widehat{J}_i\_, \_)$ holds, $\widehat{g}$ is a Riemannian metric, $\widehat{\omega}_i$ is non–degenerate and $\widehat{J}_i^2 = -1$. To show (7), split the tangent space $T_x N$ into its horizontal part H_x and its vertical part $V_x = \ker D_x p = \operatorname{im} D_e r_x$. By the definition of $\widehat{\omega}_i$ the identity (7) holds on the horizontal component. If however at least one of $X, Y \in T_x N$ is in the vertical component, say $X \in V_x$, then clearly $p^*\widehat{\omega}_i(X, Y) = 0$ but also $X = D_e r_x(\xi) = \xi^M$ for some $\xi \in \mathfrak{g}$, so

$$\omega_i(X, Y) = \omega_i(\xi^M, Y) = \langle D_x \mu_i(Y), \xi \rangle = 0.$$

This shows (7). Note however that its analogue for the metric $\widehat{g}$ only holds on the horizontal subspace. Using (7) we find that $p^*\mathrm{d}\widehat{\omega}_i = \iota^*\mathrm{d}\omega_i = 0$ and thus $\mathrm{d}\widehat{\omega}_i = 0$ since p is a submersion. So $(M /\!\!/\!\!/ G, \widehat{g}, \widehat{J}_1, \widehat{J}_2, \widehat{J}_3)$ is indeed a Hyperkähler manifold. □

3 Construction of the moduli space

3.1 Sobolev sections of vector bundles

To interpret the gauge transformations as an action of a Hilbert Lie group on a Hilbert manifold, we first have to find an appropriate Hilbert Lie group of unitary bundle automorphisms of an Hermitian vector bundle E. This can be achieved by completing $\Gamma(\mathrm{End}\ E)$ with respect to some scalar product norm and then restricting it to invertible elements preserving the Hermitian bundle metric.

The simplest candidate for such a norm would be the L^2–norm. Completing with respect to this norm would make $\Gamma(\mathrm{End}\ E)$ a Hilbert space, but it would not necessarily be closed under multiplication. As an example in Euclidean space, the real function $|x|^{-n/4}$ is L^2 over a bounded n–dimensional base containing 0, but its square $|x|^{-n/2}$ is not.

This problem can be solved by instead completing with respect to a Sobolev norm of sufficiently high order, as a consequence of Hölder's and Sobolev's inequalities. To do this, let us first define Sobolev spaces of sections of vector bundles. Manifolds and vector bundles are finite–dimensional in this section and $E \to M$ always is a real or complex vector bundle over a compact oriented Riemannian manifold (M, g) with a symmetric/Hermitian bundle metric $\langle \_\,, \_ \rangle$.

If E, F are two real or complex vector bundles both equipped with a bundle metric, there are also induced metrics on $E \oplus F$ and $E \otimes F$ defined by

$$\langle (v, w), (v', w') \rangle = \langle v, v' \rangle + \langle w, w' \rangle, \qquad \langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \langle w, w' \rangle.$$

There is further an induced metric on the dual bundle E^* given by $\langle \alpha, \beta \rangle = \langle \alpha^\sharp, \beta^\sharp \rangle$, where $\alpha^\sharp \in E_x$ is the unique vector with $\langle \alpha^\sharp, v \rangle = \alpha(v)$ for all $v \in E_x$. The induced metric on $\mathrm{Hom}(E, F) = E^* \otimes F$ can alternatively be written in the form

$$\langle A, B \rangle = \mathrm{tr}(A^* B), \qquad \forall A, B \in \mathrm{Hom}(E_x, F_x)$$

with $A^* \in \mathrm{Hom}(F_x, E_x)$ defined by $\langle A^* v, w \rangle = \langle v, Aw \rangle$. In particular, $\langle \_\,, \_ \rangle$ on E and g on TM induce bundle metrics on a variety of derived vector bundles, e.g. on $T^*M \otimes \mathrm{End}\ E$.

Furthermore, we consider the subspaces

$$\mathfrak{u}\, E = \{A \in \mathrm{End}\ E \mid A^* = -A\} \quad \text{and} \quad i\,\mathfrak{u}\, E = \{A \in \mathrm{End}\ E \mid A^* = A\}$$

of $\mathrm{End}\ E$. If E is a complex vector bundle, these are real subbundles of the complex vector bundle $\mathrm{End}\ E$. As the restriction of the above metric on $\mathrm{End}\ E$ to these bundles takes only real values, it still gives a valid scalar product on them. A slight difficulty arises however when we consider the direct sum of a real bundle E and a complex bundle F. Then $E \oplus F$ only carries the structure of a real vector bundle, so its scalar product should take on only real values. This can e.g. be solved by just taking the real part of the complex metric, i.e. by defining

$$\langle (v, w), (v', w') \rangle = \langle v, v' \rangle + \mathrm{Re}\langle w, w' \rangle \qquad \forall v, v' \in E_x, w, w' \in F_x.$$

Definition 3.1. The $W^{k,p}$ *norm* and the C^l *norm* on $\Gamma(E)$ are defined by

$$\|s\|^p_{W^{k,p}} = \sum_{i=0}^{k} \int_M |\nabla^i s|^p \,\omega, \qquad \|s\|_{C^l} = \sum_{i=0}^{l} \sup_{x \in M} |\nabla^i s(x)|,$$

where ω is the Riemannian volume form on M and $\nabla^i\colon \Gamma(E) \to \Gamma((T^*M)^{\otimes i} \otimes E)$ is the composition

$$\Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M \otimes E) \xrightarrow{\nabla^{T^*M\otimes E}} \cdots \to \Gamma((T^*M)^{\otimes i} \otimes E).$$

One also calls the $W^{0,p}$ norm L^p and the $W^{k,2}$ norm H^k. The H^k norms arise from the scalar products

$$\langle s, t\rangle_{H^k} = \sum_{i=0}^{k} \int_M \langle \nabla^i s, \nabla^i t\rangle\, \omega.$$

Definition 3.2.

- For all $l \in \mathbb{N}$ we denote the *space of l–times differentiable sections of E* by $\Gamma_{C^l}(E)$. The C^l–norm can be defined on this space as in Definition 3.1 and makes it a Banach space with $\Gamma(E)$ as a dense subspace.
- The completion $\Gamma_k(E)$ of $\Gamma(E)$ with respect to the H^k–norm is called the *space of k–times weakly differentiable sections of E*. The H^k–norm extends to this space by continuity and makes it a Hilbert space with $\Gamma(E)$ as a dense subspace.

Lemma 3.3 (Sobolev embedding theorem, [Weh04, Theorem B.2])**.** *The following inequalities hold for all $s \in \Gamma(E)$:*

- $\|s\|_{C^l} \le C\|s\|_{W^{k,p}}$ *if $l < k - n/p$.*
- $\|s\|_{W^{l,q}} \le C\|s\|_{W^{k,p}}$ *if $l < k$ and $l - n/q \le k - n/p$.*

where the constants $C > 0$ depend only on p, k, l, n and the bundle E, but not on s. Furthermore, the first inclusion is a compact operator, and the second is also compact if $l - n/q < k - n/p$.

Remark 3.4. As a consequence of Lemma 3.3, for any $s \in \Gamma_k(E)$ with $k > n/2$, defined by a H^k–Cauchy sequence in $\Gamma(E)$, there is a unique C^0–limit of this sequence in $\Gamma_{C^0}(E)$. This gives an embedding $\Gamma_k(E) \hookrightarrow \Gamma_{C^0}(E)$, which allows to evaluate H^k–sections at points. We will use identifications of this sort, sometimes without further mentioning.

Lemma 3.5 (Hölder's inequality)**.** *Let E, F, G be Hermitian vector bundles and $B\colon \Gamma(E) \times \Gamma(F) \to \Gamma(G)$ a $C^\infty(M)$–bilinear map (or equivalently a section $B \in \Gamma(\mathrm{Hom}(E, F; G))$) and let $1/r = 1/p + 1/q$. Then*

$$\|B(s,t)\|_{L^r} \le C\|s\|_{L^p}\|t\|_{L^q}$$

for some constant $C > 0$ and all $s \in \Gamma(E)$ and $t \in \Gamma(F)$.

Proof. Since $1/(p/r) + 1/(q/r) = 1$ and $|B_x(s_x, t_x)| \le |B_x||s_x||t_x|$ at every point $x \in M$, the classical Hölder inequality implies

$$\int_M |B(s,t)|^r \omega \le C^r \int_M |s|^r |t|^r \omega \le C^r \||s|^r\|_{L^{p/r}} \||t|^r\|_{L^{q/r}} = \|s\|_{L^p}^r \|t\|_{L^q}^r,$$

where $C = \sup_{x\in M} |B_x|$, which is finite since M is compact. □

Theorem 3.6 (Sobolev multiplication theorem)**.** *Let $E_1, \dots, E_r, F$ be Hermitian vector bundles and $M\colon \Gamma(E_1) \times \cdots \times \Gamma(E_r) \to \Gamma(F)$ a $C^\infty(M)$–multilinear map. Then for $k > n/2$ there is a unique continuous extension*

$$\Gamma_k(E_1) \times \cdots \times \Gamma_k(E_r) \to \Gamma_k(F)$$

of M which is in fact smooth.

Proof of Theorem 3.6. We assume that $r = 2$, $E_1 = \mathrm{Hom}(E_2, F)$ and $M(A, v) = Av$ is the usual application of linear functions. We want to prove that

$$\|(\nabla^m A)(\nabla^\ell v)\|_{L^2} \leq C\|A\|_{H^k}\|v\|_{H^k}$$

for all $m + \ell \leq k$ with $A \in \Gamma(\mathrm{Hom}(E_2, F))$, $v \in \Gamma(E_2)$ and some constant C independent of A and v.

If $k - m > n/2$, we have

$$\|(\nabla^m A)(\nabla^\ell v)\|_{L^2} \leq \|\nabla^m A\|_{C^0}\|\nabla^\ell v\|_{L^2} \leq C\|\nabla^m A\|_{H^{k-m}}\|v\|_{H^\ell} \leq C\|A\|_{H^k}\|v\|_{H^k}$$

from Lemma 3.3. If instead $k - \ell > n/2$, the same argument can be used with the roles of A and v interchanged. So assume $k - m \leq n/2$ and $k - \ell \leq n/2$. Then p, q defined by

$$\frac{1}{p} = \frac{1}{2} - \frac{k-m}{n} + \varepsilon, \quad \frac{1}{q} = \frac{1}{2} - \frac{k-\ell}{n} + \varepsilon, \quad \varepsilon = \min\left\{\frac{k}{2n} - \frac{1}{4}, \frac{1}{2} + \frac{k-m}{n}, \frac{1}{2} + \frac{k-\ell}{n}\right\}$$

are in the interval $[1, \infty)$ and $1/p + 1/q \leq 1/2$. Also this ensures $-n/p \leq (k-m) - n/2$ and $-n/q \leq (k-\ell) - n/2$. Using Hölder's inequality and Sobolev embeddings,

$$\begin{aligned}\|(\nabla^m A)(\nabla^\ell v)\|_{L^2} &\leq C_1\|(\nabla^m A)(\nabla^\ell v)\|_{L^{\frac{1}{1/p+1/q}}} \leq C_1\|\nabla^m A\|_{L^p}\|\nabla^\ell v\|_{L^q}\\ &\leq C_2\|\nabla^m A\|_{H^{k-m}}\|\nabla^\ell v\|_{H^{k-\ell}} \leq C_2\|A\|_{H^k}\|v\|_{H^k}.\end{aligned}$$

Since, for all $i \leq k$, $\nabla^i(Av)$ can be represented as a linear combination of such terms of the form $(\nabla^m A)(\nabla^\ell v)$ using the Leibniz rule, the inequality $\|Av\|_{H^k} \leq C\|A\|_{H^k}\|v\|_{H^k}$ follows. This implies the statement for the case $E_1 = \mathrm{Hom}(E_2, F)$ and yields a continuous map

$$\Gamma_k(\mathrm{Hom}(E_2, F)) \times \Gamma_k(E_2) \to \Gamma_k(F)$$

extending $(A, v) \mapsto Av$. Applying this twice gives a continuous map

$$\Gamma_k(\mathrm{Hom}(E_1, \mathrm{Hom}(E_2, F))) \times \Gamma_k(E_1) \times \Gamma_k(E_2) \to \Gamma_k(\mathrm{Hom}(E_2, F)) \times \Gamma_k(E_2) \to \Gamma_k(F)$$

and by iterating we even get a continuous map

$$\Gamma_k(\mathrm{Hom}(E_1, \dots, E_r; F)) \times \Gamma_k(E_1) \times \cdots \times \Gamma_k(E_r) \to \Gamma_k(F)$$

extending $(M, v_1, \dots, v_r) \mapsto M(v_1, \dots, v_r)$. By inserting the predefined M from the statement in the first argument, we get the desired result. As a multilinear continuous function, the resulting map is also smooth. □

Theorem 3.6 implies that $\Gamma_k(\mathrm{End}\ E)$ for $k > n/2$ is a Hilbert algebra with pointwise composition of endomorphisms as multiplication. The set of its invertible elements is therefore an appropriate completion of the space of smooth bundle automorphisms. There will however occur some bilinear bundle homomorphisms on Sobolev spaces of an order too low to be covered by Theorem 3.6.

Proposition 3.7. *Let M be a compact surface and $B\colon \Gamma(E_1) \times \Gamma(E_2) \to \Gamma(F)$ a bilinear morphism of vector bundles E_1, E_2, F on M and let $p > 2$. Then B extends to smooth maps*

$$\Gamma_1(E_1) \times \Gamma_1(E_2) \to \Gamma_0(F) \qquad \Gamma_2(E_1) \times \Gamma_1(E_2) \to \Gamma_1(F) \qquad \Gamma_{1,p}(E_1) \times \Gamma_1(E_2) \to \Gamma_1(F),$$

where $\Gamma_{1,p}(E_1)$ is the space of $W^{1,p}$–sections of E_1.

Proof. The first extension uses the continuous embedding $H^1 \hookrightarrow L^q$ for any q and Hölder's inequality. The second and third follow from the Sobolev embeddings $H^2 \hookrightarrow C^0$ and $W^{1,p} \hookrightarrow C^0$ after taking the derivative. □

Corollary 3.8. *For $k > n/2$ the set*

$$\Gamma_k(\mathrm{GL}(E)) = \{g \in \Gamma_k(\mathrm{End}\ E) \mid \exists g^{-1} \in \Gamma_k(\mathrm{End}\ E)\colon gg^{-1} = g^{-1}g = \mathrm{id}\}$$

of invertible H^k–endomorphisms of E is open in $\Gamma_k(\mathrm{End}\ E)$ and $\Gamma(\mathrm{GL}(E)) \subset \Gamma_k(\mathrm{GL}(E))$ is dense. With the composition of endomorphisms as multiplication, $\Gamma_k(\mathrm{GL}(E))$ is a Hilbert Lie group.

Proof. A standard argument using Neumann series shows that invertibility is an open condition. That $\Gamma(\mathrm{GL}(E))$ is dense follows easily from the fact that $\Gamma(\mathrm{GL}(E)) = \Gamma_k(\mathrm{GL}(E)) \cap \Gamma(\mathrm{End}\ E)$. If $g, h \in \Gamma_k(\mathrm{GL}(E))$, then $gh(gh)^{-1} = ghh^{-1}g^{-1} = \mathrm{id} = h^{-1}g^{-1}gh = (gh)^{-1}gh$, so $gh \in \Gamma_k(\mathrm{GL}(E))$. As $\Gamma_k(\mathrm{GL}(E))$ is an open subset of $\Gamma_k(\mathrm{End}\ E)$, it is a smooth manifold and multiplication is smooth. Inverses exist by definition. □

Definition 3.9. The space of *smooth unitary automorphisms* of E is

$$\Gamma(\mathrm{U}\,E) = \{g \in \Gamma(\mathrm{End}\ E) \mid gg^* = g^*g = \mathrm{id}\}.$$

For $k > n/2$ the space of *k–times weakly differentiable unitary automorphisms* of E is the set

$$\Gamma_k(\mathrm{U}\,E) = \{g \in \Gamma_k(\mathrm{End}\ E) \mid gg^* = g^*g = \mathrm{id}\}.$$

Proposition 3.10. $\Gamma_k(\mathrm{U}\,E)$ *is a closed Hilbert submanifold of* $\Gamma_k(\mathrm{End}\ E)$.

Proof. As $A \mapsto (A^* - A)$ is a continuous linear endomorphism of $\Gamma_k(\mathrm{End}\ E)$, the subspace $\Gamma_k(i\mathfrak{u}\,E) = \{A \in \Gamma_k(\mathrm{End}\ E) \mid A^* = A\}$ is closed. Consider the map

$$\Phi\colon \Gamma_k(\mathrm{GL}(E)) \to \Gamma_k(i\mathfrak{u}\,E), \quad A \mapsto A^*A.$$

As $\Gamma_k(\mathrm{GL}(E))$ is an open subset of $\Gamma_k(\mathrm{End}\ E)$, we have $T_A\Gamma_k(\mathrm{GL}(E)) \cong \Gamma_k(\mathrm{End}\ E)$ and the derivative at any $A \in \Gamma_k(\mathrm{U}(E))$ is

$$D_A\Phi\colon \Gamma_k(\mathrm{End}\ E) \to \Gamma_k(i\mathfrak{u}\,E), \quad X \mapsto X^*A + A^*X.$$

It is surjective (insert $\frac{1}{2}AY$ to get Y), so Φ is transversal over $\{\mathrm{id}\} \in \Gamma_k(i\mathfrak{u}\,E)$ and therefore $\Phi^{-1}(\{\mathrm{id}\}) = \Gamma_k(\mathrm{U}\,E)$ is an embedded submanifold of $\Gamma_k(\mathrm{GL}(E))$. □

Definition 3.11. The space $\mathscr{A}(E)$ of unitary connections on E is an affine space modeled on $\Omega^1(\mathfrak{u}\, E) = \Gamma(\Lambda^1 \otimes \mathfrak{u}\, E)$. Its H^k–completion $\mathscr{A}_k(E)$ is therefore an affine space modeled on $\Omega^1_k(\mathfrak{u}\, E) = \Gamma_k(\Lambda^1 \otimes \mathfrak{u}\, E)$ and in particular a smooth Hilbert manifold, for all $k \in \mathbb{N}$.

The spaces $\Gamma_k(\mathrm{U}\, E)$, $\Gamma_k(\mathfrak{u}\, E)$ etc. have several equivalent characterizations. They are completions of the respective spaces of smooth sections $\Gamma(\mathrm{U}\, E)$, $\Gamma(\mathfrak{u}\, E)$ in the Sobolev norm. They can also be described as the subset of $\Gamma_k(\mathrm{End}\ E)$ given by an algebraic condition like $A^*A = AA^* = 1$ or $A^* = -A$. For $k > n/2$, a third possibility is to describe them as subsets of the space of continuous sections, e.g. $\Gamma_k(\mathrm{U}\, E) = \Gamma_{C^0}(\mathrm{U}\, E) \cap \Gamma_k(\mathrm{End}\ E)$. All these definitions coincide up to obvious isomorphisms and we will use them deliberately without further mentioning.

Definition 3.12. We use the following shorthand notations:

$$\begin{aligned} \mathscr{C} &= \mathscr{A}(E) \times \Omega^{1,0}(\mathrm{End}\ E), & \mathscr{C}_k &= \mathscr{A}_k(E) \times \Omega^{1,0}_k(\mathrm{End}\ E), \\ \mathscr{D} &= \Omega^2(\mathfrak{u}\, E) \oplus \Omega^2(\mathrm{End}\ E), & \mathscr{D}_k &= \Omega^2_k(\mathfrak{u}\, E) \oplus \Omega^2_k(\mathrm{End}\ E), \\ \mathscr{G} &= \Gamma(\mathrm{U}\, E), & \mathscr{G}_k &= \Gamma_k(\mathrm{U}\, E). \end{aligned}$$

3.2 The Hyperkähler structure

Let now M be a compact connected Riemann surface (i.e. a complex manifold of real dimension two) with complex structure j, metric g and volume form ω and $E \to M$ a complex vector bundle equipped with a Hermitian bundle metric. We will describe Hyperkähler structures on section spaces of some bundles which ultimately lead to the Hyperkähler structure on the moduli space of Higgs bundles. Let $k \in \mathbb{N}$ be a fixed integer throughout this section.

3.2.1 The Hyperkähler structure on $\Omega^1(\mathrm{End}\ E)$

The space $\Omega^1(\mathrm{End}\ E)$ has a comparatively simple Hyperkähler structure. To make it a 'real' Hyperkähler manifold in the sense of Section 2 we first have to complete it with respect to a suitable Hilbert space norm. So we consider $\Omega^1_k(\mathrm{End}\ E)$. Since all relevant constructions will be pointwise (at least for sufficiently regular sections), it is worthwhile examining a single fiber $\Lambda^1 \otimes \mathrm{End}\ E_x$. The natural scalar product on $\Lambda^1 \otimes \mathrm{End}\ E$ is defined by

$$\langle \alpha_1 \otimes \psi_1, \alpha_2 \otimes \psi_2 \rangle = \langle \alpha_1, \alpha_2 \rangle \operatorname{tr}(\psi_1^* \psi_2) \qquad \forall \alpha_1, \alpha_2 \in T^*_x M,\ \psi_1, \psi_2 \in \mathrm{End}\ E_x$$

and extended bilinearly. We get a real scalar product (and therefore a Riemannian metric) G by using its real part $G = \mathrm{Re}\langle \_, \_ \rangle$.

There is an alternative expression for G. We write j for the complex structure on the Riemann surface M, to distinguish it from the Hyperkähler structure, for which we will use uppercase letters. The induced anti–involutions on T^*M, $\Lambda^1 \otimes \mathrm{End}\ E_x$, etc. will also be denoted by j. Since the metric and orientation on M together already define its complex structure, we expect j to be expressible in terms of these data. We know from Section 1.2 that $j = -\star$. This allows us to rewrite the scalar product of $\Psi_1, \Psi_2 \in \Lambda^1 \otimes \mathrm{End}\ E_x$ in the form

$$\langle \Psi_1, \Psi_2 \rangle\, \omega = -\operatorname{tr}(\Psi_1^* \wedge j\Psi_2).$$

In the decomposition $\Lambda^1 \otimes \mathrm{End}\ E_x = (\Lambda^{0,1} \otimes \mathrm{End}\ E_x) \oplus (\Lambda^{1,0} \otimes \mathrm{End}\ E_x)$ this can be written

$$\langle(\Psi_1, \Phi_1), (\Psi_2, \Phi_2)\rangle = i\,\mathrm{tr}(\Psi_1^* \wedge \Psi_2) - i\,\mathrm{tr}(\Phi_1^* \wedge \Phi_2) = i\,\mathrm{tr}(\Psi_1^* \wedge \Psi_2) - i\,\overline{\mathrm{tr}(\Phi_1 \wedge \Phi_2^*)}$$

for $\Psi_1, \Psi_2 \in \Lambda^{0,1} \otimes \mathrm{End}\ E_x$ and $\Phi_1, \Phi_2 \in \Lambda^{1,0} \otimes \mathrm{End}\ E_x$. Its real part is

$$G((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) = \mathrm{Re}\langle(\Psi_1, \Phi_1), (\Psi_2, \Phi_2)\rangle = -\,\mathrm{Im}\,\mathrm{tr}(\Psi_1^* \wedge \Psi_2 + \Phi_1 \wedge \Phi_2^*).$$

To make G a Hyperkähler metric we need 3 compatible complex structures J_1, J_2, J_3. One of them is just given by scalar multiplication with i, while the other two are defined using the decomposition into $(0,1)$– and $(1,0)$–parts:

$$J_1(\Psi, \Phi) = (i\Psi, i\Phi) \qquad J_2(\Psi, \Phi) = (\Phi^*, -\Psi^*) \qquad J_3(\Psi, \Phi) = (i\Phi^*, -i\Psi^*)$$

It is easy to check that these complex structures satisfy the quaternion relations and leave G invariant. The induced symplectic forms $\omega_i = G(J_i\_, \_)$ are trivially closed since G, J_1, J_2, J_3 are constant. So $\Lambda^1 \otimes \mathrm{End}\ E_x$ is a (linear) Hyperkähler manifold.

Now let us return to the space $\Omega^1_k(\mathrm{End}\ E)$ of Sobolev sections. The complex structures J_1, J_2, J_3 defined above combine to complex structures on $\Omega^1_k(\mathrm{End}\ E)$, also denoted by J_1, J_2, J_3. The corresponding Hyperkähler metric, which we will again call G, is just the real part of the natural L^2–scalar product

$$G((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) = \mathrm{Re}\int_M \langle(\Psi_1, \Phi_1), (\Psi_2, \Phi_2)\rangle\, \omega = -\,\mathrm{Im}\int_M \mathrm{tr}(\Psi_1^* \wedge \Psi_2 + \Phi_1 \wedge \Phi_2^*).$$

The above consideration of $\Lambda^1 \otimes \mathrm{End}\ E_x$ shows that with this metric and complex structures, $\Omega^1_k(\mathrm{End}\ E)$ is indeed an infinite–dimensional Hyperkähler manifold.

3.2.2 The Hyperkähler structure on $\mathscr{A}(E) \times \Omega^{1,0}(\mathrm{End}\ E)$

There is a Hyperkähler structure on the space $\mathscr{C} = \mathscr{A}(E) \times \Omega^{1,0}(\mathrm{End}\ E)$, or more precisely its Sobolev completion $\mathscr{C}_k = \mathscr{A}_k(E) \times \Omega^{1,0}_k(\mathrm{End}\ E)$, which is closely related to that on $\Omega^1(\mathrm{End}\ E)$ described in Section 3.2.1. The tangent space to $\mathscr{C}_k$ at any point is naturally identified with $\Omega^1_k(\mathfrak{u}\,E) \oplus \Omega^{1,0}_k(\mathrm{End}\ E)$. The projection of 1–forms to their $(0,1)$–part induces an isomorphism

$$\Omega^1_k(\mathfrak{u}\,E) \to \Omega^{0,1}_k(\mathrm{End}\ E), \quad \eta \mapsto \eta^{0,1} = \tfrac{1}{2}\eta + \tfrac{i}{2}j\eta \tag{8}$$

which is inverted by $\Psi \mapsto \Psi - \Psi^*$. This follows easily from the fact that the complex structure j commutes with taking the adjoint. Using this isomorphism, the tangent space of $\mathscr{C}_k$ can be identified with $\Omega^{0,1}_k(\mathrm{End}\ E) \times \Omega^{1,0}_k(\mathrm{End}\ E) = \Omega^1_k(\mathrm{End}\ E)$ and therefore inherits the metric and complex structures from $\Omega^1_k(\mathrm{End}\ E)$. This makes $\mathscr{C}_k$ an infinite–dimensional Hyperkähler manifold. Written out explicitly, the metric G and the complex structures J_1, J_2, J_3 are at every point in $\mathscr{A}_k(E) \oplus \Omega^{1,0}_k(\mathrm{End}\ E)$ given by

$$\begin{aligned} G((\eta_1, \Phi_1), (\eta_2, \Phi_2)) &= \frac{1}{2}\int_M \mathrm{tr}(\eta_1 \wedge j\eta_2) - \mathrm{Im}\int_M \mathrm{tr}(\Phi_1 \wedge \Phi_2^*), \\ J_1(\eta, \Phi) &= (-j\eta, i\Phi), \\ J_2(\eta, \Phi) &= (\Phi^* - \Phi, \tfrac{1}{2}\eta - \tfrac{i}{2}j\eta), \\ J_3(\eta, \Phi) &= (i\Phi^* + i\Phi, \tfrac{i}{2}\eta + \tfrac{1}{2}j\eta). \end{aligned}$$

This results in the symplectic forms

$$\begin{aligned}
\omega_1((\eta_1,\Phi_1),(\eta_2,\Phi_2)) &= -\frac{1}{2}\int_M \operatorname{tr}(\eta_1\wedge\eta_2) - \operatorname{Re}\int_M \operatorname{tr}(\Phi_1\wedge\Phi_2^*),\\
\omega_2((\eta_1,\Phi_1),(\eta_2,\Phi_2)) &= -\operatorname{Im}\int_M \operatorname{tr}(\Phi_1\wedge\eta_2+\eta_1\wedge\Phi_2),\\
\omega_3((\eta_1,\Phi_1),(\eta_2,\Phi_2)) &= \operatorname{Re}\int_M \operatorname{tr}(\Phi_1\wedge\eta_2+\eta_1\wedge\Phi_2).
\end{aligned} \tag{9}$$

The identification (8) is motivated by the fact that it is corresponds to the bijection $\mathrm{d}\mapsto\overline{\partial}$ identifying unitary connections with holomorphic structures. So G can be seen as the natural metric on pairs consisting of a holomorphic structure and a Higgs field.

3.3 The gauge group action and its moment maps

In this section, we consider the right action of the gauge group

$$\mathscr{C}\times\mathscr{G}\to\mathscr{C}, \qquad (\nabla,\Phi,g)\mapsto(g^{-1}\circ\nabla\circ g, g^{-1}\Phi g). \tag{10}$$

As it contains derivatives, we now require $k\geq 1$. Fixing $\nabla_0\in\mathscr{A}(E)$ we can extend this action to the respective Sobolev completions by

$$\mathscr{C}_k\times\mathscr{G}_{k+1}\to\mathscr{C}_k, \quad (\nabla_0+\eta,\Phi,g)\mapsto(\nabla_0+g^{-1}(\nabla_0 g)+g^{-1}\eta g, g^{-1}\Phi g) \tag{11}$$

This is well–defined and smooth by Proposition 3.7 and the fact that $g^{-1}(\nabla_0 g)\in\Omega^1_k(\mathfrak{u}\,E)$ and it clearly extends (10). It is also independent of the choice of ∇_0 by the uniqueness of continuous extensions. So (11) is a smooth right action of a Hilbert Lie group on a Hilbert manifold. So we can apply the analogues of the results of Section 2 for right actions to this action. Accordingly, we will now write $\ell_{(\nabla,\Phi)}\colon\mathscr{G}\to\mathscr{C}$ for the evaluation map and $r_g\colon\mathscr{C}\to\mathscr{C}$ for the action of $g\in\mathscr{G}$.

The Lie algebra corresponding to $\mathscr{G}_{k+1}$ is the space $\Gamma_{k+1}(\mathfrak{u}\,E)$ of $(k+1)$–fold weakly differentiable fields of skew–Hermitian endomorphisms. Every $\xi\in\Gamma_{k+1}(\mathfrak{u}\,E)$ generates a vector field $\widehat{\xi}$ of infinitesimal actions on $\mathscr{C}_k$, which is given by

$$\widehat{\xi}_{(\nabla,\Phi)} = D_e\ell_{(\nabla,\Phi)}(\xi) = (\nabla\xi,[\Phi,\xi])$$

The metric and almost complex structures defined in Section 3.2.2 are invariant under this action in the sense that $r_g^*G=G$, $J_i\circ Dr_g = Dr_g\circ J_i$ and $r_g^*\omega_i=\omega_i$ for all $g\in\mathscr{G}_{k+1}$. This is immediately obvious when inserting $Dr_g(\eta,\Psi)=(g^{-1}\eta g, g^{-1}\Psi g)$ into the definitions of these structures. In particular, the action (11) is symplectic with respect to all three symplectic structures.

In fact, the action is Hamiltonian with the moment maps $\mu_i\colon\mathscr{C}_k\to\Gamma_{k+1}(\mathfrak{u}\,E)^*$ given by

$$\langle\mu_1(\nabla,\Phi),\xi\rangle = \frac{1}{2}\int_M \operatorname{tr}\Big(\big(F^\nabla+[\Phi\wedge\Phi^*]\big)\,\xi\Big), \tag{12}$$

$$\langle\mu_2(\nabla,\Phi),\xi\rangle = \operatorname{Im}\int_M \operatorname{tr}\Big(\big(\mathrm{d}^\nabla\Phi\big)\,\xi\Big), \tag{13}$$

$$\langle\mu_3(\nabla,\Phi),\xi\rangle = -\operatorname{Re}\int_M \operatorname{tr}\Big(\big(\mathrm{d}^\nabla\Phi\big)\,\xi\Big). \tag{14}$$

It is easy to see that these maps are smooth and are $\mathscr{G}_{k+1}$–equivariant, i.e. they satisfy $\mu_i(g^*\nabla, g^*\Phi) = c_g^*\mu_i(\nabla, \Phi)$. Using the identities

$$\operatorname{tr}([\Phi,\xi]\wedge\Psi) = -\operatorname{tr}([\Phi\wedge\Psi]\xi)$$

and

$$\mathrm{d}\operatorname{tr}(\Psi\xi) = \operatorname{tr}(\mathrm{d}^\nabla(\Psi\xi)) = \operatorname{tr}\big((\mathrm{d}^\nabla\Psi)\xi\big) + \operatorname{tr}\big(\Psi\wedge\nabla\xi\big),$$

which are valid for any $\xi\in\Gamma(\operatorname{End} E)$ and $\Phi,\Psi\in\Omega^1(\operatorname{End} E)$, we can calculate for the first moment map

$$\begin{aligned}
\mathrm{d}_{(\nabla,\Phi)}\langle\mu_1,\xi\rangle(\eta,\Psi) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\frac{1}{2}\int_M \operatorname{tr}\Big(\big(F^{\nabla+t\eta} + [\Phi+t\Psi\wedge\Phi^* + t\dot{\Phi}^*]\big)\,\xi\Big) = \\
&= \frac{1}{2}\int_M \operatorname{tr}\Big(\big(\mathrm{d}^\nabla\eta + [\Phi\wedge\Psi^*] + [\Psi\wedge\Phi^*]\big)\,\xi\Big) = \\
&= \frac{1}{2}\int_M \operatorname{tr}\Big(\big(\mathrm{d}^\nabla\eta\big)\,\xi\Big) + \operatorname{Re}\int_M \operatorname{tr}\Big([\Phi\wedge\Psi^*]\,\xi\Big) = \\
&= -\frac{1}{2}\int_M \operatorname{tr}(\nabla\xi\wedge\eta) - \operatorname{Re}\int_M \operatorname{tr}([\Phi,\xi]\wedge\Psi^*) = \\
&= (\widehat{\xi}\lrcorner\,\omega_1)(\eta,\Psi).
\end{aligned}$$

So μ_1 is indeed a moment map for the action (11) with respect to the symplectic structure ω_1. A similar calculation can be performed for μ_2:

$$\begin{aligned}
\mathrm{d}_{(\nabla,\Phi)}\langle\mu_2,\xi\rangle(\eta,\Psi) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\operatorname{Im}\int_M \operatorname{tr}\Big(\big(\mathrm{d}^{\nabla+t\eta}(\Phi+t\Psi)\big)\,\xi\Big) = \\
&= \operatorname{Im}\int_M \operatorname{tr}\Big(\big(\mathrm{d}^\nabla\Psi\big)\,\xi + \big([\eta\wedge\Phi]\big)\,\xi\Big) = \\
&= -\operatorname{Im}\int_M \operatorname{tr}\Big(\nabla\xi\wedge\Psi + [\Phi,\xi]\wedge\eta\Big) = \\
&= (\widehat{\xi}\lrcorner\,\omega_2)(\eta,\Psi).
\end{aligned}$$

This calculation is still valid if we replace μ_2 and ω_2 by μ_3 and ω_3 and Im by $-\operatorname{Re}$. So all three μ_i are moment maps for their respective symplectic structures.

The common zero set $\mu_1^{-1}(0)\cap\mu_2^{-1}(0)\cap\mu_3^{-1}(0)$ of the moment maps (12), (13) and (14) is given by exactly those $(\nabla,\Phi)\in\mathscr{C}_k$ with

$$\begin{aligned}
F^\nabla + [\Phi\wedge\Phi^*] &= 0, \\
\mathrm{d}^\nabla\Phi &= 0
\end{aligned}$$

These equations look very similar to Hitchin's self–duality equations (3), (4), but they are missing the constant term $c\,\mathrm{id}\otimes\omega$ accounting for bundles E with non–zero degree. As shown in Lemma 2.19, the moment maps (12), (13), (14) are not unique, but can be modified by a constant element of $\Gamma(\mathfrak{u}\,E)^*$ fixed by the coadjoint action. Such an object can be represented by a 2–form $\Psi\in\Omega^2(\mathfrak{u}\,E)$ with values in the skew–Hermitian endomorphisms which satisfies

$$\int_M \operatorname{tr}(g\Psi g^{-1}\xi) = \int_M \operatorname{tr}(\Psi g^{-1}\xi g) = \int_M \operatorname{tr}(\Psi\xi) \quad \forall\xi\in\Gamma(\mathfrak{u}\,E).$$

This is equivalent to $[g, \Psi] = 0$ for all $\Gamma(\mathrm{U}\,E)$, which is clearly true if and only if $\Psi = if\,\mathrm{id}_E \otimes \omega$ for some real function $f\colon M \to \mathbb{R}$. Modifying the μ_i in such a way, the space $\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$ consists of all $(\nabla, \Phi) \in \mathscr{C}_k$ with

$$\begin{aligned} F^\nabla + [\Phi \wedge \Phi^*] &= if\,\mathrm{id}_E \otimes \omega, \\ \mathrm{d}^\nabla \Phi &= g\,\mathrm{id}_E \otimes \omega, \end{aligned}$$

where $f\colon M \to \mathbb{R}$ and $g\colon M \to \mathbb{C}$ are arbitrary smooth functions. The equations (3) and (4) are recovered by choosing $f = -2\pi\mu(E)/\mathrm{vol}(M)$ and $g = 0$.

3.4 The proper gauge group action

The purpose of this section is to show that the action defined above is proper, so that we can apply the results from Section 2. It turns out that only the action on the space of connections is relevant for this, and we can ignore the Higgs field for most of this section. The proof relies on the following technical lemma. Again, let $k \geq 1$ be a fixed integer. We write I for the closed unit interval $[0, 1]$ and $\mathring{I}$ for the open interval $(0, 1)$.

Lemma 3.13. *Let $R \subset M$ be a coordinate rectangle trivializing E and $\nabla_i \in \mathscr{A}(E)$ a sequence of connections converging in $\mathscr{A}_k(E)$. Then there exists a set $G \subset R$, which is the complement of a Lebesgue null set, such that for every pair $x, y \in G$ the sequence of parallel transport operators*

$$P_\gamma^{\nabla_i} \in \mathrm{Hom}(E_x, E_y)$$

along a rectangular path γ from x to y converges.

Proof. Every ∇_i is locally represented by $\eta_i\colon \mathring{I}^2 \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{C}^{m\times m})$ such that the ∇_i–parallel lift $\widehat{\gamma}_i\colon I \to \mathbb{C}^m$ is defined by the ordinary differential equation

$$\dot{\widehat{\gamma}}_i(t) = -(\eta_i)_{\gamma(t)}(\dot\gamma(t))\,\widehat{\gamma}_i(t).$$

However, we can not directly apply ODE theory, as the limit of the sequence η_i is not necessarily a continuous function, but only H^1. This can be solved by restricting it to the image of a suitably chosen curve and then applying the Sobolev embedding theorem in one dimension.

Note that $\eta_i(\partial_{x_1})\colon \mathring{I}^2 \to \mathbb{C}^{m\times m}$ converges in the H^1–norm to a limit $\eta(\partial_{x_1})$ (choosing an arbitrary representative). Therefore, the sequence

$$\int_0^1 \int_0^1 |\eta_i(x_1, x_2)(\partial_{x_1}) - \eta(x_1, x_2)(\partial_{x_1})|^2 + |\nabla\eta_i(x_1, x_2)(\partial_{x_1}) - \nabla\eta(x_1, x_2)(\partial_{x_1})|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2$$

converges to 0, so in particular the innermost integral is a null sequence for almost every value of $x_2 \in \mathring{I}$. This implies that

$$\xi_i\colon \mathring{I} \to \mathbb{C}^{m\times m}, \quad \xi_i(t) = \eta_i(t, x_2)(\partial_{x_1})$$

is an H^1–convergent sequence, so by the 1–dimensional Sobolev embedding theorem it converges in C^0. Let $G_1 \subset \mathring{I}^2$ be the set of (x_1, x_2) such that x_2 satisfies the above.

Now let $(x_1, x_2), (y_1, y_2) \in G_1$. Assume that they only differ in the first component, i.e. $x_2 = y_2$. Define the path $\gamma\colon I \to R$ by

$$\gamma(t) = (x_1 + t(y_1 - x_1), x_2),$$

then the equation for its ∇_i–parallel lift becomes

$$\dot{\widehat{\gamma}}_i(t) = (y_1 - x_1)\xi_i(x_1 + t(y_1 - x_1))\widehat{\gamma}_i(t).$$

Now the regularity theory of ODEs with parameters in a Banach space [Lan99, Section IV.1] shows that $P_\gamma^{\nabla_i} v = \widehat{\gamma}_i(1)$ for every $v \in E_x$ depends continuously on the parameter $\xi_i \in C^0(\mathbb{R}; \mathbb{C}^{m\times m})$ and therefore converges.

Repeating this procedure for the second coordinate we get a corresponding set $G_2 \subset \mathring{I}^2$. Note that for $(x_1, x_2), (y_1, y_2) \in G = G_1 \cap G_2$, the point (x_1, y_2) is also contained in G, so the parallel transports along the straight path from (x_1, x_2) to (x_1, y_2) as well as from (x_1, y_2) to (y_1, y_2) converge and so does its composition. □

Proposition 3.14. *Let $\nabla_i \in \mathscr{A}_k(E)$ be a sequence of connections and $g_i \in \mathscr{G}_{k+1}$ a sequence of gauge transformations, such that ∇_i and $g_i^*\nabla_i$ both converge in $\mathscr{A}_k(E)$. Then a subsequence of g_i converges in $\mathscr{G}_{k+1}$.*

Proof. We first argue that we can assume $\nabla_i \in \mathscr{A}(E)$ and $g_i \in \mathscr{G}$. Since $\mathscr{A}(E)$ is dense in $\mathscr{A}_k(E)$ and $\mathscr{G}$ in $\mathscr{G}_{k+1}$, we can approximate each ∇_i by a sequence $\nabla_{i,j} \in \mathscr{A}(E)$ and each g_i by a sequence $g_{i,j} \in \mathscr{G}$. By continuity $g_i^*\nabla_{i,j}$ then also converges to $g_i^*\nabla_i$ for every i. Restricting $\{\nabla_{i,j}\}_j$ to a subsequence, we can make its convergence arbitrarily fast. In particular we can achieve that $\|\nabla_{i,j} - \nabla_i\|_{H^k} \le 1/j$ and $\|g_i^*\nabla_{i,j} - g_i^*\nabla_i\|_{H^k} \le 1/j$ for all i, j. Similarly we choose a subsequence of $\{g_{i,j}\}_j$ such that $\|g_{i,j} - g_i\|_{H^{k+1}} \le 1/j$ and $\|g_{i,j}^*\nabla_{i,i} - g_i^*\nabla_{i,i}\|_{H^k} \le 1/j$ for all i, j. Now define $\widetilde{\nabla}_i = \nabla_{i,i}$ and $\widetilde{g}_i = g_{i,i}$ for all $i \in \mathbb{N}$. Then

$$\|\widetilde{g}_i - g_i\| \le 1/i, \quad \|\widetilde{\nabla}_i - \nabla_i\| \le 1/i, \quad \|\widetilde{g}_i^*\widetilde{\nabla}_i - g_i^*\nabla_i\| \le \|\widetilde{g}_i^*\widetilde{\nabla}_i - g_i^*\widetilde{\nabla}_i\| + \|g_i^*\widetilde{\nabla}_i - g_i^*\nabla_i\| \le 2/i,$$

so $\widetilde{g}_i - g_i$, $\widetilde{\nabla}_i - \nabla_i$ and $\widetilde{g}_i^*\widetilde{\nabla}_i - g_i^*\nabla_i$ are null sequences. Now let $\nabla, \nabla' \in \mathscr{A}_k(E)$ be the limits of the sequences ∇_i and $g_i^*\nabla_i$. Then $\widetilde{\nabla}_i$ and $\widetilde{g}_i^*\widetilde{\nabla}_i$ also converge to ∇ and ∇'. On the other hand, if $\widetilde{g}_i$ has a convergent subsequence, the corresponding subsequence of g_i also converges. So we can assume without loss of generality that $\nabla_i \in \mathscr{A}(E)$ and $g_i \in \mathscr{G}$.

Let now $\{(U_j, \varphi_j)\}_{j=1,\dots,N}$ be a cover of M by open coordinate rectangles and corresponding charts. By Lemma 3.13 there is a set of good points $G \subset U_j$ such that parallel transports along rectangular paths between good points with respect to the connections ∇_i converge. Analogously, there is a set $G' \subset U_j$ of good points for the sequence $\nabla'_i = g_i^*\nabla_i$. Both sets G and G' have full measure in U_j and so does their intersection $G \cap G'$. Choose any $x_j \in G \cap G' \subset U_j$. As $\mathrm{U}\, E_{x_j}$ is a compact set, the sequence $\{(g_i)_{x_j}\}_i$ has a convergent subsequence for every j. Passing to a subsequence which converges at every x_j, we now want to show that g_i converges pointwise almost everywhere.

To do this, let $y \in G \cap G'$ and let $\gamma\colon I \to U_j$ be a rectangular path from x_j to y. If $\widehat{\gamma}\colon I \to E$ is the ∇'_i–parallel lift of γ, i.e. $\nabla'_i\widehat{\gamma} = 0$, then the curve $\widetilde{\gamma}(t) = (g_i)_{\gamma(t)}\widehat{\gamma}(t)$ is ∇_i–parallel. So the parallel transports are related by

$$P_\gamma^{\nabla'_i} v = \widehat{\gamma}(1) = (g_i)_y^{-1}\widetilde{\gamma}(1) = (g_i)_y^{-1} P_\gamma^{\nabla_i}(g_i)_x v$$

and thus $(g_i)_y = P_\gamma^{\nabla_i}(g_i)_{x_j}(P_\gamma^{\nabla'_i})^{-1}$. Now by Lemma 3.13 both parallel transports converge and by the above paragraph $(g_i)_{x_j}$ also converges. So g_i converges at every point $y \in G \cap G'$. Repeating this argument for every U_j shows that g_i converges almost everywhere pointwise.

Let $\nabla \in \mathscr{A}_k(E)$ be the limit of ∇_i and $\nabla' \in \mathscr{A}_k(E)$ the limit of $\nabla'_i = g_i^* \nabla_i$. Choose a reference connection $\nabla_0 \in \mathscr{A}(E)$ and let $\eta_i = \nabla_i - \nabla_0$ and $\eta'_i = \nabla'_i - \nabla_0$. Then $\eta_i, \eta'_i \in \Omega^1(\mathfrak{u}\,E)$ satisfy

$$\nabla_0 g_i = g_i \eta'_i - \eta_i g_i \tag{15}$$

since $\nabla_0 + \eta'_i = g_i^* \nabla_i = \nabla_0 + g_i^{-1}(\nabla_0 g_i) + g_i^{-1}\eta_i g_i$.

Since $(g_i)_y$ is unitary for every $y \in M$, its (Frobenius–) norm is bounded by $\sqrt{\operatorname{rk} E}$, which is 8–integrable on the compact manifold M. So the L^8 dominated convergence theorem implies that g_i does in fact converge in L^8. Since η_i and η'_i also converge in the L^8–topology by the Sobolev embedding theorem, the right hand side of (15) then converges in L^4, so g_i even converges in $W^{1,4}$. Now by Proposition 3.7 (15) converges in H^1, so g_i converges in H^2. From now on, we can use the Sobolev multiplication theorem to iterate further, ultimately leading to a convergence of g_i in the H^{k+1}–topology, which was the statement we wanted to prove. ☐

Corollary 3.15. *The group action*

$$\mathscr{C}_k \times \mathscr{G}_{k+1} \to \mathscr{C}_k, \quad (\nabla, \Phi, g) \mapsto (g^{-1} \circ \nabla g, g^{-1} \Phi g)$$

is proper.

Proof. This follows from Proposition 3.14 since convergence $(\nabla_i, \Phi_i) \to (\nabla, \Phi)$ in $\mathscr{C}_k$ implies the convergence $\nabla_i \to \nabla$ in $\mathscr{A}_k(E)$. ☐

3.5 The infinitesimal Hitchin equations

Take any smooth solution $(\nabla, \Phi) \in \mathscr{X}$ of Hitchin's equations and consider the sequence ($\mathfrak{H}$ being Hitchin's equations)

$$T_e \mathscr{G}_{k+1} \xrightarrow{D_e \ell_{(\nabla,\Phi)}} T_{(\nabla,\Phi)} \mathscr{C}_k \xrightarrow{D_{(\nabla,\Phi)} \mathfrak{H}} T_0 \mathscr{D}_{k-1}. \tag{16}$$

This is a complex by gauge–invariance with first–order differential operators as maps. More precisely, it arises as the Sobolev extension of the complex

$$0 \to \Gamma(\mathfrak{u}\,E) \xrightarrow{\mathrm{d}_1} \Omega^1(\mathfrak{u}\,E) \oplus \Omega^{1,0}(\mathrm{End}\ E) \xrightarrow{\mathrm{d}_2} \Omega^2(\mathfrak{u}\,E) \oplus \Omega^2(\mathrm{End}\ E) \to 0 \tag{17}$$

with the operators d_1 and d_2 are defined by

$$\mathrm{d}_1 \psi = (\nabla \psi, [\Phi, \psi]) \qquad \mathrm{d}_2(\eta, \Psi) = (\mathrm{d}^\nabla \eta + [\Phi \wedge \Psi^*] + [\Psi \wedge \Phi^*], \mathrm{d}^\nabla \Psi + [\eta \wedge \Phi]).$$

These expressions can be calculated by inserting infinitesimal curves into $\ell_{(\nabla,\Phi)}$ and $\mathfrak{H}$, and then taking the derivative at $t = 0$. The associated complex of symbols is

$$0 \to \mathfrak{u}\,E \xrightarrow{(\xi \otimes -, 0)} (\Lambda^1 \otimes \mathfrak{u}\,E) \oplus (\Lambda^{1,0} \otimes \mathrm{End}\ E) \xrightarrow{(\xi \wedge -, \xi \wedge -)} \Lambda^2 \otimes \mathfrak{u}\,E \oplus \Lambda^2 \otimes \mathrm{End}\ E \to 0$$

for every $\xi \in T^*M \setminus 0$. This sequence is exact, so (17) is an elliptic complex. Elliptic regularity theory then implies that, on the appropriate Sobolev spaces, the Dirac operators

$\mathrm{d}_1 + \mathrm{d}_2^*$ and $\mathrm{d}_1^* + \mathrm{d}_2$ are Fredholm. The operators d_1^* and d_2^* are the L^2–adjoint operators of d_1 and d_2 and therefore themselves first–order differential operators. Explicitly, d_1^* is given by

$$\mathrm{d}_1^*(\eta, \Psi) = -\operatorname{tr}_g \nabla\eta + \tfrac{1}{2}\langle[\Phi^*, \Psi]\rangle + \tfrac{1}{2}\langle[\Phi, \Psi^*]\rangle.$$

where the trace in this expression comes from the Riemannian metric g on M.

Using the Atiyah–Singer index theorem, we find that the index

$$\chi = \sum_{i=0}^{2}(-1)^i \dim H^i, \qquad H^i = \ker \mathrm{d}_{i+1}/\operatorname{im}\mathrm{d}_i$$

of the complex (17) is given by $\chi = -4m^2(\gamma - 1)$ where γ is the genus of the surface M.

Now let $(\nabla, \Phi) \in \mathscr{X}_k$ be not necessarily smooth. Then, with $\eta = \nabla - \nabla_0 \in \Omega^1_k(\mathfrak{u}\,E)$ and $\Psi = \Phi - \Phi_0 \in \Omega^{1,0}_k(\mathrm{End}\ E)$, we have

$$\mathrm{d}_{1(\nabla,\Phi)} = (\nabla\psi, [\Phi, \psi]) = (\nabla_0\psi, [\Phi_0, \psi]) + ([\eta, \psi], [\Psi, \psi]) = \mathrm{d}_1\psi + K\psi.$$

The operator

$$K\colon \Gamma_{l+1}(\mathfrak{u}\,E) \to \Omega^1_l(\mathfrak{u}\,E) \oplus \Omega^{1,0}_l(\mathrm{End}\ E), \quad K\psi = ([\eta, \psi], [\Psi, \psi])$$

for any non–negative integer $0 \le l \le k$ can be written as a composition of the compact Sobolev embedding $H^{l+1} \hookrightarrow W^{l,4}$ and a multiplication operator which is continuous as a map $H^k \times W^{l,4} \to H^l$ by either Theorem 3.6 or Proposition 3.7. Therefore, K itself is a compact operator. Similarly, the operators d_2,d_1^* and d_2^* and therefore also the Dirac operators $\mathrm{d}_1 + \mathrm{d}_2^*$ and $\mathrm{d}_1^* + \mathrm{d}_2$, at non–smooth $(\nabla, \Phi) \in \mathscr{X}_k$ are just compact perturbations of such operators at a smooth pair. As such, the operators $\mathrm{d}_1 + \mathrm{d}_2^*$ and $\mathrm{d}_1^* + \mathrm{d}_2$ on H^{l+1} are still Fredholm. This implies that $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_1^*, \mathrm{d}_2^*$ all have a closed image in H^l, since $\operatorname{im}\mathrm{d}_1 \subset \ker \mathrm{d}_2 = (\operatorname{im}\mathrm{d}_2^*)^\perp$.

3.6 Regularity theory

Up to now, we have only constructed a moduli space of (irreducible) weak solutions of the Hitchin equations modulo weak gauge transformations. But what we are actually interested in is the moduli space of smooth solutions modulo smooth gauge transformations. We will now use elliptic regularity theory to show that these are in fact the same. We have already seen in Section 3.5 that Hitchin's equations are elliptic up to a non–linearity and a gauge invariance. While the non–linearity is relatively nice and does not interfere much the regularity theory, the gauge invariance must be eliminated before we can apply the standard results for elliptic operators. This is done by a gauge fixing procedure which relies on a concrete version of the slice theorem, Theorem 2.13. Figure 1 gives an overview.

Lemma 3.16. *Let $k \ge 1$ and $\mathrm{d}^*_{1(\nabla,\Phi)}\colon \Omega^1_k(\mathfrak{u}\,E) \oplus \Omega^{1,0}_k(\mathrm{End}\ E) \to \Gamma_{k-1}(\mathfrak{u}\,E)$ be the L^2–adjoint of the infinitesimal gauge transformation from Section 3.5 at the point $(\nabla, \Phi) \in \mathscr{C}_k$. Then*

1. *For all $(\nabla_1, \Phi_1), (\nabla_2, \Phi_2) \in \mathscr{C}_k$,*

$$\mathrm{d}^*_{1(\nabla_1,\Phi_1)}(\nabla_2 - \nabla_1, \Phi_2 - \Phi_1) = \mathrm{d}^*_{1(\nabla_2,\Phi_2)}(\nabla_2 - \nabla_1, \Phi_2 - \Phi_1).$$

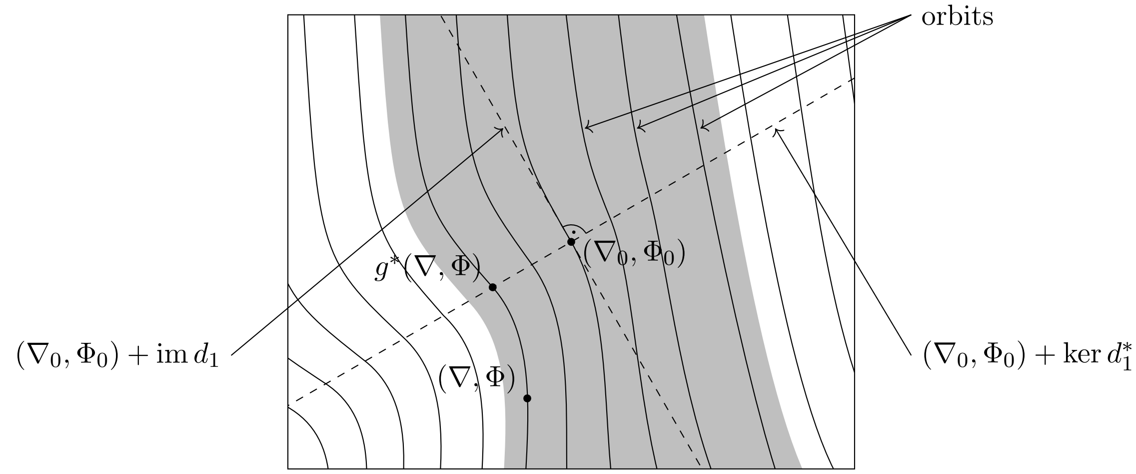

Figure 1: Near a solution $(\nabla, \Phi) \in \mathscr{X}_k$ exists a smooth pair $(\nabla_0, \Phi_0) \in \mathscr{C}$ such that in a neighborhood (gray) containing (∇, Φ) every orbit intersects $(\nabla_0, \Phi_0) + \ker \mathrm{d}_1^*$ exactly once. $g^*(\nabla, \Phi)$ then satisfies the gauge fixed equations around (∇_0, Φ_0) and is thus smooth.

2. *For all* $(\nabla, \Phi) \in \mathscr{C}_k$ *and* $g \in \mathscr{G}_{k+1}$,

$$\mathrm{d}^*_{1(g^*\nabla, g^*\Phi)}(g^*\eta, g^*\Psi) = g^{-1}\big(\mathrm{d}^*_{1(\nabla,\Phi)}(\eta, \Psi)\big)\, g.$$

3. *For all* $(\nabla_1, \Phi_1), (\nabla_2, \Phi_2) \in \mathscr{C}_k$ *and* $g \in \mathscr{G}_{k+1}$,

$$\mathrm{d}^*_{1(\nabla_1,\Phi_1)}(g^*\nabla_2 - \nabla_1, g^*\Phi_2 - \Phi_1) = 0 \;\Leftrightarrow\; \mathrm{d}^*_{1(\nabla_2,\Phi_2)}((g^{-1})^*\nabla_1 - \nabla_2, (g^{-1})^*\Phi_1 - \Phi_2) = 0.$$

Proof. Let $\psi \in \Gamma(\mathfrak{u}\, E)$ and $\eta = \nabla_2 - \nabla_1$, $\Psi = \Phi_2 - \Phi_1$. Then, for $i \in \{1, 2\}$,

$$\langle \mathrm{d}^*_{1(\nabla_i,\Phi_i)}(\eta, \Psi), \psi\rangle_{L^2} = \langle(\eta, \Psi), (\nabla_i\psi, [\Phi_i, \psi])\rangle_{L^2} = \int \langle \eta, \nabla_i\psi\rangle\, \omega + \int \mathrm{Re}\langle \Psi, [\Phi_i, \psi]\rangle\, \omega$$

Subtracting these terms for $i = 1, 2$ from each other we get

$$\int \langle \eta, [\eta, \psi]\rangle\, \omega + \int \mathrm{Re}\langle \Psi, [\Psi, \psi]\rangle\, \omega = -\int \langle\langle[\eta, \eta]\rangle, \psi\rangle\, \omega + \int \mathrm{Re}\langle\langle[\Psi^*, \Psi]\rangle, \psi\rangle\, \omega$$

The first summand vanishes since $\langle[-, -]\rangle$ is antisymmetric and the second vanishes because

$$\overline{\langle\langle[\Psi^*, \Psi]\rangle, \psi\rangle} = \langle\langle[\Psi^*, \Psi]\rangle^*, \psi^*\rangle = -\langle\langle[\Psi^*, \Psi]\rangle, \psi\rangle,$$

so $\langle\langle[\Psi^*, \Psi]\rangle, \psi\rangle$ is purely imaginary. This proves the first part of Lemma 3.16.

For the second part, we start with showing a similar equivariance property for d_1. First of all, for any $\psi \in \Gamma(\mathrm{End}\, E)$,

$$\nabla(g\psi g^{-1}) = (\nabla g)\psi g^{-1} + g(\nabla\psi)g^{-1} + g\psi(\nabla g^{-1}) = g(\nabla\psi + [g^{-1}(\nabla g), \psi])g^{-1} = g((g^*\nabla)\psi)g^{-1}$$

since $\nabla g^{-1} = -g^{-1}(\nabla g)g^{-1}$. Together with the identity $[\Phi, g\psi g^{-1}] = g[g^*\Phi, \psi]g^{-1}$ we get

$$\mathrm{d}_{1(\nabla,\Phi)}(g\psi g^{-1}) = g(\mathrm{d}_{1(g^*\nabla, g^*\Phi)}\psi)g^{-1}.$$

The desired identity follows from this simply by dualizing (with $\psi \in \Gamma(\mathfrak{u}\, E)$):

$$\begin{aligned}\langle \mathrm{d}^*_{1(g^*\nabla, g^*\Phi)}(g^*\eta, g^*\Psi), \psi\rangle &= \langle (g^*\eta, g^*\Psi), \mathrm{d}_{1(g^*\nabla, g^*\Phi)}\psi\rangle = \langle (g^*\eta, g^*\Psi), g^{-1}\mathrm{d}_{1(\nabla,\Phi)}(g\psi g^{-1})g\rangle \\ &= \langle (\eta, \Psi), \mathrm{d}_{1(\nabla,\Phi)}(g\psi g^{-1})\rangle = \langle g^{-1}\left(\mathrm{d}^*_{1(\nabla,\Phi)}(\eta,\Psi)\right) g, \psi\rangle\end{aligned}$$

The third part of Lemma 3.16 just follows from the first two parts. □

Theorem 3.17 (Regularity). *Let $k \geq 1$ and $(\nabla, \Phi) \in \mathscr{X}_k$ be a solution of Hitchin's equations. Then there exists a gauge transformation $g \in \mathscr{G}_{k+1}$ such that $g^*(\nabla, \Phi) \in \mathscr{X}$.*

If $(\nabla_i, \Phi_i) \in \mathscr{X}_k$ is a sequence of solutions converging to $(\nabla, \Phi) \in \mathscr{X}$ in the H^k–topology, there is a sequence of gauge transformations $g_i \in \mathscr{G}_{k+1}$ converging to the identity, such that $g_i^*(\nabla_i, \Phi_i) \in \mathscr{X}$ for all i and $g_i^*(\nabla_i, \Phi_i)$ converges to (∇, Φ) in the C^∞–topology.

Proof. We use Theorem 2.13 for the action $\mathscr{C}_k \times \mathscr{G}_{k+1} \to \mathscr{C}_k$ at a point $x = (\nabla, \Phi) \in \mathscr{X}_k \subset \mathscr{C}_k$ with special choices for F and Q. As a complement of $T_{(\nabla,\Phi)}(\mathscr{G}_{k+1}(\nabla, \Phi)) = \operatorname{im} \mathrm{d}_{1(\nabla,\Phi)}$ in $T_{(\nabla,\Phi)}\mathscr{C}_k$ we choose $F = \ker \mathrm{d}^*_{1(\nabla,\Phi)}$, where $\mathrm{d}^*_{1(\nabla,\Phi)}$ is seen as an operator from H^k to H^{k-1}. Then F is the L^2–orthogonal complement of $T_x(Gx)$ and clearly closed. It is also $(\mathscr{G}_{k+1})_{(\nabla,\Phi)}$–invariant due to part 2 of Lemma 3.16. The function $Q\colon T_{(\nabla,\Phi)}\mathscr{C}_k \to \mathscr{C}_k$ is just given by the affine structure of $\mathscr{C}_k$, i.e. $Q(\eta, \Psi) = (\nabla + \eta, \Phi + \Psi)$. Then Theorem 2.13 tells us that there is a $\mathscr{G}_{k+1}$–invariant neighbourhood $\widetilde{V} \subset \mathscr{C}_k$ of (∇, Φ) such that every $(\nabla_0, \Phi_0) \in \widetilde{V}$ is of the form $g^*(\nabla + \eta, \Phi + \Psi)$ for some $(\eta, \Psi) \in \ker \mathrm{d}^*_{1(\nabla,\Phi)}$ and $g \in \mathscr{G}_{k+1}$. Since $\mathscr{C}$ is dense in $\mathscr{C}_k$ we can choose $(\nabla_0, \Phi_0) \in \mathscr{C} \cap \widetilde{V}$. Then

$$\mathrm{d}^*_{1(\nabla,\Phi)}((g^{-1})^*\nabla_0 - \nabla, (g^{-1})^*\Phi_0 - \Phi) = \mathrm{d}^*_{1(\nabla,\Phi)}(\eta, \Psi) = 0$$

and therefore, using part 3 of Lemma 3.16,

$$\mathrm{d}^*_{1(\nabla_0,\Phi_0)}(g^*\nabla - \nabla_0, g^*\Phi - \Phi_0) = 0.$$

Now using the abbreviations $\eta = g^*\nabla - \nabla_0 \in \Omega^1_k(\mathfrak{u}\, E)$ and $\Psi = g^*\Phi - \Phi_0 \in \Omega^{1,0}_k(\operatorname{End} E)$ we can write this out explicitly:

$$\operatorname{tr}_g \nabla_0 \eta = \tfrac{1}{2}\langle [\Phi_0^*, \Psi]\rangle + \tfrac{1}{2}\langle [\Phi_0, \Psi^*]\rangle \tag{18}$$

Hitchin's equations for $(g^*\nabla, g^*\Phi) = (\nabla_0 + \eta, \Phi_0 + \Psi)$ take the explicit form

$$\mathrm{d}^{\nabla_0}\eta = -F^{\nabla_0} - \eta \wedge \eta - [\Phi_0 + \Psi \wedge \Phi_0^* + \Psi^*] + c\,\mathrm{id} \otimes \omega, \tag{19}$$

$$\mathrm{d}^{\nabla_0}\Psi = -\mathrm{d}^{\nabla_0}\Phi_0 - [\Phi_0 \wedge \eta] - [\Psi \wedge \eta]. \tag{20}$$

The operator $(\eta, \Psi) \mapsto (\operatorname{tr}_g \nabla_0 \eta, \mathrm{d}^{\nabla_0}\eta, \mathrm{d}^{\nabla}_0 \Psi)$ is elliptic. If $k \geq 2$ then the right hand sides of (18), (19) and (20) are H^k by the Sobolev multiplication theorem, so elliptic regularity shows that η and Ψ are in fact H^{k+1}. Repeating this argument shows that η and Ψ are even C^∞ and therefore $g^*(\nabla, \Phi) \in \mathscr{X}$. If however $k = 1$, so that η, Ψ are only H^1, then the Sobolev multiplication theorem does not apply. But both η and Ψ are L^8 by the Sobolev embedding theorem, so the right hand sides of (18), (19) and (20) are L^4 by Hölder's theorem. Now we can apply the L^p elliptic regularity theorem [Bes87, Theorem 31, Appendix] to obtain that η, Ψ are $W^{1,4}$, so their product is H^1 again by Hölder's theorem. Standard L^2 elliptic regularity then shows that η, Ψ are H^2 and we can continue with the argument above.

Now consider a sequence $(\nabla_i, \Phi_i) \in \mathscr{X}_k$ converging to $(\nabla_0, \Phi_0) \in \mathscr{X}$. Applying Theorem 2.13 with the above choices for F and Q around (∇_0, Φ_0), and neglecting finitely many elements of the sequence (∇_i, Φ_i), we get a sequence of gauge transformations $g_i \in \mathscr{G}_{k+1}$ and a sequence of solutions $(\eta_i, \Psi_i) \in \ker \mathrm{d}^*_{1(\nabla_0,\Phi_0)}$ such that

$$(\nabla_i, \Phi_i) = g_i^*(\nabla_0 + \eta_i, \Phi_0 + \Psi_i) \tag{21}$$

and $[g_i, (\eta_i, \Psi_i)]$ converges to $[e, 0]$ in $G \times_{G_x} F$. That is, by modifying g_i, η_i, Ψ_i by elements of stabilizer, we can assume that g_i converges to e and (η_i, Ψ_i) converges to 0 in H^k. Since inversion in a Lie group is continuous, $h_i = g_i^{-1}$ also converges to e and by (21) we have $\eta_i = h_i^*\nabla_i - \nabla_0$ and $\Psi_i = h_i^*\Phi_i - \Phi_0$. So (η_i, Ψ_i) is a solution of the equations (18), (19) and (20) converging to 0 in the H^k–topology. Repeating the arguments of the last paragraph, we see that the (η_i, Ψ_i) are C^∞ and converge in the C^∞–topology. So $h_i^*(\nabla_i, \Phi_i) = (\nabla_0 + \eta_i, \Phi_0 + \Psi_i)$ satisfies the statement of the theorem. □

3.7 The stabilizers

The quotient of any group action depends heavily on the stabilizers of this action. The simplest case is that of a free action, i.e. when all stabilizers are trivial. This leads to a very homogeneous quotient space which is similar to a quotient of a group by one of its subgroups. The situation for our action however is a bit more involved: For example, gauge transformations of the form $g = \lambda\,\mathrm{id}_E$ for some $\lambda \in S^1$ act trivially on all Higgs pairs. It will turn out that these are the only trivial gauge transformations for generic Higgs pairs, but on special *reducible* Higgs pairs the stabilizer can be more complex. So to have a reasonable structure on the moduli space, we will have to restrict to irreducible pairs.

Definition 3.18. The pair $(\nabla, \Phi) \in \mathscr{C}$ is *reducible* if there are a nontrivial orthogonal decomposition $E = E_1 \oplus E_2$, unitary connections ∇_1 on E_1 and ∇_2 on E_2 and endomorphism–valued forms $\Phi_1 \in \Omega^{1,0}(\mathrm{End}\ E_1)$ and $\Phi_2 \in \Omega^{1,0}(\mathrm{End}\ E_2)$ such that

$$\nabla = \begin{pmatrix} \nabla_1 & 0 \\ 0 & \nabla_2 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix}.$$

The space of all *irreducible* pairs in $\mathscr{C}$ is denoted by $\mathscr{C}^*$.

Proposition 3.19. *If $\deg(E)$ and $\mathrm{rk}(E)$ are coprime, then every solution $(\nabla, \Phi) \in \mathscr{X}$ of Hitchin's equations is irreducible.*

Proof. Suppose (∇, Φ) on $E = E_1 \oplus E_2$ is reducible to (∇_1, Φ_1) on E_1 and (∇_2, Φ_2) on E_2. Then the first of Hitchin's equations states that

$$\begin{pmatrix} F^{\nabla_1} & \\ & F^{\nabla_2} \end{pmatrix} + \begin{pmatrix} [\Phi_1 \wedge \Phi_1^*] & \\ & [\Phi_2 \wedge \Phi_2^*] \end{pmatrix} = -\frac{2\pi i}{\mathrm{vol}(M)}\mu(E) \begin{pmatrix} \mathrm{id}_{E_1}\otimes\omega & \\ & \mathrm{id}_{E_2}\otimes\omega \end{pmatrix}.$$

Taking the trace of the E_1–part and integrating, the term $[\Phi_1 \wedge \Phi_1^*]$ vanishes and we get

$$-2\pi i \deg(E_1) = -2\pi i \frac{\deg(E)}{\mathrm{rk}(E)} \mathrm{rk}(E_1),$$

so $\mu(E_1) = \mu(E)$. But this is impossible since $\mathrm{rk}(E_1) < \mathrm{rk}(E)$ and $\deg(E)/\mathrm{rk}(E)$ is already a fully reduced fraction. □

Lemma 3.20. *Let $\nabla \in \mathscr{A}(E)$ and $g \in \mathscr{G}$ be ∇–covariant constant, i.e. $\nabla g = 0$. Then g commutes with all ∇–parallel transport maps P_γ and E decomposes orthogonally as a direct sum of subbundles $E = E^{\lambda_1} \oplus E^{\lambda_2} \oplus \cdots \oplus E^{\lambda_l}$ on each of which g acts by multiplication with a constant scalar $\lambda_i \in S^1$.*

Proof. Let $\gamma\colon I \to M$ be a smooth curve from x to y and $\widehat{\gamma}\colon I \to E$ its parallel lift with $\widehat{\gamma}(0) = v \in E_x$. Then $\widetilde{\gamma}(t) = g_{\gamma(t)}\widehat{\gamma}(t)$ also defines a parallel curve with $\widetilde{\gamma}(0) = g_x v$, so $P_\gamma g_x v = \widetilde{\gamma}(1) = g_y \widehat{\gamma}(1) = g_y P_\gamma v$.

Choose a point $x \in M$. Since g_x is unitary and therefore diagonalizable it has an eigenvalue λ, i.e. the eigenspace $E_x^\lambda = \{v \in E_x \mid g_x v = \lambda v\}$ is nontrivial. For any other point $y \in M$ we define the subspace $E_y^\lambda := P_\gamma(E_x^\lambda) \subset E_y$ using an arbitrary curve $\gamma\colon I \to M$ from x to y. This definition does not depend on the choice of γ: If $\gamma'\colon I \to M$ is another such curve, then $g_x(P_{\gamma'\gamma^{-1}} v) = P_{\gamma'\gamma^{-1}} g_x v = \lambda(P_{\gamma'\gamma^{-1}} v)$ for any $v \in E_x^\lambda$, so $P_{\gamma'\gamma^{-1}} v \in E_x^\lambda$ and thus

$$P_{\gamma'}(E_x^\lambda) \ni P_{\gamma'} v = P_\gamma(P_\gamma^{-1}(P_{\gamma'}(v))) = P_\gamma(P_{\gamma'\gamma^{-1}}(v)) \in P_\gamma(E_x^\lambda).$$

Applying this construction at every point $y \in M$, the spaces E_y^λ combine to a smooth subbundle E^λ of E: In a bundle chart φ for E defined on a coordinate ball, consider the family of curves γ_y connecting y to the origin in a straight line. By the theory of ordinary differential equations their associated parallel transport maps P_{γ_y} depend smoothly on y and they linearly transform $\varphi(E_y^\lambda)$ to the same subspace of $\mathbb{C}^m$. So the composition of φ and P_{γ_-} is a bundle chart which maps E^λ locally to a fixed linear subspace.

Furthermore $g_y P_\gamma v = P_\gamma g_x v = \lambda P_\gamma v$ for all $P_\gamma v \in E_y^\lambda$, i.e. the action of g on E^λ is just multiplication by λ. Since E^λ has nonzero rank, iterating this proof on the orthogonal complement of E^λ yields the desired decomposition. □

Lemma 3.21. *Let $g \in \mathscr{G}_{k+1}$ and $(\nabla, \Phi) \in \mathscr{C}$ such that $(\nabla, \Phi) \cdot g = (\nabla, \Phi)$. Then there is an orthogonal decomposition into subbundles $E = E_1 \oplus \cdots \oplus E_l$, connections $\nabla_i \in \mathscr{A}(E_i)$, fields $\Phi_i \in \Omega^{1,0}(\mathrm{End}\, E_i)$ and distinct constants $\lambda_i \in S^1$ such that*

$$\nabla = \begin{pmatrix} \nabla_1 & & \\ & \ddots & \\ & & \nabla_l \end{pmatrix} \qquad \Phi = \begin{pmatrix} \Phi_1 & & \\ & \ddots & \\ & & \Phi_l \end{pmatrix} \qquad g = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_l \end{pmatrix}.$$

Conversely, if $\nabla \in \mathscr{A}(E)$, $\Phi \in \Omega^{1,0}(\mathrm{End}\, E)$ and $g \in \mathscr{G}_{k+1}$ can be decomposed in the above way, then $(\nabla, \Phi) \cdot g = (\nabla, \Phi)$.

Proof. The identity $(\nabla, \Phi) \cdot g = (\nabla, \Phi)$ implies that g commutes with Φ and $\nabla g = 0$. This implies in particular that $g \in \mathscr{G}$ is smooth. By Lemma 3.20 there is an orthogonal decomposition $E = E_1 \oplus \cdots \oplus E_l$ so that g is of the desired form. We have to show that the eigenspaces E_i are invariant under Φ and ∇. But since for $v \in E_i$, $X \in TM$ we have

$$g\Phi v = \Phi g v = \Phi \lambda_i v = \lambda_i \Phi v \qquad g\nabla_X v = \nabla_X(gv) = \nabla_X(\lambda_i v) = \lambda_i \nabla_X v,$$

both Φv and $\nabla_X v$ are again in E_i. The converse is trivial. □

Now the irreducible pairs are exactly those where the decomposition from Lemma 3.21 is trivial, i.e. their stabilizers consist only of gauge transformations of the form $\lambda\, \mathrm{id}_E$ for

$\lambda \in S^1$. To use the results of Section 2 we now consider Sobolev spaces and identify the open subset $\mathscr{C}_{k(S^1)} \subset \mathscr{C}_k$. It can be interpreted as the set of irreducible Higgs pairs in $\mathscr{C}_k$ due to the following theorem.

Proposition 3.22. *The subset $\mathscr{C}_k^* = \mathscr{C}_{k(S^1)} \subset \mathscr{C}_k$ of pairs with stabilizer S^1 with respect to the action $\mathscr{C}_k \times \mathscr{G}_{k+1} \to \mathscr{C}_k$ is an invariant open submanifold and satisfies $\mathscr{C}^* = \mathscr{C} \cap \mathscr{C}_k^*$.*

Proof. The subgroup $S^1 \subset \mathscr{G}_{k+1}$, consisting of all elements of the form $\lambda\,\mathrm{id}_E$, acts trivially on $\mathscr{C}_k$, so it is contained as a subgroup in every stabilizer. So by Corollary 2.14 the subset $\mathscr{C}_{k(S^1)} \subset \mathscr{C}_k$ is a $\mathscr{G}_{k+1}$–invariant open submanifold. The only thing left to show is $\mathscr{C}^* = \mathscr{C} \cap \mathscr{C}_k^*$. If $(\nabla, \Phi) \in \mathscr{C}^*$ and $(\nabla, \Phi) \cdot g = (\nabla, \Phi)$ for some $g \in \mathscr{G}_{k+1}$, then $\nabla + g^{-1}(\nabla g) = \nabla$, so $\nabla g = 0$ and in particular g is smooth. Lemma 3.21 and the irreducibility of (∇, Φ) then imply $g \in S^1$, so $(\nabla, \Phi) \in \mathscr{C}_k^*$. If on the other hand $(\nabla, \Phi) \in \mathscr{C} \cap \mathscr{C}_k^*$, then it must be irreducible, as otherwise there is a gauge transformation acting non–trivially on its irreducible components (for example, multiplication by -1 on one component, and the identity on all others). □

Though this also shows that the irreducible solutions $\mathscr{X}_k^*$ are an open set in $\mathscr{X}_k$, it does not say how common irreducible solutions are. In fact, we don't even know yet if there exist irreducible solutions at all, except if $\deg(E)$ and $\mathrm{rk}(E)$ are coprime. As irreducible solutions correspond to stable Higgs bundles via the Kobayashi–Hitchin correspondence, one can study this question using algebraic methods. But we can also use purely analytic arguments to show that $\mathscr{X}_k^*$ is dense in $\mathscr{X}_k$, i.e. that irreducibility is the generic case. This is what will be done in the remainder of this section.

Theorem 3.23. *Let $(\nabla, \Phi) \in \mathscr{X}$ be a smooth solution to Hitchin's equations and let $H = (\mathscr{G}_{k+1})_{(\nabla,\Phi)} \subset \mathscr{G}_{k+1}$ be the stabilizer of the $\mathscr{G}_{k+1}$–action at this point. Then $\mathscr{X}_{k(H)} \subset \mathscr{X}_k$ is a submanifold of codimension at least $4(l-1)(l\gamma - l - 1)$ where γ is the genus of M and l is the maximal number of irreducible components of (∇, Φ).*

Proof. By Corollary 2.14 the codimension of $(\mathscr{X}_k)_{(H)}$ in $\mathscr{X}_k$ is $\dim F - \dim F^H$, where F is a subspace of $T_{(\nabla,\Phi)}\mathscr{X}_k$ satisfying the conditions in Theorem 2.13 and H acts on it linearly via differentials. A natural choice for F is the first cohomology of the complex (17), i.e. the space of solutions $(\eta, \Psi) \in \Omega^1(\mathfrak{u}\,E) \oplus \Omega^{1,0}(\mathrm{End}\ E)$ of the equations

$$\begin{aligned} \mathrm{d}^\nabla \eta &= -[\Phi \wedge \Psi^*] - [\Psi \wedge \Phi^*], \\ \mathrm{d}^\nabla \Psi &= -[\eta \wedge \Phi], \\ \mathrm{tr}\, \nabla \eta &= \tfrac{1}{2}\langle [\Phi^*, \Psi] \rangle + \tfrac{1}{2}\langle [\Phi, \Psi^*] \rangle. \end{aligned} \tag{22}$$

Note that we can assume that η and Ψ are smooth since (22) is a linear elliptic system. The action of H on this space is by conjugation:

$$D_{(\nabla,\Phi)} r_g(\eta, \Psi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g^*(\nabla + t\eta, \Phi + t\Psi) = (g^{-1}\eta g, g^{-1}\Phi g) \qquad \forall g \in H$$

Now let $E = E_1 \oplus \cdots \oplus E_l$ be an orthogonal splitting with respect to which ∇ and Φ decompose into $\nabla_i \in \mathscr{A}(E_i)$ and $\Phi_i \in \Omega^{1,0}(\mathrm{End}\ E_i)$. By Lemma 3.21 there is a $g \in H$ which acts on each of the E_i by multiplication with a distinct $\lambda_i \in S^1$. Every $(\eta, \Psi) \in F^H$ satisfies

$g\eta = \eta g$ and $g\Psi = \Psi g$, which implies that η and Ψ also decompose into $\eta_i \in \Omega^1(\mathfrak{u}\, E_i)$ and $\Psi_i \in \Omega^{1,0}(\operatorname{End} E_i)$. It is clear that these solve the corresponding equations (22) for the bundles E_i. So F^H is contained in the product of the solution spaces of (22) for the bundles E_i, each having dimension $4m_i^2(\gamma-1)+4$ (see Section 3.8 , where $m_i = \operatorname{rk} E_i$. So

$$\dim F^H \leq \sum_{i=1}^{l}(4m_i^2(\gamma-1)+4) = 4l + 4(\gamma-1)\sum_{i=1}^{l} m_i^2$$

and thus

$$\dim F - \dim F^H \geq 4(\gamma-1)\sum_{1\leq i\neq j\leq l} m_i m_j + 4 - 4l \geq 4(\gamma-1)l(l-1) - 4(l-1). \qquad \square$$

Corollary 3.24. *If $\gamma \geq 2$, the subset $\mathscr{X}_k^* = \mathscr{X}_{k(S^1)} \subset \mathscr{X}_k$ of solutions to Hitchin's equations with stabilizer S^1 is $\mathscr{G}_{k+1}$–invariant, open and dense and satisfies $\mathscr{X}_k^* \cap \mathscr{X} = \mathscr{X}^*$.*

Proof. Since $\mathscr{X}_k^* = \mathscr{C}_k^* \cap \mathscr{X}_k$ it is clear that $\mathscr{X}_k^*$ is $\mathscr{G}_{k+1}$–invariant, open in $\mathscr{X}_k$ and $\mathscr{X}_k^* \cap \mathscr{X} = \mathscr{X}^*$. The set $\mathscr{X}_k \setminus \mathscr{X}_k^*$ of reducible solutions is the union of the sets $(\mathscr{X}_k)_{(H)}$ for all subgroups $S^1 \neq H \subset \mathscr{G}_{k+1}$ which occur as stabilizers. These sets are submanifolds of $\mathscr{X}_k$ with codimension at least $8\gamma-12$ by Theorem 3.23, so they are nowhere dense. Corollary 2.15 thus shows that the set of reducible solutions is a countable union of nowhere dense sets, so it is nowhere dense by Baire's theorem. $\square$

3.8 The moduli space

Now we have all ingredients together to obtain a smooth moduli space of solutions of Hitchin's equations (3) and (4). Fix an integer $k \geq 1$. Then by Section 3.3 there is a smooth action

$$\mathscr{C}_k \times \mathscr{G}_{k+1} \to \mathscr{C}_k \tag{23}$$

which is proper by Corollary 3.15 and splits by Section 3.5. According to Proposition 3.22, there is an open $\mathscr{G}_{k+1}$–invariant subset $\mathscr{C}_k^* \subset \mathscr{C}_k$ where the stabilizer of this action equals $S^1 \subset \mathscr{G}_{k+1}$ and the smooth Higgs pairs $\mathscr{C}^* = \mathscr{C}_k^* \cap \mathscr{C}$ in this set are precisely the irreducible ones.

Furthermore, Section 3.2.2 defined a Riemannian metric G, complex structures J_1, J_2, J_3 and symplectic structures $\omega_1, \omega_2, \omega_3$ on $\mathscr{C}_k$, which are all compatible and $\mathscr{G}_{k+1}$–invariant and with respect to which, by Section 3.3, the action (23) is Hamiltonian with moment maps μ_1, μ_2, μ_3, which split by Section 3.5 (their differentials are given by the operator d_2). The intersection of their zero sets is precisely the set $\mathscr{X}_k$ of weak solutions of Hitchin's equations and $\mathscr{X}_k \cap \mathscr{C}_k^* = \mathscr{X}_k^*$ by definition.

Theorem 2.24 then implies that the moduli space $\mathscr{M}_k^* = \mathscr{X}_k^*/\mathscr{G}_{k+1}$ naturally carries the structure of a smooth Hyperkähler manifold. The map $\pi\colon \mathscr{X}_k^* \to \mathscr{M}_k^*$ is a smooth submersion and $\mathscr{X}_k^* \subset \mathscr{X}_k$ is open, so its differential $D\pi$ induces an isomorphism $T_{(\nabla,\Phi)}\mathscr{X}_k/\ker D\pi \cong T_{[\nabla,\Phi]}\mathscr{M}_k^*$ at every $(\nabla,\Phi) \in \mathscr{X}_k^*$. From the infinitesimal picture in the proof of Corollary 2.16 it is clear that $\ker D\pi = \operatorname{im} \mathrm{d}_1 \subset \Omega_k^1(\mathfrak{u}\, E) \oplus \Omega_k^{1,0}(\operatorname{End} E)$ and since $T_{(\nabla,\Phi)}\mathscr{X}_k = \ker \mathrm{d}_2$, the tangent space $T_{[\nabla,\Phi]}\mathscr{M}_k^*$ equals the first cohomology of the complex (17). To obtain its dimension

$$\dim H^1 = \dim H^0 + \dim H^2 - \chi,$$

we use that $\dim H^0 = \dim\ker \mathrm{d}_1 = \dim \mathfrak{s}^1 = 1$ by irreducibility and $\dim H^2 = 3\dim\mathfrak{s}^1 = 3$ by the proof of Theorem 2.24. Then Section 3.5 tells us that $\chi = -4m^2(\gamma-1)$, so

$$\dim \mathscr{M}_k^* = \dim H^1 = 4m^2(\gamma-1)+4.$$

If $k,l \geq 1$ then $\mathscr{M}_k^*$ and $\mathscr{M}_{l+1}^*$ are diffeomorphic. Indeed, consider the map

$$\mathscr{M}_{k+1}^* \to \mathscr{M}_k^*, \quad [\nabla,\Phi] \mapsto [\nabla,\Phi]. \tag{24}$$

It is well–defined since $\mathscr{G}_{k+2}$–equivalent pairs are also $\mathscr{G}_{k+1}$–equivalent. To see that it is injective, let $(\nabla_2,\Phi_2) = g^*(\nabla_1,\Phi_1)$ for $(\nabla_1,\Phi_1),(\nabla_2,\Phi_2) \in \mathscr{X}_{k+1}^*$ and $g \in \mathscr{G}_{k+1}$. Then, for any $\nabla_0 \in \mathscr{A}(E)$, $\eta_1 = \nabla_1 - \nabla_0$ and $\eta_2 = \nabla_2 - \nabla_0$, we have

$$\nabla_0 g = g\eta_2 - \eta_1 g,$$

which is H^{k+1} by the Sobolev multiplication theorem, so $g \in \mathscr{G}_{k+2}$. The map (24) is also surjective by Theorem 3.17. It is smooth because the inclusion $\mathscr{X}_{k+1}^* \hookrightarrow \mathscr{X}_k^*$ is and as a smooth submersion the map $\mathscr{X}_{k+1}^* \to \mathscr{M}_{k+1}^*$ admits smooth local sections. Identifying the tangent space of $\mathscr{M}^*$ with the cohomology H^1 of the complex (17) as above, the differential of (24) is just the identity on H^1, which is an isomorphism since its dimension does not depend on k. So (24) is indeed a diffeomorphism. It is evident from the definition of all relevant structures that this is even an isomorphism of Hyperkähler manifolds.

In particular, if we endow $\mathscr{X}$ and $\mathscr{G}$ with the C^∞–topology and $\mathscr{M}^* = \mathscr{X}^*/\mathscr{G}$ with its induced quotient topology, there is a similar inclusion $\mathscr{M}^* \hookrightarrow \mathscr{M}_k^*$ which is bijective by the same argument. It is continuous by the very definition of the quotient topology and so is its inverse by Theorem 3.17. So it is a homeomorphism and we can therefore transfer the smooth Hyperkähler structure to the space $\mathscr{M}^*$ respecting its topology. This does not depend on k by the last paragraph.

In conclusion, the space $\mathscr{M}^* = \mathscr{X}^*/\mathscr{G}$ is a $(4m^2(\gamma-1)+4)$–dimensional Hyperkähler manifold arising as a Hyperkähler quotient $\mathscr{C}_k^* /\!/\!/ \mathscr{G}_{k+1}$ for any $k \geq 1$. Its tangent space at $[\nabla,\Phi] \in \mathscr{M}^*$ is the first cohomology of the complex (17) and it is an open and dense subset of $\mathscr{M} = \mathscr{X}/\mathscr{G}$. Its symplectic structures are given by the restriction of (9) to vectors tangent to $\mathscr{X}$ (i.e. solutions of d_2 in the complex (17)).

4 The Hitchin fibration

4.1 Elementary symmetric polynomials

Let V be a m–dimensional complex vector space with a scalar product. Then we can define maps $s_i\colon \operatorname{End} V \to \mathbb{C}$ for $i = 0, \dots, m$ by

$$\det(t\,\mathrm{id} + f) = \sum_{i=0}^{m} s_i(f)\, t^{m-i} \qquad \forall t \in \mathbb{C}.$$

They are called *elementary symmetric polynomials*. The limit cases are $s_0(f) = 1$ and $s_m(f) = \det(f)$.

Very similar in their properties, but easier to define concretely, are the functions

$$\operatorname{tr}^i\colon \operatorname{End} V \to \mathbb{C}, \quad f \mapsto \operatorname{tr} f^i$$

for $i = 1, \dots, m$.

Both these families of maps are invariant under conjugation, i.e. if $g \in \mathrm{GL}(V)$, then $s_i(gfg^{-1}) = s_i(f)$ and $\operatorname{tr}^i(gfg^{-1}) = \operatorname{tr}^i(f)$. By representing an endomorphism f in a basis and triangularizing the resulting matrix, it is easy to see that $s_i(f)$ and $\operatorname{tr}^i(f)$ only depend on the eigenvalues of f. In other words, adding a nilpotent endomorphism to f does not change the value of $s_i(f)$ and $\operatorname{tr}^i(f)$. The following lemma shows that every s_i can be expressed as a polynomial of the tr^j and vice–versa.

Lemma 4.1. *Let $f \in \operatorname{End} V$. The elementary symmetric polynomials $s_i(f)$ and the $\operatorname{tr}^i(f)$ are determined by each other via the recursive identity*

$$i s_i(f) = \sum_{j=1}^{i} (-1)^{j+1} \operatorname{tr}^j(f) s_{i-j}(f) \tag{25}$$

for all $i \in \{1, \dots, m\}$.

Proof. As $s_i(f)$ and $\operatorname{tr}^i(f)$ are unchanged when adding a nilpotent endomorphism to f, we can assume that f is normal with the eigenvalues $x_1, \dots, x_m \in \mathbb{C}$. In a diagonal basis representation for f we see that

$$\sum_{i=0}^{m} s_i(f) t^i = t^m \sum_{i=0}^{m} s_i(f) t^{i-m} = t^m \det(t^{-1}\,\mathrm{id} + f) = \prod_{i=1}^{m} (1 + t x_i). \tag{26}$$

Differentiating the right hand side of this for small enough values of t gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \prod_{i=1}^{m} (1 + t x_i) = \sum_{j=1}^{m} \frac{x_j}{1 + t x_j} \prod_{i=1}^{m} (1 + t x_i) = \left(\sum_{j=1}^{m} \sum_{l=0}^{\infty} (-1)^l x_j^{l+1} t^l \right) \prod_{i=1}^{m} (1 + t x_i) =$$
$$= \left(\sum_{l=0}^{\infty} (-1)^l \operatorname{tr}^{l+1}(f) t^l \right) \left(\sum_{i=0}^{m} s_i(f) t^i \right)$$

Equation (25) then follows by comparing the coefficients of this power series with the derivative of the left hand side of (26). $\square$

When expressing $s_i(f)$ by $\mathrm{tr}^1(f), \ldots, \mathrm{tr}^i(f)$ with recursive applications of (25), it becomes clear that s_i is a homogeneous polynomial of degree i. That is, there exists a multilinear map $\widetilde{s_i}\colon (\mathrm{End}\ V)^i \to \mathbb{C}$ with $s_i(f) = \widetilde{s_i}(f, \ldots, f)$ for all $f \in \mathrm{End}\ V$. This multilinear map is unique if we also require it to be symmetric.

4.2 The Hitchin fibration

Let M be a compact connected Riemann surface and $E \to M$ a complex vector bundle of rank m. The elementary symmetric polynomials s_i and the functions tr^i then induce maps $\Omega^{1,0}(\mathrm{End}\ E) \to \Gamma(K^{\otimes i})$, which we also call s_i and tr^i, as follows. First, applying $s_i\colon \mathrm{End}\ E_x \to \mathbb{C}$ pointwise, we get a map $s_i\colon \Gamma(\mathrm{End}\ E) \to \Gamma(\underline{\mathbb{C}})$. Since it is a homogeneous polynomial of degree i, there is a unique symmetric multilinear map $\widetilde{s_i}\colon \Gamma((\mathrm{End}\ E)^i) \to \Gamma(\underline{\mathbb{C}})$ such that $s_i(\varphi) = \widetilde{s_i}(\varphi, \ldots, \varphi)$. Then we define the multilinear map

$$\widehat{s_i}\colon \Gamma((T^*M \otimes \mathrm{End}\ E)^i) \to \Gamma((T^*M \otimes \underline{\mathbb{C}})^{\otimes i})$$

by multilinearly extending

$$\widehat{s_i}(\alpha_1 \otimes \varphi_1, \ldots, \alpha_i \otimes \varphi_i) = \widetilde{s_i}(\varphi_1, \ldots, \varphi_i)\ \alpha_1 \otimes \cdots \otimes \alpha_i,$$

for $\alpha_1, \ldots, \alpha_i \in \Gamma(T^*M)$ and $\varphi_1, \ldots, \varphi_i \in \Gamma(\mathrm{End}\ E)$. This in turn corresponds to a homogeneous polynomial

$$s_i\colon \Omega^1(\mathrm{End}\ E) \to \Gamma((T^*M \otimes \underline{\mathbb{C}})^{\otimes i}), \quad s_i(\Phi) = \widehat{s_i}(\Phi, \ldots, \Phi).$$

Clearly its restriction to $\Omega^{1,0}(\mathrm{End}\ E)$ maps into $\Gamma((\Lambda^{1,0})^{\otimes i}) = \Gamma(K^{\otimes i})$, where K is the canonical line bundle on M. Applying the same construction to the functions tr^i gives maps

$$s_i, \mathrm{tr}^i\colon \Omega^{1,0}(\mathrm{End}\ E) \to \Gamma(K^{\otimes i})$$

for every $i = 1, \ldots, m$. The canonical bundle K, and thus also $K^{\otimes i}$, carries a natural holomorphic structure. The next lemma gives conditions for the result of s_i and tr^i to be holomorphic with respect to this structure.

Lemma 4.2. *Let $\nabla \in \mathscr{A}(E)$ be a unitary connection and $\Phi \in \Omega^{1,0}(\mathrm{End}\ E)$ with $\mathrm{d}^\nabla \Phi = 0$. Then $s_i(\Phi)$ and $\mathrm{tr}^i(\Phi)$ are holomorphic sections of $K^{\otimes i}$ for all $i \in \{1, \ldots, m\}$.*

Proof. First we show that $\mathrm{d}\,\mathrm{tr}\,\varphi = \mathrm{tr}\,\nabla\varphi$ for all $\varphi \in \Gamma(\mathrm{End}\ E)$. Let $X \in TM$ and $\{e_i\}$ be a ∇–parallel local orthonormal frame of E. Then

$$X\,\mathrm{tr}(\varphi) = \sum_i X\langle e_i, \varphi e_i\rangle = \sum_i \langle \nabla_X e_i, e_i\rangle + \langle e_i, \nabla_X(\varphi e_i)\rangle = \sum_i \langle e_i, (\nabla_X \varphi)e_i\rangle = \mathrm{tr}(\nabla_X \varphi),$$

so $\mathrm{d}\,\mathrm{tr}\,\varphi = \mathrm{tr}\,\nabla\varphi$.

Now let $\Phi \in \Omega^{1,0}(\mathrm{End}\ E)$ be holomorphic. In local holomorphic coordinates on an open set $U \subset M$, we can write $\Phi = \mathrm{d}z \otimes \varphi$, where φ is a local holomorphic section of $\mathrm{End}\ E$ with its holomorphic structure given by ∇, i.e. $\nabla\varphi \in \Omega^{1,0}(\mathrm{End}\ E|_U)$. Then $\nabla\varphi^i$ is also a $(1,0)$–form and so is $\mathrm{d}\,\mathrm{tr}\,\varphi^i = \mathrm{tr}\,\nabla\varphi^i$, hence $\mathrm{tr}\,\varphi^i$ is a holomorphic function on M. This in turn implies that $\mathrm{tr}^i(\Phi) = (\mathrm{tr}\,\varphi^i)(\mathrm{d}z)^{\otimes i}$ is a holomorphic section of $K^{\otimes i}$. By Lemma 4.1 $s_i(\Phi)$ is then holomorphic for all $i \in \{1, \ldots, m\}$. □

In particular this lets tr^i and s_i induce maps

$$s_i\colon \mathscr{X} \to H^0(M, K^{\otimes i}) \qquad \mathrm{tr}^i\colon \mathscr{X} \to H^0(M, K^{\otimes i}),$$

where $H^0(M, K^{\otimes i})$ is the space of holomorphic sections of $K^{\otimes i}$. Since s_i and tr^i are invariant under conjugation, these maps descend to the moduli space $\mathscr{M}$ and together assemble to maps

$$\rho\colon \mathscr{M} \to \bigoplus_{i=1}^{m} H^0(M, K^{\otimes i}), \quad [\nabla, \Phi] \mapsto (s_1(\Phi), \ldots, s_m(\Phi)),$$
$$\widetilde{\rho}\colon \mathscr{M} \to \bigoplus_{i=1}^{m} H^0(M, K^{\otimes i}), \quad [\nabla, \Phi] \mapsto (\mathrm{tr}^1(\Phi), \ldots, \mathrm{tr}^m(\Phi)).$$

When restricting to the moduli space $\mathscr{M}^*$ of irreducible Higgs bundles, the maps ρ and $\widetilde{\rho}$ are smooth and even holomorphic with respect to J_1. The map ρ is often called the *Hitchin fibration.*

Lemma 4.3. *The space*

$$\mathscr{B} = \bigoplus_{i=1}^{m} H^0(M, K^{\otimes i})$$

is a $m^2(\gamma - 1) + 1$–dimensional complex vector space, where γ is the genus of M.

Proof. This follows from the Riemann–Roch theorem, which states that for every holomorphic line bundle $L \to M$ we have

$$\dim H^0(M, L) - \dim H^0(M, K \otimes L^*) = \deg(L) - \gamma + 1.$$

When we insert the trivial line bundle for L, which has degree 0 and satisfies $\dim H^0(M, L) = 1$, since its only global sections are the constant functions, we obtain

$$\dim H^0(M, K) = \gamma. \tag{27}$$

Now we apply Riemann–Roch again with $L = K$. Since $K \otimes K^*$ is isomorphic to the trivial bundle we get

$$\gamma - 1 = \deg(K) - \gamma + 1,$$

so $\deg(K) = 2\gamma - 2$. Finally, we can use the Riemann–Roch theorem with $L = K^{\otimes i}$ using that $\deg(K^{\otimes i}) = i \deg(K)$ and that $K^{\otimes j}$ for negative j has no holomorphic sections to obtain

$$\dim H^0(M, K^{\otimes i}) = i(2\gamma - 2) - \gamma + 1 = (\gamma - 1)(2i - 1)$$

for $i \geq 2$, which added up together with (27) gives the statement of the lemma. □

4.3 Properness

An important property of the Hitchin fibration, which we will prove now, is that it is proper, i.e.the preimage of every compact set is compact. In particular this implies that every fiber is a compact subspace of $\mathscr{M}$. First, we need some estimates.

Lemma 4.4. *Every nilpotent matrix $A \in \mathbb{C}^{m\times m}$ satisfies*

$$\|A\|^2 \leq c\,\|[A, A^*]\|$$

for some $c \geq 0$ depending only on m.

Proof. For an arbitrary matrix $A \in \mathbb{C}^{m\times m}$ the Frobenius norm satisfies the inequality

$$\|A\|^2 = |\operatorname{tr} A^*A| = |\langle 1, A^*A\rangle| \leq \|1\|\|A^*A\| = \sqrt{n}\|A^*A\|, \tag{28}$$

the right hand side of which can be expressed using

$$\|[A, A^*]\|^2 = \operatorname{tr}(AA^*AA^* - AA^*A^*A - A^*AAA^* + A^*AA^*A) = 2\|A^*A\|^2 - 2\|A^2\|^2. \tag{29}$$

Plugging (28) and (29) together then gives the inequality

$$\|A\|^4 \leq \tfrac{1}{2}m\|[A, A^*]\|^2 + m\|A^2\|^2. \tag{30}$$

From the expansion

$$[A^2, A^{*2}] = A[A, A^*]A^* + [A, A^*]AA^* + A^*A[A, A^*] + A^*[A, A^*]A,$$

sub–multiplicativity of the Frobenius norm and the elementary fact that $\|A\| = \|A^*\|$, we then get the estimate

$$\|[A^2, A^{*2}]\| \leq 4\|A\|^2\|[A, A^*]\|. \tag{31}$$

We now show the lemma for every A such that $A^{2^k} = 0$, using induction over $k \in \mathbb{N}$. The case $k = 0$ is trivial, so let $A \in \mathbb{C}^{m\times m}$ such that $A^{2^{k+1}} = 0$. Using the induction hypothesis for A^2 and then (31), we get

$$\|A^2\|^2 \leq c\|[A^2, A^{*2}]\| \leq 4c\|A\|^2\|[A, A^*]\| \tag{32}$$

and therefore, combining this with (30),

$$\|A\|^4 \leq \tfrac{1}{2}m\|[A, A^*]\|^2 + 4mc\|A\|^2\|[A, A^*]\|. \tag{33}$$

This can easily be solved to

$$\|A\|^2 \leq (\sqrt{m/2 + 4m^2c^2} + 2mc)\|[A, A^*]\|,$$

proving the lemma, since all nilpotent matrices are covered as soon as $2^k \geq m$. □

Lemma 4.5. *Let $A \in \mathbb{C}^{m\times m}$ have the eigenvalues $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. Then*

$$\|A\|^2 \leq c_1\|[A, A^*]\| + c_2 \sum_{i=1}^{m} |\lambda_i|^2$$

for some constants $c_1, c_2 \geq 0$ depending only on m.

Proof. Any matrix can be decomposed into a sum $A = D + N$ such that D is normal and N is nilpotent. Since D is normal, Lemma 4.4 and sub–multiplicativity of the Frobenius norm yield the inequality

$$\|N\|^2 \leq c\|[N,N^*]\| = c\|[D+N,D^*+N^*] - [D,N^*] - [N,D^*]\| \leq c\|[A,A^*]\| + 4c\|D\|\|N\|.$$

Solving this for $\|N\|$ gives the estimate

$$\|N\| \leq \sqrt{c\|[A,A^*]\| + 4c^2\|D\|^2} + 2c\|D\|$$

which can be rearranged into

$$\|A\|^2 \leq (\|D\| + \|N\|)^2 \leq 2c\|[A,A^*]\| + (16c^2 + 8c + 2)\|D\|^2.$$

This is the desired result since $\|D\|^2$ is the sum of the absolute squares of the eigenvalues of D, which are equal to those of A. □

A direct consequence is the following:

Corollary 4.6. *Using any norm on the finite–dimensional space $H^0(M, K^{\otimes i})$, there are constants c_1, c_2 such that, for all $\Phi \in \Omega^{1,0}(\operatorname{End} E)$,*

$$\|\Phi\|_{L^4}^4 \leq c_1\|[\Phi \wedge \Phi^*]\|_{L^2}^2 + c_2 \sum_{i=1}^{m} \|s_i(\Phi)\|^4,$$

where $m = \operatorname{rk} E$ and $s_1, \ldots, s_m$ are the elementary symmetric polynomials.

Lemma 4.7. *Let ∇ be a unitary connection on E, $\widetilde{\nabla}$ the connection on $T^*M \otimes E$ induced by ∇ and the Levi–Civita– or Chern connection ∇_{C} and let $\Phi \in \Omega^{1,0}(E)$. Then $\overline{\partial}^{\nabla}\Phi = 0 \in \Omega^{1,1}(E)$ is equivalent to $\overline{\partial}^{\widetilde{\nabla}}\Phi = 0 \in \Omega^{0,1}(T^*M \otimes E)$.*

Proof. Let $\varphi_1, \ldots, \varphi_m$ be a local $\overline{\partial}^{\nabla}$–holomorphic frame of E and write $\Phi = \alpha^i \otimes \varphi_i$ for some $\alpha^i \in \Omega^1$. Then

$$\begin{aligned} \mathrm{d}^{\nabla}(\alpha^i \otimes \varphi_i) &= \mathrm{d}\alpha^i \otimes \varphi_i - \alpha^i \wedge \nabla\varphi_i \\ \widetilde{\nabla}(\alpha^i \otimes \varphi_i) &= \nabla_{\mathrm{C}}\,\alpha^i \otimes \varphi_i + \alpha^i \otimes \nabla\varphi_i. \end{aligned}$$

Taking the $(0,1)$–parts of these equations, the last terms vanish, since φ_i is a holomorphic frame. Moreover, the $(0,1)$–part of $\nabla_{\mathrm{C}}\,\alpha^i$ equals $\overline{\partial}\alpha^i$ by the definition of the Chern connection. □

Lemma 4.8. *Let (∇, Φ) be a solution of the Hitchin equations. Then Φ satisfies the inequality*

$$\|[\Phi \wedge \Phi^*]\|_{L^2} \leq \sqrt{-\kappa_{\min}}\|\Phi\|_{L^2} \tag{34}$$

where $\kappa_{\min} \leq 0$ is the minimum of the Gaussian curvature of M.

Proof. Let $\widetilde{\nabla}$ be the unitary connection on $T^*M \otimes \text{End } E$ induced by the Levi–Civita resp. Chern connection on TM and the connection ∇ on E. Lemma 4.7 together with $\overline{\partial}^{\nabla}\Phi = \mathrm{d}^{\nabla}\Phi = 0$ implies that $\widetilde{\nabla}\Phi \in \Omega^{1,0}(T^*M \otimes \text{End } E)$. Consider the 1–form on M given by $X \mapsto \langle \Phi, \widetilde{\nabla}_X \Phi\rangle_{T^*M\otimes\text{End } E}$. The statement of the lemma will follow by calculating its derivative

$$\mathrm{d}\langle \Phi, \widetilde{\nabla}\Phi\rangle = \langle \widetilde{\nabla}\Phi \wedge \widetilde{\nabla}\Phi\rangle + \langle \Phi, F^{\widetilde{\nabla}}\Phi\rangle. \tag{35}$$

The first summand is equal to

$$\langle \widetilde{\nabla}\Phi \wedge \widetilde{\nabla}\Phi\rangle = -\langle \widetilde{\nabla}\Phi, \star\widetilde{\nabla}\Phi\rangle\omega = i\|\widetilde{\nabla}\Phi\|^2\omega \tag{36}$$

since $\star = -j$ and $\widetilde{\nabla}\Phi$ is a $(1,0)$–form. The second summand in (35) can be expressed in terms of the curvatures of ∇ and the Levi–Civita connection.

$$\langle \Phi, F^{\widetilde{\nabla}}\Phi\rangle(X,Y) = \langle \Phi, -\Phi F^{\mathrm{LC}}_{X,Y} + [F^{\nabla}_{X,Y}, \Phi]\rangle = \langle \Phi^\sharp, -F^{\mathrm{LC}*}_{X,Y}\Phi^\sharp\rangle + \langle \Phi, [F^{\nabla,\perp}_{X,Y}, \Phi]\rangle \tag{37}$$

Now suppose $(X, Y) = (e_1, e_2)$ is a positively oriented orthonormal basis of the tangent space (at some point) and let $\varphi_i = \Phi(e_i) \in \text{End } E$. Then

$$\langle \Phi^\sharp, F^{\mathrm{LC}\,*}_{e_1,e_2}\Phi^\sharp\rangle = \sum_{i,j}\langle e_i \otimes \varphi_i, F^{\mathrm{LC}\,*}_{e_1,e_2}e_j \otimes \varphi_j\rangle = \sum_{i,j}\langle \varphi_i, \varphi_j\rangle\langle e_j, F^{\mathrm{LC}}_{e_1,e_2}e_i\rangle = -2\kappa i\,\mathrm{Im}\langle \varphi_1, \varphi_2\rangle$$

where κ is the Gaussian curvature of M. But we also have

$$\langle \Phi \wedge \Phi\rangle = \sum_{i,j}\langle \varphi_i, \varphi_j\rangle e^i \wedge e^j = (\langle \varphi_1, \varphi_2\rangle - \langle \varphi_2, \varphi_1\rangle)e^1 \wedge e^2 = 2i\,\mathrm{Im}\langle \varphi_1, \varphi_2\rangle\,\omega$$

so the first summand in (37) equals $\kappa\langle \Phi \wedge \Phi\rangle(e_1, e_2) = -\kappa\langle \Phi, \star\Phi\rangle = i\kappa\|\Phi\|^2$. The second summand in (37) can be calculated using

$$F^{\nabla,\perp}_{e_1,e_2} = -[\Phi \wedge \Phi^*](e_1, e_2) = \langle [\Phi, \star\Phi^*]\rangle = i\langle [\Phi, \Phi^*]\rangle$$

which implies

$$\langle \Phi, [F^{\nabla,\perp}_{e_1,e_2}, \Phi]\rangle = \langle \langle [\Phi, \Phi^*]\rangle, F^{\nabla,\perp}_{e_1,e_2}\rangle = i\|\langle [\Phi, \Phi^*]\rangle\|^2 = i\|[\Phi \wedge \Phi^*]\|^2, \tag{38}$$

where the last equality again follows from $\star = -j$. Now we can combine (35), (36), (37) and (38) and obtain

$$\mathrm{d}\langle \Phi, \widetilde{\nabla}\Phi\rangle = i\|\widetilde{\nabla}\Phi\|^2\omega + i\kappa\|\Phi\|^2\omega + i\|[\Phi \wedge \Phi^*]\|^2\omega.$$

Integrating this yields

$$\|\widetilde{\nabla}\Phi\|^2_{L^2} + \int_M \kappa\|\Phi\|^2\omega + \|[\Phi \wedge \Phi^*]\|^2_{L^2} = 0,$$

from which (34) follows. □

Theorem 4.9. *The map*

$$\rho\colon \mathscr{M} \to \mathscr{B}, \quad [\nabla, \Phi] \mapsto (s_1(\Phi), \ldots, s_m(\Phi))$$

is proper.

Proof. Choose any norms on the finite–dimensional spaces $H^0(M, K^{\otimes i})$, let C be a compact subset of $\mathscr{B}$ and let $(\nabla_i, \Phi_i) \in \mathscr{X}$ represent a sequence in the preimage of C. We want to show that a subsequence converges up to gauge transformations. Since C is bounded, by Corollary 4.6 there are $M, c \geq 0$ such that

$$\|\Phi_i\|_{L^4}^4 \leq c\|[\Phi_i \wedge \Phi_i^*]\|_{L^2}^2 + M \tag{39}$$

for all i. But using Lemma 4.8 we have

$$\|[\Phi_i \wedge \Phi_i^*]\|_{L^2}^4 \leq |\kappa_{\min}|^2\|\Phi_i\|_{L^2}^4 \leq c|\kappa_{\min}|^2\|[\Phi_i \wedge \Phi_i^*]\|_{L^2}^2 + |\kappa_{\min}|^2 M.$$

Solving by $\|[\Phi_i \wedge \Phi_i^*]\|_{L^2}$, we see that this is uniformly bounded in L^2 by some constant, and by the first of Hitchin's equations so is F^{∇_i}. By Uhlenbeck's weak compactness theorem [Weh04, Theorem 7.1] there is a sequence of gauge transformations $g_i \in \mathscr{G}_2$ such that a subsequence of $g_i^*\nabla_i$ converges weakly in the H^1–topology to some $\nabla \in \mathscr{X}_1$. Passing to this subsequence, [Weh04, Theorem 8.3] implies that there is a further sequence of gauge transformations $h_i \in \mathscr{G}_{1,4} \subset \mathscr{G}_1$ with, for high enough i,

$$\nabla^*\eta_i = 0 \quad \text{and} \quad \|\eta_i\|_{L^4} \leq c\|g_i^*\nabla_i - \nabla\|_{L^4}.$$

where we wrote $\eta_i = h_i^* g_i^* \nabla_i - \nabla \in \Omega_0^1(\mathfrak{u}\, E)$. The embedding $H^1 \hookrightarrow L^4$ is compact, so this shows that η_i converges to 0 in the L^4 topology. Now consider the equation

$$0 = \mathrm{d}^{h_i^* g_i^* \nabla_i}\Phi_i' = \mathrm{d}^{\nabla}\Phi_i' + \eta_i \wedge \Phi_i'. \tag{40}$$

Because $\Phi_i' = h_i^* g_i^* \Phi_i$ is L^4–bounded by (39) and $\|\eta_i \wedge \Phi_i'\|_{L^2} \leq c\|\eta_i\|_{L^4}\|\Phi_i'\|_{L^4}$ for some constant c, we see that $\mathrm{d}^{\nabla}\Phi_i'$ converges to 0 in L^2. But $\mathrm{d}^{\nabla} \colon \Omega^{1,0}(\mathrm{End}\, E) \to \Omega^2(\mathrm{End}\, E)$ is an elliptic operator, so elliptic regularity implies that Φ_i' converges in H^1 to some element in the kernel of d^{∇}. Now the gauge fixing $\nabla^*\eta_i = 0$ together with Equation 40 and the first of Hitchin's equations

$$\mathrm{d}^{\nabla}\eta_i = -F^{\nabla} - \eta_i \wedge \eta_i - [\Phi_i' \wedge \Phi_i'^*] + c\,\mathrm{id} \otimes \omega$$

gives an elliptic system with the inhomogeneity converging in L^2 (since η_i and Φ_i' both converge in the L^4–topology), so η_i also converges in H^1.

So we now have an H^1–convergent sequence $(\nabla + \eta_i, \Phi_i')$ which is $\mathscr{G}_1$–gauge equivalent to (∇_i, Φ_i). By Theorem 3.17 the limit is gauge equivalent via $g \in \mathscr{G}_2$ to a smooth solution. The second part of Theorem 3.17 applied to $g^*(\nabla + \eta_i, \Phi_i')$ then shows that a $\mathscr{G}_2$–equivalent sequence of smooth solutions converges in C^∞. Clearly, the limit is again a solution. So $(\nabla_i, \Phi_i) \in \mathscr{X}$ is $\mathscr{G}_1$–equivalent to a C^∞–convergent sequence $(\widetilde{\nabla}_i, \widetilde{\Phi}_i) \in \mathscr{X}$. But if $g_i \in \mathscr{G}_1$ is a sequence such that $g_i^*\nabla_i = \widetilde{\nabla}_i$ and $\widetilde{\eta}_i = \widetilde{\nabla}_i - \nabla_i \in \Omega^1(\mathfrak{u}\, E)$, then $\nabla_i g_i = g_i \widetilde{\eta}_i$ for all i, which implies $g_i \in \mathscr{G}$. So $[\nabla_i, \Phi_i] \in \mathscr{M}$ converges. This shows that $\rho^{-1}(C)$ is compact. □

4.4 Symplectic structure

In this section we will examine the symplectic structure of the Hitchin fibration and show that its regular fibers are Lagrangian submanifolds. This requires a way to find tangent vectors to the solution space satisfying certain relations. The following lemma gives a way to deform a vector tangent to only the second of Hitchin's equations to one that is also tangent to the first equation. It can be seen as an infinitesimal version of the Kobayashi–Hitchin correspondence, where a holomorphic Higgs field (i.e. a solution of the second equation) is transformed into a solution of both equations.

Lemma 4.10. *Let* $(\nabla, \Phi) \in \mathscr{X}$ *and* $(\eta, \Psi) \in \Omega^1(\mathfrak{u}\, E) \oplus \Omega^{1,0}(\mathrm{End}\ E)$ *be tangential to the equation* $\mathrm{d}^\nabla \Phi = 0$*, i.e.*

$$\mathrm{d}^\nabla \Psi + [\eta \wedge \Phi] = 0. \tag{41}$$

Then there exists $\xi \in \Gamma(i\, \mathfrak{u}\, E)$ *such that* $(\eta', \Psi') = (\eta - i \star \nabla\xi, \Psi + [\Phi, \xi])$ *in addition to (41) satisfies*

$$\mathrm{d}^\nabla \eta' + [\Phi \wedge \Psi'^*] + [\Psi' \wedge \Phi^*] = 0, \tag{42}$$

so that $(\eta', \Psi') \in T_{(\nabla,\Phi)}\mathscr{X}$*.*

Proof. Let us first show that (η', Ψ') still satisfies (41). This property is independent of ξ and follows from

$$\mathrm{d}^\nabla[\Phi, \xi] = [\mathrm{d}^\nabla \Phi, \xi] - [\nabla\xi \wedge \Phi] = -[\bar\partial^\nabla \xi \wedge \Phi] = -[-i \star \bar\partial^\nabla \xi \wedge \Phi] = -[-i \star \nabla\xi \wedge \Phi],$$

where we have used that $i = -j = \star$ on $\Omega^{0,1}(\mathrm{End}\ E)$ and that the $(1,0)$–part of $\nabla\xi$ does not enter.

Now we require ξ to be a solution of the equation

$$\nabla^*\nabla\xi + \langle[\Phi, [\Phi^*, \xi]]\rangle + \langle[\Phi^*, [\Phi, \xi]]\rangle = i \star \mathrm{d}^\nabla \eta + i \star [\Phi \wedge \Psi^*] + i \star [\Psi \wedge \Phi^*]. \tag{43}$$

The left hand side of (43) is a self–adjoint elliptic operator $\Gamma(i\, \mathfrak{u}\, E) \to \Gamma(i\, \mathfrak{u}\, E)$, so a solution exists if and only if the right hand side is L^2–orthogonal to every solution θ of the homogeneous equation

$$\nabla^*\nabla\theta + \langle[\Phi, [\Phi^*, \theta]]\rangle + \langle[\Phi^*, [\Phi, \theta]]\rangle = 0 \tag{44}$$

If $\theta \in \Omega^1(i\, \mathfrak{u}\, E)$ solves (44), then

$$\|\nabla\theta\|^2_{L^2} + \|[\Phi^*, \theta]\|^2_{L^2} + \|[\Phi, \theta]\|^2_{L^2} = 0,$$

so integrating the right hand side of (43) against θ and using the identities $\langle \mathrm{d}^\nabla \eta, \theta\rangle = \mathrm{d}\langle \eta, \theta\rangle + \langle \eta \wedge \nabla\theta\rangle$ and $\langle[\Phi^* \wedge \Psi], \theta\rangle = -\langle \Psi \wedge [\Phi, \theta]\rangle$ yields

$$-i \int_M \mathrm{d}\langle \eta, \theta\rangle + \langle \eta \wedge \nabla\theta\rangle - \langle \Psi^* \wedge [\Phi^*, \theta]\rangle - \langle \Psi \wedge [\Phi, \theta]\rangle = 0.$$

This shows that a solution $\xi \in \Gamma(i\, \mathfrak{u}\, E)$ of (43) exists. Next observe that $\nabla^* = -\star \mathrm{d}^\nabla \star$ as an operator from $\Omega^1(\mathrm{End}\ E)$ to $\Gamma(\mathrm{End}\ E)$. This follows by integrating

$$\langle \Psi, \nabla\varphi\rangle\omega = -\langle \star \Psi \wedge \nabla\varphi\rangle = \mathrm{d}\langle \star \Psi, \varphi\rangle - \langle \mathrm{d}^\nabla \star \Psi, \varphi\rangle = \mathrm{d}\langle \star \Psi, \varphi\rangle - \langle \star \mathrm{d}^\nabla \star \Psi, \varphi\rangle\omega$$

for $\Psi \in \Omega^1(\mathrm{End}\ E)$ and $\varphi \in \Gamma(\mathrm{End}\ E)$. Furthermore, the second term on the left hand side of (43) equals

$$\langle[\Phi, [\Phi^*, \xi]]\rangle = \star[\Phi \wedge \star[\Phi^*, \xi]] = i \star[\Phi \wedge [\Phi^*, \xi]] = -i \star[\Phi \wedge [\Phi, \xi]^*]$$

and a similar transformation for the third term shows that (43) is equivalent to

$$\mathrm{d}^\nabla(i \star \nabla\xi) - [\Phi \wedge [\Phi, \xi]^*] - [[\Phi, \xi] \wedge \Phi^*] = \mathrm{d}^\nabla \eta + [\Phi \wedge \Psi^*] + [\Psi \wedge \Phi^*],$$

which is just (42). □

Proposition 4.11. *Let $\alpha_1, \alpha_2 \in \mathscr{B}^*$ be linear functionals on the complex vector space $\mathscr{B}$. Then the complex functions $f_1 = \alpha_1 \circ \widetilde{\rho}$ and $f_2 = \beta_2 \circ \widetilde{\rho}$ on $\mathscr{M}^*$ are Poisson–commuting with respect to the complex symplectic structure $\omega_+ = \omega_2 + i\omega_3$ on $\mathscr{M}^*$.*

Proof. Let without loss of generality $\alpha \in H^0(M, K^{\otimes i})^*$ and $f = \alpha \circ \widetilde{\rho}$. By Serre duality, there is a $K^{\otimes -i}$–valued 2–form β such that α is of the form $\alpha(q) = \int_M \beta q$ and thus

$$f([\nabla, \Phi]) = \int_M \beta \operatorname{tr}(\Phi^i).$$

Let $p\colon \mathscr{X}^* \to \mathscr{M}^*$ be the projection map and $\iota\colon \mathscr{X}^* \to \mathscr{C}$ the inclusion. Then Theorem 2.24 defines the complex symplectic form ω_+ by $p^*\omega_+ = \iota^*\widehat{\omega}_+$, where

$$\widehat{\omega}_+((\eta_1, \Psi_1), (\eta_2, \Psi_2)) = i \int_M \operatorname{tr}(\Psi_1 \wedge \eta_2 + \eta_1 \wedge \Psi_2).$$

Let X_f be the Hamiltonian vector field for f. It is defined by $\mathrm{d}f(Y) = \omega_+(X_f, Y)$ for all $Y \in T\mathscr{M}^*$. Let $\widehat{X}_f \in T\mathscr{X}^*$ such that $Dp(\widehat{X}_f) = X_f$, then $\mathrm{d}(f \circ p)(Y) = p^*\omega_+(\widehat{X}_f, Y) = \widehat{\omega}_+(\widehat{X}_f, Y)$ for all $Y \in T\mathscr{X}^*$. With $\widehat{X}_f = (\eta_1, \Psi_1)$, this yields

$$\int_M \beta' \operatorname{tr}(\Phi^{i-1}\Psi_2) = i \int_M \operatorname{tr}(\Psi_1 \wedge \eta_2 + \eta_1 \wedge \Psi_2) \quad \forall (\eta_2, \Psi_2) \in T_{(\nabla,\Phi)}\mathscr{X}^*, \tag{45}$$

where β' is a suitable symmetrization of β. Now consider $\beta'\Phi^{i-1} \in \Omega^2(K^{\otimes(-1)} \otimes \operatorname{End} E)$ and its contraction $\operatorname{tr}_K \beta'\Phi^{i-1} \in \Omega^{0,1}(\operatorname{End} E)$. Since the projection $\operatorname{pr}^{0,1}$ from $\Omega^1(\mathfrak{u}\, E)$ to $\Omega^{0,1}(\operatorname{End} E)$ is an isomorphism, there is a unique $\eta \in \Omega^1(\mathfrak{u}\, E)$ such that $\eta^{0,1} = -i \operatorname{tr}_K \beta'\Phi^{i-1}$. Then $(\eta, 0) \in \Omega^1(\mathfrak{u}\, E) \oplus \Omega^{1,0}(\operatorname{End} E)$ satisfies (45). To see this, let z be a local holomorphic coordinate and write locally $\Phi = \varphi\, \mathrm{d}z$ and $\Psi_2 = \psi\, \mathrm{d}z$ and $\omega = ir\, \mathrm{d}\overline{z} \wedge \mathrm{d}z$ for $\varphi, \psi \in \operatorname{End} E$ and a real–valued function r on M. Then

$$\eta^{0,1} = -i \operatorname{tr}_K \beta'(dz)^{i-1} \otimes \varphi^{i-1} = -i(\star\, \beta'(dz)^i) \operatorname{tr}_K \omega \otimes \partial_z \otimes \varphi^{i-1} = qr\varphi^{i-1}\mathrm{d}\overline{z}$$

with the complex function $q = \star\, \beta'(dz)^i$. Furthermore,

$$\beta' \operatorname{tr}(\Phi^{i-1}\Psi_2) = \operatorname{tr}(\varphi^{i-1}\psi)q\omega = iqr \operatorname{tr}(\varphi^{i-1}\mathrm{d}\overline{z} \wedge \psi\, \mathrm{d}z) = i \operatorname{tr}(\eta^{0,1} \wedge \Psi_2)$$

and $(\eta, 0)$ thus satisfies (45). But $(\eta, 0)$ is not necessarily an element of $T_{(\nabla,\Phi)}\mathscr{X}^*$. It is however tangent to $\mathrm{d}^\nabla\Phi = 0$ since $[\eta \wedge \Phi] = 0$. So by Lemma 4.10 there is a $\xi \in \Gamma(i\, \mathfrak{u}\, E)$ such that $(\eta - i \star \nabla\xi, [\Phi, \xi]) \in T_{(\nabla,\Phi)}\mathscr{X}^*$. This modification does not change the right hand side of (45) since

$$\operatorname{tr}([\Phi, \xi] \wedge \eta_2 - i \star \nabla\xi \wedge \Psi_2) = -\operatorname{tr}([\eta_2 \wedge \Phi]\xi) + \mathrm{d} \operatorname{tr}(\Psi_2\xi) - \operatorname{tr}(\mathrm{d}^\nabla\Psi_2\xi),$$

which integrates to 0. So $X_f = Dp(\eta - i \star \nabla\xi, [\Phi, \xi])$. If g is another function arising in the same way as f, with its differential represented by β'', then

$$\{g, f\} = \mathrm{d}g(X_f) = \mathrm{d}(g \circ p)(\widehat{X}_f) = \int_M \beta'' \operatorname{tr}(\Phi^{i-1}[\Phi, \xi]) = 0. \qquad \square$$

Corollary 4.12. *Let $F \subset \mathscr{M}^*$ be a regular fiber of $\widetilde{\rho}\colon \mathscr{M}^* \to \mathscr{B}$. Then F is a Lagrangian submanifold with respect to ω_+ and for all $x \in F$ the map $\mathscr{B}^* \to T_xF, \alpha \mapsto X_{\alpha\circ\widetilde{\rho}}$ is an isomorphism with $X_{i\alpha\circ\widetilde{\rho}} = J_1 X_{\alpha\circ\widetilde{\rho}}$.*

Proof. Let $x \in F$ and $H_x = \{(X_{\alpha\circ\widetilde{\rho}})_x \mid \alpha \in \mathscr{B}^*\} \subset T_x\mathscr{M}^*$ be the subspace of Hamiltonian vector fields of the form as in Proposition 4.11. Then its symplectic complement $H_x^{\omega_+}$ consists of all tangent vectors $Y \in T_x\mathscr{M}^*$ with $\alpha(D\widetilde{\rho}(Y)) = D(\alpha \circ \widetilde{\rho})(Y) = \omega(X_{\alpha\circ\widetilde{\rho}}, Y) = 0$ for all $\alpha \in \mathscr{B}^*$, so $H_x^{\omega_+} = \ker D_x\widetilde{\rho} = T_xF$ is the tangent space along the fiber $F \subset \mathscr{M}^*$. Proposition 4.11 shows that $\omega_+(X_{\alpha\circ\widetilde{\rho}}, X_{\beta\circ\widetilde{\rho}}) = 0$ for all $\alpha, \beta \in \mathscr{B}$, so $H_x \subset H_x^{\omega_+} = T_xF$, i.e. F is a coisotropic submanifold of $\mathscr{M}^*$. Since $D_x\widetilde{\rho}$ is surjective, we have $\dim F = \frac{1}{2}\dim \mathscr{M}^*$, so F is indeed a Lagrangian submanifold and $H_x = H_x^{\omega_+} = T_xF$. It follows that the map $\mathscr{B}^* \to T_xF$ is surjective and thus an isomorphism by dimension. For all $Y \in T_x\mathscr{M}^*$, it satisfies

$$\omega_+(X_{i\alpha\circ\widetilde{\rho}}, Y) = i\alpha(D\widetilde{\rho}(Y)) = \alpha(D\widetilde{\rho}(J_1Y)) = \omega_+(X_{\alpha\circ\widetilde{\rho}}, J_1Y) = \omega_+(J_1X_{\alpha\circ\widetilde{\rho}}, Y). \qquad \square$$

Since we know by Theorem 4.9 that the Hitchin fibration is a proper map, every fiber not containing any reducible solutions is a compact submanifold. On these fibers we can integrate the Hamiltonian vector fields $X_{\alpha\circ\widetilde{\rho}}$ to get a group action of $\mathscr{B}^*$ on the fiber. This gives the following result.

Corollary 4.13. *Let $b \in \mathscr{B}$ such that the fiber $F = \rho^{-1}(b)$ of $\rho\colon \mathscr{M} \to \mathscr{B}$ over b contains only irreducible solutions and assume b is a regular value of $\rho\colon \mathscr{M}^* \to \mathscr{B}$. Then every connected component of F is biholomorphic to a complex torus $\mathbb{C}^d/\Lambda$ of complex dimension $d = m^2(\gamma - 1) + 1$.*

Proof. Let F be a regular fiber of ρ. Since the maps ρ and $\widetilde{\rho}$ have the same fibers, we can use $\widetilde{\rho}$ instead for this proof. By Corollary 4.12 every $\alpha \in \mathscr{B}^*$ induces a vector field $X_{\alpha\circ\widetilde{\rho}}$ tangent to the fiber F. Proposition 4.11 shows that any two of these vector fields commute. Let $\varphi_\alpha\colon \mathbb{R} \times \mathscr{M}^*_{\mathrm{reg}} \to \mathscr{M}^*_{\mathrm{reg}}$ be the flow of $X_{\alpha\circ\widetilde{\rho}}$, which exists for all times since F is compact and all integral curves stay in a single fiber. Then $(\alpha, x) \mapsto \varphi_\alpha(1, x)$ is a fiber–preserving action of the Abelian Lie group $\mathscr{B}^*$ on $\mathscr{M}^*_{\mathrm{reg}}$.

Now for a single regular fiber F and $x \in F$ let $r_x\colon \mathscr{B}^* \to F$ be the evaluation map. Its differential $D_0r_x\colon \mathscr{B}^* \to T_xF$ is the isomorphism $\alpha \mapsto X_{\alpha\circ\widetilde{\rho}}$. Therefore, the image of r_x, which is the $\mathscr{B}^*$–orbit containing x, is open in F. Since F is a disjoint union of such open orbits and $\mathscr{B}^*$ is connected, every connected component of F is exactly one orbit. Let $F' \subset F$ be the component containing x. Then $\mathscr{B}^*$ acts transitively on F'. Now consider the stabilizer subgroup $\Lambda = (\mathscr{B}^*)_x = r_x^{-1}(x)$. Since $D_\alpha r_x$ is an isomorphism for all $\alpha \in \Lambda$, Λ is a discrete subgroup of $\mathscr{B}^*$. It does not depend on the choice of $x \in F'$, as for Abelian groups all stabilizers along an orbit are equal.

The map $r_x\colon \mathscr{B}^*/\Lambda \to F'$ is clearly bijective. Its differential at $\beta \in \mathscr{B}^*$ is the map $D_\beta r_x(\alpha) = (X_{\alpha\circ\widetilde{\rho}})_{\beta x}$, which is an isomorphism and satisfies $D_\beta r_x(i\alpha) = J_1 D_\beta r_x(\alpha)$, so r_x is biholomorphic. It is only left to show that $\mathscr{B}^*/\Lambda$ is a torus, that is Λ is $2d$–dimensional as a $\mathbb{Z}$–vector space. If its dimension was less than $2d$, then $\mathscr{B}^*/\Lambda$ would not be compact, but F' is compact. On the other hand, if the dimension of Λ was greater than $2d$, we could take a $2d$–dimensional $\mathbb{Z}$–subspace $\Lambda' \subset \Lambda$ and consider the action of $\mathbb{Z}\alpha$ on the compact manifold $\mathscr{B}^*/\Lambda'$ for some $\alpha \in \Lambda \setminus \Lambda'$. Its stabilizer must be trivial, as otherwise α and a basis of Λ' would be $\mathbb{Z}$–linearly dependent. But this implies that the $\mathbb{Z}\alpha$–orbit has a limit point, which is impossible since Λ is discrete. $\square$

We have not addressed the question whether the Hitchin fibration ρ has any regular values. But it can be shown by an algebraic argument [Hit87a, Section 5.1] that ρ is a surjective map. First of all, this implies that solutions to Hitchin's equations always exist, i.e. $\mathscr{M}$ is not empty. Moreover, by Sard's Theorem there is an open dense set $\mathscr{B}_{\mathrm{reg}} \subset \mathscr{B}$ of regular values of ρ. If in addition there exist no reducible solutions in $\mathscr{M}$ (for example if $\deg(E)$ and $\mathrm{rk}(E)$ are coprime), then $\mathscr{M}_{\mathrm{reg}} = \rho^{-1}(\mathscr{B}_{\mathrm{reg}})$ is locally a product of $\mathscr{B}$ and a complex torus, i.e. a fiber bundle over $\mathscr{B}_{\mathrm{reg}}$ with tori as fibers.

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