On the hypothesis of isotropy

Gauge invariant formulation of the cosmological backreaction

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In this work a covariant and gauge invariant averaging formalism for finite volumes will be developed. This averaging will be applied to the scalar parts of Einstein's equations. For this purpose "dust" as a physical laboratory will be coupled to the gravitating system. The goal is to study the deviation from the homogeneous universe and the impact of this deviation on the dynamics of our universe.

At first, the standard homogeneous cosmological model will be presented. Then, the so called backreaction as proposed in [3] will be introduced. We will cite [8] and show that the averaging procedure used in [3] is not gauge invariant. Furthermore, a remedy to this problem will be presented. Fields of physical observables (dust) will be included in the studied system and used to construct a reference frame to perform the averaging without a formal gauge fixing.

The derived equations resolve the question whether backreaction is gauge dependent. The outcome of our approach will be compared to the results of [3]. Those will be reinterpreted in our set up. In the last chapter we will make suggestions for experimental methods for studying the inhomogeneities and their effect on the dynamics of the universe.

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Chapter 1

Cosmological backreaction

1.1 Introduction

Einstein's theory of general relativity enabled the humankind to describe the state of the entire universe in a mathematically precise way. The question is which solution of the Einstein field equations describes the universe we observe. As usual in physics, one observes the sources and searches for a solution for the field which would be generated by them. But even if one knows the sources, i.e. the energy momentum distribution, the theory can not be straightforwardly solved as would be the case for Maxwell's electromagnetism. Einstein's equations have a highly complicated structure and what makes the task especially involved is the fact that the source is dependent on the field (the metric) which one wishes to solve for. So, to have a solution it is necessary to reduce the degrees of freedom by introducing symmetries. The most drastic reduction of degrees of freedom was performed by Alexander Friedmann who used the assumption of the spatially isotropic universe. This leaves just one free parameter -the scale factor of the universe- and Einstein's theory becomes solvable. The homogeneity assumption was at the times of A. Friedmann rather motivated by philosophy, the mediocrity- or Copernican principle. This principle states that our position in the universe is not preferred in any way.

Nowadays the discovery of the Cosmic Microwave Background radiation is the strongest evidence for isotropy. The microwave signal is extremely close to black body radiation with minimal fluctuations of 10^{-5} relative magnitude. Therefore, we know that the universe used to be highly isotropic at the age of 380,000 years when electrons and protons formed hydrogen atoms and allowed the photons to travel freely. At this time the FRW universe, which will be shortly discussed in detail, was an excellent description of our universe. However, the very fact that today this thesis can be written violates the isotropy. More precisely, the energy density inside a galaxy is by a factor of 10^5 greater than the average energy density of the universe. So how is the homogeneity hypothesis meant at present?

Hawking and Ellis write in their book "Large scale structure of space-time": "...the universe, when viewed on a suitable scale, is approximately spatially homogeneous." This statement is imprecise in two ways. First, according to which averaging procedure is spatial homogeneity approximate? Second, what is this suitable scale? The goal of this work will be to make this into a more precise mathematical statement.

We will proceed in the following way: At first, the formalism used by Thomas Buchert will be introduced. In this formalism the Einstein equations of an inhomogeneous universe are studied. The author defines a space-time slicing using the velocity flow of an ideal fluid. The scalar parts of the Einstein equations are projected out using the normal vectors of this slicing and they are averaged over a space-like domain. The astonishing observation is that the resulting equations have a similar structure to the Friedmann equations, but also contain extra terms. These terms are a quantitative measure of the inhomogeneity effect on the dynamics of the domain under consideration. The author calls the newly found terms the "backreaction". These terms are of great interest to quantitatively analyze the deviation from an FRW universe.

Buchert's derivation has been criticized in the literature. Gabriele Veneziano has shown that the averaging functional used by T. Buchert breaks gauge invariance. This is a point worth worrying about, since we can only observe quantities which are independent of the chosen gauge. Veneziano proposes a remedy for the problem concerning the choice of foliation, but he fails to construct a gauge independent averaging scheme for finite volumes. Veneziano's argument for the difficulty of constructing such a finite-volume method, is the absence of spatially inhomogeneous scalar fields in an FRW universe. This is of course correct, but why should one average an FRW universe? The case we are interested in, is spatially not homogeneous and we wish to investigate whether this feature appears in the averaged case. So, in our model such spatially inhomogeneous physical scalars do exist and those will be used to construct the gauge independent averaging formalism for finite volumes.

In the third chapter we will introduce the method of deparametrization using dust. This formalism has been developed to address the problem of time in general relativity. We will also discuss this problem in detail and how adding matter to the system helps address it. The system under consideration will be enlarged by the specific type of matter, namely the pressure-less fluid (called dust). The coordinates of the dust will serve as a physical coordinate system and as the desired scalar fields with a spatial gradient. We will perform the averaging of the scalar parts of Einstein's equations in a manifestly gauge invariant way. As always in physics, it is legitimate to fix a gauge when the whole formalism used is gauge covariant. We will do so and rediscover Buchert's equations in a certain gauge and under certain conditions. Hence, we will show that the

Buchert equations for the dynamics of finite volumes are valid under certain constraints and the backreaction is, in principle, an observable quantity. This finally resolves the question of the gauge dependence of the backreaction. In the last chapter we will make suggestions how to estimate the desired values experimentally with the final goal to give an exact statement concerning the isotropy of the universe.

1.2 Introduction to homogeneous cosmology

Einstein's theory of gravity makes incredible predictions for our universe under the assumption of isotropy. So at first let us follow A. Friedmann's path, assume an isotropic universe and derive the Friedmann equations as in [19]. Those will be compared to Buchert's equations later on.

First the notions of isotropy and homogeneity have to be defined in a precise way.

***Definition of homogeneity*:**

A space-time is called spatially homogeneous if there exists an isometry of h_{ab} on Σ_t which keeps h_{ab} and U^α fixed and carries a point $p \in \Sigma_t$ into $q \in \Sigma_t$.

Definition of isotropy:

A space-time is said to be spatially isotropic at each point, if there exists a congruence of time-like curves (i.e observers), with tangents U^α, filling the space-time and satisfying the following property: Given any point p and any two unit "spatial" tangent vectors S_1^a, $S_2^a \in V_p$ (orthogonal to U^α), there exists an isometry of $g_{\mu\nu}$ which leaves p and U^α at p fixed, but rotates S_1^a into S_2^a. Thus, in an isotropic universe it is impossible to construct a geometrically preferred tangent vector orthogonal to U^α.

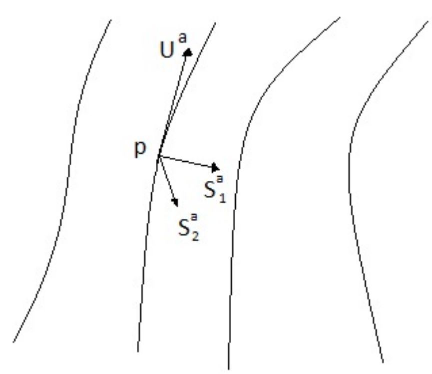

Figure 1.1: congruence of time-like curves

Claim:

Under the assumption of isotropy it is possible to foliate the space-time into isotropic hypersurfaces Σ_t which are orthogonal to U^α.

***Proof*:**

Assume the contrary. Furthermore assume Σ_t and the isotropic observers are unique. If now the tangent subspace to p orthogonal to U^α does not coincide with Σ_t , it is possible to find a preferred spatial direction, by projecting U^α on Σ_t. The vector $U^\alpha_\perp$ then represents a spatially preferred direction. This contradicts the assumption.

We assume only isotropy. Then within the foliation, $g_{\mu\nu}$ induces a metric $h(t)_{ab}$ on Σ_t by restricting the action of $g_{\mu\nu}$ at p to vectors tangent to Σ_t. Keep in mind that because of isotropy, it must be impossible to construct any geometrically preferred vectors on Σ_t. Consider now the Riemann tensor ${}^{(3)}R^d_{abc}$ constructed from h_{ab} on Σ_t. The construction $R_{abe}{}^d h^{ec} = R_{ab}{}^{cd}$ at p, is a linear map L of the space W of two forms into itself $R : W \to W$. By the symmetry of the R-tensor, L is also a symmetric map. Therefore, W has on orthogonal basis of eigenvectors of L. From isotropy we conclude that all eigenvalues of L must be equal (otherwise we could construct a preferred spatial vector) i.e.

$$L = K\, id$$

$${}^{(3)}R_{ab}{}^{cd} = K\, \delta^c_{[d}\delta^d_{b]} \Rightarrow R_{abcd} = K\, h_{c[a}h_{b]d}$$

use now the Bianchi identity for the Riemann tensor.

$$0 = D^{(3)}_{[e} R_{ab]cd} = (D_{[e}K) h_{|c|a} h_{b]d}$$

(Here D is the derivative operator associated with h on Σ_t) From this we conclude:

$$D_e K = 0 \Rightarrow K = const.$$

That means the eigenvalues of L, L being the map associated with the Riemann tensor, are equal and constant all over Σ_t (so called constant curvature). This implies homogeneity for the spatial hypersurface. Remarkable is that homogeneity is a consequence of isotropy and was not assumed a priori. So the crucial assumption is isotropy.

Now the problem has been reduced to identifying spaces of constant curvature. It turns out that there are just three distinct choices. $K > 0$, $K < 0$ and $K = 0$ spherical- , hyperbolic- and flat-space respectively. The metric splits into $g_{\mu\nu} = -u_\mu u_\nu + h_{\mu\nu}(t)$ where at each t, $h(t)$

is the metric of either a sphere, a hyperboloid or the euclidean space on Σ_t. In the Friedman-Robertson-Walker cosmological model the metric has only one free parameter a. In the positive spatial curvature case it is the radius of the three sphere. In general we call it the scale factor. The time τ is the proper time measured by the isotropic observers. The metric has the following structure, where $k \in \{-1, 0, 1\}$ denotes the spatial curvature:

$$ds^2 = -d\tau^2 + a(\tau)^2 \left[\frac{dr^2}{1 - k\, r^2} + r^2 \left(d\theta^2 + sin^2(\theta)\, d\phi^2 \right) \right]$$

Note that for $k = 1$ this coordinatization has a singularity and so it covers only the $r < 1$ fraction of the 3-sphere.

The Einstein Tensor computed from the above metric and written as a matrix has the following form:

$$G^{\mu}_{\ \nu} = - \left\{ \begin{array}{cccc} \frac{3\left(k+\dot{a}^2\right)}{a^2} & 0 & 0 & 0 \\ 0 & \frac{k+2a\ddot{a}+\dot{a}^2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{k+2a\ddot{a}+\dot{a}^2}{a^2} & 0 \\ 0 & 0 & 0 & \frac{k+2a\ddot{a}+\dot{a}^2}{a^2} \end{array} \right\} \tag{1.1}$$

To solve the Einstein equation of the universe we also need the source i.e. the energy momentum tensor. The one for non relativistic matter, is to a good approximation $T^{\mu\nu} = \rho u^{\mu} u^{\nu}$ an energy momentum tensor of a pressure-less fluid. The situation is different if one includes radiation and needs to take the pressure into account as well. The most general tensor which is compatible with the isometries of the FRW model is the one of the perfect fluid $T^{\mu\nu} = \rho(t) u^{\mu} u^{\nu} + p(t)(u^{\mu} u^{\nu} + g^{\mu\nu}) = \rho u^{\mu} u^{\nu} + p\, P_{\perp}$, with ρ the energy density, p the pressure and $P_{\perp}$ the projection tensor on the orthogonal plane to u^{μ}. The velocity flow of the fluid is normalized i.e. $u^{\mu} u_{\mu} = -1$ (the minus appears due to our metric convention $(-, +, +, +)$). The different types of considered tensors are characterized by the equation of state $\omega = p/\rho$. The tensor can be written as a matrix, in analogy to the Einstein tensor above:

$$T^{\mu}_{\ \nu} = T^{\mu\lambda} g_{\lambda\nu} = \rho(t) u^{\mu} u^{\lambda} g_{\lambda\nu} + p(t)(u^{\mu} u^{\lambda} g_{\lambda\nu} + g^{\mu\lambda} g_{\lambda\nu}) = \rho(t)\, u^{\mu} u_{\nu} + p(t)\, (u^{\mu} u_{\nu} + g^{\mu}_{\ \nu})$$

In the fluid's rest frame $u^{\mu} = (1, 0, 0, 0)$ and hence the matrix can be explicitly written as:

$$T^{\mu}_{\ \nu} = \left\{ \begin{array}{cccc} -\rho(t) & 0 & 0 & 0 \\ 0 & p(t) & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & p(t) \end{array} \right\} \tag{1.2}$$

Note that we have chosen coordinates in which we expressed the metric and we also have decided to use the fluid's rest frame, since this is a convenient choice. This is absolutely legitimate, since all the equations we have are covariant tensor equations and hence transform correctly under coordinate changes.

Now we can use the velocity flow to project the Einstein equation on it:

$$G_{\mu\nu} u^{\mu} u^{\nu} = 8\pi G_N\, \rho(t)\, (u^{\mu} u_{\mu})^2 + p(t) \left((u^{\mu} u_{\mu})^2 + (u^{\mu} u_{\mu}) \right) = 8\pi G_N\, \rho(t)$$

As mentioned above in the fluid's rest frame $u^{\mu} = (1, 0, 0, 0)$ and therefore the left hand side is just the temporal-temporal component of the Einstein tensor. Comparing to (1.1) and taking:

$$G^{\mu}_{\ 0}\, g_{\mu 0} = \frac{3\,(k + \dot{a}^2)}{a^2}$$

We obtain the first Friedmann equation, which can be formulated with the the Hubble expansion rate:

$$H = \dot{a}/a$$

$$\boxed{\begin{gathered} 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G_N \rho - 3\frac{k}{a^2} \qquad (1.3) \\[2em] 3H^2 = 8\pi G_N \rho - 3\frac{k}{a^2} \end{gathered}}$$

An amazing observation is that assuming $\rho > 0$ (which is a reasonable assumption based on our experience) from the first Friedmann equation it follows that the universe can not be static. The only possibility to enforce a vanishing H is to assume the scale factor to have a specific fine-tuned value, which would still yield an unstable solution. Einstein himself has realized this property, but found it unphysical and tried to "cure" this feature by introducing a cosmological constant term, which will be discussed below in more detail.

To compute the trace of Einstein's equation we can use the explicit matrix notation above (1.1) and (1.2) :

$$Tr(G^{\mu}_{\ \nu}) = 8\pi G_N \, Tr(T^{\mu}_{\ \nu})$$

This yields the second Friedmann equation:

$$\boxed{\begin{aligned} -6\frac{\ddot{a}}{a} &= 8\pi G_N(\rho + 3p) \qquad (1.4) \\ \left(\dot{H} + H^2\right) &= -\frac{4\pi}{3} G_N(\rho + 3p) \end{aligned}}$$

Of course there are also other projections possible, as: $G_{\mu\nu}h^{\mu a}h^{\nu b}$ and $G_{\mu\nu}h^{\mu a}u^{\nu}$ (where a, $b \in \{1, 2, 3\}$) but those are identically zero in an isotropic universe. This can be seen directly from the equation (1.1). On the other hand it is clear that such "space-space" components, independent from the identical diagonal components would violate isotropy.

The total energy momentum tensor of the universe can now be a linear superposition of tensors of ideal fluids with different equations of state. The standard model matter would be to a good approximation described by ideal fluids, with $\omega = 0$ for fermionic, non-relativistic matter and $\omega = 1/3$ for bosons (radiation). Given the equation of state for a perfect fluid i.e. ω, the Friedmann equations can be solved and give an evolution equation for the scale factor, with a constant a_0 determined by initial conditions:

$$a(t) = a_0 \, t^{\left(\frac{2}{3(\omega+1)}\right)} \qquad (1.5)$$

Note: This solution is not valid for $\omega = -1$, which would result in a constant energy density and an exponential growth of the scale factor. It is remarkable that $\omega = -1$ would correspond to a cosmological constant Λ which can also be included in the most general Einstein tensor (Lovelock).

In cosmology the evolution of the universe is of great interest. The energy budget is the crucial quantity influencing the universe's dynamics. Given certain initial conditions, i.e. fractions of the different energy-momentum tensors in the total energy budget at one point of time, the question could be asked: How will the energy be diluted with the change of the scale factor? To solve this problem we differentiate (1.3) and setting here $k = 0$ insert (1.4), this results in:

$$\dot{\rho} = -3\,H\;(\rho + p) = -3\,H\,\rho\,(1 + \omega) \Leftrightarrow d\rho = -3\frac{1}{a}da\,(1 + \omega)$$

Integrating this, we obtain:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+\omega)}$$

- For non relativistic matter with $\omega = 0$ the scaling is: $\rho(a) \propto a^{-3}$
- For radiation with $\omega = \frac{1}{3}$ the scaling is: $\rho(a) \propto a^{-4}$
- For $\omega = -1$, which is the case for the cosmological constant the energy density is constant. This case will be discussed later more carefully.

Our result applied to the cosmological hot big bang model, which is the most accepted at the present times, shows the following. The early universe which was extremely hot and radiation dominated, expanded according to (1.5) as $a(t) \sim t^{\frac{1}{2}}$, but the energy density of relativistic matter was diluted during the expansion with the fourth inverse power of a, while the non relativistic matter was diluted with the third inverse power of a. Therefore, at some point the non relativistic matter must have taken over. In the non-relativistic (dust) era the scale factors behavior was dominantly $a(t) \sim t^{\frac{2}{3}}$. This scenario matches with the observational data to a high accuracy.

A new fascinating observation has been made in the nineties. Supernova surveys indicate that our era is dominated by the cosmological constant. To this evolution the simple solution (1.5) does not apply. To discuss the cosmological constant problem, let us first derive an expression for the total energy budget and discuss its value at present. Rewriting the first Friedmann equation, including the cosmological constant and rearranging terms, one obtains:

$$1 = H^{-2}\left(\frac{8\pi G_N}{3}\rho(t) - \frac{k}{a^2} + \frac{\Lambda}{3}\right)$$

With definitions of the partial energy budgets Ω_i :

$$\Omega_m := \frac{8\pi G_N}{3\,H^2}\rho(t)$$

$$\Omega_k := -\frac{k}{H^2}$$

$$\Omega_\Lambda := -\frac{\Lambda}{3H^2}$$

We can write down the total energy budget of the universe:

$$\Omega_m + \Omega_k + \Omega_\Lambda = 1$$

The observations of the cosmic microwave background (CMB), supernovae explosions (SNe) and barion acoustic oscillations (BAO) give very strong evidence that the universe is flat, the $\Omega_m \backsimeq 0,3$ and the $\Omega_\Lambda \backsimeq 0,7$. It is the overlap of this three independent observations, which makes this data so credible and provides an extremely strong test for any model which claims to explain nature. The so called Λ-CDM model is the most successful today, since it explains observations of all this experiments to a high accuracy. The evidence for a non radiating (dark) matter as well as the fascinating physics of the observed effects, can not be discussed at this point, but we will come back to some of them in the last chapter.

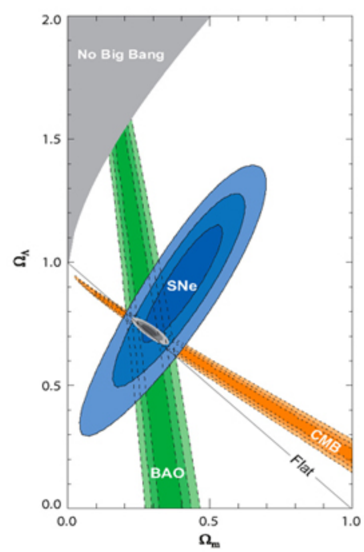

Figure 1.2: Cosmological budget

The cosmological constant Λ_{CC} is a natural term in the Einstein equation. On the other hand from quantum field theory we know an object called vacuum energy. This object has the equation of state $\omega = -1$. According to QFT if there are any quantum fields present in the theory the whole space is filled with this ground state. Since QFT is the best theory we know to understand the fundamental properties of matter, we have to take it seriously. This means that the value of Λ_{QFT} will renormalize the the bare value of Λ_{CC} and result in an effective Λ_{Eff}. So far only shifts in the vacuum energy, as in the Casimir effect, have been observed. The absolute value seems to be only accessible via cosmological observations. This absolute value is predicted by the theory, but the predicted value of Λ_{QFT} and the observed Λ_{Eff} differ by at least 50 orders of magnitude. This discrepancy indicates that we do not understand some mechanism very profoundly or have to accept a fine tuning up to 50 decimal orders. Such a fine tuning shows by itself some fundamental problem in the theory. In the fourth chapter we will discuss whether studying the effect of inhomogeneities can open a new window of opportunity and help come closer to the solution of this challenging riddle.

Briefly summarizing, the up-to-date observational results from supernovae explosions and the cosmic microwave background, indicate that the universe is flat $k = 0$ (or very close to it) and that its expansion accelerates, which would mean a domination of a quantity with constant energy density.

1.3 Inhomogeneous Cosmology

From the CMB we know that the universe used to be highly isotropic, but today this isotropy has been violated. Since this was an evolutionary process we expect it to happen in a continuous way, starting from an FRW universe and smoothly developing inhomogeneities. One is interested in effective equations describing the dynamics of space-time and expects them to have a similar structure as the Friedmann equations and corrections, small compared to the Friedmann background.

The question we could ask now is, whether this assumption of an FRW background is an unnecessary limitation, or whether it can be avoided and the Einstein equations can be solved generally. For this purpose, let us compare GR and Maxwell's Electromagnetism. Let us assume that we have the external sources (the currents) in Maxwell's theory and want to solve the field equation for A^μ:

$$\partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \Box A^\nu - \partial^\nu \partial \cdot A = J^\nu$$

Choosing the Lorentz gauge by shifting $A^\nu \to A^\nu_\perp = A^\nu + \partial^\nu \chi$ with $\Box\chi = 0$ s.t. $\partial \cdot A_\perp = 0$ we rewrite the equation as:

$$\Box A^\nu_\perp = J^\nu \text{ which has the solution: } \; A^\nu_\perp = \frac{J^\nu}{\Box}$$

Where the inverse of Box is understood as the appropriate Green's function. This is possible only because J has no dependence on A.

The situation in GR is drastically different. Consider the field equation for the gravitational field:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

Now one is interested in the solution for $g_{\mu\nu}$. The Einstein tensor depends on the metric and its derivatives up to second order but the problem is that also the source has a g dependence.

Suppose the matter is described by a Lagrange function $\mathcal{L}_M$, then the energy momentum tensor would be:

$$T_{\mu\nu}(g) = \frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}}$$

In this case we can not write down a Green's function as a working recipe for the solution. The only way to solve the equation is to impose symmetries or to choose a background.

For instance choosing Minkowski background and expanding in small perturbations around it one would get $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and can solve the dynamics for a field containing $h_{\mu\nu}$, the so called de Donder fiel. Defined by:

$$\Psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} Tr(h_{\mu\nu})$$

We choose a gauge where $\partial^\mu \Psi_{\mu\nu} = 0$, the de Donder gauge. The diffeomorphism invariance of the theory (which will be discussed in more detail in the second chapter) is the freedom to transform coordinates as $x^\mu \to x^\mu + \xi^\mu$ which results in $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$. This makes it possible to impose the de Donder gauge condition. So the equation of motion for Ψ in the de Donder gauge reads:

$$\Box \Psi_{\mu\nu} = 8\pi G_N T_{\mu\nu}(\eta) \;\Rightarrow \Psi_{\mu\nu} = 8\pi G_N \frac{T_{\mu\nu}(\eta)}{\Box}$$

Note that the source here only depends on the background and the d'Lambert operator is defined on Minkowski. This shows that if we do not want to make symmetry assumptions about the full metric, we are forced to choose a background. Coming back to the question in the beginning of this section, we answer: Yes, we have to assume an FRW background and this is also a physical and natural way to do it. Exactly because of the evolutionary process, which started in an homogeneous era.

There are different approaches possible to estimate the effect of inhomogeneities on the cosmic evolution. One possibility to study local perturbations of the metric and they time development, as it is done in [13]. An other way would be to develop an averaging procedure and find equations governing the dynamics of the averaged domains. The averaging scheme has a big physical significance, since cosmological observations are often strongly coarse grained and rather correspond to averages than local measurements. Nevertheless it is interesting to study the local perturbative approach to get an idea, for instance which components of Einstein's equations will be most interesting to study. A detailed discussion of the cosmological perturbation theory is beyond the scope of this work. We will briefly summarize some results from [13].

It has been shown that perturbations of the energy momentum tensor of an ideal fluid can be written to first order in gauge invariant variables as:

$$\delta \bar{T}^0_0 = \bar{\delta\epsilon}, \qquad \delta \bar{T}^0_i = \frac{1}{a}\left(\epsilon_0 + p_0\right)\left(\delta \bar{u}_{\parallel i} + \delta u_{\perp i}\right), \qquad \delta \bar{T}^i_j = -\bar{\delta p}\,\delta^i_j \tag{1.6}$$

The metric perturbations in a flat FRW universe can be written in the so called Longitudinal gauge, with ϕ and ψ being the scalar perturbations and η the conformal time, as:

$$ds^2 = a(\eta)^2\left[-\left(1+2\phi\right)d\eta^2 + \left(1-2\psi\right)\delta_{ij}dx^i dx^j\right]$$

The Einstein tensor calculated from the above metric is diagonal. Equations (1.6) show that the perturbations of the energy momentum tensor are diagonal in the spatial-spatial parts. If one furthermore assumes a pressure-less fluid of low energy density s.t. $\epsilon_0/a \ll 1$, the whole perturbed Einstein equation will be diagonal. Moreover, the structure of the equations is such that the zero-zero component and the trace carry the full information of the perturbed system. This is of course not a rigorous proof, but a motivation to concentrate the analysis of the averaged, perturbed Einstein equations on their scalar parts.

1.4 Backreaction as proposed by T. Buchert

In [3] a formalism for averaging traces of Einstein's equations was proposed. This formalism will be introduced in the following section.

1.4.1 Choice of foliation and basic equations

As a model system, a pressure free fluid called "dust" (which obeys the isometries of an isotropic cosmology) living on a manifold $\mathcal{M}$ of topology $\mathbb{R} \times \Sigma$ is studied. Furthermore a cosmological constant Λ is included. Already at this point we know that the averaged space-time is not general. For instance it can not be a Minkowski background, since it does not support an energy density present in the whole space and of course the presence of Λ also forbids Minkowski. The system, as mentioned above, should be rather viewed as a space-time, which deviates from FRW in a continuous way.

The Einstein equation reads:

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 8\pi G_N \rho u_\mu u_\nu + \Lambda g_{\mu\nu} = 0$$

To obtain the zero-zero component and the trace of this equation, projections on the velocity flow of the fluid as well as on the hypersurfaces defined by it are used. The co-moving frame is chosen immediately at the beginning. The rest frame of the fluid, which corresponds to a geodesic observer is a physically relevant choice. It represents our position on the earth when we observe distant objects. The problem is that if the gauge is fixed before averaging the averaging procedure will break the diffeomorphism invariance and the result can be questioned. This is exactly the point Veneziano has criticized about Buchert's averaging scheme. We will come back to this issue later.

For now we will proceed and present the calculations by T. Buchert. First a flow orthogonal coordinate system is chosen, such that $x^\mu = (t, X^k) = f^\mu(X^k, t)$ and therefore:

$$u^\mu = \partial_t f^\mu =: \dot{f}^\mu = (1, 0, 0, 0)$$

These coordinates label geodesics in space-time i.e. $\nabla_u u = 0$. Together with the choice of vanishing 3-velocity they are also co-moving. Since in that case $\dot{X}^k = 0$, they can be called Lagrangian coordinates of the fluid elements. The fluid filling the space-time obeys the mass conservation law:

$$\nabla_\mu (\rho u^\mu u^\nu) = 0$$

The velocity flow of the dust foliates the space-time into hypersurfaces Σ_t with the induced metric h_{ij} , which is the pull-back of the projection tensor $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ on the hypersurfaces of constant t . The full metric can be written in the following way:

$$ds^2 = -dt^2 + h_{ij} dX^i dX^j$$

With the definition of the projection tensor $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and its pull-back, one can express the extrinsic curvature:

$$K_{ij} = -h_i^\mu h_j^\nu \nabla_\nu u_\mu$$

Here the Latin indices represent coordinates on Σ_t and the Greek indices the coordinates on $\mathcal{M}$.

Intermezzo:

Here a few words on the extrinsic curvature. This object is the second fundamental form on the hypersurface Σ_t and can be viewed as the time derivative of the metric h_{ij}, in the following sense: The extrisic curvature is a tensor, entirely in Σ i.e $K_{\mu\nu} u^\mu = K_{\mu\nu} u^\nu = 0$ and therefore $K_{\mu\nu} = K_{\nu\mu}$. Using this symmetry one obtains:

$$K_{\mu\nu} = \frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) = \frac{1}{2}\mathcal{L}_u g_{\mu\nu} = \frac{1}{2}(\mathcal{L}_u (h_{\mu\nu} - u_\mu u_\nu)) = \frac{1}{2}\mathcal{L}_u h_{\mu\nu}$$

In case the coordinates are adapted to u_μ (Gaussian coordinates):

$$K_{\mu\nu} = \frac{1}{2}\partial_t h_{\mu\nu}$$

So, the intuitive picture one should have for the role of the extrinsic curvature is the bending of the hypersurface Σ in the manifold $\mathcal{M}$.

With the extrinsic curvature, Einstein's equations can be rewritten in a way which will be more convenient for later discussion[1]. Projecting the Einstein equation written as one tensor $E_{\mu\nu} = 0$

[1] A detailed derivation of the Einstein tensor in terms of the extrinsic curvature and the 3-scalar-curvature can be found in Appendix A in the derivation of the ADM formalism.

on the velocity flow of the fluid gives the zero-zero component:

$$E_{\mu\nu}u^{\mu}u^{\nu} = G_{\mu\nu}u^{\mu}u^{\nu} - 8\pi G_N \rho \left(u_{\mu}u^{\mu}\right)^2 - \Lambda \left(u_{\mu}u^{\mu}\right) = 0 \Rightarrow R_{00} + \frac{1}{2}R = 8\pi G_N \rho + \Lambda$$

where

$$R_{00} = \frac{1}{2}((K^a_a)^2 - K^i_j K^j_i)$$

and therefore the Hamiltonian constraint density reads:

$$\frac{1}{2}(R + K^2 - K^i_j K^j_i) = 8\pi G_N \rho + \Lambda$$

The projection of $E_{\mu\nu}u^{\mu}h^{\nu}_i = 0$ gives the space-time components, called momentum constraints (where D_j is the covariant derivative operator related to the space metric h and ∂_{X_j} is the derivative w.r.t. the Lagrangian coordinates X^i of the dust):

$$D_i K^i_j - \partial_{X_j} K = 0$$

Except for the four constraints, there are three evolution equations. Using the mass conservation one can compute:

$$0 = u_{\nu} \cdot \nabla_{\mu}(\rho u^{\mu}u^{\nu}) = u_{\nu}(u^{\nu}u^{\mu}\nabla_{\mu}\rho + \rho u^{\nu}\nabla_{\mu}u^{\mu} + \rho\nabla_{u}u) = -u^{\mu}\nabla_{\mu}\rho - \rho\nabla_{\mu}u^{\mu} = \partial_t \rho - \rho K$$

$$\Rightarrow \dot{\rho} = K\rho \tag{1.7}$$

The projection of $E_{\mu\nu}h^{\mu}_j h^{\nu}_i = 0$ gives the spatial-spatial components of Einstein's equations and using those one gets:

$$\dot{(h_{ij})} = -2h_{ik}K^k_j \tag{1.8}$$

$$\dot{(K^i_j)} = K\,K^i_j + R^i_j - (4\pi\rho G_N + \Lambda)\delta^i_j$$

An other important scalar equation is the Raychaudhuri-Landau equation. It basically describes the motion of nearby bits of matter due to gravity. With the definition of the shear tensor:

$$\sigma_{ij} := \theta_{ij} - \frac{1}{3}\theta h_{ij}$$

where $\theta_{ij} = -K_{ij}$ is the expansion tensor and $\theta = tr(\theta_{ij})$ the expansion rate. Note that θ can be viewed as the local analog of the Hubble rate, since we have observed that the extrinsic curvature is a time derivative of the metric. Defining the square of the shear tensor as:

$$\sigma^2 := \frac{1}{2}\sigma^i_j\sigma^j_i$$

The Raychaudhuri-Landau [2] equation can be formulated as:

$$\boxed{\dot{\theta} + \frac{1}{3}\theta^2 + 2\sigma^2 + 4\pi G_N\rho - \Lambda = 0} \tag{1.9}$$

This equation is the local analog to the second Friedmann equation. It describes the change of the local expansion rate. With the definition of the shear tensor at hand we can rewrite the Hamiltonian constraint density somewhat more conveniently:

$$\boxed{\frac{1}{3}\theta^2 = 8\pi G_N\rho + \Lambda - \frac{1}{2}R + \sigma^2} \tag{1.10}$$

This would be the analog of the first Friedmann equation, which contains the square of the expansion rate. In the local case the deviation from the Friedmann equations is due to the shear, which is of course not present in an isotropic universe. This set of equations has also been discussed in connection with perturbation theory by Kasai (1995), Matarrese (1996, and ref. therein) and by Matarrese & Terranova (1996), as well as in the papers by Russ et al. (1996, 1997).

One more useful relation can be obtained from the projections. Taking the trace of (1.8), written in the form:

$$K^i_j = -\frac{1}{2}h^{ik}(\dot{h_{kj}})$$

[2]The Raychaudhuri-Landau equation governs the motion of nearby bits of matter. A sketch of the derivation will be given in the Appendix B.

and defining:

$$J(t,\, X^i) := \sqrt{det(h_{ij})}$$

One obtains with:

$$\frac{1}{2}h^{ik}\dot{h_{ki}} = (ln\dot{(}J))$$

the following identity:

$$\dot{J} = -K\,J\; = \theta J$$

Using this, the continuity equation (1.7) can be integrated along the flow lines:

$$\rho(t,\, X^i) = (\rho(t_0,\, X^i)J(t_0,\, X^i))\, J^{-1} \tag{1.11}$$

This equation shows that mass is conserved along the flow lines, which is a natural result. The equations derived in this section and their averages will be studied to address the backreaction problem.

1.4.2 Averaging the traces of Einstein's equations

The averaging procedure as proposed in [3] is a spatial averaging of scalar quantities. The foliation into spatial hypersurfaces Σ_t is defined by the choice of the rest frame of the dust, which corresponds to the choice of a geodesic observer. The spatial average of a scalar Ψ over a domain D, located on the hypersurface of constant time Σ_t, is defined by:

$$\left\langle \Psi(t,\, X^i) \right\rangle_D := \frac{1}{V_D}\int_D J\, d^3X\, \Psi(t,\, X^i)$$

with the volume element $dV := \sqrt{det(h_{ij})}\, d^3X \, = J\, d^3X$ of the spatial hypersurfaces of constant time. So the volume naturally is:

$$V_D := \int J\, d^3X$$

Furthermore an effective scale factor is defined as:

$$a_D(t) := \left(\frac{V_D(t)}{V_{D_0}}\right)^{\frac{1}{3}}$$

Thus the averaged expansion rate can be written in terms of the scale factor:

$$\langle\theta\rangle_D = \frac{1}{V_D}\int J\, d^3X\, \theta = \frac{1}{V_D}\int J\, d^3X\, \frac{\dot{J}}{J} = \frac{1}{V_D}\int \dot{J}\, d^3X = \frac{\dot{V}_D}{V_D} = 3\frac{\dot{a}_D}{a_D} \tag{1.12}$$

The dot denotes partial derivative w.r. t. time and hence commutes with the integral.

The integral (1.11) states the conservation of the total rest mass M_D of the dust as transported along the flow lines, the integral over a spatial volume gives the conserved mass included in the domain:

$$M_D = \int_D J\, d^3X\, \rho = const \;\Leftrightarrow \langle\rho\rangle_D = \frac{M_D}{V_{D_0} a_D^3}$$

In the future discussion the subscript D is not going to be mentioned explicitly but always assumed when averages are considered.

Commutation Rule for the time derivative:

$$\frac{\partial}{\partial t}\langle\Psi\rangle - \left\langle\dot{\Psi}\right\rangle = \langle\Psi\theta\rangle + \langle\Psi\rangle\langle\theta\rangle$$

Proof:

$$\frac{\partial}{\partial t}\langle\Psi\rangle = \frac{\frac{\partial}{\partial t}\int\Psi J\, d^3x}{V_D} - \int\Psi J\, d^3x\frac{1}{V^2}\dot{V} = \int\dot{\Psi}J\, d^3x\frac{1}{V} + \int\Psi\dot{J}\, d^3x\frac{1}{V} - \int\Psi J\, d^3x\frac{1}{V}\frac{\dot{V}}{V} =$$

$$= \left\langle\dot{\Psi}\right\rangle + \langle\Psi\theta\rangle - \langle\Psi\rangle\langle\theta\rangle$$

At this point we prove another equation which is going to be of use later. With the commutation rule one gets:

$$\left\langle\dot{\theta}\right\rangle = \frac{\partial}{\partial t}\langle\theta\rangle + \langle\theta\rangle^2 - \left\langle\theta^2\right\rangle$$

Furthermore compute:

$$\frac{\partial}{\partial t}\langle\theta\rangle = 3\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right) = 3\frac{\ddot{a}}{a} - 3\frac{\dot{a}^2}{a^2} = 3\frac{\ddot{a}}{a} - \frac{1}{3}\langle\theta\rangle^2$$

Combining the above expressions we get:

$$\frac{1}{3}\left\langle\theta^2\right\rangle + \left\langle\dot{\theta}\right\rangle = 3\frac{\ddot{a}}{a} + \frac{2}{3}\langle\theta\rangle^2 - \frac{2}{3}\left\langle\theta^2\right\rangle \tag{1.13}$$

This relation will prove to be useful in the later discussion.

1.4.3 The averaged Einstein equations

In this section the averaging procedure described above will be applied to the scalar parts of Einstein's equations. To use the scalar equations derived from the trace and zero-zero components of Einstein's equations was motivated in 1.3. First the Hamiltonian constraint density will be averaged. Second the Raychaudhuri equation, which is derived from the trace of Einstein's equations, will be studied after performing the averaging. The resulting equations, called Buchert equations, will be discussed in the end of this section. Λ is assumed to be zero since it is not of interest for the averaging.

Hamiltonian constraint density:

After applying the spatial averaging to (1.10) one obtains:

$$8\pi G_N \langle\rho\rangle = \frac{1}{3}\left\langle\theta^2\right\rangle + \frac{1}{2}\langle R\rangle - \left\langle\sigma^2\right\rangle$$

Inserting unity and transforming it according to (1.12), the equation containing the backreaction term reads:

$$8\pi G_N \langle\rho\rangle = \frac{1}{3}\left\langle\theta^2\right\rangle - \frac{1}{3}\langle\theta\rangle^2 + \frac{1}{3}\langle\theta\rangle^2 + \frac{1}{2}\langle R\rangle - \left\langle\sigma^2\right\rangle =$$

$$= \frac{1}{3}\left\langle\theta^2\right\rangle - \frac{1}{3}\langle\theta\rangle^2 + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{2}\langle R\rangle - \left\langle\sigma^2\right\rangle = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{2}\langle R\rangle + \frac{Q_D}{2}$$

Here the quantity $Q_D := \frac{2}{3}\langle\theta^2\rangle - \frac{2}{3}\langle\theta\rangle^2 - 2\langle\sigma^2\rangle = \frac{2}{3}(\Delta\theta)^2 - 2\langle\sigma^2\rangle$ (called backreaction) was introduced. Note that a here denotes the scale factor of the averaging domain, as introduced above. Furthermore, it is obvious that Q_D vanishes if we have a homogeneous universe, since there is no shear and also no variance of the expansion rate. In the case of vanishing backreaction the first Friedmann equation is rediscovered. The subtle difference is that this equation was obtained by averaging a local quantity and would also hold for a finite volume.

Raychaudhuri's equation:

After averaging (1.9) the relation reads:

$$\left\langle\dot{\theta}\right\rangle + \frac{1}{3}\left\langle\theta^2\right\rangle + 2\left\langle\sigma^2\right\rangle + 8\pi G_N\langle\rho\rangle = 0$$

Using the computed identity (1.13) one gets:

$$3\frac{\ddot{a}}{a} + \frac{2}{3}\langle\theta\rangle^2 - \frac{2}{3}\left\langle\theta^2\right\rangle + 2\left\langle\sigma^2\right\rangle + 8\pi G_N\langle\rho\rangle =$$

$$= 3\frac{\ddot{a}}{a} - Q_D + 8\pi G_N \langle \rho \rangle = 0$$

Which is the equivalent of the second Friedmann equation for a finite volume. Again the correction term is the domain dependent backreaction Q_D. In both cases the deviation of the Buchert equations from Friedmann's equations contain the averaged shear. This was expected since also the local analoga of the Friedmann equations deviate only be the shear term. The amazing and surprising result is that Buchert's equations also contain a correction, which is connected to the variance of the local expansion rate. This is a purely statistical quantity, which can be only obtained in the averaging process.

***Cosmological balance*:**

Defining:

$$H_D := \frac{\dot{a}}{a}$$

$$\Omega_m := \frac{8\pi G_N M_D}{3 V_0 a^3 H_D^2}$$

$$\Omega_k := -\frac{\langle R \rangle}{6 H_D^2}$$

$$\Omega_Q := -\frac{Q_D}{6 H_D^2}$$

With the above definitions the Hamiltonian constraint density reads:

$$\Omega_m + \Omega_k + \Omega_Q = 1$$

If from the start a presence of a cosmological constant Λ was allowed, it would not be affected by the averaging procedure and result in an analogous term Ω_Λ in the cosmological balance equation:

$$\Omega_m + \Omega_k + \Omega_Q + \Omega_\Lambda = 1$$

Compare this with the cosmological balance equation from the Λ-CDM model. An interesting question is, whether the effect of inhomogeneities expressed in Q_D opens a possibility to substitute or change the cosmological constant usually assumed in the standard Λ-CDM model.

To understand the nature of the accelerated expansion is of great physical interest, since there is gigantic mismatch between the value of Λ_{Eff} renormalized by the quantum vacuum energy and the Λ value we expect in the Λ-CDM model, in order to explain the accelerating expansion. To address this question, we have to ensure that the effect of Q_D can be observed in an experiment. All the derived domain dependent quantities have to be physical observables for this purpose. We will discuss in the next section that the crucial feature for a quantity to be observable is gauge invariance and in the case of general relativity diffeomorphism invariance.

1.5 Summary Chapter 1

In this chapter we have seen what amazing predictions GR gives us under the assumption of isotropy for the history of our universe and how many of this predictions can by verified in experiments. The problem of accelerated expansion, associated with the cosmological constant, and its connection to the quantum vacuum has been presented. Then the isotropy hypothesis has been questioned. Especially at present times it can only be understood in a statistical way. Hence the necessity of an averaging prescription arose naturally. We have analyzed perturbations about FRW and conjectured that the most relevant parts of Einstein's equations will be the scalars. We presented the averaging procedure proposed in [3] and found that effective Friedmann equations for domains can be derived from averages of the traces of Einstein's equations. The effective equations contain correction terms due to the deviation from FRW. Those terms seem to open a new window of opportunity to address the question of accelerated expansion. In order to take the predictions seriously we have to ensure gauge invariance. This will be the topic of the following chapters.

Chapter 2

On diffeomorphism invariance and observables

In this chapter the general covariance of Eistein's theory and its implications for physics will be discussed. Furthermore it will be shown that the averaging functional used in [3] violates this principle. At last a remedy for the breaking of gauge invariance will be proposed.

2.1 Active and passive diffeomorphisms

Before elaborating on the general covariance of Einstein's equations, we discuss the difference between active and passive diffeomorphisms. We will start with an illustrative example to understand this in a similar manner as in [16].

Consider the surface of the earth and call it $\mathcal{M}$. The temperature on the earth is given by a scalar function $T : \mathcal{M} \to \mathbb{R}$. So the temperature in London would be $T(L)$ and the temperature in Paris $T(P)$. Imagine now a weather model where the temperature is changed due to winds only. Therefore during a time-interval wind can carry the air from London to Paris and hence:

$$T(P) \to \tilde{T}(P) = T(f(P)) = T(L) \tag{2.1}$$

The function $f : \mathcal{M} \to \mathcal{M}$ represents the displacement of the air by wind. This corresponds to an *active diffeomorphism.*

The other situation could be the following: Describing the coordinates on earth with longitude and latitude one could express the temperature in London (which is close to Greenwich) by $T(\theta = \theta_L, \phi = 0)$ and the temperature in Paris by $T(\theta = \theta_P, \phi \backsimeq -2°20')$. The French could disagree with this coordinate choice and identify the origin of the polar angle with Paris. The temperature in Paris would be certainly not affected by such transformations:

$$\tilde{T}(\theta = \theta_P, \phi = 0) = T(\theta = \theta_P, \phi + 2°20')$$

Such change of coordinates can also be described by a function $f : \mathcal{M} \to \mathcal{M}$ and the equation reads $\tilde{T}(x) = T(f(x))$ which is a formula of exactly the same structure as (2.1). This is called a *passive diffeomorphism.* Even if the mathematical formulation is the same, nevertheless the the processes are very different from the physical point of view.

Now let us define the notions in a mathematically precise way:

Active diffeomorphism :

Given a Manifold $\mathcal{M}$, an active diffeomorphism ϕ is a smooth invertible map from $\mathcal{M}$ to $\mathcal{M}$. A scalar field T on $\mathcal{M}$ is a map $T : \mathcal{M} \to \mathbb{R}$. Given an active diffeomorphism f, we define the new scalar field $\tilde{T}$ transformed by f as:

$$\tilde{T}(P) = T(f(P))$$

This means the field has been pushed forward to a new space-time point and in the next step the coordinates have been adopted in such a way that the fields are evaluated at the same coordinate values. In the later discussion of the diffeomorphism invariance of the averaging formalism we will refer to active diffeomorphisms as local field redefinitions or gauge transformations (GT).

Passive diffeomorphism:

Given a coordinate system[1] a passive diffeomorphism is an invertible differentiable map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ that defines a new coordinate system $\tilde{x}$ on $\mathcal{M}$ by $x(P) = f(\tilde{x}(P))$. The value of the field T in coordinate system $\tilde{x}$ is given by:

$$\tilde{t}(\tilde{x}) = t(x) = t(f(\tilde{x}))$$

In the later discussion we will call passive diffeomorphisms, general coordinate transformations (GCT).

On active diffeomorphisms and the Lie derivative:

Even though in the case of a scalar field the physical interpretation of active and passive diffeomorphisms is clear, it is not a good example to point out the difference between them. To illustrate this we will use the transformation of a vector and tensor field under a gauge transformation, similar to [1]. Let us start with a co-vector field A_μ. Consider an infinitesimal active diffeomorphism generated by a vector field ξ by $x \rightarrow \tilde{x} = x - \xi d\lambda$, hence:

$$\tilde{A}_\mu(x) = A_\alpha(x + \xi d\lambda)\,\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = A_\mu(x) + \{\xi^\alpha \partial_a A_\mu(x) + A_\alpha(x)\partial_\mu \xi^\alpha\}\, d\lambda \tag{2.2}$$

Consider the difference to a passive diffeomorphism generated by the same transformation $x \rightarrow \tilde{x} = x - \xi d\lambda$:

$$\tilde{A}_\mu(\tilde{x}) = A_\alpha(x)\,\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = A_\mu(x) + \{A_\alpha(x)\partial_\mu \xi^\alpha\}\, d\lambda \tag{2.3}$$

Note that the active diffeomorphism has a similar structure, but contains an extra term, which we will discuss later.

Now we study the transformation of a tensor on $\mathcal{M}$, we take the metric tensor useful for later

[1] A coordinate system x on a d-dim manifold $\mathcal{M}$ is an invertible differentiable map from $\mathcal{M}$ to $\mathbb{R}^d$. Given a field T on $\mathcal{M}$, this map determines the function $t : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $t(x) = T(P(x))$, called "the field T in coordinates x". Many times t and T are not distinguished.

discussion.

$$\tilde{g}_{\mu\nu}(x) = g_{\alpha\beta}(x+\xi d\lambda)\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu} = g_{\mu\nu}(x) + \{\xi^\alpha\partial_a g_{\mu\nu}(x) + g_{\alpha\nu}(x)\partial_\mu\xi^\alpha + g_{\mu\alpha}(x)\partial_\nu\xi^\alpha\}\, d\lambda \tag{2.4}$$

For a metric tensor, fulfiling the metric compatibility requirement we have:

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \{g_{\alpha\nu}(x)\partial_\mu\xi^\alpha + g_{\mu\alpha}(x)\partial_\nu\xi^\alpha\}\, d\lambda \tag{2.5}$$

We introduce the notion of a Lie derivative, as known from differential geometry. With T being a tensor field changed by an infinitesimal diffeomorphism $\tilde{T} := f_{\Delta\lambda}T$ along the vector field ξ (as defined above) by an amount first-order in $\Delta\lambda$ and linear in ξ , the Lie derivative is defined as:

$$\mathcal{L}_\xi T := \lim_{\Delta\lambda\to 0}\frac{f_{\Delta\lambda}T(x) - T(x)}{\Delta\lambda}$$

The Lie derivatives of the above fields (2.2) and (2.5) read therefore:

$$\mathcal{L}_\xi A_\mu(x) = \xi^\alpha\partial_a A_\mu(x) + A_\alpha(x)\partial_\mu\xi^\alpha \quad \text{and} \quad \mathcal{L}_\xi g_{\mu\nu}(x) = \xi^\alpha\partial_a g_{\mu\nu}(x) + g_{\alpha\nu}(x)\partial_\mu\xi^\alpha + g_{\mu\alpha}(x)\partial_\nu\xi^\alpha$$

We see that in fact an infinitesimal active diffeomorphism is generated by a Lie derivative as: $\tilde{T} \to T + \mathcal{L}_\xi T\, d\lambda$. Considering the vector field, we can interpret the first part of the Lie derivative generating the infinitesimal diffeomorphism as a push forward to an other point of the manifold. The second term represents the subsequent coordinate change (hence the similarity to (2.3)), such that the fields are evaluated at the same coordinate values. Since the Lie- and covariant derivative of a scalar coincide, the transformation equations of scalar fields under active and passive diffeomorphisms are the same.

Transformation of the volume element:

Before proceeding to the next section, we study the transformation properties of the volume element $d^n x \sqrt{-g}$ under active and passive diffeomorphisms, since this will be useful later. We start with a passive diffeomorphism, or general coordinate transformation GCT generated by $\tilde{x} = f(x)$:

$$d^n x \to d^n \tilde{x} = d^n x \left| \frac{\partial f}{\partial x} \right|_{f^{-1}(x)}$$

Where $\left|\frac{\partial f}{\partial x}\right|_{f^{-1}(x)}$ is the Jacobian matrix of the transformation. Resulting from the transformation of the metric tensor, the metric determinant transforms as following:

$$\sqrt{-g(x)} \to \sqrt{-\tilde{g}(\tilde{x})} = \left| \frac{\partial x}{\partial f} \right|_{f^{-1}(x)} \sqrt{-g(x)}$$

With $\left|\frac{\partial x}{\partial f}\right|_{f^{-1}(x)}$ being the inverse Jacobian matrix of the transformation. Combining this elements we observe that the total volume element is a scalar under GCTs:

$$d^n x \sqrt{-g(x)} \to d^n \tilde{x} \sqrt{-\tilde{g}(\tilde{x})} = d^n x \sqrt{-g(x)}$$

On the other hand $d^n x$ is invariant under an active diffeomorphism or gauge transformation (since the push forward combined with the coordinate redefinition result in an identity), while the metric determinant transforms as follows:

$$\sqrt{-g(x)} \to \sqrt{-\tilde{g}(x)} = \left| \frac{\partial x}{\partial f} \right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))}$$

This results in the transformation of the volume element as:

$$d^n x \sqrt{-g(x)} \to d^n \tilde{x} \sqrt{-\tilde{g}(x)} = d^n x \left| \frac{\partial x}{\partial f} \right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))}$$

Hence, it is not a scalar under an active diffemorphism (We will apply a GCT to fix this in the next section).

2.2 General covariance

In this section we will give a short outlook on the role of general covariance for GR and its history. This will show us again the important difference between active and passive diffeomorphisms.

Already in 1912 Einstein has understood the role of the gravitational field as an entity which defines the local inertial frame and also describes how these inertial frames "fall" w.r.t each other. But it took Einstein three years to discover the field equations of this object[2]. Einstein believed that physics must not depend on the choice of coordinates and therefore he wanted the field equations to be covariant under passive diffeomorphisms. This first impulse though was stopped by himself. Why did the genius hesitate to follow his philosophy and derive generally covariant equations? And even worse, in 1914 Einstein was convinced that the equations must not be generally covariant. But why? The reason is exactly the similarity between the equations for active and passive diffeomorphisms, which implies that a generally covariant theory also has the property that solutions are mapped to solutions by active diffeomorphisms[3]. This was used by Einstein as an argument against the general covariance in his famous "hole" argument. We will present this argument briefly:

Consider a space-time represented my a manifold $\mathcal{M}$ which contains matter everywhere except for one region H , the hole.

Figure 2.1: The hole is the region in M without matter

[2]Einstein himself called this time: "my struggle with the meaning of coordinates". At the same time David Hilbert was working on the same problem of finding the GR field equations which has put even more pressure on Einstein.

[3]We will make this into an exact mathematical statement in the next section.

Assume that there is a point A inside H s.t. the geometry is flat around A and a point B in H s.t. the geometry is not flat around B. If the equations are generally covariant, there is a diffeomorphism f which carries A into B, but does not affect any point outside the hole. Further consider a solution of the gravitational field equations and call it g and its pull-back under f called $\tilde{g}$. Take a space-like hypersurface entirely in the past of H. In that case g and $\tilde{g}$ are identical on that hypersurface, since both are solutions of the equations of motion. However, g and $\tilde{g}$ have different properties in the hole, obviously the scalar curvature is different in point A . Therefore the equations of GR seem not deterministic. This was the fact that has shocked Einstein and made him and Hilbert look for non-generally covariant equations. The genius epiphany which finally led to the equations of GR had Einstein ,who understood the full gravity of relativity. He saw that physics is not about distinguishable points on the manifold, but about "space-time coincidences[4]". This resolves the "hole problem" in an elegant way. Consider the previous construction, but additionally two particles whose world-lines are geodesics determined by the geometry and intersect at the point B (where a gravitational wave is present).

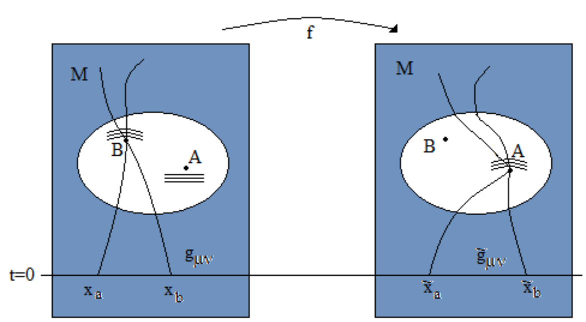

Figure 2.2: The hole with two particles traveling through it

Now an active diffeomorphism can carry B into A but leave the points outside the hole unaf-

[4]In his own words: "All our space-time verifications inevitably amount to determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meeting of two or more of this points. Moreover, the results of our measuring are nothing but verifications of such meetings of the material points of our instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time. The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such conditions." [A. Einstein: Grundlage der Allgemeinen Relativitätstheorie]

fected. The transformed solutions are again solutions of the equations of motion, but the world-lines intersect in A now. The important realization is that the gravitational wave is now also present at A i.e. at the point where the particles' world-lines cross. The equations describe the same physical reality (observable events) and this is manifested in the fact that they are invariant under diffeomorphisms active and passive. This feature also gives the name to the Dirac observables. Those are quantities which do not depend on the gauge chosen and in our case on the coordinate system. At the same time Einstein realized that the points A and B a priori are not physical. In the empty hole a flat region can be mapped on a non-flat region and this means that asking what is the geometry around a point in the manifold without further specifications, is not legitimate. This is an extremely strong statement. Expressed in the most radical form we can say that, the points on the space-time manifold are not physical, the only observable events are relative relations between material quantities. We will make use of this brilliant idea, which was established by Einstein and developed further in the course of time, by introducing a physical coordinate system and using it for the averaging procedure. The idea of relative coordinatisation will be discussed in detail in chapter 3.

2.3 The gauge principle in physics

In this section we will give the example of quantum electrodynamics which shows how powerful the gauge principle in physics is. Using this as a motivation we will identify the gauge symmetry of general relativity.

Gauge principle in QED:

Dirac discovered an equation which governs the dynamics of spin-1/2 femions. This Dirac equation can be reproduced by the variation of a Lagrange function:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi \tag{2.6}$$

Where γ^μ denotes a vector of gamma matrices, m the mass of the fermion and ψ the spinor wave-function describing the fermion (with $\bar{\psi} = \psi^\dagger \gamma^0$ being the adjoint spinor). The square of the spinor wave-function has as usually in quantum mechanics a probabilistic interpretation. And therefore a transformation $\psi \to e^{i\theta}\psi$ would not affect our observations. In fact we do not observe a dependence of the fermions on a continuous parameter like θ . This observational fact lets us believe that this must be a symmetry of the theory and therefore any formal description of the considered phenomena, should be invariant under this symmetry. We observe that (2.6) is indeed invariant under this (global) transformation.

The fact that θ does not depend on the coordinates means that the phase change happens everywehre at the same time. This is unphysical. A change of phase of the wave-function, if we view it as a physical process, must happen subluminal. Therefore, the parameter θ has to have a space-time dependence. Consider a transformation $\psi \to e^{i\theta(x)}\psi$ of (2.6), the term $-\bar{\psi}\gamma^\mu\partial_\mu\theta\psi$ breaks the symmetry.

Now we will present a technique which restores gauge invariance and gives more amazing results. Introduce the so called covaraint derivative:

$$D_\mu = \partial_\mu + iA_\mu$$

Where the so called gauge field A_μ has a gauge symmetry $A_\mu \to A_\mu + \partial_\mu\theta$ (observe similarity to classical EM). This property allows to cancel the gauge breaking term and the new Lagrangian is gauge invariant under the local phase change:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}\gamma^\mu A_\mu\psi - m\bar{\psi}\psi \tag{2.7}$$

The new term $\bar{\psi}\gamma^\mu A_\mu\psi$ is a coupling between fermions and the gauge field. It describes correctly the interaction of photons and charged fermions observed in experiments. Furthermore, the symmetry gives rise to a conserved current, since (2.7) does not depend on a time derivative of A_μ:

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = -\bar{\psi}\gamma^\mu\psi = J^\mu$$

This current is a physical quantity and can be measured. This is one of the examples illustrating the power of the gauge principle in physics. Let us summarize:

- The observation of an independence of the observed phenomena under certain transformations makes us assume a symmetry of nature.
- The implementation of those symmetries on the level of the action leads to conservation laws and gives rise to gauge fields responsible for the interactions.
- Once we are convinced of the presence of a symmetry in the theory it can serve as a criterion for the choice of observables, since those have to be gauge invariant.

General relativity:

Einstein conjectured that physics has to be coordinate independent and therefore the equations have to be covariant under general coordinate transformations. General covariance means that the action has to be a scalar under passive diffeomorphisms. This is the case for the Einstein-Hilbert action:

$$S = \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}$$

The contracted Ricci-tensor is a scalar under passive diffeomorphism, as can be easily checked. The invariance of the volume element under those has been already shown. The action is therefore a passive diffeomorphism scalar and its variation yields the Einstein vacuum equation:

$$\delta S = \int d^4x \sqrt{-g}\, G_{\mu\nu} \delta g^{\mu\nu} = 0$$

We will prove that any action which is a scalar under passive diffeomorphisms is also a scalar under active diffeomorphisms, given that the diffeomorphism generator vanishes at the boundary. This is the mathematical formulation of the fact which shocked Einstein at first.

Assume that Ψ is a scalar under passive diffeomorphisms. Since for a scalar the Lie derivative is identical with the covariant derivative we have: $\mathcal{L}_\xi \Psi = \xi^\mu \nabla_\mu \Psi$. Compute first using (2.5):

$$\mathcal{L}_\xi \sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu} = \left(\nabla_\mu \xi^\mu\right)\sqrt{-g}$$

Now consider the transformation of the action under an active diffeomorphism in the region Ω (generated by a Lie derivative):

$$\delta S = \int_\Omega \mathcal{L}_\xi \left(\Psi\sqrt{-g}\right) d^4x = \int_\Omega \left(\xi^\mu \nabla_\mu \Psi + \Psi \nabla_\mu \xi^\mu\right)\sqrt{-g}\, d^4x =$$

$$= \int_{\partial\Omega} \left(\Psi \xi^\mu\right) d^3\sigma_\mu = 0$$

This transformation vanishes, since we assumed that $\xi = 0$ on $\partial\Omega$. In a similar way Einstein realized that solutions of generally covariant equations of motion for the gravitational field will be mapped to valid solutions by active diffeomorphisms. This led to the hole argument described above. We have learned in the previous section that this is not a paradox if we stop considering points on the manifold as physical observables.

Before drawing any conclusions, let us study the implications of the active diffeomorphism invariance of the gravitational action:

$$\delta S = \int_\Omega d^4x\,\sqrt{-g}\left(G_{\mu\nu}\right)\delta g^{\mu\nu} = \int_\Omega d^4x\,\sqrt{-g}\left(G_{\mu\nu}\right)\mathcal{L}_\xi g^{\mu\nu} = \int_\Omega d^4x\,\sqrt{-g}\left(G^{\mu\nu}\right)\left(\nabla_\mu \xi_\nu\right) =$$

$$= \int_{\partial\Omega} \left(G^{\mu\nu}\xi_\nu\right) d^3\sigma_\mu - \int_\Omega d^4x\,\sqrt{-g}\left(\xi_\nu\right)\left(\nabla_\mu G^{\mu\nu}\right) = -\int_\Omega d^4x\,\sqrt{-g}\left(\xi_\nu\right)\left(\nabla_\mu G^{\mu\nu}\right) = 0 \Rightarrow \nabla_\mu G^{\mu\nu} = 0$$

Investigating the symmetry of the gravitational action under the active diffeomorphisms we observe that we get the identity (it is an identity and not a conservation law, since it holds also of shell), known as the Bianchi identity. It is a fundamental property arising from the form of the Riemann tensor and makes the Einstein tensor consistent with local energy momentum conservation.

To summarize, we followed a logic of the previous section and implemented the symmetry we conjectured (general covariance) into our theory. We observed that this leads to an even bigger symmetry (active diffeomorphism covariance) and this symmetry implies consistent identities for our theory. Even if our procedure is similar to the one in QED (a theory with internal symmetries) general relativity seems to be very different (since it is based on external symmetries). Therefore, we can not write down a gauge field which would give rise to interactions in the same way as in other QFTs and we have no clear idea how to bring time into the formalism (to be discussed later). Nevertheless, the fact that active diffeomorphisms give rise to consistent identities, suggests that we have to identify the true gauge transformations of GR with active diffeomorphisms and following the above logic use the invariance under those as a quality management to construct physical observables.

The last sections motivated why an observable in GR has to be invariant under active diffeomorphisms (now referred to as gauge transformations) and the equations we use, have to be covariant under those. Having established this we turn to the averaging functional and study its gauge transformation properties.

2.4 Transformations of the averaging functional

Notation:

We define the notation for the following section. Let the transformation $x \to \tilde{x} = f(x)$ be a diffeomorphism and S a scalar which transforms under this diffeomorphism.

We will evaluate the scalar quantity at different coordinate points when we address general coordinate transformations.

- GCT:

$$\tilde{S}(\tilde{x}) = S(x)$$

And we will evaluate the transformation law at the same coordinate point when we address gauge transformations. This corresponds to a local field redefinition.

- GT:

$$\tilde{S}(x) = S(f^{-1}(x))$$

Remark: The metric is a tensor field on $\mathcal{M}$ and undergoes a local redefinition as well. Its behavior under transformations (as discussed above) leads to the following transformation property of the determinant of the metric with the inverse Jacobian:

$$\tilde{g}_{\mu\nu}(x) = \left[\frac{\partial x^\alpha}{\partial f^\mu}\frac{\partial x^\beta}{\partial f^\nu}\right]_{f^{-1}(x)} g_{\alpha\beta}(f^{-1}(x)) \Rightarrow \sqrt{-\tilde{g}(x)} = \left|\frac{\partial x}{\partial f}\right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))}$$

2.4.1 The averaging functional applied by Buchert

At first we make a naive attempt of averaging a scalar field S on a space-time manifold $\mathcal{M}_n$ over a region Ω. We will see that this will not lead to a gauge invariant prescription.

$$F(S,\Omega) = \int_{\Omega_x} \sqrt{-g}\, S(x) d^n x$$

The label x of the region Ω_x denotes that this region depends on the coordinate choice. Regard the transformation property of this expression under a gauge transformation, using the equations from the last section we have:

$$F(S,\Omega) \rightarrow \tilde{F}(\tilde{S},\Omega) = \int_{\Omega_x} \sqrt{-\tilde{g}}\tilde{S}(x) d^n x =$$

$$= \int_{\Omega_x} \left|\frac{\partial x}{\partial f}\right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))} S(f^{-1}(x)) d^n x$$

This is the consequence of the volume element not being a scalar under a gauge transformation. Hence, we rewrite $F(S,\Omega)$ and introduce a general coordinate transformation $\hat{x} = f^{-1}(x)$ (This transformation is the inverse of the internal coordinate redefinition of the gauge transformation), noticing that its Jacobian cancels the inverse Jacobian from the transformation of the metrics' determinant:

$$\tilde{F}(\tilde{S},\Omega) = \int_{\Omega_x} \left|\frac{\partial x}{\partial f}\right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))} S(f^{-1}(x)) d^n x =$$

$$= \int_{\Omega_{\hat{x}}} \sqrt{-g(\hat{x})} S(\hat{x}) d^n \hat{x} = F(S, \tilde{\Omega})$$

This did not cure the problem, just made it less obvious. The gauge invariance is still broken by the integration region. To see the gauge breaking more explicitly introduce a “window” function $W_\Omega(x)$ (can be thought as a collection of theta step functions) ,such that it cuts out the region Ω out of the space-time.

$$\tilde{F}(\tilde{S}, \Omega) = \int_{\mathcal{M}} \left| \frac{\partial x}{\partial f} \right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))} S(f^{-1}(x))\, W_\Omega(x) d^n x =$$

$$= \int_{\mathcal{M}} \sqrt{-g(\hat{x})} S(\hat{x}) W_\Omega(f(\hat{x})) d^n \hat{x} = \int_{\mathcal{M}} \sqrt{-g(\hat{x})} S(\hat{x}) W_{\tilde{\Omega}}(\hat{x}) d^n \hat{x} = F(S, \tilde{\Omega})$$

The observation we gain from this exercise, is that the transformation property of the integration region -or better to say its non-transformation- is the reason for the functional to be gauge dependent. The above functional is exactly the one used in [3] to perform the averaging. It was furthermore restricted to a 3-D hypersurface of constant “time” which was itself defined through the choice of foliation and therefore by gauge fixing. This approach has been criticized in the literature[5], because in Buchert’s formalism, presented in 1.4, the gauge is fixed before the averaging. It is not clear though whether this gauge fixing and the averaging will commute. Furthermore a gauge can be only chosen safely when the whole formalism is covariant. The breaking of gauge invariance by the averaging functional violates this property and has led to the criticism and rose doubts concerning the validity of Buchert’s equations. Our goal will be to clarify this issue.

2.4.2 Gauge invariant functional:

We present here an idea proposed in [8] and developed further in [7]. In the previous subsection it has been shown that the gauge is broken due to the properties of the integration region, which we encoded by the window function W_Ω as the support of the integration domain Ω. The problem is that W_Ω is an unphysical quantity and not affected by the gauge transformation. If we promote it to a field the situation changes and it transforms as a scalar under the local field redefinition.

$$W \to \tilde{W}(x) = W(f^{-1}(x)) \tag{2.8}$$

[5]In [8] Veneziano points out this problem and also proposes a possible solution to it, on which we will elaborate later.

Now the functional is set up in the following way:

$$F(S,\Omega) = \int_{\mathcal{M}_4} \sqrt{-g(x)}S(x)W_\Omega(x)d^4x$$

And it transforms as follows:

$$F(S,\Omega) \to \tilde{F}(\tilde{S},\Omega) = \int_{\mathcal{M}_4} \sqrt{-\tilde{g}(x)}\tilde{S}(x)\tilde{W}_\Omega(x)d^4x =$$

$$= \int_{\mathcal{M}_4} \left|\frac{\partial x}{\partial f}\right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))}S(f^{-1}(x))W_\Omega(f^{-1}(x))d^4x$$

Introducing $\hat{x} = f^{-1}(x)$ we obtain:

$$\tilde{F}(\tilde{S},\Omega) = \int_{\mathcal{M}_4} \sqrt{-g(\hat{x})}S(\hat{x})W_\Omega(\hat{x})d^4\hat{x} = F(S,\Omega)$$

The gauge functional is obviously gauge invariant. And the crucial property is (2.8), which promotes W_Ω to scalar field. The question is if it is possible to construct the window-function with the desired transformation property.

2.4.3 The window function

The window function is a scalar if it is a function constructed out of scalar fields. Consider a scalar field B with a space-like gradient and a scalar field A with a time-like gradient. The window function can be constructed by building a cylinder-like region in space time (here Θ is the Heaviside function):

$$W_\Omega(x) = \Theta(A(x) - A_1)\Theta(A_2 - A(x))\Theta(r_0 - B(x))$$

Now consider the limit of $A_2 \to A_1 = A$:

$$W_\Omega(x) \to W_\Omega(x) = \Theta(r_0 - B(x))\delta(A - A(x))$$

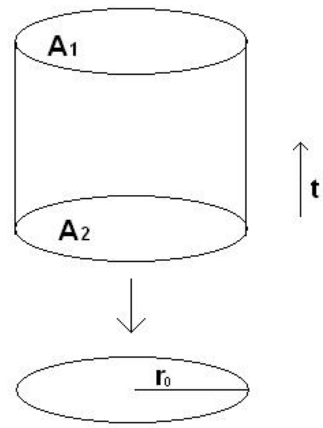

Figure 2.3: The Window function

Since A and B are scalars i.e. they transform according to $\tilde{S}(x) = S(f^{-1}(x))$, the constructed window function will have the desired transformation property (2.8). And hence the averaging functional constructed with W is gauge invariant. It can be defined as:

$$\langle S \rangle_{\{A,r_0\}} = \frac{F(S,\Omega)}{F(S,1)} = \frac{\int_{\mathcal{M}_4} \sqrt{-g(x)} S(x) \Theta(r_0 - B(x)) \delta(A - A(x)) d^4x}{\int_{\mathcal{M}_4} \sqrt{-g(x)} \Theta(r_0 - B(x)) \delta(A - A(x)) d^4x}$$

Modified window function:

The window function can be constructed in a more formal way. Instead of taking the limit of $A_2 \to A_1 = A$ in the construction of the window function, it is possible to formulate the choice of the spatial slice as the derivative of the Heavyside function along the time generating vector field

$$n^\mu = \frac{-\partial^\mu A}{\sqrt{-\partial_\mu A \partial^\mu A}}$$

i.e.:

$$W_\Omega = n^\mu \nabla_\mu \theta(A(x) - A_0) \theta(r_0 - B(x)) = -\frac{-\partial^\mu A}{\sqrt{-\partial_\mu A \partial^\mu A}} \delta(A(x) - A_0) \partial_\mu A \, \theta(r_0 - B(x)) =$$

$$= \delta(A(x) - A_0)\,\sqrt{-\partial_\mu A \partial^\mu A}\,\theta(r_0 - B(x))$$

This mechanism is analogue to the above one, but more formal since it avoids taking the limit of the theta function to obtain the delta distribution. The averaging functional will be defined in the same way displayed above just using this new window function.

Remark:

The problem of construction of the gauge invariant functional transforms into the problem of finding these inhomogeneous fields for the construction of the window function. In [8] it is proposed to use a physical field for A, but the author claims that there are no fields with a spatial gradient in an FRW universe and therefore the spatial boundary has to be extended to infinity. Because of that, his approach can not be applied to finite volumes. It is clear that there are no spatially inhomogeneous fields in an FRW universe. On the other hand the studied system is not a perfect FRW universe, but an inhomogeneous model which contains inhomogeneous fields. From the physical point of view there must be such fields in nature since averaged quantities can be observed in experiments and hence must have a gauge invariant description.

2.5 Summary Chapter 2

In this Chapter we have seen what crucial role diffeomorphism invariance plays for GR. In a historical section Einstein's path to the theory has been sketched. Also we have found that for a quantity to be a physical observable it has to be active diffeomorphism invariant and hence gauge invariant in our language. The gauge transformation property of Buchert's averaging functional has been studied and it was found that it actually breaks the gauge invariance, as has been criticized by Veneziano. Furthermore an alternative gauge invariant functional has been presented. Veneziano's idea of introducing a window function to maintain gauge invariance has been discussed and also the problem of finding scalar fields for the construction of the window function was stated. In [7] a scalar field A with a time-like gradient is used, which is a scalar but is not interpreted in the physical sense in his approach. Furthermore the B field in Veneziano's scheme is not a physical scalar and breaks gauge invariance at the boundary. Therefore the spatial boundary of the domain

Σ_{A_0} has to be extended to infinity. In the following chapters we will construct a gauge invariant averaging functional for finite volumes as well as give physical meaning to the fields A and B.

Chapter 3

Observables and deparametrisation

In the following chapter we will discuss the difference between relative and absolute coordinatization. The key idea of relative coordinatization will be useful to address the problem of constructing physical observables as well as the problem of time in general relativity. The Brown-Kuchar method will be presented where the studied system is enlarged by a compatible system, which serves as a coordinate system and defines the time flow [2].

3.1 On relative coordinatization and compatibility

In the previous chapter we have seen that the a priori choice of points on a manifold has no physical meaning. The properties of the gravitational field around a chosen point A are not determined and can change by application of diffeomorphisms as was shown in the hole argument. The solution to this problem was also found by Einstein who introduced particles with their world-lines into the system and postulated that only the relative events w.r.t those particles , like the intersections of the world-lines, are of physical significance. Keeping this idea in mind we will discuss two ways of coordinatizing a manifold. Let us begin again with an illustrative example.

Consider the surface of the earth as a manifold. One way of coordinatizing it would correspond to drawing the longitude and latitude lines on a globe and give positions of points relative to them. This way is abstract and artificial since we don't see the lines existing on our earth. This procedure would correspond to a formal gauge fixing with respect to an unphysical coordinate system and the coordinate values on earth obtained in such a way would of course depend on the arbitrary choice of coordinates and hence be not physical observables.

The other method would be to consider all rivers on earth and use this grid (assuming that it covers the surface in a dense enough way) to give the relative position of any point with respect to this grid. A convenient reference are the intersection points of the rivers. This is not a formal gauge fixing, since the rivers themselves are physical objects. Therefore the statement that a city is located on earth at the site where the Elbe flows into the North sea is gauge invariant.

This principle described above allows us to make measurements and predictions without a formal gauge fixing. We describe all events as coincidences in space-time. Our system has to describe the coordinate system as part of it in a dynamical way and consider its backreaction. The backreaction has to be smaller than the effect of interest, we call the laboratory in this case compatible with the measurement. To elaborate on the problem of compatibility, let us consider an example. Let's think of the measurement of the field in electrodynamics. In order to measure or even describe a field we need a test charge. A real physical or a mind experiment would use a charge for this purpose which is small enough so that its backreaction on the field could be neglected. Mathematically one would like to take the limit $q \to 0$ (note that the charge is quantized in nature and one has to be careful with such limit statements). However, in the case of a classical field it is possible to make sense of this idea. So once one places the probe charge into the field, one wishes to observe it with respect to the position of an observer. This observer has to be electrically neutral in order to avoid backreaction and also to distinguish the action of the field on the test charge from the action on him. Actually, the very notion of a field arises from the fact that there is a neutral observer.

Now, let us consider a gravitating system. In this case the situation is more subtle, since there are no neutral observers with respect to gravity. This was the fundamental idea which led to relativity and distinguishes gravity from all our field theories. Nevertheless, it is possible to find compatible measurement systems for certain GR experiments. In this thesis we deal with the cosmological problem of isotropy. We know that at the time of decoupling of photons the space used to be isotropic to a very high precision and so was well described be an FRW metric. Now the inhomogeneities give rise to perturbations about this background, but we have the belief that on horizon scales the metric is still close to FRW. Therefore for the measurement we need a system compatible with the FRW symmetries. The dust which will be used in the following section fulfills this property since its energy momentum tensor is the special case of vanishing pressure of the most general tensor compatible with the FRW symmetries: the tensor of a perfect fluid. Hence, if one would add a dust of very low energy density the FRW space-time would not be strongly affected. A Minkowski background, for instance, would be immediately destroyed, no matter how low the dust density is. The problem that the dust is added everywhere and hence there is no smooth limit to empty space.

In the course of this work we will go even one step further, we conjecture that in the cosmological context it is reasonable to assume an energy momentum tensor of the universe, the energy density of which never vanishes completely. This enables us to separate from the existing tensor a pressureless fluid tensor of low energy density without changing the system at all. In chapter 4 this dust will be used as a reference frame to perform the averaging without a formal gauge fixing.

3.2 Explicit construction of observables

As Einstein pointed out it in his hole argument the notion of a point in GR is a priori meaningless. It is washed away by general covariance. However, mostly in physics one is interested in describing the nature of some interacting objects and not just the geometry. The matter helps to solve this paradox. In this section we will describe how matter can be used to construct physical, local observables.

As already mentioned, GR has redundant degrees of freedom to ensure general covariance. This is represented by the presence of constrains which restrict the trajectories in the phase space to the physical ones. This constraints are also generators of diffeomorphisms which leave physics invariant. This is explicitly formulated as:

$$\frac{\partial A}{\partial X_\mu} = \{ H_\mu ,\ A \}$$

Hence, if we want to construct physical observables i.e. quantities which do not depend on the coordinate choice, we have to find quantities which commute with the constraints. We will use matter to achieve that. First we will discuss the oldest example which helped Einstein understanding the meaning of general covariance.

3.2.1 Single particle as reference frame

Following the example of [15], we take a single particle X freely falling in space-time, coupled to gravity and also carrying a clock. There is a variable T attached to the particle and monotonically growing along its trajectory. In a certain coordinate system O the universe will be described by

$$\{\mathcal{M},\, g_{\mu\nu}(y)\,,\, X^\mu(\tau)\,,\, T(\tau)\}$$

Where $y = (t\,,\, \vec{y})$. We can construct the function $\tilde{X}$ from X and T by:

$$\tilde{X}^\mu(T(\tau)) := X^\mu(\tau)$$

If we take $\tilde{X}(0)$ in the universe, this point is a physical one. We could for instance say that the geometry in this point is flat and it will be a gauge invariant statement. Consider a scalar function of the metric and let $R(y)$ be its values in the origin O. Then the quantity $\tilde{R} := R(\tilde{X}(0))$ is a physical observable. To see that, regard an infinitesimal change of coordinates induced by a vector field f. We will have:

$$\delta R(y) = -f^\mu(y)\partial_\mu R(y)$$

$$\delta X^\mu(\tau) = f^\mu(X(\tau))$$

$$\delta T(\tau) = 0$$

from that we get:

$$\delta\tilde{R} = -f^\mu(\tilde{X}(0))\partial_\mu R(\tilde{X}(0)) + \partial_\mu R(\tilde{X}(0))\delta\tilde{X}^\mu(0) = \partial_\mu R(\tilde{X}(0))\left[f^\mu(X(0)) - f^\mu(X(0))\right] = 0$$

Now we would like to implement the idea in a more formal way, as was done in [15]. First starting with the simplest system. A freely falling particle with a clock. Therefore the particles trajectory has to fulfill the geodesic equation.

$$\frac{d^2X^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}(X)\frac{dX^\nu}{ds}\frac{dX^\mu}{ds} = 0$$

$$\frac{d^2T}{ds^2} = 0$$

and the proper time $s(\tau)$ given by:

$$\frac{ds(\tau)}{d\tau} = \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}g_{\mu\nu}(X(\tau))} =: r(\tau)$$

This equations have to be added to the gravitational equations, which have to be Einstein equations with the energy momentum tensor of the particle on the right side. So it means we have to include the particle's Lagrangian into the total Lagrangian. There are several Lagrange functions yielding this equations. Several have been proposed by Kuchar. A possible choice would be:

$$S_{part} = -m \int \left[-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\, g_{\mu\nu}(X(\tau)) - \left(\frac{dT}{d\tau}\right)^2 \right]^{1/2} d\tau =: -m \int r_T(\tau)\, d\tau$$

So the total action is then:

$$S[g\,,\,X\,,\,T] = \frac{1}{8\pi G_N} \int dt \int d^3y \sqrt{g}\, R - m \int r_T(\tau)\, d\tau \tag{3.1}$$

In order to find the Dirac observables Hamiltonian analysis of the system is needed. The technique to perform a Hamiltonian analysis of a GR system was developed by Richard Arnowitt, Staneley Deser and Charles Misner and is called the ADM formalism[1].

The problem is that both terms in (3.1) have different evolution parameters. To express the second evolution parameter in terms of the first it is convenient to fix the parametrization invariance by requiring that $X^0(\tau) = \tau$. Now we can identify τ with t the particles degrees of freedom will be described by $X^a(t),\ a = 1, 2, 3$ without redundancy. The action reads as following:

$$S[g\,,\,X\,,\,T] = \frac{1}{8\pi G_N} \int dt \int d^3y \sqrt{g}\, R - m \left[-\dot{X}^a \dot{X}^b g_{ab}(X(t)) - 2\dot{X}^a g_{a0} - g_{00}(X(t)) - \dot{T}^2 \right]^{1/2}$$

In ADM coordinates $\{h_{ab}(y),\, N^\mu(y)\}$, the metric takes the form:

$$ds^2 = -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt)$$

The action expressed in these coordinates is:

$$S = \int dt \left[\int d^3y\, L_{ADM}[g, \dot{g}, N] - m \left((N(X))^2 - (\dot{X}^a + N^b(X))^2 - \dot{T}^2 \right)^{1/2} \right]$$

To compute the Hamiltonian we need to perform a Legendre transform and therefore first to compute the conjugate momenta:

$$p^{ab}(y) := \frac{\partial L}{\partial \dot{g}_{ab}(y)} = p^{ab}_{ADM}(y)[h, \dot{h}, N]$$

$$p_a := \frac{dL}{d\dot{X}^a} = -m \frac{(X^a \dot{+} N^a)}{\left[N^2 - (\dot{X}^a + N^a)^2 - \dot{T}^2 \right]^{1/2}}$$

---

[1]Consult appendix A for a brief description of the ADM formalism.

$$P := \frac{dL}{d\dot{T}} = -m \frac{\dot{T}}{\left[N^2 - (\dot{X}^a + N^a)^2 - \dot{T}^2\right]^{1/2}}$$

$$\pi_\mu(y) := \frac{\partial L}{\partial \dot{N}^\mu(y)} = 0$$

Here $p^{ab}_{ADM}[h, \dot{h}, N]$ are the momenta of the ADM theory as for instance described in [19]. So the Hamiltonian is:

$$H = \frac{1}{8\pi G_N} \int d^3y \, N^\mu(y) \, H^{ADM}_\mu[h,p](y) - N^a(X)p_a - N(X)\sqrt{m^2 + p^2 + P^2}$$

By considering the quantities multiplying the lapse and shift $(N,\ N^a)$ we get the constraints:

$$H_a(y) = H^{ADM}_a(y) - \delta^3(y - X)p_a$$

$$H_0(y) = H^{ADM}_0(y) - \delta^3(y - X)\sqrt{m^2 + p^2 + P^2}$$

As discussed above the Hamiltonian vanishes on shell and the dynamics is described by the constraints. Having finished the Hamiltonian analysis of the system we can turn to the construction of the Dirac variables. We can consider any scalar function of the metric, like the Ricci scalar:

$$\tilde{R} := R(X)$$

This quantity commutes with the momentum constraints. To prove it write $H(f) = \int d^3y f^a(y) H_a(y)$ and compute the Poisson brackets:

$$\left\{\tilde{R},\, H(f)\right\} = \left\{R(X),\, H^{ADM}(f)\right\} - \left\{R(X),\, f^a(X)p_a\right\} =$$

$$= \delta_f R(y)_{y=X} - \partial_b R(X)\left\{X^b,\, f^a(X)p_a\right\} =$$

$$= -f^a(X)\partial_a R(X) + f^a(X)\partial_a R(X) = 0$$

Here a class of variables has been found invariant under the three-dimensional diffeomorphisms.

In the next step we are going to deal with the Hamiltonian constraint. Consider the two observables $\tilde{R}$ and T on the phase space $\left\{h_{ab}\, p^{ab},\, X^a,\, p_a\,, T,\, P\right\}$ with any scalar density f, the

Hamiltonian flow of the constraint $H(f) = \int d^3y\, f(y)\, H_0(y)$ generates orbits in the phase space. Along those the evolution of the observables is given by:

$$\frac{d\tilde{R}}{dt} = \left\{\tilde{R}, H(f)\right\}$$

$$\frac{dT}{dt} = \{T,\, H(f)\}$$

Assume $\tilde{R}(t)$ and $T(t)$ are solutions of the above equations. For every solution we define $\bar{R}(T(t)) = \tilde{R}(t)$ or $\bar{R} = \tilde{R} \circ T^{-1}$ so by construction the quantity $\bar{R}$ for every value of T is t-invariant and hence commutes with $H(f)$. Furthermore $\bar{R}$ does not depend on f.

Consider:

$$\left\{\bar{R}\, H(f+\delta f)\right\} = \left\{\bar{R}, H(f)\right\} + \left\{\bar{R},\, H(\delta f)\right\} = \left\{\bar{R},\, H(\delta f)\right\}$$

We know:

$$\left\{H(f),\, H(\vec{f})\right\} = H(\mathcal{L}_{\vec{f}}\, f)$$

Where $\vec{f}$ is a vector field, which induces the Lie derivative of the scalar density f and $H(\vec{f}) = \int d^3y\, f^a H_a(y)$ with H_a being the three-diffeomorphism constraints. We have always such a vector field that:

$$\left\{\bar{R},\, H(f+\delta f)\right\} = \left\{\bar{R},\, \left\{H(f),\, H(\vec{f})\right\}\right\}$$

Since $\bar{R}$ commutes with $H(\vec{f})$ (it is defined in terms of 3-diff invariant quantities) and it commutes with $H(f)$ by construction, it follows that $\bar{R}$ also commutes with $H(f+\delta f)$ and hence does not depend on the choice of f. So given any scalar function of the metric R and any real number T. $\bar{R}(T)$ defines an observable which commutes with all constraints and is a Dirac observable. The interpretation of the quantity is very physical. It is the value of the scalar R (for instance the curvature) at the point where the particle is and at the moment the clock shows the value T. We could also take just one of the particles coordinate functions and the values it assumes are also in the same way Dirac observables. We will use this idea later not only for a single particle, but for a continuum of geodesic observers. It will be the analogy to the introduced dust in the Brown-Kuchar model. Now let us generalize the above discussion to a continuum of dust.

3.2.2 Observables with respect to dust

Imagine we would take many particles and use their world lines as references. The particles' coordinates would carry a particle index. Now let us take the continuum limit and take arbitrary many particles being arbitrary close together in such a way that the particle density is constant. We arrive at the notion of a field with the particles' continuous index Z_k $k \in \{1,2,3\}$, in fact three functions assigning the particle-labels uniquely. The Lagrangian variables are now:

$$\{g_{\mu\nu}(\vec{y},t),\ X^\mu(Z,\,\tau)\,,\,T(Z,\,\tau)\}$$

The action reads (We will analyze the action for a dust field in the next section. At this point we just take it as a given):

$$S = \int dt \left[\int d^3y\, L_{ADM}[g,\dot{g},N] - m\int d^3Z \left((N(X(Z)))^2 - \left[\dot{X}^a(Z) + N^b(X\,(Z))\right]^2 - \dot{T}(Z)^2\right)^{1/2}\right]$$

Here L_{ADM} is the standard ADM Lagrangian for GR. The corresponding constraints read:

$$H(\vec{f}) = H^{ADM}(\vec{f}) - \int d^3Z\, f^a(X(Z))p_a(Z)$$

$$H(f) = H^{ADM}(f) - \int d^3Z\, f(X(Z))\,\sqrt{m^2 + p(Z)^2 + P(Z)^2}$$

Now we generalize the observations made for a single particle. Analogously a scalar quantity $\tilde{R}(Z) := R(X(Z))$ commutes with the three-diff constraints. It is also possible to construct an other 3-diff. invariant quantity, namely:

$$\tilde{h}_{ab}(Z) = \partial_a X^c(Z)\partial_b X^d(Z)h_{cd}(X(Z))$$

This satisfies:

$$\left\{\tilde{h}_{ab}(Z),\, H(\vec{f})\right\} = 0$$

In the same way as for the singe particle, let us define $\bar{h}_{ab}(Z,T)$:

$$\bar{h}_{ab}(Z,\,T(Z,t) := \tilde{h}_{ab}(Z\,,\,t)$$

Here the following equations define the functions $\tilde{h}$ and T:

$$\frac{d}{dt}T(Z,t) = \{T(Z,t)\,,\, H(f)\}$$

$$\frac{d}{dt}\tilde{h}_{ab}(Z,t) = \left\{\tilde{h}_{ab}(Z,t)\,,\, H(f)\right\}$$

The quantity $\bar{h}_{ab}(Z,T)$ is a physical observable for any given Z and T which commutes with all constraints, as has been also shown in [15]:

$$\left\{\bar{h}_{ab}(Z,T)\,,\, H_\mu(y)\right\} = 0$$

The physical interpretation is here also very intuitive. This quantity expresses the geometry of the manifold of the points defined by the particles at the moment when all clocks display the value T. In the next chapter we will use this geometry to form scalars out of the coordinates of the dust particles.

3.3 The problem of time

By the "problem of time" in GR it is meant that the theory is completely parametrized. The space-time manifold describes everything in the space-time, there is no natural time evolution in the theory. The discussed diffeomorphism invariance is the reason why there is apparently no time. If the canonical Hamiltonian is derived in the ADM formalism one discovers that it is a linear combination of the Hamiltonian- and diffeomorphism constraints and generates infinitesimal space-time diffeomorphisms on shell. Since all the constraints vanish, so does the canonical Hamiltonian and is therefore not a generator of physical time evolution. Furthermore we already discussed that observables have to commute with the constraints and will also commute with the physical Hamiltonian. This is especially problematic if one wishes to set up a quantum mechanical formalism, since it seems that "nothing is happening in quantum gravity". An other interesting observation is that the Friedmann equations contain a time evolution. The time dependent quantities are represented by functions on the phase space which do not Poisson commute with the constraints[2] and hence should not be observable. At this point we have two options:

[2] T. Thiemann pointed out this difficulty in [17] and proposed a deparametrization mechanism based on one extra scalar field, "counting" the time. The scalar fields have similar properties as dust, if the potential term is appropriate.

- Drop Dirac's postulate, that observable quantities should be gauge invariant. This postulate worked astonishingly well in theories known so far.
- Admit that there is something in gravitational physics what we do not understand and use this hint to look for knew physics and new ideas to describe gravity.

We will follow the second trail and present a formalism which allows us to introduce the notion of time in GR.

3.4 Canonical description of dust

Before describing the Brown Kuchar mechanism, we will introduce the Lagrangian method to describe dust. There are many Lagrangian functions which describe a pressure-less fluid, the one presented here is most natural for the later discussion. Furthermore the notation for the dust coordinates will be established.

The Action:

First we show that the proposed action indeed yields the correct equations for a self gravitating, pressure-free fluid. The same action has been used in [9] to perform gauge independent perturbation theory. Furthermore the variation will enable us to give physical meaning to the introduced fields.

The action reads:

$$S = \int d^4x\,\sqrt{-g}\left\{-\frac{1}{8\pi G_N}R + \rho\left(u^\mu u^\nu g_{\mu\nu} + 1\right)\right\}$$

The velocity flow U of the dust is defined in components as:

$$u^\mu = -\partial^\mu T + W_j \partial^\mu Z^j$$

The action is a functional of the fields: $\{T\,, W_j\,, Z^j\,, g_{\mu\nu}\,, \rho\}$.

Now we vary the action S w.r.t the fields:

$$\delta S = \int d^4x\sqrt{-g}\left\{\left(-\frac{1}{2}\,g_{\mu\nu}\rho\left(u^\mu u^\nu g_{\mu\nu} + 1\right) + \frac{-1}{8\pi G_N}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \rho u_\mu u_\nu\right)\delta g^{\mu\nu}\right\} + \tag{3.2}$$

$$+\int d^4x\sqrt{-g}\left\{\left(u^\mu u^\nu g_{\mu\nu} + 1\right)\delta\rho + 2\rho u^\mu g_{\mu\nu}\delta u^\nu\right\} = 0$$

The Lagrange multiplier (variation w.r.t ρ) ensures that $u^\mu u_\mu = -1$ and so the first term in the variation w.r.t. $g_{\mu\nu}$ vanishes, resulting in:

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G_N\, \rho u_\mu u_\nu = 8\pi G_N\, T_{\mu\nu}$$

Which is Einstein's equation for a pressure-less fluid. This justifies already the choice of the action. We will study the variation with respect to the other fields to understand how the dust fields furnish a suitable coordinate system. We analyze further the last term of (3.2):

$$\int d^4x \sqrt{-g}\left\{2\rho u^\mu g_{\mu\nu}\delta u^\nu\right\} = \tag{3.3}$$

$$= \int d^4x \,\left\{\partial^\nu\left(\sqrt{-g}\, 2\rho u^\mu g_{\mu\nu}\right)\delta T - \partial^\nu\left(\sqrt{-g}\, 2\rho u^\mu g_{\mu\nu} W_j\right)\delta Z^j + \left(\sqrt{-g}\,\rho u^\mu g_{\mu\nu}\partial^\nu Z^j\right)\,\delta W_j\right\} = 0$$

This results in three independent equations. Assuming $\rho \neq 0$ we take the third term:

$$u^\mu g_{\mu\nu}\partial^\nu Z^j = \mathcal{L}_U Z^j = 0 \tag{3.4}$$

The Lie derivative of Z^j along U vanishes, this indicates that the fields Z^j are constant along the integral curves of U . Furthermore (3.4) implies:

$$\mathcal{L}_U T = u^\mu\left(\partial_\mu T - W_j \partial_\mu Z^j\right) = -u^\mu u_\mu = 1$$

We observe that T is the proper time along the dust flow lines.

The second term of (3.3) gives:

$$\partial^\nu\left(\sqrt{-g}\,\rho u^\mu g_{\mu\nu} W_j\right) = \sqrt{-g}\,\nabla_\nu\left(\rho u^\nu W_j\right) = 0 \tag{3.5}$$

And the first reads:

$$\partial^\nu\left(\sqrt{-g}\,\rho u^\mu g_{\mu\nu}\right) = \sqrt{-g}\,\nabla^\nu\left(\rho u_\nu\right) = 0 \tag{3.6}$$

Which is the continuity equation for the ideal fluid. Inserting (3.6) into (3.5) one arrives at:

$$u^{\nu}\nabla_{\nu}\left(W_j\right) = \nabla_U W_j = 0 \tag{3.7}$$

Together with (3.4), (3.7) shows that U is a time-like field and hence the following holds:

$$u^{\nu}\delta\nabla_{\nu}\left(x^{\mu}\right) = \nabla_u \delta x^{\mu} = \delta u^{\mu}$$

So we can rewrite $\rho u^{\mu} g_{\mu\nu}\delta u^{\nu} = 0$ and since this expression is still under the integral, partial integration can be performed:

$$\rho u^{\mu} g_{\mu\nu}\delta u^{\nu} = \rho u^{\mu} u^{\lambda} g_{\mu\nu}\delta\nabla\left(x^{\nu}\right)_{\lambda} = -\nabla_{\lambda}\left(\rho u^{\mu} u^{\lambda} g_{\mu\nu}\right)\delta x^{\nu} = 0$$

Now using the continuity equation (3.6) one obtains:

$$\nabla_{\lambda}\left(\rho u^{\mu} u^{\lambda} g_{\mu\nu}\right) = u_{\nu}\nabla_{\lambda}\left(\rho u^{\lambda}\right) + \rho u^{\lambda}\nabla_{\lambda}\left(u_{\nu}\right) = \rho\nabla_u u_{\nu} = 0 \Rightarrow \nabla_U U = 0$$

Obviously the integral curves of U are affinely parametrized geodesics. T is the proper time along the geodesics and each integral curve is determined by a constant value of Z^j. So using the variational principle we have found the physical meaning of all the relevant fields. Now the dust coordinates can be set up.

The dust coordinates:

The dust is completely described by eight scalar fields: The eigentime T, the density field ρ , the spatial coordinates Z^k and the velocity fields W_l $k, l \in \{1, 2, 3\}$. The condition $det(Z^K, \alpha) \neq 0$ (where $Z^K = (T, Z^k)$) , ensures that the coordinate lines do not intersect. The velocity flow is defined as: $u_{\alpha} = -\partial_{\alpha}T + W_k\partial_{\alpha}Z^k$.

We have seen that U is a future directed unit vector field and the scalar fields Z^k are constant along the flow lines. Which are geodesics parametrized by T. The congruence of the dust flow lines introduces a preferred reference frame into the space-time manifold $\mathcal{M}$ which is a physical space-time and hence has topology $R \times \Sigma$. So to represent any point y on the manifold it is sufficient to know the coordinate Z^k on Σ of the dust flow line which goes through y and the coordinate T, representing the eigentime when the dust flow line intersects with y.

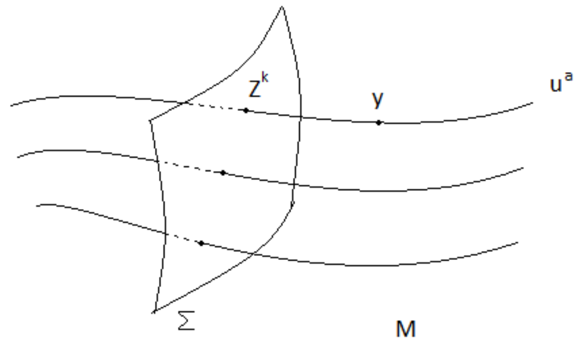

Figure 3.1: Dust flow

This gives us the mappings $\tau = T(y)$ as well as $z = Z(y)$. This maps can be inverted (since the non zero determinant condition from above holds) and one gets a map $(\tau,\, z) \to y = Y(\tau,\, z)$. The vectors $u^\alpha := \partial_\tau Y^\alpha$ and $\partial_k Y^\alpha$ form a basis of the tangent space $T\mathcal{M}$.

To point out the important facts once more: Any solution of the dust equations, coming from the dust action, describes the motion of the dust that allows the space time manifold $\mathcal{M}$ to be split into the space and time manifolds, S and $\mathcal{T}$. An instant of $\mathcal{T}$ and a flow line of S intersect at a unique event $y \in \mathcal{M}$. The space-time manifold $\mathcal{M}$ is thus a Cartesian product $\mathcal{T} \times S$ of the space and time dust manifolds. The mappings are summarized as:

$$T(y) \times Z(y) : \ \mathcal{M} \to \mathcal{T} \times S$$

$$Y(\tau,\, z) = \mathcal{T} \times S \to \mathcal{M}$$

This is also the reason why dust can serve as a standard of space-time, as shown by Brown and Kuchar in [2].

3.5 The Brown Kuchar mechanism

As discussed above the canonical Hamiltonian in GR is constraint to vanish and hence can not serve as a true time translation generator. In this section we will show how a system can be enlarged by a suitable subsystem (the dust). This makes it possible to construct an abelian algebra of constraints. Furthermore observables, which Poisson commute with the constraints can be constructed and evolve in time under the action of a physical Hamiltonian. This method is called the Brown Kuchar mechanism and was first published in [2]. Especially we are interested in the physical Hamiltonian to give a meaning to the notion of time inside the theory.

The procedure is as follows. We study the dust, described by the above action, minimally coupled to gravity. Consider the map Y as introduced in the previous section. It defines the foliation of the space-time as well as a lapse function N and a shift vector N^i. After introducing $N^\alpha := \partial_t Y^\alpha$ which describes the transition from one leaf of the foliation to the next, we read off the lapse and shift:

$$N^\alpha := N\, n^\alpha + N^i \partial_i Y^\alpha$$

Here n^α is the unit normal to the spatial hypersurface and $\partial_i Y^\alpha$ are the tangents. With this definition the pull back under Y of a scalar-field from the manifold to $\mathbb{R} \times \Sigma$ can be written as:

$$\partial_\alpha Z^k\, n^\alpha = \frac{\dot{Z}^k - \partial_\alpha Z^k\, N^\alpha}{N}$$

Now the dust Lagrangian can be written as:

$$L^D = \frac{1}{2}\sqrt{g}\rho \left[\frac{1}{N} \left(-\dot{T} + W_k \dot{Z}^k - u_i N^i \right)^2 - N \left(g^{ij} u_i u_j + 1 \right) \right]$$

Derive the conjugate momentum to $\dot{T}$ (where dot denotes derivative w.r.t. coordinate time t):

$$P := \frac{\partial L^D}{\partial \dot{T}} = \sqrt{g}\rho \frac{1}{N} \left(\dot{T} - W_k \dot{Z}^k + u_i N^i \right)$$

Perform the Legendre transformation:

$$P\dot{T} - L^D = -P_k \dot{Z}^k + N\, H^D_\perp + N^i H^D_i$$

Here $P_k := -PW_k$, the momentum constraint is $H^D_i := -Pu_i = P\partial_i T + P_k \partial_i Z^k$ and the Hamiltonian constraint reads:

$$H^D_\perp := \frac{1}{2}\frac{P^2}{\rho\sqrt{g}} + \frac{1}{2}\frac{\rho\sqrt{g}}{P^2}\left(P^2 + g^{ij} H^D_i H^D_j \right)$$

The dust action has now the following form, and its variation w.r.t ρ makes it possible to eliminate the density from the Hamiltonian constraint:

$$S^D = \int_R dt \int_\Sigma d^3x \left(P\dot{T} + P_k \dot{Z}^k - N\, H^D_\perp - N^i H^D_i \right)$$

$$0 = \frac{\delta S^D}{\delta \rho} = -N \frac{\partial H^D_\perp}{\partial \rho} \Rightarrow \rho = \frac{1}{\sqrt{g}} \frac{P^2}{\sqrt{P^2 + g^{ij} H^D_i H^D_j}}$$

$$H^D_\perp = \sqrt{P^2 + g^{ij} H^D_i H^D_j}$$

The total constraints, in our case the gravity plus dust constraints, are equal to zero.

$$H_\perp = H^G_\perp + H^D_\perp = 0 \tag{3.8}$$

$$H_i = H^G_i + H^D_i = 0 \, \forall i \tag{3.9}$$

The constraints (3.8) and (3.9) of dust coupled to gravity satisfy the following algebra:

$$\{H_\perp(x)\,,\, H_\perp(x')\} = g^{ij}(x) H_i(x) \partial_j \delta(x, x') - g^{ij}(x') H_i(x') \partial_j \delta(x', x) \tag{3.10}$$

$$\{H_\perp(x)\,,\, H_i(x')\} = -H_\perp(x') \partial_{i'} \delta(x, x') \tag{3.11}$$

$$\{H_i(x)\,,\, H_j(x')\} = H_j(x) \partial_i \delta(x, x') - H_i(x') \partial_j \delta(x', x) \tag{3.12}$$

Equation (3.12) shows that $H_i(x)$ are the generators of spatial diffeomorphisms (DiffΣ). Equation (3.11) tells that $H_\perp(x)$ is a scalar density of weight 1 under spatial diffeomorphisms. From the fact that the right hand side of (3.10) contains the dynamical variable g_{ij} we conclude that the total system is not a true Lie algebra. And especially $H_\perp$ is not the true generator for time translations.

To resolve this problem the equations (3.8) and (3.9) can be used to get new operators, which we will call new constraints. The new constraints are completely equivalent to the old constraints, but have a physical behavior which we will discuss below.

Define the new Hamiltonian constraint:

$$H_\uparrow = P - \sqrt{G} = 0 \tag{3.13}$$

with:

$$G = (H^G_\perp)^2 - g^{ij} H^G_i H^G_j$$

And the new Momentum constraint:

$$H_{\uparrow k} = P_k + h_k = 0 \tag{3.14}$$

with:

$$h_k = Z^i_k H^G_i + \sqrt{G} \partial_i T \, Z^i_k$$

where Z^i_k is the inverse matrix of $\partial_i Z^k$ i.e. $\partial_i Z^l \, Z^i_k = \delta^l_k$.

The old constraints and the new constraints are equivalent. Nevertheless there is a difference. The momenta $P_K = (P, \, P_k)$ in (3.13) and (3.14) are separated from the rest of the canonical variables. Now, since the old and new constraints are equivalent and the Poisson brackets of the old constraints vanish on shell, so must do the Poisson brackets of the new constraints. However, since the momenta P_K appear in the new constraints without coefficients, their Poisson brackets can not depend on P_K. Therefore the new constraints can not help to make these brackets vanish and hence they must vanish strongly i.e. even off shell. So we see that the new constraints Poisson commute:

$$\{H_{\uparrow J}, \, H_{\uparrow K}\} = 0 \text{ for } H_{\uparrow K} := (H_\uparrow, \, H_{\uparrow k})$$

The new constraints generate an abelian algebra. Now we smear the new constraints with scalar functions $N^{\uparrow K}(x) = \big(N^\uparrow(x), \, N^{\uparrow k}(x)\big)$:

$$H[N^\uparrow] = \int_\Sigma d^3x \, N^\uparrow(x) \, H_\uparrow(x)$$

$$H[\vec{N}^\uparrow] = \int_\Sigma d^3x \, N^{\uparrow k}(x) \, H_{\uparrow k}(x)$$

The smeared constraints generate through their Poisson brackets the changes of the fields:

$$\big\{T, \, Z^k, \, P, \, P_k, \, g_{ij}, \, p^{ij}\big\}$$

Of particular interest is $H[N^{\uparrow}]$, since it displaces the hypersurface of proper time by $N^{\uparrow}$ along the flow lines of the dust. This can be seen from the following equations:

$$\dot{T}(x) := \left\{T(x),\, H[N^{\uparrow}]\right\} = N^{\uparrow}$$

$$\dot{Z}^k(x) := \left\{Z^k(x),\, H[N^{\uparrow}]\right\} = 0$$

From the vanishing of the Poisson brackets of the new constraints and (3.13) it follows that the quantities $\sqrt{q\,G}(x)$ (with q being the metrics' determinant) mutually Poisson commute at different space time points. Similarly as in [17] we can construct Observables, which experience time evolution trough the operator $H = \int d^3x\, H(x)$, where $H(x) = \sqrt{q\,G}(x)$.

Define using a function f (invariant under spatial diffeomorphisms):

$$O_f(\tau) := \sum_{n=0}^{\infty} \frac{1}{n!} \{H_\tau,\, f\,\}_{(n)}$$

with:

$$H_\tau = \int_\Sigma d^3x\, [\tau - T(x)]\, H(x)$$

It has been shown in [17] that H is the non vanishing Hamilton which describes the time evolution of O_f, namely:

$$\frac{dO_f(\tau)}{d\tau} = \{H,\, O_f(\tau)\}$$

The details of the proof of this relation are beyond the scope of our discussion. The important message is that dust can introduce a physical coordinate system and can also be used to construct observables which evolve w.r.t a non vanishing physical Hamiltonian.

3.6 Summary Chapter 3

In this Chapter it has been shown, how matter coupled to gravity can be used to construct observable quantities. This idea which goes back to the father of relativity has been developed in the recent years, since it is of great interest for quantum gravity models.

A single freely falling particle has been studied. This idea has been generalized to a field of freely falling observers, the dust field. In the last part of this Chapter the Brown Kuchar mechanism has been discussed. It was shown that dust can introduce a preferred coordinate system into the manifold, with the help of geodesic trajectories of the dust particles. The crucial property here of course is that the particles do not interact. In that manner the theory can be deparametrized and a physical Hamiltonian can be found. Furthermore with this method gauge independent observables can be constructed, which evolve in time under the action of the physical Hamiltonian.

In the following chapter, the philosophy will be to introduce the dust into the gravitating system and use the dust coordinates and their combinations to construct physical scalars needed for the averaging procedure.

Chapter 4

Gauge invariant averaging for finite volumes

In this chapter an obviously gauge invariant formalism for averaging over finite volume regions of space-time will be presented and applied to traces of Einstein's equations. This model will use dust to represent a physical coordinate system. At this point it is important to mention the compatibility of the dust as coordinate system with a deformed FRW universe, as already discussed in 3.1. We know that our universe used to be homogeneous to a very high precision and hence was described well by the FRW universe. However, nowadays there are large inhomogeneities locally present in the universe. Nevertheless we belief that the underlying geometry is not drastically different on horizon scales. Therefore it is sensible to use a laboratory which does not destroy the FRW space-time, since it has the appropriate symmetries. The dust we are going to use fulfills this criterion.

4.1 The model

We consider a space-time filled with matter and demand that the energy density of this matter should never vanish. This assumption helps, to avoid the hole problem discussed in 2.2. Demanding a non vanishing matter content we are able to give physical meaning to every point in space-time. This is also a reasonable assumption on physical grounds, since even the voids contain little amounts of cosmic dust or radiation. Furthermore it is speculated that there is matter present in the universe, which interacts only (or almost only) with gravity. The dark matter is considered to be "cold" i.e. non- relativistic in the standard model of cosmology, hence it can be well described by dust. There are no observations which would contradict the assumption that this dark dust is spread everywhere in the universe.

The distribution of matter in our model though is not homogeneous, since it is supposed to represent an anisotropic universe. This is the difference to the approach in [8, 7], we allow for matter to exist which is distributed inhomogeneously and hence it is possible to find physical fields with a space-like gradient.

Within our model the energy momentum tensor can be decomposed in the way described below, since we have demanded for the energy density never to vanish. We can view this in two ways. Either the dust of low energy is formally separated from the total energy momentum tensor, or it is added to it, which would not affect the system strongly.

$$T^{\mu\nu} = T^{\mu\nu}_{\text{dust}} + \tilde{T}^{\mu\nu}$$

Where the $T^{\mu\nu}_{\text{dust}} = \epsilon\, u^\mu u^\nu$ is the dust energy momentum with infinitesimal ϵ. And $\tilde{T}^{\mu\nu}$ is the energy momentum tensor of the rest of the system.

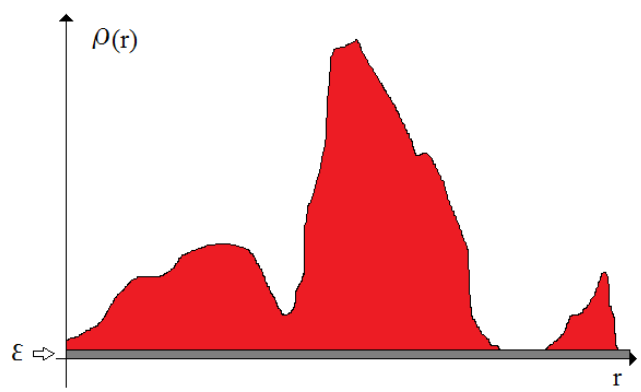

Figure 4.1: Matter distribution

Consequently the Lagrangian of the theory is as follows:

$$L = L_{\text{gravity}} + \tilde{L} + L_{\text{dust}}$$

The gravitational Lagrangian is the Ricci scalar multiplied by the square root of the metric determinant. The dust Lagrangian is as presented in Chapter 3.

$$L = \sqrt{-g}\,\epsilon\,\{v^\mu v^\nu g_{\mu\nu} + 1\}$$

With the following eight fields, ϵ being the energy density, T eigentime , W_l velocity field and Z^k coordinate field, where $k, l \in \{1, 2, 3\}$:

$$v^\mu = -\partial^\mu T + W_k \partial^\mu Z^k$$

T has a time-like gradient and the Z^k 's have obviously space-like gradients. This fields will be used for the construction of a window function which transforms as a scalar. We will restrict our analysis to the Lagrangian formalism and not be concerned with the form of the physical Hamiltonian. On the other hand we know that a physical Hamiltonian exists, since the system with dust deparametrizes, as has been shown in [9, 17, 6].

4.1.1 Time

The eigentime of the dust particles is a map into the dust-time manifold:

$$T : (\mathbb{R} \times \Sigma) \to \mathcal{T}$$

As was discussed in 3.2 at each fixed value $T = \tau$, T is a scalar w.r.t. coordinate changes on the manifold $\mathcal{M}$ in the sense:

$$\tilde{T}(X) = T(\tilde{X})$$

where $\tilde{X} = f^{-1}(X)$. On the dust-time manifold T is trivially a scalar. This will be scalar with the space-like gradient which will be used for the construction of the window function. We take the eigentime T of the dust particles as the scalar field which generates the time flow. Therefore the unit time-like vector is:

$$n^\mu = \frac{-\partial^\mu T}{\sqrt{-\partial_\mu T \partial^\mu T}}$$

The foliation is defined as the hypersurfaces on which T takes constant values. It is the physical time of the dust field. This defines a time-space split with the projector orthogonal to the time flow:

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

4.1.2 Space

The dust is a collection of geodesic observers, freely falling on $\mathcal{M}$. Each particle carries a label, at any instant of time τ a particle has a position, so this position of the labeled particle is a physical value and hence a Dirac observable. Taking the continuum limit, while keeping the particle density constant we obtain a field, where the indices of the particles can be viewed now as three dimensional coordinates Z^i , $i \in \{1, 2, 3\}$. Z is a map from the hypersurface of constant time into the dust-space manifold of constant $T = \tau$:

$$Z : \Sigma \rightarrow S(\tau)$$

Define the quantity: $\bar{Z}^i(T(\tau)) := Z^i(\tau)$. The $\bar{Z}^i$ are for evry fixed pair of values Z^i and T scalars on $\mathcal{M}$, as shown in chapter 3 (omit the bar in the coming discussion):

$$\tilde{Z}^k(X) = Z^k(\tilde{X})$$

where $\tilde{X} = f^{-1}(X)$. On the other hand they are three vectors on S. To construct a quantity which is a scalar on $\mathcal{M}$ and S, more work is required. The basic idea is to construct a scalar on S out of the coordinate fields of the dust, a quantity like $Z_k Z^k$. Therefore we need the induced metric on the hypersurfaces h_{ij}. To get this object we pull back the orthogonal projector $h_{\mu\nu}$ on the hypersurface of constant T denoted by Σ. To stay general we introduce coordinates on Σ called x^a which represent a map:

$$Z^{-1} : S(\tau) \rightarrow \Sigma$$

Furthermore, the coordinates on $\mathcal{M}$ are denoted by $X^\mu(x^a)$ and represent a map:

$$X : \Sigma \rightarrow \mathcal{M}$$

The combined map which we are going to use for the pull-back is ρ, defined as:

$$\rho := X \circ Z^{-1} \Rightarrow \rho : S(\tau) \rightarrow \mathcal{M}$$

We could visualize the interrelations as follows:

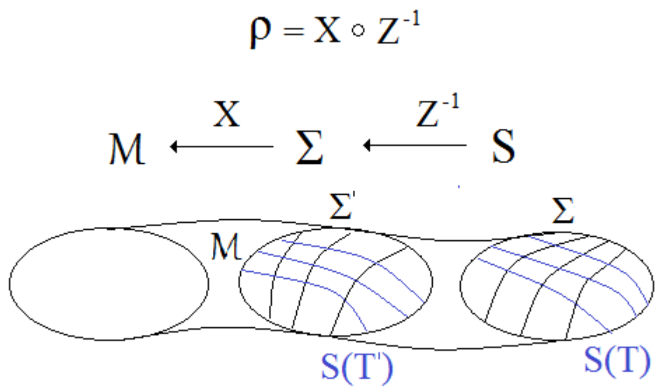

Figure 4.2: Interrelations

With the pull backs of the maps given as:

$$X_* = \frac{\partial X^\mu}{\partial x^a} =: X^\mu_a$$

$$Z^{-1}_* = \frac{\partial x^i}{\partial Z^j} =: (Z^{-1})^a_j$$

$$\rho_* = \frac{\partial X^\mu}{\partial Z^j} = X^\mu_a (Z^{-1})^a_j$$

The choice of the X and Z^{-1} maps corresponds to the gauge choice. Later we will pick the so called ADM gauge where $\rho = X$ and $Z^{-1} = id$ therefore $\Sigma = S$. After clarifying the maps we come back to the main goal and pull-back on the dust space:

$$h_{ij}(Z) = \rho_* h_{\mu\nu} = X^\mu_a (Z^{-1})^a_i X^\nu_b (Z^{-1})^b_j \, h_{\mu\nu} = \frac{\partial X^\mu}{\partial Z^i} \frac{\partial X^\nu}{\partial Z^j} h_{\mu\nu} =$$

$$=: X^\mu_i X^\nu_j h_{\mu\nu} = X^\mu_i X^\nu_j g_{\mu\nu} + X^\mu_i X^\nu_j n_\mu n_\nu =$$

$$= X^\mu_i X^\nu_j g_{\mu\nu} = X^\mu_a (Z^{-1})^a_i X^\nu_b (Z^{-1})^b_j \, g_{\mu\nu}$$

Here the property has been used that n^μ is orthogonal to the tangents of $S:\ \partial_i X^\mu$, i.e. $X^\mu_i n_\mu = 0 \; \forall \, i$.

Note: For the construction we have used the orthogonality of the time flow to the spatial hypersurfaces. To ensure this we can pick the dust velocity field W_k to vanish. This is simply assuming that the dust we consider is rotation free, which is reasonable for all practical purposes.

This quantity commutes with the 3-diffeomorphism constraints and in the next step following the logic from chapter 3 we construct the observable, using the dynamic equation with t being X^0 from the ADM split:

$$\frac{\partial h_{ij}(Z)}{\partial t} = \{h_{ij}(Z)\,,\, H(f)\}$$

Here $H(f) = \int d^3x\, H_0(x) f(x)$ with f some scalar density and H_0 the total Hamiltonian constraint of the system (Compare to section 3.2.2). Even if an analytic solution to this equation can not be obtained, we can set initial values by field re-definition and perform a series expansion. The solution of this equation $h_{ij}(Z,t)$ is used to define the observable $\bar{h}_{ij}(Z,T)$:

$$\bar{h}_{ij}(Z\,,\, T(Z,t)) := h_{ij}(Z,t)$$

This quantity commutes for fixed T and Z with all constraints and is hence an observable (as has been shown in chapter 3). In the following we will omit the bar for simplicity. So we have found the tensor quantity on S which is needed for the contraction with the vectors.

Now we can write:

$$Z^2 = Z^k h_{kl}\, Z^l =: B$$

So the dust vectors are contracted with a tensor on the dust manifold and one obtains a physical scalar on $\mathcal{M}$ and on S. It transforms as:

$$\tilde{B}(X) = B(\tilde{X})$$

and:

$$\tilde{B}(x,T) = B(\tilde{x},T)$$

where on $\mathcal{M}$: $\tilde{X} = f^{-1}(X)$ and correspondingly on S: $\tilde{x} = f^{-1}(x)$.

Proof:

- The induced metric on the hypersurface $X^\mu_a (Z^{-1})^a_i X^\nu_b (Z^{-1})^b_j \, g_{\mu\nu}$ is a scalar under coordinate changes on the manifold by construction, an explicit calculation shows it in components:

$$\tilde{h}_{ij}(X) = \frac{\partial}{\partial Z^i}\frac{\partial \tilde{X}^\mu}{\partial X^\lambda} X^\lambda \frac{\partial X^\lambda}{\partial \tilde{X}^\mu}\frac{\partial}{\partial Z^j}\frac{\partial \tilde{X}^\nu}{\partial X^\rho} X^\rho \frac{\partial X^\rho}{\partial \tilde{X}^\nu} g_{\lambda\rho} = \frac{\partial X^\lambda}{\partial Z^i}\frac{\partial X^\rho}{\partial Z^j} g_{\lambda\rho} = h_{ij}(\tilde{X})$$

- The constructed quantity $B = Z^2$ is a scalar on S since the 3-vectors Z^k are contracted with a rank two tensor on S. Again explicit calculation shows:

$$B(\vec{Z}) = Z^k h_{kl} \, Z^l = \left(\frac{\partial Z^k}{\partial x^a} x^a\right) \tilde{h}_{kl} \left(\frac{\partial Z^l}{\partial x^b} x^b\right) =$$

$$= \left(\frac{\partial Z^k}{\partial x^a} x^a\right) \left(h_{ab}\frac{\partial x^a}{\partial Z^k}\frac{\partial x^b}{\partial Z^l}\right) \left(\frac{\partial Z^l}{\partial x^b} x^b\right) = x^a \, h_{ab} \, x^b = B(\vec{x})$$

- The connecting relation which makes the whole object invariant under coordinate transformations on the manifold is that the Z^k s are the dust labels and hence scalars on $\mathcal{M}$ as discussed above:

$$\tilde{Z}^k(X) = Z^k(\tilde{X})$$

  where $\tilde{X} = f^{-1}(X)$.

Now we have shown that $B = Z^2$ is a scalar on $\mathcal{M}$ and on the dust manifold S. Since the Z^2 is a scalar on S we can express it in arbitrary coordinates x^a on Σ, the ADM split hypersurface of constant T. This coordinate representation is also convenient for the Window function discussed in the following section.

4.1.3 Window function

Having the desired physical scalars at hand we write the window function as:

$$W_\Omega = \delta(T(x) - T_0) \, \sqrt{-\partial_\mu T \partial^\mu T} \, \theta(r_0 - B(x))$$

Which is again a scalar on $\mathcal{M}$ and S, so the integral of a scalar over a domain Ω is gauge invariant as has been shown in 2.4.2. Write the integral as:

$$F(\Omega) = \int_{\mathcal{M}_4} \sqrt{-g(x)}\,\sqrt{-\partial_\mu T \partial^\mu T}\,\Theta(r_0 - B(x))\delta(T_0 - T(x))d^4x$$

Now the averaging functional can be defined in the following way and simplified integrating out the delta function:

$$\langle S\rangle_{\{A,r_0\}} = \frac{F(S,\Omega)}{F(S,1)} = \frac{\int_{\mathcal{M}_4} \sqrt{-g(x)}\,\sqrt{-\partial_\mu T \partial^\mu T} S(x)\Theta(r_0 - B(x))\delta(T_0 - T(x))d^4x}{\int_{\mathcal{M}_4} \sqrt{-g(x)}\,\sqrt{-\partial_\mu T \partial^\mu T}\Theta(r_0 - B(x))\delta(T_0 - T(x))d^4x} =$$

$$= \frac{\int_{\Sigma_0}\int_{\mathbb{R}} N\,\sqrt{h(\vec{x})}\,\sqrt{-\partial_\mu T \partial^\mu T} S(x)\Theta(r_0 - B(x))\delta(T_0 - T(x))d^3x\,dT}{\int_{\Sigma_0}\int_{\mathbb{R}} N\,\sqrt{h(\vec{x})}\,\sqrt{-\partial_\mu T \partial^\mu T}\Theta(r_0 - B(x))\delta(T_0 - T(x))d^3x\,dT} =$$

$$= \frac{\int_{\Sigma_0} N\,\sqrt{h(x)}\,\sqrt{-\partial_\mu T \partial^\mu T}|_{T=T_0} S(x)\Theta(r_0 - B(x))d^3x}{\int_{\Sigma_0} N\,\sqrt{h(x)}\,\sqrt{-\partial_\mu T \partial^\mu T}|_{T=T_0}\Theta(r_0 - B(x))d^3x}$$

Where Σ_0 is the hypersurface on which $T(x) = T_0$, h the determinant of the induced spatial metric discussed above and N the lapse function for a general not synchronous gauge. The average is manifestly a gauge independent quantity, since $T(x)$ and $B(x)$ are scalars under Diff($\mathcal{M}$) and also Diff(S) and so the window function is also a scalar. Hence, the average of S is obviously an observable.

Equipped with this functional we can address the problem of averaging the scalar parts of Einstein's equations of the model of the universe presented above, where the coordinate system, the dust, is included into the studied system. We will perform a similar analysis as in [7], but in a more general way, since our formalism applies also to finite volumes.

4.1.4 Time derivative of the average

To average the dynamic equations of our model it is also necessary to compute time derivatives of the averaged quantities. In this case the time derivative means partial derivative w.r.t T_0 the eigentime of the homogeneous dust used for the deparametrisation. Let us begin with the derivative of $F(S,\Omega)$. The strategy will be to choose certain coordinates in which the computation will be easier and then to generalize our result to an arbitrary frame.

The eigentime T is the function that defines the space as hypersurfaces on which it assumes a constant value. Therefore it is spatially homogeneous. In this case we can choose coordinates, in future referred to as ADM coordinates, such that: $n_\mu = N\,(-1,0,0,0)$ and $n^\mu = \frac{1}{N}(1,\ -N^i)$

where N is the lapse and N^i the shift. The metric reads:

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Furthermore since we have set the velocity field W^k of the coordinate dust to zero, which implies the orthogonality of n^μ to the spatial hypersurfaces, the shift vectors vanish. In this coordinates we compute the time derivative. The partial derivative commutes past the integral and the derivative of the δ distribution we understand in the weak sense.

$$\frac{\partial F(S,\Omega)}{\partial T_0} = -\int_{\mathcal{M}_4} \sqrt{-g(x)}\,\sqrt{-\partial_\mu T\partial^\mu T} S(x)\Theta(r_0 - B(x))\frac{\partial}{\partial T}\{\delta(T_0 - T(x))\}d^4x =$$

$$= -\int_{\mathcal{M}_4} \sqrt{-g(x)}\,\sqrt{-\partial_\mu T\partial^\mu T} S(x)\Theta(r_0 - B(x))(\partial_0 T)^{-1}\partial_0\{\delta(T_0 - T(x))\}d^4x$$

In the coordinates chosen we can read of from the form of n_μ the only component of $\partial_\mu T$ which is not zero: $\partial_0 T$ and $-\partial_\mu T g^{\mu\nu}\partial_\nu T = -(\partial_0 T)^2 g^{00}$ with $g^{00} = -N^{-2}$ therefore the above equation simplifies to:

$$-\int_{\mathcal{M}_4} \sqrt{-g(x)}\; S(x)\Theta(r_0 - B(x))\sqrt{-g^{00}}\partial_0\{\delta(T_0 - T(x))\}d^4x =$$

$$= \int_{\mathcal{M}_4} \partial_0\left\{\sqrt{-g(x)}\; S(x)\Theta(r_0 - B(x))\sqrt{-g^{00}}\right\}\delta(T_0 - T(x))d^4x =$$

$$= \int_{\mathcal{M}_4} \sqrt{|h|}\,\{\,\Theta(r_0 - B(x))\,(N\theta S + \partial_0 S) - \delta(r_0 - B(x))S\partial_0 B(x)\}\,\delta(T_0 - T(x))d^4x$$

Here $h = det(h_{ij})$ and $\theta = \frac{1}{N}\partial_0 log(\sqrt{h})$, since $\theta_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij} = \partial_T h_{ij} = \frac{1}{N}\partial_0 h_{ij}$ in this gauge and the trace $\theta = \frac{1}{N}h^{ij}\dot{h}_{ij} = \frac{1}{N}\partial_0 log(\sqrt{h})$. Having this we can rewrite the expression in a covariant way, so that it reduces to our result when the ADM coordinates are fixed.

$$\frac{\partial F(S,\Omega)}{\partial T_0} = F\left(\frac{\partial_\mu T\partial^\mu S}{\sqrt{-\partial_\mu T\partial^\mu T}},\Omega\right) + F\left(\frac{S\theta}{\sqrt{-\partial_\mu T\partial^\mu T}},\Omega\right) - 2\,F\left(\frac{\partial_\mu T\partial^\mu B}{\sqrt{-\partial_\mu T\partial^\mu T}}\,S\,\delta(r_0 - B),\Omega\right)$$

The last term vanishes in case $n^\mu \partial_\mu B = 0$ i.e. the spatial coordinates do not depend on the time variable. This is indeed the case in our model since n^μ is orthogonal to the space such that $\nabla_n Z^k = 0$ which implies the above relation. Hence the time derivative reads:

$$\frac{\partial F(S,\Omega)}{\partial T_0} = F\left(\frac{\partial_\mu T \partial^\mu S}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega\right) + F\left(\frac{S\theta}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega\right) \tag{4.1}$$

Now we can calculate the time derivative of the average of a scalar $\langle S, \Omega\rangle = \frac{F(S,\Omega)}{F(1,\Omega)}$.

$$\frac{\partial \langle S,\Omega\rangle}{\partial T_0} = \frac{\partial F(S,\Omega)}{\partial T_0}\frac{1}{F(1,\Omega)} - \langle S,\Omega\rangle \frac{1}{F(S,\Omega)}\frac{\partial F(1,\Omega)}{\partial T_0} =$$

$$= \left(F(\partial_{T_0} S,\Omega) + F(\frac{N\,S\,\theta}{\partial_0 T},\Omega)\right)\frac{1}{F(1,\Omega)} - \langle S,\Omega\rangle \frac{1}{F(1,\Omega)} F(\frac{N\,\theta}{\partial_0 T},\Omega) =$$

$$= \langle \partial_{T_0} S,\Omega\rangle + \left\langle \frac{N\,S\,\theta}{\partial_0 T},\Omega\right\rangle - \langle S,\Omega\rangle \left\langle \frac{N\,\theta}{\partial_0 T},\Omega\right\rangle$$

Covariantly expressed we obtain the general form of the Buchert, Ehlers (compare to [3]) commutation rule:

$$\boxed{\frac{\partial \langle S,\Omega\rangle}{\partial T_0} = \left\langle \frac{\partial_\mu T \partial^\mu S}{\partial_\mu T \partial^\mu T},\Omega\right\rangle + \left\langle \frac{S\,\theta}{\sqrt{-\partial_\mu T \partial^\mu T}},\Omega\right\rangle - \langle S,\Omega\rangle \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}},\Omega\right\rangle \qquad (4.2)}$$

4.1.5 The effective scale factor

In order to be interpreted as a domain dependent scale factor analogous to the a of the FRW model, a quantity has to be considered which is gauge independent. As any physical observable must not depend on the coordinates chosen.

To accomplish this we use the gauge independent averaging functional to define the quantity s to be the effective scale factor:

$$\frac{1}{s}\frac{\partial s}{\partial T_0} := \frac{1}{3}\frac{1}{F(1,\Omega)}\frac{\partial F(1,\Omega)}{\partial T_0}$$

Using (4.1) one sees that

$$\frac{\partial F(1,\Omega)}{\partial T_0} = F\left(\frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega\right)$$

and hence:

$$\frac{1}{s}\frac{\partial s}{\partial T_0} = \frac{1}{3}\left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle \tag{4.3}$$

Remark:

A more pictorial way of looking at this, is to write:

$$s = \left(\frac{F(1,\Omega)_{T_0}}{F(1,\Omega)_0}\right)^{\frac{1}{3}}$$

Therefore:

$$\frac{\partial s}{\partial T_0} = \frac{1}{3}(s^3)^{-\frac{2}{3}}\frac{1}{F_0}\frac{\partial F(1,\Omega)_{T_0}}{\partial T_0} = \frac{1}{3}s\left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle$$

We see from that calculation that s has indeed a property of a scale factor, but its definition above circumvent the necessity of introducing a normalization at a certain time zero.

4.2 Einstein's equations

This part will be similar to 1.4.3, but in this case the n^μ which chooses the time is the flow of the coordinate dust already present in the model and not an arbitrary choice. The time arises natural and there is no need of artificial gauge fixing. We describe the time flow without formally choosing a gauge.

4.2.1 Gauge independent averaging

Normal projection of the equations of motion:

The Einstein tensor multiplied with the time flow vectors (which is equivalent to the Hamiltonian constraint density) gives:

$$G^{\mu\nu}n_\mu n_\nu = \frac{1}{2}R_s + \frac{1}{3}\theta^2 - \sigma^2$$

Where the expansion rate and the shear are defined w.r.t n_μ, the R_s is the spatial curvature associated with the projector on the hypersurface orthogonal to n_μ.

On the other hand one has from the matter contribution:

$$T^{\mu\nu}n_\mu n_\nu = (T^{\mu\nu}_{\text{dust}} + \tilde{T}^{\mu\nu})n_\mu n_\nu$$

$$\tilde{T}^{\mu\nu}n_\mu n_\nu$$

$$T^{\mu\nu}_{\text{dust}}n_\mu n_\nu = \epsilon(v_\mu n^\mu)^2 = \epsilon(-\partial_\mu T n^\mu)^2 = \epsilon(-\partial_\mu T\partial^\mu T)$$

With $v^\mu = \sqrt{-\partial_\mu T\partial^\mu T}\, n^\mu$ since we said that the velocity field W^k vanishes. Now the Hamiltonian constraint density can be averaged. For normalization divide it first by $-3\,\partial_\mu T\partial^\mu T$ apply the averaging functional introduced above and insert the identity by adding and subtracting the average of θ squared:

$$\left\langle \frac{1}{6}\frac{R_s}{-\partial_\mu T\partial^\mu T} + \frac{1}{9}\frac{\theta^2}{-\partial_\mu T\partial^\mu T} - \frac{1}{3}\frac{\sigma^2}{-\partial_\mu T\partial^\mu T}\right\rangle + \frac{1}{9}\left\langle \frac{\theta}{\sqrt{-\partial_\mu T\partial^\mu T}}\right\rangle^2 - \frac{1}{9}\left\langle \frac{\theta}{\sqrt{-\partial_\mu T\partial^\mu T}}\right\rangle^2 = \tag{4.4}$$

$$= \frac{8\pi G_N}{3}\left\langle \frac{\tilde{T}^{\mu\nu}n_\mu n_\nu}{-\partial_\mu T\partial^\mu T} + \epsilon\right\rangle$$

Using:

$$\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 = \frac{1}{9}\left\langle \frac{\theta}{\sqrt{-\partial_\mu T\partial^\mu T}}\right\rangle^2$$

And defining the backreaction:

$$Q_D = \frac{1}{3}\left(\left\langle \frac{\theta^2}{-\partial_\mu T\partial^\mu T}\right\rangle - \left\langle \frac{\theta}{\sqrt{-\partial_\mu T\partial^\mu T}}\right\rangle^2\right) - 2\left\langle \frac{\sigma}{-\partial_\mu T\partial^\mu T}\right\rangle$$

We rewrite (4.4) as:

$$\boxed{\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 = -\frac{1}{6}\left\langle \frac{R_s}{-\partial_\mu T\partial^\mu T}\right\rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\left\langle \frac{\tilde{T}^{\mu\nu}n_\mu n_\nu}{-\partial_\mu T\partial^\mu T} + \epsilon\right\rangle}$$

This is the effective equation describing the velocity of the expansion of the domain dependent scale factor. It is the analogue to the first Friedmann equation. With the definitions comparable to those in [3] we can write the cosmological energy balance:

$$H_D := \frac{1}{s}\frac{\partial s}{\partial T_0}$$

$$\Omega_m := \frac{8\pi G_N}{3\,H_D^2}\left\langle \frac{\tilde{T}^{\mu\nu}n_\mu n_\nu}{-\partial_\mu T\partial^\mu T}\right\rangle$$

$$\Omega_\epsilon := \frac{8\pi G_N}{3\,H_D^2}\ \langle\epsilon\rangle$$

$$\Omega_k := -\frac{\langle R\rangle}{6H_D^2}$$

$$\Omega_Q := -\frac{Q_D}{6H_D^2}$$

$$\Omega_k + \Omega_Q + \Omega_\epsilon + \Omega_m = 1$$

From the energy balance equation we see that the backreaction term Ω_Q as well as the coordinate mass term Ω_ϵ enter the balance and the backreaction term opens the possibility of contributing to an effective cosmological constant term.

Raychaudhuri's equation:

Combining the Hamiltonian constraint density with the trace of the Einstein equations projected orthogonal to the time flow, one gets[1]:

$$R_{\mu\nu}n^\mu n^\nu = T_{\mu\nu}h^{\mu\nu} - \frac{1}{2}T$$

Which is equivalent to:

$$-n^\mu\nabla_\mu\theta = 2\sigma^2 + \frac{1}{3}\theta^2 - \nabla^\nu(n^\mu\nabla_\mu n_\nu) + (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)n^\mu n^\nu \tag{4.5}$$

In our model the last term gives:

[1] The interested reader can consult the appendix B for a derivation

$$(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T)n^\mu n^\nu = T^{dust}_{\mu\nu}n^\mu n^\nu + \tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}T = \tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}\tilde{T} + \epsilon(-\partial_\mu T\partial^\mu T) + \frac{1}{2}(-\epsilon)$$

Now we will need the second time derivative of the scale factor s:

$$\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = \frac{\partial}{\partial T_0}\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right) + \left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2$$

We will use the commutator (4.2) and eqn. (4.3) to get:

$$\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = \frac{1}{3}\left\langle \frac{\partial_\mu T}{-\partial_\mu T\partial^\mu T}\,\partial^\mu\left(\frac{\theta}{(-\partial_\mu T\partial^\mu T)^{1/2}}\right)\right\rangle + \frac{1}{3}\left\langle\frac{\theta^2}{-\partial_\mu T\partial^\mu T}\right\rangle - \frac{3}{9}\left\langle\frac{\theta}{\sqrt{-\partial_\mu T\partial^\mu T}}\right\rangle^2 =$$

$$= -\frac{1}{3}\left\langle\frac{\partial_\mu T\,\partial^\mu\theta}{(-\partial_\mu T\partial^\mu T)^{3/2}}\right\rangle + \frac{1}{3}\left\langle\frac{\theta^2}{-\partial_\mu T\partial^\mu T}\right\rangle - \frac{2}{9}\left\langle\frac{\theta}{(-\partial_\mu T\partial^\mu T)^{1/2}}\right\rangle^2 - \frac{1}{6}\left\langle\frac{\partial_\mu T\partial^\mu(\partial_\nu T\partial^\nu T)}{(-\partial_\mu T\partial^\mu T)^{5/2}}\right\rangle$$

Rewriting :

$$\partial^\mu T\partial_\mu\theta = \nabla_T\theta = -(-\partial_\mu T\partial^\mu T)^{1/2}n^\mu\nabla_\mu\theta$$

We can insert (4.5) into the first term:

$$\boxed{\begin{aligned}\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} &= \frac{1}{3}\left\langle\frac{\theta^2}{-\partial_\mu T\partial^\mu T}\right\rangle - \frac{2}{9}\left\langle\frac{\theta}{(-\partial_\mu T\partial^\mu T)^{1/2}}\right\rangle^2 - \frac{1}{6}\left\langle\frac{\partial_\mu T\partial^\mu(\partial_\mu T\partial^\mu T)}{(-\partial_\mu T\partial^\mu T)^{5/2}}\right\rangle + \\ &-\frac{1}{3}\left\{2\left\langle\frac{\sigma^2}{-\partial_\mu T\partial^\mu T}\right\rangle + \frac{1}{3}\left\langle\frac{\theta^2}{-\partial_\mu T\partial^\mu T}\right\rangle - \left\langle\frac{\nabla_\mu(n^\mu\nabla_\mu n_\nu)}{-\partial_\mu T\partial^\mu T}\right\rangle\right\} + \\ &+\frac{8\pi G_N}{3}\left\{\left\langle\frac{\tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}\tilde{T}}{(-\partial_\mu T\partial^\mu T)^{1/2}}\right\rangle + \frac{1}{2}\left\langle\frac{\epsilon}{\partial_\mu T\partial^\mu T} + 2\epsilon\right\rangle\right\}\end{aligned}}$$

This is the effective Raychaudhuri equation for the domain. There is an analogy to the second Friedman equation, since it describes the acceleration of the domain dependent scale factor s.

4.2.2 Evaluation in the ADM gauge

In this subsection we will evaluate the general equations in the ADM gauge, which was introduced above, and compare them to the effective equations obtained by T. Buchert. As a reminder in the ADM gauge $n_\mu = N(-1,0,0,0)$ and $n^\mu = \frac{1}{N}(1,\ -N^i)$ where N is the lapse and N^i the shift. The metric reads:

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Since we have chosen n_μ to be orthogonal to the spatial hypersurfaces the shift vectors vanish i.e. the metric reads: $ds^2 = -N^2dt^2 + h_{ij}dx^i dx^j$. Especially we get for:

$$-\partial_\mu T \partial^\mu T = -\partial_0 T\, g^{00} \partial_0 T = \frac{1}{N^2}(\partial_0 T)^2$$

and:

$$\partial_0 T_0 = 1$$

Since T_0 defines the hypersurface of constant physical time.

The Hamiltonian constraint reduces to:

$$\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 = -\frac{1}{6}\left\langle R_s N^2\right\rangle - \frac{1}{6}\left\{\frac{2}{3}\left(\left\langle N^2\theta^2\right\rangle - \left\langle N\,\theta\right\rangle^2\right) - 2\left\langle N^2\sigma^2\right\rangle\right\} +$$

$$+\frac{8\pi G_N}{3}\left\langle N^2\,\tilde{T}_{\mu\nu}n^\mu n^\nu + \epsilon\right\rangle =$$

$$= -\frac{1}{6}\left\langle R_s N^2\right\rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\left\langle N^2\,\tilde{T}_{\mu\nu}n^\mu n^\nu + \epsilon\right\rangle$$

And the Raychaudhuri equation to:

$$-\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = -\frac{2}{9}\left(\left\langle N^2\theta^2\right\rangle - \left\langle N\theta\right\rangle^2\right) + \frac{2}{3}\left\langle N^2\sigma^2\right\rangle - \frac{1}{3}\left\langle\theta\partial_0 N\right\rangle - \frac{1}{3}\left\langle N\,h^{ij}\nabla_i\nabla_j N\right\rangle +$$

$$+\frac{8\pi G_N}{3}\left\langle N^2\left(\tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}\tilde{T}\right) + \epsilon\right\rangle =$$

$$= -\frac{1}{3}Q_D - \frac{1}{3}\left\langle\theta\partial_0 N\right\rangle - \frac{1}{3}\left\langle N\,h^{ij}\nabla_i\nabla_j N\right\rangle + \frac{8\pi G_N}{3}\left\langle\left(\tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}\tilde{T}\right) + \epsilon\right\rangle$$

Setting $N=1$ we obtain:

The Hamilton constraint density:

$$\boxed{\begin{aligned}\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 &= -\frac{1}{6}\left\langle R_s\right\rangle - \frac{1}{6}\left\{\frac{2}{3}\left(\left\langle\theta^2\right\rangle - \left\langle\theta\right\rangle^2\right) - 2\left\langle\sigma^2\right\rangle\right\} + \\ &+\frac{8\pi G_N}{3}\left\langle\tilde{T}_{\mu\nu}n^\mu n^\nu + \epsilon\right\rangle = \\ &= -\frac{1}{6}\left\langle R_s\right\rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\left\langle\tilde{T}_{\mu\nu}n^\mu n^\nu + \epsilon\right\rangle\end{aligned}}$$

And the Raychaudhuri equation to:

$$\boxed{\begin{aligned}-\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} &= -\frac{2}{9}\left(\left\langle\theta^2\right\rangle - \left\langle\theta\right\rangle^2\right) + \frac{2}{3}\left\langle\sigma^2\right\rangle + \\ &+\frac{8\pi G_N}{3}\left\langle\left(\tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}\tilde{T}\right) + \epsilon\right\rangle = \\ &= -\frac{1}{3}Q_D + \frac{8\pi G_N}{3}\left\langle\left(\tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}\tilde{T}\right) + \epsilon\right\rangle\end{aligned}}$$

Note: The coordinate dust energy ϵ has a negative effect on the acceleration.
The averaging functional with the window function reduces in this gauge to:

$$\langle S, \Omega\rangle = \frac{\int_{\Sigma_0}\sqrt{h(x)}S(x)\Theta(r_0 - B(x))d^3x}{\int_{\Sigma_0}\sqrt{h(x)}\Theta(r_0 - B(x))d^3x}$$

4.2.3 The average functional in the ADM gauge:

To perform integration in general we need a push forward from the manifold into the $\mathbb{R}^n$, in our case a map from the dust space into $\mathbb{R}^n$ is required. Define: $\Psi := \Phi \circ X \circ Z^{-1}$ s.t. $\Psi : S(T) \to \mathbb{R}^n$ where Φ is the map $\Phi : M \to \mathbb{R}^n$, Z^{-1} the map $Z^{-1} : S(T) \to \Sigma$, X the map $X : \Sigma \to M$ and $S(T)$ the dust space slice.

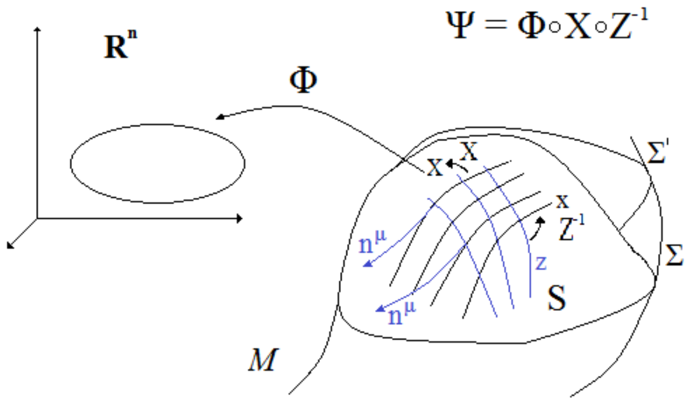

Figure 4.3: Dust Map

It was already shown that Z^2 is gauge invariant so now it is legitimate to fix a gauge. A legitimate choice would be Z^{-1}=id the identity, X the map from the hypersurface coordinates $x^a = Z^i$ to the manifold coordinates X^μ and therefore $S = \Sigma$. In the ADM gauge the coordinates are t and x^a as read of from the metric $ds^2 = -dt^2 + h_{ab}dx^a dx^b$. This leads to:

$$Z^2(x) = Z^i h_{ij} Z^j = x^a h_{ab}\, x^b$$

Rewriting the average functional one obtains:

$$\langle S,\Omega\rangle = \frac{\int_{\Sigma_0}\sqrt{h(x)}S(x)\Theta(r_0 - Z^2(x))d^3x}{\int_{\Sigma_0}\sqrt{h(x)}\Theta(r_0 - Z^2(x))d^3x} = \frac{\int_0^\infty\int_0^{2\pi}\int_0^\pi\, drd\phi d\theta sin(\theta)r^2\sqrt{h}S(r,\phi,\theta)\Theta(r_0 - r^2)}{\int_0^\infty\int_0^{2\pi}\int_0^\pi\, drd\phi d\theta sin(\theta)r^2\sqrt{h}\Theta(r_0 - r^2)} =$$

$$= \frac{\int_0^{r_0^2}\int_0^{2\pi}\int_0^\pi\, drd\phi d\theta sin(\theta)r^2\sqrt{h}S(r,\phi,\theta)}{\int_0^{r_0^2}\int_0^{2\pi}\int_0^\pi\, drd\phi d\theta sin(\theta)r^2\sqrt{h}}$$

Which is a volume integral over a sphere of radius r_0^2.

4.2.4 Comparison to Buchert equations:

The difference between our formalism and the averaging applied by T. Buchert in [3] is that we did not choose a gauge from the beginning but deparametrized the manifold with dust. This introduced a time in the theory. Furthermore we found a way to perform gauge invariant averages over finite volumes. Hence the averaged quantities $\{s\, \dot{s}\, \ddot{s}\ \langle\epsilon\rangle\}$ became physical observables and we obtained equations governing their dynamics. The equations we get after fixing the ADM gauge are slightly different to the ones in Buchert's paper. We rediscover Buchert's equations in our general formalism if we set the rest energy momentum $\tilde{T}_{\mu\nu}$ to zero. Therefore one could think that in Buchert's formalism just the geometry is averaged. The problem with this point of view is that even an infinitesimal energy density ϵ leads to a geometry arbitrary far away from the vacuum geometry, since there is no smooth limit $\epsilon \to 0$.

Hence Buchert's equations describe correctly the evolution of a cosmology filled with one type of pressure-free fluid in the rest frame of the fluid. We see that in this special case the procedures of gauge fixing and averaging commute. Our formalism allows to generalize the averaging to an almost arbitrary energy momentum tensor. The only restriction is that the energy density must not vanish anywhere. Especially it is not possible to apply this formalism to vacuum.

4.2.5 Example: Ideal fluid cosmology

A cosmology equipped with an energy momentum tensor of an ideal fluid will be studied. This is a reasonable choice for the energy momentum, since it is the most general tensor fulfilling the symmetries of an isotropic universe and since we know from the CMB that the universe has been extremely isotropic this model is a reasonable choice. This procedure can be easily generalized to an arbitrary number of ideal non interacting fluids.

Normal projection of the equations of motion:

The Einstein tensor multiplied with the time flow vectors (which is equivalent to the Hamiltonian constraint density) gives:

$$G^{\mu\nu} n_\mu n_\nu = \frac{1}{2} R_s + \frac{1}{3} \theta^2 - \sigma^2$$

Where the expansion rate and the shear are defined w.r.t n_μ, the R_s is the spatial curvature associated with the projector on the hypersurface orthogonal to n_μ. On the other hand one has from the matter contribution:

$$T^{\mu\nu} n_\mu n_\nu = (T^{\mu\nu}_{\text{fluid}} + T^{\mu\nu}_{\text{dust}}) n_\mu n_\nu$$

$$T^{\mu\nu}_{\text{fluid}} n_\mu n_\nu = (\rho + p)(u^\mu n_\mu)^2 - p = \rho + (\rho + p)\, sinh^2(\alpha_T)$$

$$T^{\mu\nu}_{\text{dust}} n_\mu n_\nu = \epsilon (v_\mu n^\mu)^2 = \epsilon (-\partial_\mu T n^\mu)^2 = \epsilon (-\partial_\mu T \partial^\mu T)$$

With the tilt angle α_T defined via $sinh^2(\alpha_T) := (u^\mu n_\mu)^2 - 1$. Furthermore as discussed above $v^\mu = \sqrt{-\partial_\mu T \partial^\mu T}\; n^\mu$ since we said that the velocity field W^k vanishes. Now the Hamiltonian constraint can be averaged. For normalization divide it first by $-3\,\partial_\mu T \partial^\mu T$ and apply the averaging functional introduced above:

$$\left\langle \frac{1}{6}\frac{R_s}{-\partial_\mu T\partial^\mu T} + \frac{1}{9}\frac{\theta^{\mathbf{2}}}{-\partial_\mu T\partial^\mu T} - \frac{1}{3}\frac{\sigma^{\mathbf{2}}}{-\partial_\mu T\partial^\mu T}\right\rangle + \frac{1}{9}\left\langle \frac{\theta}{\sqrt{-\partial_\mu T\partial^\mu T}}\right\rangle^2 - \frac{1}{9}\left\langle \frac{\theta}{\sqrt{-\partial_\mu T\partial^\mu T}}\right\rangle^2 = \tag{4.6}$$

$$= \frac{8\pi G_N}{3}\left\langle \frac{\rho + (\rho+p)sinh^{\mathbf{2}}(\alpha_T)}{-\partial_\mu T\partial^\mu T} + \epsilon\right\rangle$$

As above we rewrite (4.6) as:

$$\boxed{\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 = -\frac{1}{6}\left\langle \frac{R_s}{-\partial_\mu T\partial^\mu T}\right\rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\left\langle \frac{\rho + (\rho+p)sinh^{\mathbf{2}}(\alpha_T)}{-\partial_\mu T\partial^\mu T} + \epsilon\right\rangle}$$

This is the effective equation describing the velocity of the expansion of our domain-dependent scale factor. With the definitions as above we can write the cosmological balance equation as:

$$H_D := \frac{1}{s}\frac{\partial s}{\partial T_0}$$

$$\Omega_m := \frac{8\pi G_N}{3\,H_D^2}\left\langle \frac{\rho + (\rho+p)sinh^{\mathbf{2}}(\alpha_T)}{-\partial_\mu T\partial^\mu T}\right\rangle$$

$$\Omega_\epsilon := \frac{8\pi G_N}{3\,H_D^2}\,\langle\epsilon\rangle$$

$$\Omega_k := -\frac{\langle R\rangle}{6H_D^2}$$

$$\Omega_Q := -\frac{Q_D}{6H_D^2}$$

$$\Omega_k + \Omega_Q + \Omega_\epsilon + \Omega_m = 1$$

From the energy balance equation we see that the backreaction term Ω_Q opens the possibility of contributing to an effective cosmological constant term in this model, which is in principle close to Λ-CDM.

Raychaudhuri equation:

Combining the Hamiltonian constraint with the trace of the Einstein equations projected orthogonal to the time flow, one gets:

$$R_{\mu\nu}n^{\mu}n^{\nu} = T_{\mu\nu}h^{\mu\nu} - \frac{1}{2}T$$

Which is equivalent to:

$$-n^{\mu}\nabla_{\mu}\theta = 2\sigma^{\mathbf{2}} + \frac{1}{3}\theta^{\mathbf{2}} - \nabla^{\nu}(n^{\mu}\nabla_{\mu}n_{\nu}) + (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)n^{\mu}n^{\nu} \tag{4.7}$$

In our model the last term gives:

$$(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T)n^{\mu}n^{\nu} = T^{\text{fluid}}_{\mu\nu}n^{\mu}n^{\nu}+T^{\text{dust}}_{\mu\nu}n^{\mu}n^{\nu}+\frac{1}{2}T = \rho+sinh^2(\alpha_T)(\rho+p)+\epsilon(-\partial_{\mu}T\partial^{\mu}T)+\frac{1}{2}(3p-\rho-\epsilon)$$

Again we will use the commutator (4.2) and eqn. (4.3) to get:

$$\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = \frac{1}{3}\left\langle\frac{\partial_{\mu}T}{-\partial_{\mu}T\partial^{\mu}T}\,\partial^{\mu}\left(\frac{\theta}{(-\partial_{\mu}T\partial^{\mu}T)^{1/2}}\right)\right\rangle + \frac{1}{3}\left\langle\frac{\theta^2}{-\partial_{\mu}T\partial^{\mu}T}\right\rangle - \frac{3}{9}\left\langle\frac{\theta}{\sqrt{-\partial_{\mu}T\partial^{\mu}T}}\right\rangle^2 =$$

$$= -\frac{1}{3}\left\langle\frac{\partial_{\mu}T\,\partial^{\mu}\theta}{(-\partial_{\mu}T\partial^{\mu}T)^{3/2}}\right\rangle + \frac{1}{3}\left\langle\frac{\theta^2}{-\partial_{\mu}T\partial^{\mu}T}\right\rangle - \frac{2}{9}\left\langle\frac{\theta}{(-\partial_{\mu}T\partial^{\mu}T)^{1/2}}\right\rangle^2 - \frac{1}{6}\left\langle\frac{\partial_{\mu}T\partial^{\mu}(\partial_{\nu}T\partial^{\nu}T)}{(-\partial_{\mu}T\partial^{\mu}T)^{5/2}}\right\rangle$$

Rewriting :

$$\partial^{\mu}T\partial_{\mu}\theta = \nabla_T\theta = -(-\partial_{\mu}T\partial^{\mu}T)^{1/2}n^{\mu}\nabla_{\mu}\theta$$

We can insert (4.7) into the first term:

$$\boxed{\begin{aligned}\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = \frac{1}{3}\left\langle\frac{\theta^2}{-\partial_\mu T\partial^\mu T}\right\rangle - \frac{2}{9}\left\langle\frac{\theta}{(-\partial_\mu T\partial^\mu T)^{1/2}}\right\rangle^2 - \frac{1}{6}\left\langle\frac{\partial_\mu T\partial^\mu(\partial_\mu T\partial^\mu T)}{(-\partial_\mu T\partial^\mu T)^{5/2}}\right\rangle + \\ -\frac{1}{3}\left\{2\left\langle\frac{\sigma^2}{-\partial_\mu T\partial^\mu T}\right\rangle + \frac{1}{3}\left\langle\frac{\theta^2}{-\partial_\mu T\partial^\mu T}\right\rangle - \left\langle\frac{\nabla_\mu(n^\mu\nabla_\mu n_\nu)}{-\partial_\mu T\partial^\mu T}\right\rangle\right\} + \\ +\frac{8\pi G_N}{3}\left\{\left\langle\frac{(\rho+p)sinh^2(\alpha_T)+\rho}{(-\partial_\mu T\partial^\mu T)^{1/2}}\right\rangle + \frac{1}{2}\left\langle\frac{3p-\rho-\epsilon}{-\partial_\mu T\partial^\mu T}+2\epsilon\right\rangle\right\}\end{aligned}}$$

This is the effective Raychaudhuri equation for the domain in the case of the energy momentum tensor being that of an ideal fluid.

4.2.6 Evaluation in the ADM gauge

In this subsection we will evaluate the equations of 4.2.5 in the ADM gauge, which was introduced above.

The Hamiltonian constraint density reduces to:

$$\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 = -\frac{1}{6}\left\langle R_s N^2\right\rangle - \frac{1}{6}\left\{\frac{2}{3}\left(\left\langle N^2\theta^2\right\rangle - \left\langle N\,\theta\right\rangle^2\right) - 2\left\langle N^2\sigma^2\right\rangle\right\} +$$

$$+\frac{8\pi G_N}{3}\left\langle N^2\left(\rho+(\rho+p)sinh^{\mathbf{2}}(\alpha_T)\right)+\epsilon\right\rangle =$$

$$= -\frac{1}{6}\left\langle R_s N^2\right\rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\left\langle N^2\left(\rho+(\rho+p)sinh^{\mathbf{2}}(\alpha_T)\right)+\epsilon\right\rangle$$

And the Raychaudhuri equation to:

$$-\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = -\frac{2}{9}\left(\left\langle N^2\theta^2\right\rangle - \left\langle N\theta\right\rangle^2\right) + \frac{2}{3}\left\langle N^2\sigma^2\right\rangle - \frac{1}{3}\left\langle\theta\partial_0 N\right\rangle - \frac{1}{3}\left\langle N\,h^{ij}\nabla_i\nabla_j N\right\rangle +$$

$$+\frac{4\pi G_N}{3}\left\langle 2\,N^2((\rho+p)sinh^2(\alpha_T)) + N^2(\rho+3p)+2\epsilon\right\rangle =$$

$$= -\frac{1}{3}Q_D - \frac{1}{3}\left\langle\theta\partial_0 N\right\rangle - \frac{1}{3}\left\langle N\,h^{ij}\nabla_i\nabla_j N\right\rangle + \frac{4\pi G_N}{3}\left\langle 2\,N^2((\rho+p)sinh^2(\alpha_T)) + N^2(\rho+3p)+2\epsilon\right\rangle$$

Setting $N=1$ we obtain:

The Hamilton constraint density:

$$\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 = -\frac{1}{6}\left\langle R_s\right\rangle - \frac{1}{6}\left\{\frac{2}{3}\left(\left\langle\theta^2\right\rangle - \left\langle\theta\right\rangle^2\right) - 2\left\langle\sigma^2\right\rangle\right\} +$$

$$+\frac{8\pi G_N}{3}\left\langle(\rho + (\rho + p)sinh^{\mathbf{2}}(\alpha_T)) + \epsilon\right\rangle =$$

$$= -\frac{1}{6}\left\langle R_s\right\rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\left\langle\rho + (\rho + p)sinh^{\mathbf{2}}(\alpha_T) + \epsilon\right\rangle$$

And the Raychaudhuri equation:

$$-\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = -\frac{2}{9}\left(\left\langle\theta^2\right\rangle - \left\langle\theta\right\rangle^2\right) + \frac{2}{3}\left\langle\sigma^2\right\rangle +$$

$$+\frac{8\pi G_N}{3}\left\langle 2\left((\rho + p)sinh^2(\alpha_T)\right) + (\rho + 3p) + 2\epsilon\right\rangle =$$

$$= \frac{8\pi G_N}{3}\left\langle 2\left((\rho + p)sinh^2(\alpha_T)\right) + (\rho + 3p) + 2\epsilon\right\rangle - \frac{1}{3}Q_D$$

Observations:

- Both the Hamiltonian constraint and the Raychaudhuri equation contain the energy density of the coordinate dust. In the second equation it contributes negatively to the acceleration of the scale factor.

- The fact that generically the flow of the coordinate dust is not parallel to the flow of the fluid which is averaged, is manifested in the $sinh^2(\alpha_T)$ term. This as well contributes negatively to the acceleration of the scale factor s.

- After the gauge choice the spatial integral is an ordinary 3-dimensional integral.

- The backreaction Q_D has the potential to contribute to the acceleration. This has been addressed in the literature [4, 10, 19, 14] and will be discussed in more detail later on.

4.3 Summary Chapter 3

A manifestly gauge invariant averaging formalism for finite volumes was presented in this chapter. At this point the problem of the gauge dependence of the backreaction was solved. The model used just required the assumption of a non-vanishing energy density. In this case a dust energy momentum tensor can be separated from the total energy density. The coordinate fields of the dust were used as a reference frame to perform averaging.

The so derived equations for the average of an arbitrary energy momentum tensor were evaluated in the ADM gauge, which corresponds to choosing the rest frame of the dust. Furthermore as an example the equation is evaluated for an energy momentum tensor of an ideal fluid. The resulting equation was also studied in the ADM gauge. An interesting phenomenon was observed. It appears that "tilt" effects (i.e. non-co-linearity of the dust flow and the fluid's flow) contribute negatively to the acceleration pf the domain dependent scale factor.

The derived equations in the ADM gauge were compared to the Buchert equations. So we are able to interpret the Buchert equations in the gauge independent framework. It turned out that they describe the averaged behavior of a pressure-free fluid, using this fluid itself as a reference and choosing it's rest frame. It turns out that in this special case gauge fixing and averaging commute.

By deriving gauge independent equations for the cosmological backreaction, we proved that it is in principle an observable. Now we are ready to study cosmological data and can trust the derived equations to correctly describe the observations. Of course the final goal is to test the results experimentally, since a theory which has no physical evidence might be not even wrong, but is irrelevant.

Chapter 5

Experimental Opportunities

After developing a gauge invariant formalism and showing that now the quantities are observable and a gauge can be safely chosen, the question which is really interesting to address is, whether we can observe any of the effects due to the inhomogeneities.

5.1 Redshifts and introduction to observational methods

One of the striking realizations while thinking about relativity is that we can not experience volumes as we think of them i.e. space-like volumes on cosmological scales. Since we have to wait until light arrives at our position from every point of the observed volume we would have to wait for thousands of years to get relevant information about the cosmos. However for relatively close events we can neglect the effect of the light-cone and assume that the observed light-like volumes are close to they space-like version. This will be discussed in more detail below. In this section we will discuss how the light propagates in a dynamical FRW space-time, following the pedagogical introduction of [13]. Since in the previous sections we have seen that there are Friedmann type equations governing the domain dynamics the FRW light propagation is a sensible approximation for observational studies.

5.1.1 Light Geodesics:

Light travels on geodesics i.e. it defines the shortest distance connecting two points on the space-time manifold. In terms of the proper time it means that:

$$ds = 0$$

The space-time interval along the trajectory vanishes. This statement holds in any local inertial frame and since ds is invariant it should be true along the light geodesics in any space-time. In most cases radial propagation is studied and hence it is convenient to introduce the so called conformal time. Defined as:

$$\eta := \int \frac{dt}{a(t)}$$

With this coordinate change the FRW metric takes the form:

$$ds^2 = a^2(\eta)\,(d\eta^2 - d\chi^2 - \Phi^2(\chi)\,(d\theta^2 + sin^2\theta\,d\phi^2))$$

with:

$$\Phi^2(\chi) = \begin{cases} sinh^2\chi & k = -1 \\ \chi^2 & k = 0 \\ sin^2\chi & k = 1 \end{cases}$$

The trajectory of constant ϕ, θ is a geodesic. The radial coordinate along the trajectory is then entirely determined by $d\eta^2 = d\chi^2$ so the radial light geodesic is described by :

$$\chi(\eta) = \pm\eta + const$$

5.1.2 Redshifts:

The space dynamics leads to a change of the wavelength of a photon. For example in an expanding space the photon experiences a reddening. We consider a source with comoving coordinate χ_{em}, emitting the signal at η_{em} of conformal duration $\Delta\eta$ therefore the trajectory of the signal is $\chi(\eta) = \chi_{em} - (\eta - \eta_{em})$. It reaches the detector at $\chi_{obs} = 0$ at time $\eta_{obs} = \eta_{em} + \chi_{em}$. The conformal duration of the signal is the same at the point of emission and detection, but the physical length differs.

$$\Delta t_{em} = a(\eta_{em})\,\Delta\eta$$

$$\Delta t_{obs} = a(\eta_{obs})\Delta\eta$$

If light is emitted with the wavelength $\lambda_{em} = \Delta t_{em}$ and it is observed with $\lambda_{obs} = \Delta t_{obs}$ such that:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(\eta_{obs})}{a(\eta_{em})}$$

We observe that the wavelength of the light changed proportional to the scale factor and its frequency proportional to the inverse of the scale factor. This effect can be viewed at distances much smaller than the Hubble scale as a Doppler shift, resulting from the velocity of the source moving away from the ob sever due to the expansion of space. So if the distance between the source and observer is $\Delta l \ll H^{-1}$, then there is a local inertial frame in which space-time can be considered quasi flat. Now according to the Hubble law, the relative recessional speed of the objects is $v = H(t)\Delta l \ll 1$. Hence we can introduce the notion of the Doppler shift by:

$$\Delta\omega := \omega(t_1) - \omega(t_2) \backsimeq \omega(t_1)H(t_1)\Delta l$$

The time delay between measurements is $\Delta t = \Delta l$ and we can rewrite the equation in a differential form:

$$\dot{\omega} = -H(t)\,\omega = -\frac{\dot{a}(t)}{a(t)}\,\omega$$

This has the solution:

$$\omega \backsim \frac{1}{a}$$

The above derivation has been performed in a local inertial frame, but it can also be applied piecewise to any space-time. The difference is that the interpretation of the red-shift as Doppler shift becomes ill defined.

5.1.3 The redshift parameter

The redshift parameter is defined as the fractional shift in wavelength of the light emitted by a distant source at time t_{em} and observed on Earth today:

$$1 + z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$

Therefore:

$$1 + z = \frac{a_0}{a(t_{em})} \tag{5.1}$$

Where a_0 is the scale factor at present time. The redshift can be used instead of time to parametrize the history of the universe. A value of z corresponds to a time when the universe was $1 + z$ times smaller than now.

For later purposes a differential equation between time and the redshift will be very useful. Differentiating (5.1) one obtains:

$$dz = -\frac{a_0}{a^2(t)}\dot{a}(t)\,dt = -(1 + z)H(t)\,dt \tag{5.2}$$

And the integral relation is:

$$t = \int_z^\infty \frac{dz}{H(z)(1+z)}$$

A constant of integration has been chosen in such a way that $z \to \infty$ corresponds to the initial moment of time $t = 0$. Obviously we need the red shift history of the scale factor to perform this integration and this is the big problem in the end of the day.

However, knowing the redshift from a distant source one can determine its separation from us. The comoving distance to a source that emitted a light at time t_{em} which arrives today is:

$$\chi = \eta_0 - \eta_{em} = \int_{t_{em}}^{t_0} \frac{dt}{a(t)}$$

Substituting $a(t) = a_0/(1+z)$ and (5.2) one finds:

$$\chi(z) = \frac{1}{a_0} \int_0^z \frac{dz}{H(z)}$$

So observing characteristic lines from spectral transitions say in Hydrogen atoms one can measure the redshift of a distant star with great accuracy. Of course assuming the local physical laws governing the micro physics of a distant star are the same as on Earth. So we assume a universality of the laws of physics. On the other hand if would let this assumption go we could not discuss anything in cosmology and therefore it is the only chance we have to assume this universality.

5.1.4 Luminosity-redshift relation

An other important method to study the expansion history of the universe is the Luminosity-redshift relation. Consider a source of radiation of total luminosity (emitted energy per unit time) L located at comoving distance χ_{em} from us. The total energy released by the source at time t_{em} within a conformal time interval $\Delta\eta$ is:

$$\Delta E_{em} = L \Delta t_{em}(\Delta\eta) = L\, a(t_{em}) \Delta\eta$$

All the photons are located in a shell of constant conformal width $\Delta\eta = \Delta\chi$. The radius of the shell grows with time and the photons' frequencies are reddened. Therefore when the photons reach an observer at a time t_0, the total energy in the shell is:

$$\Delta E_{obs} = \Delta E_{em} \frac{a(t_{em})}{a_0} = L \frac{a^2(t_{em})}{a_0} \Delta\eta$$

The area of the surface of the shell is at this moment:

$$S_{sh}(t_0) = 4\pi a^{\mathbf{2}}{}_0 \Phi^{\mathbf{2}}(\chi_{em})$$

(With the Φ defined above.) And the physical width:

$$\Delta l_{sh} = a_0 \Delta\chi = a_0 \Delta\eta$$

The shell passes the observer's position over a time interval $\Delta t_{sh} = a_0\Delta\eta$. And the bolometric flux energy (energy per unit area per unit time) is:

$$F := \frac{\Delta E_{obs}}{S_{sh}(t_0)\Delta t_{sh}} = \frac{L}{4\pi\Phi^{\mathbf{2}}(\chi_{em})} \frac{a^2(t_{em})}{a_0^4}$$

And as a function of redshift:

$$F := \frac{\Delta E_{obs}}{S_{sh}(t_0)\Delta t_{sh}} = \frac{L}{4\pi a_0^2 \Phi^2(\chi_{em}(z))(1+z)^2}$$

Where $d_L(z) = \Phi(\chi_{em}(z))(1+z)$ is the luminosity distance. Instead of F in astronomy the apparent bolometric magnitude is often used. It is defined as:

$$m_{bol}(z) = -2.5\, log_{10} F = 5\, log_{10}(1+z) + 5\, log_{10}(\Phi(\chi_{em}(z))) + const$$

For the $z \ll 1$ one can expand the expression and finds that irrespective of the curvature and matter composition of the universe the result is:

$$m_{bol}(z) = 5\, log_{10}(z) + \frac{2.5}{ln(10)}(1 - q_0) + O(z^{\mathbf{2}}) + const \tag{5.3}$$

With:

$$q_0 := -\left(\frac{\ddot{a}}{aH^{\mathbf{2}}}\right)_0$$

Therefore the second derivative of the scale factor becomes experimentally assessable. The measurements of Type IA supernovae (stellar events with well known luminosity) have shown that the universe is accelerating and expanding. This indicates a domination of the cosmological constant in the Λ-CDM model, discussed in the first chapter. To obtain more precise results especially distant sources are of interest and there the above expansion becomes meaningless as soon as one approaches redshifts of order one. This problem will be also discussed later in greater detail.

5.2 Model dependence of our interpretations

The cosmological data are interpreted assuming the FRW model and this means isotropy. For instance if we refer to standard candles we mean that the collapses of binary star systems have been well studied at low red shifts and a certain averaging has been performed to obtain the "standard" value. At this point we should mention that those red shift distances are below the isotropy scale, which is estimated to be about $\simeq$ 100 Mpc. Therefore, this sample might be atypical. We need to identify the valid isotropy scale and reconsider the calibration.

Not only the calibration suffers from the model dependence, the observation of the distant supernovae is the most relevant for the detection of the accelerated expansion. To interpret the red shift luminosity relations of distant supernovae at red shifts larger than one, we need to assume a certain expansion history in the FRW model. The luminosity distance is related to the comoving distance $r(z)$ as $d_L(z) = a_0 r(z)(1+z)$. Let us rewrite the first Friedmann equation, taking into account the different scaling properties for different equations of state, as:

$$\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_\Lambda + \Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_k \left(\frac{a_0}{a}\right)^2}$$

Hence we can write:

$$t(z) = \int_0^{\frac{1}{1+z}} \frac{1}{H_0} \left[\frac{da}{a\sqrt{\Omega_\Lambda + \Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_k \left(\frac{a_0}{a}\right)^2}} \right]$$

To compute $r(z)$ we have to pick the parameters $\{\Omega_\Lambda, \Omega_m, \Omega_R, \Omega_k\}$ as defined above. Define first:

$$F_k(r_1) = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} r_1 & k=0 \\ \arcsin(r_1) & k=1 \\ \mathrm{arcsh}(r_1) & k=-1 \end{cases}$$

From the light propagation we know that $F_k(r) = t(z) + c$, we solve for r, setting $x = \frac{a}{a_0}$:

$$r(z) = F_k^{-1}\left[\frac{1}{a_0 H_0}\int_{\frac{1}{1+z}}^{1}\frac{dx}{x\left(\Omega_\Lambda + \Omega_k x^{-2} + \Omega_m x^{-3} + \Omega_R x^{-4}\right)^{\frac{1}{2}}}\right]$$

So, we can predict the bolometric flux energy of a distant source according to the energy "budget" of the universe:

$$F := \frac{L}{4\pi a_0^2 d_L^2(z)}$$

The observations of the apparent bolometric flux energy of standard candles at different red-shifts and the theoretical predictions allow to restrict the parameter space of the omegas. This observations give rise to the blue contures in the graph below.

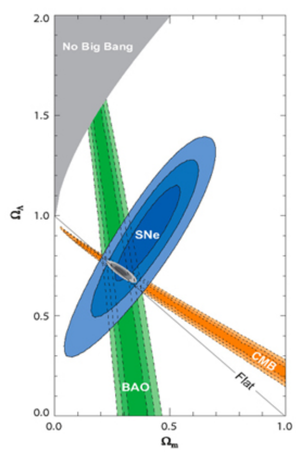

Figure 5.1: Observations of the CMB, BAO and SNe

The blue and green areas are deduced from experiments independent from the supernovae. The CMB inhomogeneities and typical distances in the distribution of the galaxies are thought to be manifestations of the same phenomenon, the barion acoustic oscillations. The BAOs have their name from the similarity to sound waves. In the primordial plasma the barions felt essentially two forces, the gravitational force pointing inwards a region of space and the photon pressure pointing outwards of this region. The oscillatory counter play of these forces caused wavelike propagations in the plasma. Furthermore the matter, which was not interacting strongly with light and did not feel the photon pressure, collapsed faster in the regions and enhanced the gravitational attraction.

At the time of recombination of electrons and protons the photon pressure vanished and the oscillations stopped, leaving behind a "dark matter" core and a barionic shell. This process leads to a very specific shape of the power spectrum of the CMB. From this spectrum it can be concluded, that the space was very close to flat and furthermore it gives an estimate of the contribution of non relativistic matter to the energy budget, also under the FRW assumption. The study of the CMB anisotropies gives rise to the yellow band above. It is assumed that these early inhomogeneities have provided seeds for the structure formation in the course of time and therefore the specific distance (radius of the barionic shell) is still imprinted in the universe. This distance can be estimated studying in a statistical way the intergalactic separations. The results of this study are represented by the green band above. Also the study of galactic distances, sometimes called "standard rulers" , is a process where for the estimation of large distances an FRW model has to be assumed.

To summarize, we have seen what important role the assumption of isotropy has for the interpretation of cosmological observations. Now that we have developed a method to estimate corrections to the homogeneous model and the possibility to determine the isotropy scale we have to reconsider the observations and their interpretation. Furthermore the Ω_Q parameter can contribute to the total budget equation and the above graph might drastically change after taking into account the backreaction. We have seen that studying inhomogeneities opens a wide field of research within conservative physics.

5.3 Effective theory approach

In the course of chapter 4 it has been sown that an averaging can be performed in a gauge independent manner. This is a result which could be expected, since it means that averaged quantities are physical observables and can be measured. The daily experience suggests exactly the same result. So after formally showing this statement it is important to underline the fact that now such quantities as $\{s, \dot{s}\,, \ddot{s}\,, \langle\epsilon\rangle\,, Q_D\}$ can be viewed as (gauge independent) fields governed by effective equations (5.4) and therefore must be observable. To test the theory, especially the dynamics predicted by the effective equations, experiments would be desired. In the following sections we will consider possible experimental opportunities and try to find those which are most physically significant. Note that in [4] the above fields are also viewed as physical fields of an effective theory, even though they were not derived in a gauge independent way. With our results [4] has a better foundation concerning the gauge invariance.

5.4 Averaging of space-like domains

Since it has been observed that in the universe at the present time the contribution from radiation is only very small, it is reasonable to view all the luminescent matter as a pressure free fluid and hence dust. Since we as observers sitting in the solar system, which also can be viewed as a dust grain, are geodesic observers we can choose the dust coordinate system as a reasonable system to perform our measurements in. So in this case we are averaging the dust filled universe using this dust itself as a reference frame.

Now domains over which the averaging will be performed have to be defined. This can be done by choosing angles under which the observations will take place furthermore a splitting in red shifts has to be performed to distinguish distances, since so far we deal with averages over space-like domains and therefore we have to average over domains of nearly constant red shift.

The combination of the observations of the Microwave background with the Barion Acoustic oscillations and the Supernovae data suggest that the total curvature of our visible universe is flat, or very close to flat[12]. We can conjecture that if the scale of averaging allows it, the averaged spatial curvature of the domain in the above equations can be set to zero. In the following work we will also suggest a test for this hypothesis using a light cone construction. Here setting $\langle R \rangle_D = 0$ one finds:

$$Q_D = -6 \left(\frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 + \frac{8\pi G_N}{3} \langle \epsilon \rangle \tag{5.4}$$

$$Q_D = 3 \left(\frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} \right) + 16\pi G_N \langle \epsilon \rangle$$

The observation of the luminescent matter gives now an estimate for the energy density. A factor may be needed to get close to the real value of the energy density since not all matter emits radiation. So this correction to “dark” matter might become unavoidable. The expansion rate in the domain under observation can be computed from the observations of the redshifts. For redshifts sufficiently smaller one, one can use the measurement of apparent bolometric flux (5.3) to estimate the acceleration of the domain. It is important to notice that this procedure is restricted to small redshifts.

So in principle it is possible to measure the domain dependent parameters on the right hand sides of the equations (5.4). Interesting questions one could ask would be:

1. Do these equations give the same result for Q_D, the backreaction parameter?
2. How does this results depend on the choice of the domain size and position?
3. Measuring the averaged values at different red shifts, how big has the domain size has to be to get the same averaged values in all domains? How big is the isotropy scale?
4. How does the isotropy scale depend on the red shift?

If the results of both equations are compatible and disappear in the limit of observing large domains, we could conclude that indeed there is a geometrical, domain dependent quantity which is a possible source for the observed dynamics in the universe. Furthermore if we observe that the equations are closed and Q_D is a physical quantity we can ask the question, whether there is a scale where Q_D is the same in all domains and whether there is a scale where Q_D vanishes. This we could define as the isotropy scale, which would be a mathematically exact statement. Unfortunately the above results are restricted to small z as will be discussed below.

5.5 The problem of space-like domains

As mentioned above the observation of a space-like domain on cosmological scales is impossible, so one always has to deal with approximations. At redshifts much smaller than one, the observation of a light-like domain gives a result which is close to the result one would obtain from observing the real space-like domain, since the effect of the light-come and the expansion of the space during the signal transmission are relatively small. At large redshifts and big domain sizes though the situation is different. For instance as pointed out in [10] the distribution of stars in a domain approaches a stable, domain independent value at domain sizes about 100 Mpc, this is just two orders of magnitude smaller than the Hubble patch, i.e. the observable universe and therefore the light-cone and expansion history effect have to be taken into account if one actually wants to make isotropy statements. At this point it is important to mention that our results strengthen the theoretical foundation [10] is based on.

Chapter 6

Conclusion:

In this work the technique of relational coordinatization was used to perform gauge independent averaging of a deformed FRW space-time with inhomogeneous matter distribution. The added physical coordinate system is a pressure-less fluid -the dust- , which does not disturb the FRW geometry and is hence a compatible "laboratory". A motivation was given to study the traces of Einstein's equations with our averaging framework. By performing the averaging two equations were derived. The averaged version of the Hamiltonian constraint and the Raychaudhuri equation. Both contain a term, which appears as a result of averaging and was called backreaction in [3]. We have hence solved the dispute about gauge dependence of the backreaction, since our formalism is not perturbative and manifestly gauge independent.

The Buchert equations were rediscovered as a special case of averaged equations of a pressure less fluid which are evaluated in the fluid's rest frame. Therefore we have shown that in this case gauge fixing and averaging commute and hence the proceeding in [3] is legitimate, even if this was not obvious so far, as pointed out in [8]. Our work shows the correctness of Buchert's equations (under the described conditions) and gives theoretical back up to the papers, which made use of them, as [10] ,[4] ,[18] and [14]. Of course it has to be ensured that the prerequisites of a single ideal fluid are fulfilled in this cases. Otherwise one has to make use of our, more general approach. These matters are, which have to be investigated.

It is worth noticing that the dust system corresponds to freely falling geodesic observer and such an observer has a privileged position from our point of view, since it represents to a good approximation our position on earth. We observed that the property of the observer to follow a geodesic is a gauge independent quality, as was also shown in [11] in a perturbative method (up to second order). This is a hint that our general result is correct.

After showing the gauge independence of the backreaction, it is of greatest interest to develop experimental techniques to study its effects on the dynamics of space-time. At this point it is important to underline, that the backreaction as it appears as a pure averaging effect, is the correct measure of inhomogeneity and worth studying to identify the isotropy-scale of the universe. Some observational methods have been proposed in chapter 5 . Also it was pointed out that we will have to expect corrections to our space-like averaging procedure due to light-cone effects on large scales. A technique to address those has to be developed.

The question whether backreaction can account for the accelerated expansion of our universe can not yet be answered. It has been shown in toy-models in [14], that the backreaction has the potential to induce an apparent acceleration. A possible mechanism as described in [14] is the following: Suppose an expanding universe with one over-dense region A with volume $V(A)$ and one under-dense region B with volume $V(B)$. The average expansion velocity would be roughly $\dot{a} = V(A+B)^{-1}\left(\dot{a}_A V(A) + \dot{a}_B V(B)\right)$. Now due to attractive forces of matter the expansion of the over-dense region A is slowed down. So, since B expands faster than A its volume will become larger than the volume of A and also its relative contribution to the average expansion rate $\dot{a}$ will grow. This effect can result in an acceleration of the averaged scale factor, with gravity acting purely attractive. Unfortunately, so far no realistic models for the universe have been developed to test this hypothesis in a simulation. This will be a big challenge for computational physics in the future.

The bottom line is that backreaction remains a hot topic and its effects have to be studied theoretically, numerically and of course with the help of observations. This work has proven that backreaction can be studied in a gauge independent way and hence is an observable from the theoretical point of view. The next step, regarding the progress of this work, will be to apply our averaging functional to observational data and investigate whether the effects of backreaction can be detected in nature.

Appendix A

The ADM formalism

The Hamiltonian approach to general relativity was developed by Richard Arnowitt, Staneley Deser and Charles Misner and is called the ADM formalism. The crucial difference to the obviously covaraint Lagrangian method and the great difficulty is the necessity to define a preffered time direction. This choice is not unique and we will see its consequences at the end of our discussion. For now we assume that the manifold is physical in the sense, that it has topology $\mathbb{R} \times \Sigma$. So a coordinate function t can be defined s.t. it determines the Σ slices as the sets on which t takes a constant value. The "time flow" corresponding to t is the vector field which obeys the relation:

$$t^{\mu}\nabla_{\mu}t = \nabla_{\vec{t}}t \; = 1$$

At this point t^{μ} and t can not be interpreted in physical terms, since the metric is unknown. It is the dynamical field the equations of motion have to be solved for.

It is convenient to split the "time" vectors into its parts normal to the hypersurfaces and its projections on them. Therefore we define the lapse function and shift vector respectively:

$$N = -g_{\mu\nu}t^{\mu}n^{\nu}$$

$$N^{\mu} = h^{\nu}_{\ \mu}t^{\mu}$$

Here n^ν is the unit normal to the hypersurfaces of constant t and $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ the projector orthogonal to n^ν. The interpretation of the introduced notions is as following; Consider an infinitesimal step from one leaf of the foliation to the next. In this case N measures the (physical) proper time τ that lapses during this step. And N^a measures the shift one experiences inside the hypersurface flowing the flow of the coordinate time t .

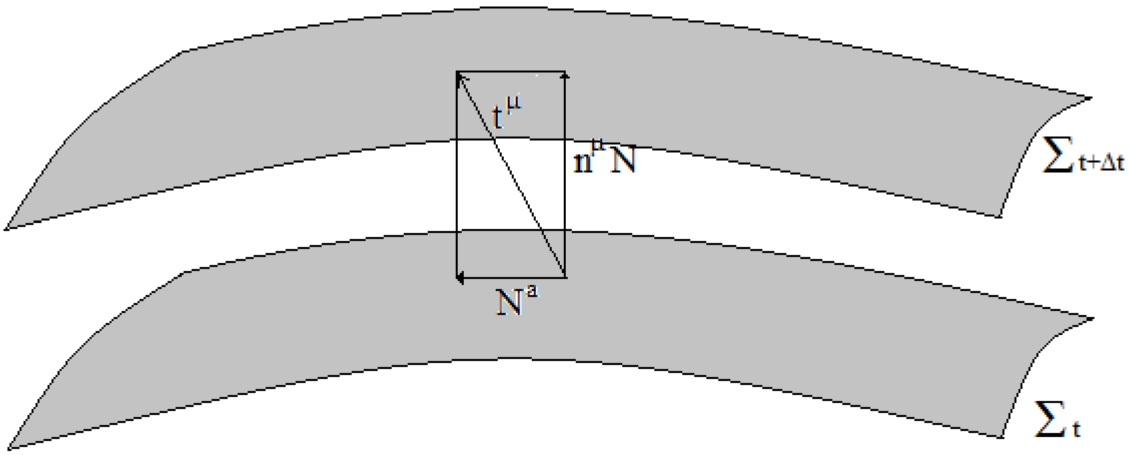

Figure A.1: Lapse and Shift

The leafs of the foliation are submanifolds with an induced metric, which can be derived from $h_{\mu\nu}$ by pulling it back under the map $X^\mu : \Sigma \to \mathcal{M}$:

$$h_{ab} := \frac{\partial X^\mu}{\partial x^a}\frac{\partial X^\nu}{\partial x^b} h_{\mu\nu} = X^\mu_a X^\nu_b \, h_{\mu\nu}$$

Now the shift can be rewritten and solved for n^ν:

$$N^a = h^\nu{}_\mu X^a_\nu t^\mu = g^\nu{}_\mu X^a_\nu t^\mu - n^\nu n_\mu X^a_\nu t^\mu = t^a - n^a \, N \Rightarrow n^a = \frac{1}{N}\,(t^a - N^a) \tag{A.1}$$

With the new definitions the metric can be written as:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(-N^2 + N_a N^a\right) dt^2 + 2N_a dx^a\, dt + h_{ab}dx^a dx^b \tag{A.2}$$

Note that from now on we will omit the explicit pull back notation, but it is always implicitly meant when objects with 3 indices multiply 4 index objects (so we do not distinguish Latin and Greek indices in the following).

Viewing $\{h_{ab},\, N,\, N_a\}$ as fields we see that they contain the same information as the metric, which is usually solved for. The reason is that $h_{ab}h^{bd} = \delta_a^d$ and $h^{ab}\nabla_b t = 0$ allows knowing h_{ab} to solve for h^{ab} and hence compute $N^a = h^{ab}N_b$. We observe that indeed the considered fields allow to compute (A.2). Note further that $\sqrt{-g} = N\sqrt{h}$.

Having defined those notions, we can proceed towards the Hamiltonian. The first step would be to express the Lagrange function in the new variables, (we neglect the boundary terms in the Lagrangian, since this would only complicate the discussion), hence we regard the Hilbert action:

$$S = \int d^4x\, \mathcal{L}_G \quad \text{with} \quad \mathcal{L}_G = \sqrt{-g}\, R$$

.

To express the action in the new variables a de tour is needed to connect the curvature of the 3-surfaces to the total curvature. To calculate this relation a Lemma is handy:

Lemma:

Let $\{\mathcal{M},\, g_{\mu\nu}\}$ be a space-time and let Σ be a smooth spacelike hypersurface in $\mathcal{M}$. Let h_{ab} denote the induced metric on Σ, and let D_a denote the derivative operator associated with h_{ab}. Then D_a is given by the displayed formula, where ∇_μ is associated with $g_{\mu\nu}$:

$$D_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = h^{a_1}_{\mu_1} \ldots h^{\nu_l}_{b_l} h_c^{\lambda} \nabla_\lambda T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l}$$

Or using the implicit pull-back notation:

$$D_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = h^{a_1}_{d_1} \ldots h^{e_l}_{b_l} h_c^{f} \nabla_f T^{d_1\ldots d_k}{}_{e_1\ldots e_l}$$

Proof:

It is straight forward to verify that D_a satisfies Linearity, Leibniz rule, Commutativity with contractions, consistency with tangents and torsion freedom. Remains to show the metric compatibility. Check, using $\nabla_\mu g_{\nu\lambda} = 0$ and $h_{a\mu} n^\mu = 0$ the compatibility:

$$D_a h_{bc} = h_a^{\mu} h_b^{\ \nu} h_c^{\ \lambda} \nabla_\mu \left(g_{\nu\lambda} + n_\nu n_\lambda \right) = 0$$

So, D_a is the unique derivative associated with h_{ab}.

Coming to the task of relating the 3 curvature to the total curvature, write the 3 curvature with the help of the new derivative operator, with ω being a dual vector on Σ, as:

$$^{(3)}R_{abc}{}^{d} \omega_d = D_a D_b \omega_c - D_b D_a \omega_c \tag{A.3}$$

Furthermore we have using the Lemma and the fact that $h_a^{\ b} h_c^{\ d} \nabla_b h_d^{\ e} = h_a^{\ b} h_c^{\ d} \nabla_b \left(g_d^{\ e} + n_d n^e \right) = h_a^{\ b} h_c^{\ d} \nabla_b \left(n_d \right) n^e = K_{ac} n^e$:

$$D_a D_b \omega_c = D_a \left(h_b^{\ d} h_c^{\ e} \nabla_d \omega_e \right) = h_a^{\ f} h_b^{\ g} h_c^{\ k} \nabla_f \left(h_g^{\ d} h_k^{\ e} \nabla_d \omega_e \right) = \tag{A.4}$$

$$= h_a^{\ f} h_b^{\ d} h_c^{\ e} \nabla_f \nabla_d \omega_e + h_c^{\ e} K_{ab} n^d \nabla_d \omega_e + h_b^{\ d} K_{ac} n^e \nabla_d \omega_e$$

Noting that $h_b^{\ d} n^e \nabla_d \omega_e = h_b^{\ d} \nabla_d \left(\omega_e n^e \right) - \omega_e h_b^{\ d} \nabla_d n^e = -\omega_e h_b^{\ d} \nabla_d n^e = -K_b^{\ e} \omega_e$ we insert (A.4) in (A.3) and rewrite simplifying terms:

$$^{(3)}R_{abc}{}^{d} = h_a^{\ f} h_b^{\ g} h_c^{\ k} h_j^{\ d} R_{fgh}{}^{j} - K_{ac} K_b^{\ d} + K_{bc} K_a^{\ d}$$

Using this result we compute the following quantity:

$$2G_{ab} n^a n^b =^{(4)} R + 2R_{ac} n^a n^c = R_{abcd} \left(g^{ac} + n^a n^c \right) \left(g^{bd} + n^b n^d \right) = R_{abcd} h^{ac} h^{bd} = \left(K_a^{\ a} \right)^2 - K_b^{\ a} K_a^{\ b} + ^{(3)}R$$

Additionally from the definition of the Riemann tensor we have:

$$R_{ab}n^a n^b = R_{acb}{}^c n^a n^b = -n^a \left(\nabla_a \nabla_c - \nabla_c \nabla_a\right) n^c = \left(\nabla_a n^a\right)\left(\nabla_c n^c\right) - \left(\nabla_c n^a\right)\left(\nabla_a n^c\right) +$$

$$-\nabla_a \left(n^a \nabla_c n^c\right) + \nabla_c \left(n^a \nabla_a n^c\right) = \left(K_a{}^a\right)^2 - K_{ac}K^{ac} - \nabla_a \left(n^a \nabla_c n^c\right) + \nabla_c \left(n^a \nabla_a n^c\right)$$

Note that the last two terms are divergences and hence can be neglected under an integral. For later purpose we will need the local expression and those therms can not be dropped any longer. We rewrite the equation in terms of the local expansion rate and the shear (introduced in chapter 1) for the derivations in the next section:

$$R_{ab}n^a n^b = 2\sigma^2 + \frac{1}{3}\theta^2 + n^a \nabla_a \theta - \nabla_c \left(n^a \nabla_a n^c\right) \tag{A.5}$$

Having developed the above formulae we can express the total scalar curvature, defining $K := \left(K_a{}^a\right)$, as:

$${}^{(4)}R = 2\left(G_{ab}n^a n^b - R_{ab}n^a n^b\right) =^{(3)} R + K_{ab}K^{ab} - K^2$$

The Hilbert action reads in terms of this:

$$S_G = \int d^4x\, \sqrt{h}\, N \left\{{}^{(3)}R + K_{ab}K^{ab} - K^2\right\} \tag{A.6}$$

Recalling the definition of the extrinsic curvature in the first chapter we rewrite it in terms of the "time" derivative of the metric using A.1:

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab} = \frac{1}{2}\left\{n^c \nabla_c h_{ab} + h_{ac}\nabla_b n^c + h_{cb}\nabla_a n^c\right\} = \frac{1}{2N}\left\{N\, n^c \nabla_c h_{ab} + h_{ac}\nabla_b \left(N\, n^c\right) + h_{cb}\nabla_a \left(N\, n^c\right)\right\} =$$

$$= \frac{1}{2} N^{-1} h_a^{\ c} h_b^{\ d} \left\{ \mathcal{L}_t h_{cd} - \mathcal{L}_N h_{cd} \right\} = \frac{1}{2} N^{-1} \left\{ \dot{h}_{ab} - D_a N_b - D_b N_a \right\}$$

Substituting this form of the extrinsic curvature the in (A.6) we obtain the Gravitational action in the ADM form. First we calculate the conjugate momenta:

$$\pi^{ab} = \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{ab}} = \sqrt{h} \left(K^{ab} - K\, h^{ab} \right)$$

Note that the conjugate momenta of N and N^a are zero, this already indicates that those are not true dynamical variables.

Now we are ready to compute the Hamiltonian using the Legendre transformation:

$$H_G = \pi^{ab} \dot{h}_{ab} - L_G =$$

$$= \sqrt{h} \left\{ N \left[-^{(3)}R + \frac{1}{h} \pi^{ab} \pi_{ab} - \frac{1}{2h} \pi^2 \right] - 2 N_b \left[D_a \left(\sqrt{h}^{-1} \pi^{ab} \right) \right] + 2 D_a \left(\sqrt{h}^{-1} N_b \pi^{ab} \right) \right\}$$

The last term is a divergence and will be therefore dropped.

Interesting results can be obtained by varying the action w.r.t N and N_a, this gives the constraint equations:

$$-^{(3)}R + \frac{1}{h} \pi^{ab} \pi_{ab} - \frac{1}{2h} \pi^2 = 0 \tag{A.7}$$

$$D_a \left(\sqrt{h}^{-1} \pi^{ab} \right) = 0 \tag{A.8}$$

Those are exactly the Hamiltonian and Diffeomorphism constraints discussed in the first chapter. Here we see explicitly that the canonical Hamiltonian is a linear combination of the constraints and therefore it vanishes. The Hamiltonian constraint density (A.7) expresses the freedom of choice of the time slicing and generates an infinitesimal gauge transformation which connects one slicing to an other.

The Hamilton equations are equivalent to the Einstein vacuum equations, here again divergence terms have been dropped and (A.8) has been used:

$$\dot{h}_{ab} = \frac{\delta H_G}{\delta \pi^{ab}} = 2\sqrt{h}^{-1} N \left(\pi_{ab} - \frac{1}{2} h_{ab} \pi \right) + 2D_{(a}N_{b)}$$

$$\dot{\pi}^{ab} = -\frac{\delta H_G}{\delta h_{ab}} = -N\sqrt{h} \left({}^{(3)}R^{ab} - \frac{1}{2}{}^{(3)}R\, h^{ab} \right) + \frac{1}{2} N \sqrt{h}^{-1} h^{ab} \left(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2 \right) +$$

$$-2N\sqrt{h}^{-1} \left(\pi^{ac}\pi_c^{\ b} - \frac{1}{2}\pi_{ab}\pi \right) + \sqrt{h} \left(D^a D^b N - h^{ab} D^c D_c N \right) + \sqrt{h} D_c \left(\sqrt{h}^{-1} N^c \pi^{ab} \right) - 2\pi^{c(a} D_c N^{b)}$$

These equations have been derived by Arnowitt, Deser and Misner in 1962. The astonishing observation is that even though these equations are equivalent to Einstein's equations, the Hamiltonian vanishes and can not generate true time evolution. This situation seems to be similar to Maxwell's theory where constraints appear as a result of a gauge freedom. So, similarly to Electromagnetism we can redefine the configuration space on which these equations are operating. The new space consists of equivalence classes of Riemann metrics $\tilde{h}_{ab}$ and is called the super-space. All metrics which can be converted from one into an other by pull-backs under spatial diffeomorphisms belong in one equivalence class. Therefore the elements of the super-space automatically obey (A.8). This is a good news, unfortunately the Hamiltonian constraint remains. This shows that the time was included into the theory in an artificial manner. The question of time is a great challenge and is discussed in detail in chapter three. For a more detailed discussion of the ADM formalism consult [19].

Appendix B

The Raychaudhuri equation

The equation we will derive in this section is a general result of Einstein's theory of relativity and was discovered independently by Amal Kumar Raychaudhuri and Lev Landau. It describes the motion of nearby bits of matter in general relativity and has the intuitive interpretation that gravity acts as an attractive "force" between any two units of mass-energy. The derivation is kept general, except for neglecting the torque in the system. Consider a foliation of space-time as described above, where n^a is the unit normal to the hypersurfaces Σ and h_{ab} the projector orthogonal to n^a. First setting $8\pi G_N = 1$ we trace Einstein's equation:

$$Tr\left(R_{ab} - \frac{1}{2}R\, g_{ab} = T_{ab}\right) \Rightarrow T = -R$$

Now rewrite the Einstein equation as:

$$R_{ab} = T_{ab} - \frac{1}{2}T\, g_{ab}$$

Project it on the the unit normal n^a and use to rewrite the lhs. (A.5) from the last section:

$$R_{ab}n^a n^b = 2\sigma^2 + \frac{1}{3}\theta^2 + n^a\nabla_a\theta - \nabla_c\left(n^a\nabla_a n^c\right)$$

Combining this we obtain the final result, the Raychaudhuri equation (4.5) as used in chapter 4:

$$2\sigma^2 + \frac{1}{3}\theta^2 + n^a\nabla_a\theta - \nabla_c\left(n^a\nabla_a n^c\right) = \left(T_{ab} - \frac{1}{2}g_{ab}T\right)n^a n^b \tag{B.1}$$

Now suppose we are studying a system of non interacting dust particles, freely falling in the gravitational field. So, u^a is the flow of proper time for the dust and the energy-momentum tensor reads $T_{ab} = \rho u_a u_b$. Evaluate (B.1) with the dust energy-momentum tensor :

$$2\sigma^2 + \frac{1}{3}\theta^2 + n^a\nabla_a\theta - \nabla_c\left(n^a\nabla_a n^c\right) = \left(\rho u_a u_b + \frac{1}{2}g_{ab}\rho\right)n^a n^b = \rho\left(u_a n^a\right)^2 - \frac{1}{2}\rho$$

We now identify n^a and u^a, what physically means to choose the rest frame of the dust. Denoting with dot the derivative w.r.t proper time of the dust we rewrite:

$$2\sigma^2 + \frac{1}{3}\theta^2 + \dot{\theta} - \nabla_c\left(\dot{u}^c\right) = 4\pi G_N\,\rho$$

(Here units have been restored.) Furthermore assuming that the dust is irrotational, which is consistent with the claim that its flow is orthogonal to the hypersurfaces, we get:

$$2\sigma^2 + \frac{1}{3}\theta^2 + \dot{\theta} = 4\pi G_N\,\rho$$

This is the form of the Raychaudhuri equation used in [3] and described in the first chapter as equation (1.9). For a more detailed discussion of the physical interpretation of the Raychaudhuri equation consult [5].

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Erklärung

Hiermit versichere ich an Eides statt, diese Arbeit selbständig angefertigt zu haben und keine anderen als die angegebenen Hilfsmittel verwendet zu haben.

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