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Master's Thesis

Non-Perturbative Approach to Pure Yang-Mills Theory in $2+1$ Dimensions

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A great deal more is known
than has been proved.

(Richard P. Feynman)

Abstract

In this thesis a proposal by Leigh, Minic and Yelnikov for the analytic computation of the glueball mass spectrum of pure Yang-Mills theory in $2+1$ spacetime dimensions is critically analysed. It is based on the construction of local gauge-invariant variables in the Hamiltonian formulation of the theory, that was developed by Karabali, Kim and Nair, and which is also reviewed here in detail.

The results by Leigh et al. are based on a conjectured approximation to the vacuum wave functional that is quasi-Gaussian in the gauge-invariant variable, but with a non-trivial kernel that interpolates between the UV where asymptotically free gluons are recovered and the IR where it leads to confinement. However, its derivation relies on an assumption that could not be verified and, strictly speaking, is even violated.

It is shown that the proposed vacuum wave functional has exactly the same form when re-expressed in the original gauge field variables and the calculations are then repeated in this framework. The key ingredient is the regularisation of the kinetic part of the Hamiltonian, which is performed here using a well-known method first introduced by Fujikawa in the calculation of the axial anomaly. The quantitative results contradict those found by Leigh et al. and the reason for this discrepancy remains unclear. Finally, the physical consequences of the values obtained here are discussed.

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1. Introduction

In today's standard model of particle physics, the strong interaction which is responsible for binding quarks to form hadrons like the proton or neutron, is described by quantum chromodynamics (QCD). It is a non-abelian (or Yang-Mills) gauge theory based on the symmetry group SU(3), the corresponding charge is called colour charge and the gauge bosons mediating the force are called gluons. Despite its great success in quantitatively predicting observations made at collider experiments at high energies, the low energy sector of the theory is still not understood very well. The reason for that originates from the peculiar behaviour of the coupling constant: It becomes smaller at high energies (this phenomenon is referred to as *asymptotic freedom*), allowing for standard perturbative calculations in this regime, but at low energies it increases, making perturbation theory inapplicable.

Two of the most important non-perturbative properties of QCD, which must be satisfied in order to make it a viable theory describing our world, are the existence of a *mass gap* and the *confinement of quarks*. The former states that there is a finite energy gap between the vacuum and the first excited state, which is necessary to explain why the strong interaction is very short ranged despite the fact that gluons are massless. The latter means that it is impossible to separate a single quark from a colour-neutral bound state, which would explain why no isolated quark has ever been observed. Even though there exists a lot of evidence in favour of both of these important properties from computer simulations performed in lattice QCD, where the continuous spacetime is replaced by a discrete lattice, still nobody has succeeded in giving an analytic proof of them. Therefore, even almost 50 years after the proposal of QCD as the theory of strong interactions, the questions of the mass gap and confinement still remain outstanding unsolved problems in theoretical physics.

In view of the technically complicated nature of non-perturbative problems, it might be useful to first study a model that is somehow simpler, but that still incorporates the main features of the theory. One possible prototype is pure Yang-Mills theory in $2+1$ spacetime dimensions, as for example discussed on a qualitative level by Feynman in [1].

"Pure" means that it contains no quarks, but as Wilson argued in [2], the question of confinement can be answered without quarks by the vacuum expectation value of the gauge field phase factor around a closed loop. Furthermore, even without quarks there remains the confinement problem for gluons, meaning that the first excited states above the vacuum must be colour-neutral bound states of gluons, so-called *glueballs*. The mass gap hypothesis then states that the lightest glueball mass must be greater than zero.

The reduction of the system by one spatial dimension is a more severe change and it is a priori not clear that the qualitative behaviour in this theory will still be the same. But the possibility to make any progress in this challenging field of research should be worth the risk. Of course the hope is that once the $2+1$ dimensional theory is understood, the methods and insights can be carried over to the original higher dimensional theory, and

at the moment there is in fact some evidence for how this might work ([3], [4], [5], [6]).

One particular approach to analytically attack the mass gap and confinement problem for pure Yang-Mills theory in $2 + 1$ dimension was developed by Karabali, Kim and Nair (KKN) ([7], [8], [9], [10]), where they constructed the theory in Hamiltonian formulation in terms of new local gauge-invariant variables, similar to the "corner variables" introduced earlier by Bars [11]. Based on this formalism, Leigh, Minic and Yelnikov (LMY) in 2007 proposed an approximation to the vacuum wave functional, from which they could compute the glueball mass spectrum for this theory [12]. Their strategy was to directly solve the Schrödinger equation in terms of the KKN variables up to quadratic order in the relevant variable (which is roughly the same as the colour-magnetic field).

The good thing about the LMY proposal is that it asymptotes the expected form corresponding to asymptotically free gluons in the high energy limit as well as a confining vacuum in the low energy limit. Furthermore, the deduced string tension and glueball masses are in good agreement with lattice simulation data, even though the agreement of the mass spectrum is not as "excellent" as claimed by the authors.

But the major flaw is that in order to solve the Schrödinger equation they needed a certain assumption about the spectrum of the kinetic energy operator, which they could not prove. In fact, LMY were not only unable to prove their conjecture, they even found disagreement by direct calculations. They used it anyway to derive their result, mainly with the justification that in the end it nicely agrees with most expectations. The other justification is that in terms of the KKN variables the conjecture seems rather natural at a heuristic level, and the observed disagreement presents only small deviations from it, which are closely related to the very subtle regularisation procedure of the kinetic energy operator that is perhaps not yet completely understood. This situation is of course very unsatisfying, and it would be desirable to have an independent approach to verify (or falsify, and possibly correct) the LMY conjecture.

The main objective of this thesis is to work out all the details of the KKN and LMY approach and to give all calculations in a thorough way. After this has been done, it is shown how the LMY vacuum wave functional can be re-expressed in terms of the original Yang-Mills variables, and that it has exactly the same quasi-Gaussian form. This raises the question why the change of variables was necessary or helpful at all for the LMY calculations, and they are repeated in the original variables, providing an independent test of the key conjecture. Those calculations lead to divergent expressions, that are dealt with similar to a well-known and widely accepted regularisation procedure, first used by Fujikawa for the calculation of the axial anomaly [13]. The outcome is that although the LMY conjecture is confirmed at a qualitative level, the quantitative results are different. Finally, it is shown that a modification of the conjecture according to these results leads to a vacuum wave functional that still has the correct UV limit but fails to reproduce the confining IR behaviour. The discrepancy with the LMY results remains a puzzle that has to be left for future research.

The thesis is organised as follows. In chapter 2 the Hamiltonian formulation of the theory is reviewed. Chapter 3 explains the change to the KKN variables, with the main focus on the derivation of the regularised Hamiltonian, which is of major importance for the LMY calculations that are presented in chapter 4. Chapter 5 is devoted to testing the LMY conjecture by direct calculations, emphasising the delicate dependence on the exact regularisation procedure. In chapter 6 the LMY calculations are repeated in terms of the

original variables, possible regularisation issues are discussed and the consequences of the altered results for the vacuum wave functional are analysed. Chapter 7 gives a conclusion and the appendix shows some detailed calculations that were omitted before.

2. Formalism

In this chapter, after explaining some notational conventions, classical pure Yang-Mills theory in $2+1$ dimensions is defined and its quantisation in temporal gauge in the Hamiltonian formulation is briefly reviewed.

2.1. Conventions

Throughout this thesis Latin letters from the beginning of the alphabet are used as $su(N)$ indices, Greek letters denote spacetime indices in $2+1$ dimensions and Latin letters from the middle of the alphabet indicate purely spatial components:

$$a, b, \ldots = 1, \ldots, N^2 - 1 \tag{2.1a}$$
$$\mu, \nu, \ldots = 0, 1, 2 \tag{2.1b}$$
$$i, j, \ldots = 1, 2. \tag{2.1c}$$

Repeated indices of any kind should be summed over unless stated otherwise, and Greek indices are lowered with the Minkowski-metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1). \tag{2.2}$$

A boldface notation of real variables is used for spatial vectors — for example $\boldsymbol{x} \in \mathbb{R}^2$ — to distinguish them from the corresponding vectors in Minkowski space — $x \in \mathbb{R}^{2+1}$. In the context of complex variables $z \in \mathbb{C}$, boldface emphasises the general dependence of functions on both z and its complex conjugate $\bar{z}$ — for example $f(\boldsymbol{z})$. Arguments z (or $\bar{z}$) indicate holomorphic (or antiholomorphic) functions of the complex variable, i.e. $\bar{\partial} f(z) = 0 = \partial f(\bar{z})$.

Spatial integrals over a real variable $\boldsymbol{x} \in \mathbb{R}^2$ are written as

$$\int_x := \int \mathrm{d}^2 x \equiv \int \mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{2.3}$$

and similarly for a complex variable $z \in \mathbb{C}$, $z_1 = \mathrm{Re}(z)$, $z_2 = \mathrm{Im}(z)$:

$$\int_z := \int \mathrm{d}^2 z \equiv \int \mathrm{d}z_1 \wedge \mathrm{d}z_2 \equiv \int \frac{1}{2i} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{2.4}$$

To minimize the use of parentheses we adopt (until chapter 6) the convention that derivative operators like ∂_μ act only on the first function to their right. In contrast, when they should be understood as operators acting on everything to their right, they will be supplemented by a hat, $\hat{\partial}_\mu$. When there is no function at all to their right, it is clear that

they are meant as operators and the hat will sometimes be omitted in those cases. So for example we have

$$\partial_\mu f g = (\partial_\mu f) g \tag{2.5a}$$
$$\partial_\mu (fg) = \partial_\mu f g + f \partial_\mu g \tag{2.5b}$$
$$\hat{\partial}_\mu f = \partial_\mu f + f \hat{\partial}_\mu \quad \Leftrightarrow \quad [\hat{\partial}_\mu, f] = \partial_\mu f \tag{2.5c}$$
$$\hat{\partial}_\mu f g = \partial_\mu f g + f \partial_\mu g + f g \hat{\partial}_\mu. \tag{2.5d}$$

The conventions for the $SU(N)$ generators t^a and the gauge field A_μ (in the fundamental representation) will be adopted from [12]:

$$[t^a, t^b] = i f^{abc} t^c \tag{2.6a}$$
$$\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab} \tag{2.6b}$$
$$A_\mu = -i A_\mu^a t^a, \quad A_\mu^a \in \mathbb{R}. \tag{2.6c}$$

Note that in this convention the t^a's are hermitian and the A_μ's anti-hermitian, traceless $N \times N$ matrices. Also, the coupling constant g is included in the field A_μ.

The covariant derivative ∇_μ and the field strength tensor $F_{\mu\nu}$ are given by

$$\nabla_\mu = \partial_\mu + A_\mu \tag{2.7a}$$
$$F_{\mu\nu} = [\hat{\nabla}_\mu, \hat{\nabla}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = -i F_{\mu\nu}^a t^a, \tag{2.7b}$$
$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c. \tag{2.7c}$$

Gauge transformations take the form

$$A_\mu \mapsto g A_\mu g^{-1} - \partial_\mu g g^{-1}, \quad g(x) \in SU(N). \tag{2.8}$$

2.2. The Hamiltonian Formalism

Since the Hamiltonian formulation (especially in the Schrödinger picture) is not very commonly used to describe a quantum field theory, it is worthwhile to give a brief review. A more detailed presentation can be found for example in the text book by B. Hatfield [14].

Pure Yang-Mills theory in $2+1$ dimensions is defined by a Lagrangian density for the non-abelian gauge field A_μ that reads

$$\mathcal{L} = \frac{1}{2g^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} = \frac{1}{2g^2} \left[(F_{0i}^a)^2 - (F_{12}^a)^2 \right] \tag{2.9}$$

giving rise to the classical field equations

$$\nabla_\mu^{ab} F^{b\mu\nu} = 0 \tag{2.10}$$

where $\nabla_\mu^{ab} = \delta^{ab} \partial_\mu + f^{acb} A_\mu^c$ is the covariant derivative in adjoint representation.

For the Hamiltonian formalism that will be used it is convenient to adopt the *temporal gauge*[1]

$$A_0 = 0 \tag{2.11}$$

so only the spatial field components A_i are left and the Lagrangian density simplifies to

$$\mathcal{L} = \frac{1}{2g^2} \left[(\partial_0 A_i^a)^2 - (B^a)^2 \right] \tag{2.12}$$

where

$$B^a = F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c \tag{2.13}$$

is the colour-magnetic field which in $2+1$ dimensions has only one (spatial) component. The canonically conjugate momenta to the field A_i^a are given by

$$\Pi_i^a = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i^a)} = \frac{1}{g^2} \partial_0 A_i^a = -E_i^a \tag{2.14}$$

and represent (apart from sign) the colour electric field. So the pure Yang-Mills Hamiltonian in $2+1$ dimensions in temporal gauge becomes

$$\mathrm{H} = \int_x \left(\Pi_i^a \partial_0 A_i^a - \mathcal{L} \right) = \int_x \left(\frac{g^2}{2} (E_i^a)^2 + \frac{1}{2g^2} (B^a)^2 \right). \tag{2.15}$$

In the quantised version of the theory (in the Heisenberg picture) the state of a system is described by a (time independent) state-vector $|\Psi\rangle$, and the field variables and their conjugate momenta are replaced by (time dependent) operators satisfying the equal time commutation relations

$$[A_i^a(\boldsymbol{x}, t), \Pi_j^b(\boldsymbol{y}, t)] = -[A_i^a(\boldsymbol{x}, t), E_j^b(\boldsymbol{y}, t)] = i\delta^{ab} \delta_{ij} \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{2.16}$$

Their equations of motion are[2]

$$\partial_0 A_i^a(\boldsymbol{x}) = i \left[\mathrm{H}, A_i^a(\boldsymbol{x}) \right] = ig^2 \int_y E_i^b(\boldsymbol{y}) \left[E_i^b(\boldsymbol{y}), A_i^a(\boldsymbol{x}) \right] = -g^2 E_i^a(\boldsymbol{x}) \tag{2.17a}$$

$$\partial_0 E_i^a(\boldsymbol{x}) = i \left[\mathrm{H}, E_i^a(\boldsymbol{x}) \right] = \frac{i}{g^2} \int_y B^b(\boldsymbol{y}) \left[\partial_1 A_2^b - \partial_2 A_1^b + f^{bcd} A_1^c A_2^d, E_i^a(\boldsymbol{x}) \right] =$$

$$= \frac{1}{g^2} \left(-\delta_{i2} \partial_1 B^a + \delta_{i1} \partial_2 B^a + \delta_{i2} f^{bca} A_1^c B^b + \delta_{i1} f^{bad} A_2^d B^b \right)(\boldsymbol{x}) =$$

$$= -\frac{1}{g^2} \left(\delta_{i2} \nabla_1^{ab} B^b - \delta_{i1} \nabla_2^{ab} B^b \right)(\boldsymbol{x}) = \epsilon_{ij} \frac{1}{g^2} \nabla_j^{ab} B^b(\boldsymbol{x}) \tag{2.17b}$$

and so the Hamiltonian system indeed reproduces the original field equations (2.10) in temporal gauge for $\nu = 1, 2$. But the $\nu = 0$ field equation is missing, which in temporal gauge becomes the "Gauss law" constraint

$$\nabla_i^{ab} E_i^b \equiv \partial_i E_i^a + f^{abc} A_i^b E_i^c = 0. \tag{2.18}$$

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[1]It is also known as *Hamiltonian*- or *Weyl gauge*.
[2]For the sake of a compact notation we omit time arguments. All operators are to be taken at the same time.

This equation, however, is inconsistent as an operator identity with the commutation relations (2.16) because acting with $\nabla_j^{cb}(\boldsymbol{y})$ on (2.16) does not give zero on the right hand side. Nevertheless, it is enough to require the Gauss law operator (2.18) to vanish when acting on physical states $|\Psi\rangle$:

$$\nabla_i^{ab} E_i^b |\Psi\rangle = 0. \tag{2.19}$$

In the Schrödinger picture, the state is now described by a time dependent state-vector $|\Psi(t)\rangle$ satisfying the Schrödinger equation

$$i\partial_t |\Psi(t)\rangle = \mathrm{H}|\Psi(t)\rangle \tag{2.20}$$

with the Hamiltonian to be understood as an operator valued functional of the canonical variables A_i^a and their corresponding momenta $-E_i^a$. Those are now time independent operators (and hence so is the Hamiltonian) satisfying the commutation relations

$$- [A_i^a(\boldsymbol{x}), E_j^b(\boldsymbol{y})] = i\delta^{ab}\delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{2.21}$$

In A-representation (i.e. the representation in which the operator A is diagonal with eigenvectors denoted by $|A\rangle$) the state is described by a wave-functional $\langle A|\Psi(t)\rangle \equiv \Psi(t)[A]$, the operator A_i^a acts as a multiplication operator and the momentum operator becomes a functional derivative with respect to the gauge field:

$$E_i^a(\boldsymbol{x}) = i\frac{\delta}{\delta A_i^a(\boldsymbol{x})}. \tag{2.22}$$

So in this basis the Hamiltonian takes the form

$$\mathrm{H}[A] = \mathrm{T}[A] + \mathrm{V}[A], \tag{2.23a}$$

$$\mathrm{T}[A] := \int_x \left(-\frac{g^2}{2}\frac{\delta^2}{\delta A_i^a(\boldsymbol{x})^2} \right) \tag{2.23b}$$

$$\mathrm{V}[A] := \int_x \left(\frac{1}{2g^2} B^a(\boldsymbol{x})^2 \right) \tag{2.23c}$$

with T and V denoting the kinetic and potential energy operators, respectively, in analogy to ordinary quantum mechanics. Since H is time independent, the time dependent Schrödinger equation (2.20) can be solved just like in ordinary quantum mechanics by a separation Ansatz $\Psi_E(t)[A] = \mathrm{e}^{-iEt}\Psi_E[A]$ with $\Psi_E[A]$ satisfying the time independent Schrödinger equation

$$\mathrm{H}[A]\Psi_E[A] = E\Psi_E[A] \tag{2.24}$$

which for a field theory is a functional differential equation.

The Gauss law constraint (2.19) now reads

$$\nabla_i^{ab}\frac{\delta}{\delta A_i^b(\boldsymbol{x})}\Psi(t)[A] = 0. \tag{2.25}$$

This is in fact equivalent to requiring all physical states to be invariant under residual gauge transformations with respect to the gauge condition $A_0 = 0$, i.e. gauge transformations which are independent of time:

$$A_i \mapsto gA_ig^{-1} - \partial_i gg^{-1}, \quad g(\boldsymbol{x}) \in \mathrm{SU}(N). \tag{2.26}$$

The equivalence can be seen as follows [1]. For infinitesimal[3] residual gauge transformations $g = 1 - i\theta^a(\boldsymbol{x})t^a + \mathcal{O}(\theta^2)$ the transformation (2.26) becomes (terms of higher order in θ are suppressed)

$$A_i^a \mapsto A_i^a - \nabla_i^{ab}\theta^b = A_i^a - (\partial_i\theta^a + f^{abc}A_i^b\theta^c) \tag{2.27}$$

and the resulting change of the state functional $\Psi[A]$ is

$$\delta\Psi[A] = \int_x \delta A_i^a(\boldsymbol{x}) \frac{\delta\Psi[A]}{\delta A_i^a(\boldsymbol{x})} \tag{2.28a}$$

$$= -\int_x \left(\partial_i\theta^a(\boldsymbol{x}) + f^{abc}A_i^b(\boldsymbol{x})\theta^c(\boldsymbol{x})\right) \frac{\delta}{\delta A_i^a(\boldsymbol{x})}\Psi[A] \tag{2.28b}$$

$$= \int_x \theta^a(\boldsymbol{x}) \left(\delta^{ac}\partial_i + f^{abc}A_i^b(\boldsymbol{x})\right) \frac{\delta}{\delta A_i^c(\boldsymbol{x})}\Psi[A] \tag{2.28c}$$

$$= \int_x \theta^a(\boldsymbol{x})\nabla_i^{ab} \frac{\delta}{\delta A_i^b(\boldsymbol{x})}\Psi[A] \stackrel{!}{=} 0 \tag{2.28d}$$

which, because the θ^a are arbitrary functions of $\boldsymbol{x}$, is indeed equivalent to (2.25). Therefore the quantum system can be described by (2.23)-(2.24) together with the requirement on physical states to be invariant under time independent gauge transformations.

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[3]Note that in two space dimensions *any* gauge transformation can be reached by a continuous series of infinitesimal ones, since $\pi_2(\mathrm{SU}(N)) = 0$. See for example [15, chapter 23].

3. The KKN Approach

The calculations by Leigh et al. are based on a reformulation of the theory which was developed by Karabali, Kim and Nair (KKN) in a series of papers ([7], [8], [9], [10]) and will be reviewed in detail in this chapter. The emphasis will lie on those aspects that are most important for the calculations by Leigh et al. in [12] and so the way they are presented will to some extend also follow the discussion therein.

The main idea is to change from the original gauge field variables A_i to some new *local gauge-invariant* variables and view the physical states as functionals of those. Moreover, this approach uses the fact that in $2 + 1$ spacetime dimensions the two spatial coordinates can be grouped into one complex coordinate.[1] But as will be seen in section 3.4, there is a price to pay for obtaining those gauge-invariant variables: the introduction of a new "holomorphic" invariance that has to be taken care of. Nevertheless, the hope is that the new variables offer a possibility to explore the non-perturbative features of Yang-Mills theory.

3.1. New Field Variables

The first step in the KKN construction is to combine the two spatial coordinates into one complex coordinate:

$$z = x_1 - ix_2 \qquad\qquad \partial \equiv \partial_z = \frac{1}{2}(\partial_1 + i\partial_2) \qquad\qquad (3.1\text{a})$$

$$\bar{z} = x_1 + ix_2 \qquad\qquad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_1 - i\partial_2) \qquad\qquad (3.1\text{b})$$

Since the A_i are the components of a 1-form, their corresponding transformation is

$$A \equiv A_z = \frac{1}{2}(A_1 + iA_2) \qquad\qquad \mathrm{d}z = \mathrm{d}x_1 - i\mathrm{d}x_2 \qquad\qquad (3.2\text{a})$$

$$\bar{A} \equiv A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2) = -A^{\dagger} \qquad\qquad \mathrm{d}\bar{z} = \mathrm{d}x_1 + i\mathrm{d}x_2. \qquad\qquad (3.2\text{b})$$

So instead of two traceless anti-hermitian matrices we now have one traceless but otherwise general complex matrix. The complexified version of the covariant derivative becomes

$$\nabla = \partial + A \qquad\qquad \overline{\nabla} = \bar{\partial} + \bar{A}. \qquad\qquad (3.3)$$

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[1]There is, however, an analogous way of defining gauge-invariant variables in $3 + 1$ dimensions without introducing any complex structure, which can be found in [5] and is closely related to the formalism of Bars [11].

Now the crucial step is the introduction of the new field variable M by the defining equation

$$A = -\partial M M^{-1} \qquad \Rightarrow \qquad \bar{A} = M^{\dagger -1} \bar{\partial} M^{\dagger} \tag{3.4}$$

which can be solved for M (and $M^{\dagger}$) by means of the Green's functions for the operator ∂ (and $\bar{\partial}$)[2]:

$$\partial_z G(\boldsymbol{z}, \boldsymbol{z}') = \delta^{(2)}(\boldsymbol{z} - \boldsymbol{z}') = \bar{\partial}_z \bar{G}(\boldsymbol{z}, \boldsymbol{z}') \tag{3.5a}$$

$$\Rightarrow G(\boldsymbol{z}, \boldsymbol{z}') = \frac{1}{\pi(\bar{z} - \bar{z}')}, \quad \bar{G}(\boldsymbol{z}, \boldsymbol{z}') = \frac{1}{\pi(z - z')}. \tag{3.5b}$$

(See section 2.1 for the use of boldface complex variables.) Equation (3.4) now yields

$$M(\boldsymbol{z}) = h^{\dagger}(\bar{z}) + \int_{z'} A(\boldsymbol{z}') M(\boldsymbol{z}') G(\boldsymbol{z}', \boldsymbol{z}) \tag{3.6a}$$

$$M^{\dagger}(\boldsymbol{z}) = h(z) + \int_{z'} \bar{G}(\boldsymbol{z}, \boldsymbol{z}') M^{\dagger}(\boldsymbol{z}') \bar{A}(\boldsymbol{z}') \tag{3.6b}$$

where $h^{\dagger}$ and h are matrices in the kernel of ∂ and $\bar{\partial}$, respectively. These implicit expressions can be solved iteratively, e.g. for $h = 1$ one gets:

$$M(\boldsymbol{z}) = 1 + \int_{z'} A(\boldsymbol{z}') G(\boldsymbol{z}', \boldsymbol{z}) + \int_{z'} \int_{z''} A(\boldsymbol{z}') A(\boldsymbol{z}'') G(\boldsymbol{z}'', \boldsymbol{z}') G(\boldsymbol{z}', \boldsymbol{z}) + \dots \tag{3.7a}$$

$$M^{\dagger}(\boldsymbol{z}) = 1 + \int_{z'} \bar{G}(\boldsymbol{z}, \boldsymbol{z}') \bar{A}(\boldsymbol{z}') + \int_{z'} \int_{z''} \bar{G}(\boldsymbol{z}, \boldsymbol{z}') \bar{G}(\boldsymbol{z}', \boldsymbol{z}'') \bar{A}(\boldsymbol{z}'') \bar{A}(\boldsymbol{z}') + \dots \tag{3.7b}$$

Other choices of h would give different M's. This is the origin of the new *holomorphic* invariance examined below.

The tracelessness of A implies

$$0 = \det(M) \operatorname{Tr}(\partial M M^{-1}) = \partial \det(M) \tag{3.8}$$

and so we can choose $\det(M) = 1$, i.e.

$$M \in SL(N, \mathbb{C}). \tag{3.9}$$

(If $\det(M^{\text{old}}) \neq 1$ just define $M := M^{\text{old}} / \det(M^{\text{old}})$ which has determinant one and can be inserted in (3.4) instead of M^{old} because of (3.8).)

3.2. Gauge-Invariant Objects

The effect of a gauge transformation[3] (2.26) of A_i with $g(\boldsymbol{x}) \in \mathrm{SU}(N)$ on the complexified filed variable A is obviously

$$A \mapsto g A g^{-1} - \partial g g^{-1} \tag{3.10}$$

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[2]For a derivation of the Green's functions see appendix A.

[3]From now on the adjective "residual" will be dropped and it will be understood that gauge transformations are time-independent.

which can be expressed in terms of M as

$$-\partial M M^{-1} \mapsto -(g\partial M M^{-1}g^{-1} + \partial g g^{-1}) = -\partial(gM)(gM)^{-1} \tag{3.11}$$

and thus translates to

$$M \mapsto gM. \tag{3.12}$$

This simple behaviour under gauge transformations can be viewed as a motivation for introducing the new variable M according to (3.4), and it allows us to define the *local gauge-invariant variable*

$$H := M^\dagger M. \tag{3.13}$$

It will be convenient also to introduce a gauge-invariant *current*[4]

$$J := \partial H H^{-1}. \tag{3.14}$$

Note that since H has determinant one, J is traceless:

$$\operatorname{Tr}(J) = \operatorname{Tr}(\partial H H^{-1}) = \partial \det(H) = 0 \tag{3.15}$$

and can thus be expanded in the $\mathrm{SU}(N)$ generators with *complex* components J^a

$$J = J^a t^a, \quad J^a = 2\operatorname{Tr}(t^a J). \tag{3.16}$$

3.3. Fields in Adjoint Representation

Later on, the adjoint components of several field variables will be used, and so in this section it will be explained how they are defined.

The components of the $\mathrm{SU}(N)$ generators in the adjoint representation are

$$(T^a)^{bc} = -if^{abc}, \tag{3.17}$$

and since the adjoint representation of A in our conventions is

$$A^{\mathrm{adj}} = -iA^a\, T^a, \tag{3.18}$$

its components are given by[5]

$$A^{bc} = -A^a f^{abc} = A^a f^{bac}. \tag{3.19}$$

The covariant derivative in the adjoint representation acts on a Lie algebra valued field $\Phi(\boldsymbol{x}) = -i\Phi^a(\boldsymbol{x})t^a$ as

$$\nabla^{\mathrm{adj}}\,\Phi(\boldsymbol{x}) = \partial\Phi + A^{\mathrm{adj}}\,\Phi = \partial\Phi - iA^cT^c\,\Phi \tag{3.20}$$

---

[4]This definition differs from the one given in [7] or [12] by a constant factor of $\frac{c_A}{\pi} \equiv \frac{N}{\pi}$ to simplify some formulas.

[5]Note that by a slight abuse of notation we use the same symbol for objects in the adjoint and in the fundamental representation. Which one is meant will always be clear from the number of indices — one for the fundamental and two for the adjoint.

or in components

$$\left(\nabla^{\text{adj}}\,\Phi\right)^a = \partial\Phi^a - iA^c(T^c)^{ab}\Phi^b = \partial\Phi^a + A^{ab}\Phi^b = \nabla^{ab}\Phi^b, \tag{3.21}$$

$$\nabla^{ab} := \delta^{ab}\partial + A^{ab}. \tag{3.22}$$

It can also be written in terms of the covariant derivative in the fundamental representation as

$$\nabla^{\text{adj}}\,\Phi(\boldsymbol{x}) = [\nabla^{\text{fun}}, \Phi(\boldsymbol{x})]. \tag{3.23}$$

Now the new field variable M is not an algebra element, but rather an element of the (complexified) group. Therefore its adjoint components are defined as

$$M^{ab} := 2\operatorname{Tr}\left(t^a M t^b M^{-1}\right). \tag{3.24}$$

First note that this prescription for general $N \times N$ matrices A, B yields a simple multiplication rule:

$$A^{ab}B^{bc} = 2\operatorname{Tr}\left(t^a A t^b A^{-1}\right) 2\operatorname{Tr}\left(t^b B t^c B^{-1}\right) \tag{3.25a}$$

$$= 2\left(A^{-1}t^a A\right)_{\alpha\beta} 2\left(t^b\right)_{\beta\alpha}\left(t^b\right)_{\gamma\delta}\left(Bt^c B^{-1}\right)_{\delta\gamma} \tag{3.25b}$$

$$= 2\left(A^{-1}t^a A\right)_{\alpha\beta}\left(\delta_{\beta\delta}\delta_{\alpha\gamma} - \frac{1}{N}\delta_{\beta\alpha}\delta_{\gamma\delta}\right)\left(Bt^c B^{-1}\right)_{\delta\gamma} \tag{3.25c}$$

$$= 2\operatorname{Tr}\left(t^a A B t^c B^{-1} A^{-1}\right) \tag{3.25d}$$

$$= (AB)^{ac} \tag{3.25e}$$

where the second term in (3.25c) gives vanishing contribution because the t^a are traceless. In particular this identity shows that the inverse of M^{ab} is in fact given by

$$\left(M^{-1}\right)^{ab} = 2\operatorname{Tr}\left(t^a M^{-1}t^b M\right) = M^{ba}. \tag{3.26}$$

To justify definition (3.24), one can show that A^{ab} and M^{ab} defined as above fulfil the adjoint version of (3.4):

$$-\left(\partial M M^{-1}\right)^{ab} = -\partial M^{ac}\left(M^{-1}\right)^{cb} = -2\,\partial\operatorname{Tr}\left[t^a M t^c M^{-1}\right] 2\operatorname{Tr}\left[t^c M^{-1}t^b M\right] \tag{3.27a}$$

$$= -2\operatorname{Tr}\left[\left(\partial M^{-1}t^a M + M^{-1}t^a\partial M\right)t^c\right] 2\operatorname{Tr}\left[t^c M^{-1}t^b M\right] \tag{3.27b}$$

$$= -2\operatorname{Tr}\left[\left(\partial M^{-1}t^a M + M^{-1}t^a\partial M\right)M^{-1}t^b M\right] \tag{3.27c}$$

$$= 2\operatorname{Tr}\left(\partial M M^{-1}[t^a, t^b]\right) = 2iA^c\operatorname{Tr}\left([t^a, t^b]t^c\right) = -A^c f^{abc} \tag{3.27d}$$

$$= A^{ab} \tag{3.27e}$$

where (3.27c) follows in the same way as (3.25c). Furthermore, since

$$M^{ac\dagger} = (M^{ca})^* = 2\operatorname{Tr}\left[(t^c)^{\text{T}} M^* (t^a)^{\text{T}}\left(M^{-1}\right)^*\right] = 2\operatorname{Tr}\left[\left(M^{-1}\right)^{\dagger}t^a M^{\dagger}t^c\right] = (M^{\dagger})^{ac} \tag{3.28}$$

(here * denotes complex conjugation), the adjoint version of H is given by

$$H^{ab} = M^{ac\dagger}M^{cb} = (M^{\dagger}M)^{ab} = 2\operatorname{Tr}(t^a H t^b H^{-1}). \tag{3.29}$$

Finally, repeating the calculation done in (3.27), one finds the adjoint components of J to be

$$J^{ab} = (\partial H H^{-1})^{ab} = \partial H^{ac} H^{-1cb} = \ldots = i f^{acb} J^c \tag{3.30}$$

as one might have expected. (Note that there is an extra factor of i here as compared to the corresponding formula for A, (3.19). This is because the components of J were defined in (3.16) *without* a factor of $-i$ as opposed to the convention for A.)

3.4. Holomorphic Invariance

As already mentioned above, the definition of the new field variables M by equation (3.4) is not unique. In fact, the corresponding A and $\bar{A}$ will be left unchanged by the transformation

$$M(\boldsymbol{z}) \mapsto M(\boldsymbol{z}) h^\dagger(\bar{z}), \qquad M^\dagger(\boldsymbol{z}) \mapsto h(z) M^\dagger(\boldsymbol{z}) \tag{3.31}$$

where h is some unimodular complex matrix with $\bar{\partial} h = 0$ ($\Leftrightarrow \partial h^\dagger = 0$), i.e. the matrix elements of h are holomorphic functions of the complex space-variable $\boldsymbol{z}$. Therefore this new invariance is called *holomorphic invariance*. Note that this transformation of M is equivalent to keeping $h^\dagger(\bar{z})$ in (3.6) arbitrary (and regarding the M's appearing in (3.6) as the *transformed* ones), and at the same time imposing the following transformation rules for the Green's functions:

$$G(\boldsymbol{z}, \boldsymbol{z}') \mapsto h^{\dagger -1}(\bar{z}) G(\boldsymbol{z}, \boldsymbol{z}') h^\dagger(\bar{z}') \tag{3.32a}$$

$$\bar{G}(\boldsymbol{z}, \boldsymbol{z}') \mapsto h(z) \bar{G}(\boldsymbol{z}, \boldsymbol{z}') h^{-1}(z') \tag{3.32b}$$

Because of the (anti-)holomorphicity of the transformation matrix, the new Green's functions will still fulfil the defining equations $\partial_z G^{(h)}(\boldsymbol{z}, \boldsymbol{z}') = \delta(\boldsymbol{z} - \boldsymbol{z}') = \bar{\partial}_z \bar{G}^{(h)}(\boldsymbol{z}, \boldsymbol{z}')$, but it has to be kept in mind that they have nevertheless now become matrices, and so their ordering becomes important. In particular, the transformation rule (3.32) only makes sense if the ordering of Green's functions in (3.7) is the way it is written there, i.e. the Green's function with, say $\tilde{\boldsymbol{z}}$ as its second argument is adjacent to the left of the Green's function with $\tilde{\boldsymbol{z}}$ as its first argument. The ordering of other expressions involving Green's functions will always be determined by requiring the correct behaviour of the whole expression under holomorphic transformations.

Next, let us investigate the transformation behaviour of the gauge-invariant variables defined above under this new symmetry. The H field is not invariant but transforms under h as

$$H(\boldsymbol{z}) \mapsto h(z) H(\boldsymbol{z}) h^\dagger(\bar{z}). \tag{3.33}$$

This means that H is still not a suitable variable for the construction of physical states, which must of course be invariant under h-transformations. The current J transforms as

$$\begin{aligned} J = \partial H H^{-1} &\mapsto \partial(h H h^\dagger)(h H h^\dagger)^{-1} \\ &= \partial h h^{-1} + h \partial H H^{-1} h^{-1} + 0 \\ &= h J h^{-1} + \partial h h^{-1}. \end{aligned} \tag{3.34}$$

This shows that J is a connection for holomorphic transformations (just like the original A-field is a connection for gauge transformations) and so there is a corresponding h-covariant

derivative[6]

$$\hat{D} := \hat{\partial} - J \mapsto \hat{\partial} - hJh^{-1} - \partial h h^{-1}$$
$$= (\hat{\partial} - hJh^{-1} - \partial h h^{-1})hh^{-1} = h\hat{D}h^{-1} \tag{3.35}$$

which in the adjoint representation acts on a matrix-valued field $\Phi = \Phi^a t^a$ as

$$D\Phi = [\hat{D}, \Phi] = \partial\Phi - [J, \Phi] = \left(D^{ab}\Phi^b\right) t^a, \tag{3.36a}$$
$$D^{ab} = \delta^{ab}\partial - i f^{acb} J^c = \delta^{ab}\partial - J^{ab}, \tag{3.36b}$$

in complete analogy to the gauge-covariant derivative (3.21). Furthermore, one can define a h-covariant Laplacian[7]

$$\Delta := \frac{1}{2} \left(\hat{\bar{\partial}}\hat{D} + \hat{D}\hat{\bar{\partial}} \right) \tag{3.37}$$

acting in the adjoint representation on Φ as

$$\Delta\Phi = \frac{1}{2}\left([\hat{\bar{\partial}}, [\hat{D}, \Phi]] + [\hat{D}, [\hat{\bar{\partial}}, \Phi]] \right) = \frac{1}{2}\left(\bar{\partial}[\hat{D}, \Phi] + [\hat{D}, \bar{\partial}\Phi] \right)$$
$$= \partial\bar{\partial}\Phi - \frac{1}{2}\left(\bar{\partial}[J, \Phi] + [J, \bar{\partial}\Phi] \right) = \left(\Delta^{ab}\Phi^b \right) t^a, \tag{3.38a}$$
$$\Delta^{ab} = \delta^{ab}\partial\bar{\partial} - \frac{1}{2}i f^{acb}\left(\hat{\bar{\partial}}J^c + J^c\bar{\partial} \right) \tag{3.38b}$$
$$= \delta^{ab}\partial\bar{\partial} - \frac{1}{2}\left(\hat{\bar{\partial}}J^{ab} + J^{ab}\bar{\partial} \right) \tag{3.38c}$$
$$= \delta^{ab}\partial\bar{\partial} - \frac{1}{2}\bar{\partial}J^{ab} - J^{ab}\bar{\partial}. \tag{3.38d}$$

The point of these definitions is that if Φ is h-covariant, i.e. $\Phi \mapsto h\Phi h^{-1}$, then the same is true for $D\Phi$ and $\Delta\Phi$, as can immediately be seen by using the transformation (3.35) in the corresponding commutators. Thus, these h-covariant derivative operators can be used (instead of the usual partial derivative ∂ and Laplacian $\bar{\partial}\partial$) in the construction of physical states. Finally, note that such a Φ is given by $\bar{\partial}J$, because h is holomorphic and therefore

$$\bar{\partial}J \mapsto \bar{\partial}(hJh^{-1} + \partial h h^{-1}) = h\bar{\partial}Jh^{-1}. \tag{3.39}$$

Furthermore, $\bar{\partial}J$ is essentially real in the following sense:

$$(\bar{\partial}J)^\dagger = \partial(H^{-1}\bar{\partial}H) = \partial H^{-1}\bar{\partial}H + H^{-1}\bar{\partial}\partial H$$
$$= -H^{-1}\partial H H^{-1}\bar{\partial}H + H^{-1}\bar{\partial}(\partial H H^{-1}H)$$
$$= -H^{-1}J\bar{\partial}H + H^{-1}\bar{\partial}(JH)$$
$$= H^{-1}\bar{\partial}JH \tag{3.40}$$

where in the second line we inserted $1 = H^{-1}H$ and used the fact that $\partial H H^{-1} = -H\partial H^{-1}$. These two properties of $\bar{\partial}J$ already indicate that it will be a very useful variable for the construction of the vacuum wave functional of the theory.

---

[6]The minus sign in front of J is due to the plus sign of the inhomogeneous part in the h-transformation behaviour of J in (3.34) as opposed to the g-transformation behaviour of A.

[7]Any other linear combination of $\hat{\bar{\partial}}$ and $\hat{D}$ would also be h-covariant, simply because $\hat{\bar{\partial}}$ and h commute. The reason for taking this particular form for Δ is that it is parity-even as will be shown in section 3.5.

3.5. Quantum Numbers

Since the glueball spectrum of the theory will later be probed by taking two point functions of operators with the desired quantum numbers J^{PC}, it is worthwhile to discuss those quantum numbers for the new field variables now. The main attention will be paid to the transformation behaviour of $\bar{\partial}J$ and the h-covariant Laplacian Δ, which as just argued are good candidates for the construction of the vacuum state.

3.5.1. Spin J

This is the quantum number corresponding to rotations in the 1-2-plane, i.e. the spatial $SO(2)$ subgroup of the Lorentz group. Since A_i and ∂_i are the components of a (co-)vector field we have:

$$J_A = J_\partial = -1, \quad J_{\bar{A}} = J_{\bar{\partial}} = +1 \tag{3.41}$$

From equation (3.7) we see that M is invariant under rotations and so

$$J_M = J_H = 0, \tag{3.42a}$$
$$J_J = -1, \tag{3.42b}$$
$$J_{\bar{\partial}J} = J_\Delta = 0. \tag{3.42c}$$

The last line shows that a state that is constructed only from $\bar{\partial}J$ and Δ will have $J = 0$, as required for the vacuum state.

3.5.2. Parity P

Parity in $2+1$ dimensions can be defined as the transformation[8]

$$\mathbf{P} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}. \tag{3.43}$$

For the complex coordinate z this implies

$$z = x_1 - ix_2 \overset{\mathbf{P}}{\mapsto} x_1 + ix_2 = \bar{z} \tag{3.44}$$

which is simply complex conjugation. Thus under $\mathbf{P}$:

$$A \mapsto \bar{A}, \quad \hat{\partial} \mapsto \hat{\bar{\partial}} \overset{(3.4)}{\Rightarrow} M \mapsto M^{\dagger -1} \overset{(3.13)}{\Rightarrow} H \mapsto H^{-1} \tag{3.45}$$

and

$$
\begin{aligned}
\bar{\partial}J \equiv \bar{\partial}(\partial H H^{-1}) &\mapsto \partial(\bar{\partial}H^{-1}H) \\
&= \partial\bar{\partial}H^{-1}H + \bar{\partial}H^{-1}\partial H \\
&= H^{-1}(H\partial\bar{\partial}H^{-1} + H\bar{\partial}H^{-1}\partial H H^{-1})H \\
&= H^{-1}(H\partial\bar{\partial}H^{-1} + \bar{\partial}H H^{-1}H\partial H^{-1})H \\
&= H^{-1}\bar{\partial}(H\partial H^{-1})H \\
&= -H^{-1}\bar{\partial}J H.
\end{aligned}
\tag{3.46}
$$

---

[8]The crucial point is that it induces an odd number of minus signs, i.e. that the determinant of the corresponding matrix is -1.

To simplify the calculation for the behaviour of $\hat{\Delta}$ under $\mathbf{P}$ first note that one can express the h-covariant derivative as

$$\hat{D} \equiv \hat{\partial} - J = \hat{\partial} - \partial H H^{-1} = \hat{\partial} + H \partial H^{-1} = H \hat{\partial} H^{-1}. \tag{3.47}$$

Then, for a field Φ transforming under parity as

$$\Phi \overset{\mathbf{P}}{\mapsto} \alpha_\Phi H^{-1} \Phi H, \qquad \alpha_\Phi = \pm 1, \tag{3.48}$$

one finds:

$$\left[\hat{\bar{\partial}}, [\hat{D}, \Phi]\right] \overset{\mathbf{P}}{\mapsto} \alpha_\Phi \left[\hat{\partial}, H^{-1}[\hat{\bar{\partial}}, \Phi]H\right] = \alpha_\Phi H^{-1} \left[H\hat{\partial}H^{-1}, [\hat{\bar{\partial}}, \Phi]\right] H = H^{-1}\left[\hat{D}, [\hat{\bar{\partial}}, \Phi]\right]H,$$

$$\tag{3.49a}$$

$$\left[\hat{D}, [\hat{\bar{\partial}}, \Phi]\right] \overset{\mathbf{P}}{\mapsto} \alpha_\Phi H^{-1}\left[\hat{\bar{\partial}}, [\hat{D}, \Phi]\right]H \tag{3.49b}$$

and thus, as promised in section 3.4, the sum of those is parity even in the sense that

$$\Delta\Phi \equiv \frac{1}{2}\left(\left[\hat{\bar{\partial}}, [\hat{D}, \Phi]\right] + \left[\hat{D}, [\hat{\bar{\partial}}, \Phi]\right]\right) \overset{\mathbf{P}}{\mapsto} +\alpha_\Phi H^{-1}\Delta\Phi H. \tag{3.50}$$

3.5.3. Charge Conjugation C

The transformation of the original gluon field under charge conjugation is[9]

$$\mathbf{C} : A_i \mapsto -A_i^{\mathrm{T}} \tag{3.51}$$

where T denotes the transposed matrix. So the complex field A also transforms as

$$\mathbf{C} : A \mapsto -A^{\mathrm{T}} \tag{3.52}$$

which in terms of the new field variable M reads

$$-\partial M M^{-1} \overset{\mathbf{C}}{\mapsto} (\partial M M^{-1})^{\mathrm{T}} = M^{-1\mathrm{T}}\partial M^{\mathrm{T}} = -\partial M^{-1\mathrm{T}} M^{\mathrm{T}} \tag{3.53}$$

and thus

$$M \overset{\mathbf{C}}{\mapsto} M^{-1\mathrm{T}} \tag{3.54a}$$

$$\Rightarrow H \equiv M^\dagger M \overset{\mathbf{C}}{\mapsto} H^{-1\mathrm{T}} \tag{3.54b}$$

$$\Rightarrow J \equiv \partial H H^{-1} \overset{\mathbf{C}}{\mapsto} \partial H^{-1\mathrm{T}} H^{\mathrm{T}} = -H^{-1\mathrm{T}}\partial H^{\mathrm{T}} = -(\partial H H^{-1})^{\mathrm{T}} = -J^{\mathrm{T}} \tag{3.54c}$$

i.e. J is $\mathbf{C}$-odd whereas D is $\mathbf{C}$-even in the sense that for a field transforming as $\Phi \overset{\mathbf{C}}{\mapsto} \Phi^{c\mathrm{T}}$

$$D\Phi \equiv [\hat{D}, \Phi] \overset{\mathbf{C}}{\mapsto} [\hat{\partial} + J^{\mathrm{T}}, \Phi^{c\mathrm{T}}] = [\hat{\partial} - J, \Phi^c]^{\mathrm{T}} = +(D\Phi^c)^{\mathrm{T}}. \tag{3.55}$$

---

[9]In QCD this follows from demanding invariance of the quark-gluon coupling $\propto \overline{\Psi}A_\mu^a t^a \gamma^\mu \Psi$ under $\mathbf{C}$, under which the quark field Ψ transforms as $\Psi \mapsto \mathbf{C}\Psi\mathbf{C}^{-1} = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$ where $\mathcal{C}$ is the charge conjugation matrix acting on the spinor indices , see e.g. [16].

3.6. Jacobian

The change of variables from the original Yang-Mills gauge field A to the new gauge-invariant variable H implies a Jacobian relating the corresponding path-integral measures which was explicitly computed by KKN [10]:

First consider the change of variables from $(A, \bar{A})$ to $(M, M^\dagger)$. From the definition (3.4) it follows that

$$
\begin{aligned}
\mathrm{d}A &= -\partial \mathrm{d}M M^{-1} - \partial M \mathrm{d}M^{-1} \\
&= -\partial(\mathrm{d}M M^{-1}) + \mathrm{d}M(M^{-1}M)\partial M^{-1} - \partial M(M^{-1}M)\mathrm{d}M^{-1} \\
&= -\partial(\mathrm{d}M M^{-1}) + \mathrm{d}M M^{-1}A - A\mathrm{d}M M^{-1} \\
&= -[\hat{\nabla}, \mathrm{d}M M^{-1}] \tag{3.56a} \\
\mathrm{d}\bar{A} &= [\hat{\bar{\nabla}}, M^{\dagger^{-1}}\mathrm{d}M^\dagger]. \tag{3.56b}
\end{aligned}
$$

and so their Haar measures are related by

$$
\mathrm{d}\mu[A] = [\mathrm{d}A\mathrm{d}\bar{A}] = \det(\nabla\overline{\nabla})\,\mathrm{d}\mu[M, M^\dagger] \tag{3.57}
$$

where ∇ and $\overline{\nabla}$ are in the adjoint representation. Now M can be uniquely decomposed as $M = U\rho$ where U is unitary and ρ is hermitian, related to the field variable H by $\rho^2 = H$. The measure factorises accordingly (for the explicit calculation see [10]),

$$
\mathrm{d}\mu[M, M^\dagger] = \mathrm{d}\mu[H]\mathrm{Vol}(\mathcal{G}_*), \tag{3.58}
$$

where $\mathcal{G}_* = \{g(\boldsymbol{x}) \in \mathrm{SU}(N), g \to 1 \text{ as } |\boldsymbol{x}| \to \infty\}$, and so the path integral measure on the space of gauge-invariant field configurations $\mathcal{C} = \{A_i^a\}/\mathcal{G}_*$ can be written as

$$
\mathrm{d}\mu[\mathcal{C}] = \frac{\mathrm{d}\mu[A]}{\mathrm{Vol}(\mathcal{G}_*)} = \det(\nabla\overline{\nabla})\mathrm{d}\mu[H]. \tag{3.59}
$$

The determinant can be computed (see [10] and references therein) with the result

$$
\det(\nabla\overline{\nabla}) = \sigma e^{2NS_{\mathrm{WZW}}[H]}, \tag{3.60}
$$

where σ is a constant determinant factor and S_{WZW} is the Wess-Zumino-Witten (WZW) action [17]

$$
S_{\mathrm{WZW}}[H] = \frac{1}{2\pi}\int \mathrm{d}^2z\,\mathrm{Tr}(\partial H\bar{\partial}H^{-1}) + \frac{i}{12\pi}\int \mathrm{d}^3x\,\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}(H^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\lambda H) \tag{3.61}
$$

which is gauge- and holomorphic-invariant[10]. The inner product of two wave-functionals now takes the form:

$$
\langle \Psi_1 | \Psi_2 \rangle = \int \mathrm{d}\mu[H]e^{2NS_{\mathrm{WZW}}[H]}\Psi_1^*\Psi_2. \tag{3.62}
$$

It should be noted at this point that the appearance of the WZW-term in the Jacobian is an artefact that stems from the complex parametrisation that was chosen. As already mentioned above, one could also keep the original real coordinates and gauge fields and introduce new M-variables in a similar way. In that case the Jacobian would only contain a field independent term and would thus be trivial. [5]

---

[10]The holomorphic invariance can be seen by using the Polyakov-Wiegmann formula, i.e. equation (7) in [18].

3.7. Hamiltonian in New Variables

As already argued, the simple behaviour (3.39) of $\bar{\partial}J$ under holomorphic transformations suggests that this might be a good variable to express physical states in, as it is not only gauge-invariant but also holomorphic-covariant. Therefore, also the Hamiltonian (2.23) should be expressed in terms of $\bar{\partial}J$ (or J).

Just as H[A] is invariant under gauge transformations, we have to make sure that H[J] will be invariant under holomorphic transformations. This consistency requirement will be of crucial importance for the transformation of the kinetic part T.

3.7.1. Potential Term

The potential part V is easy to handle, because $\bar{\partial}J$ is nearly the same as the colour-magnetic field. To see this, first note that the complex covariant derivative and its conjugate can be written as

$$\hat{\nabla} := \hat{\partial} + A = \hat{\partial} + M\partial M^{-1} = M\hat{\partial}M^{-1} \tag{3.63a}$$

$$\hat{\bar{\nabla}} := \hat{\bar{\partial}} + \bar{A} = \hat{\bar{\partial}} + M^{\dagger-1}\bar{\partial}M^{\dagger} = M^{\dagger-1}\hat{\bar{\partial}}M^{\dagger}. \tag{3.63b}$$

Now we use this and the definition $H = M^{\dagger}M$ to express B in terms of the gauge-invariant variables:

$$
\begin{aligned}
B = F_{12} &= [\hat{\nabla}_1, \hat{\nabla}_2] = -i[\hat{\nabla} + \hat{\bar{\nabla}}, \hat{\nabla} - \hat{\bar{\nabla}}] = 2i[\hat{\nabla}, \hat{\bar{\nabla}}] \\
&= 2i\left(M\hat{\partial}M^{-1}M^{\dagger-1}\hat{\bar{\partial}}M^{\dagger} - M^{\dagger-1}\hat{\bar{\partial}}M^{\dagger}M\hat{\partial}M^{-1}\right) \\
&= 2iM^{\dagger-1}\left(H\hat{\partial}H^{-1}\hat{\bar{\partial}} - \hat{\bar{\partial}}H\hat{\partial}H^{-1}\right)M^{\dagger} \\
&= 2iM^{\dagger-1}\left[-\bar{\partial}(H\hat{\partial}H^{-1})\right]M^{\dagger} \\
&= 2iM^{\dagger-1}\left[-\bar{\partial}(H\partial H^{-1})\right]M^{\dagger} \\
&= 2iM^{\dagger-1}\bar{\partial}JM^{\dagger}.
\end{aligned} \tag{3.64}
$$

(The factor of i is due to the fact that we chose the convention $B = -iB^a t^a$ for the magnetic field components in analogy to A and F, but $J = J^a t^a$ for the current.) Therefore, we can rewrite the potential energy operator as

$$\text{V} \equiv \frac{1}{2g^2}\int (B^a)^2 = -\frac{1}{g^2}\int \text{Tr}(B^2) = \frac{4}{g^2}\int \text{Tr}[(\bar{\partial}J)^2] = \frac{2}{g^2}\int (\bar{\partial}J^a)^2. \tag{3.65}$$

Since $\bar{\partial}J$ transforms homogeneously under h, we immediately see that this expression is h-invariant, as it should be.

3.7.2. Kinetic Term

The kinetic energy operator T, which involves functional derivatives with respect to A, is more difficult to deal with. The problem is that because it involves two functional derivatives acting at the same point, it needs to be regularised. The maintenance of the

holomorphic invariance will be crucial for this regularisation procedure. This calculation was first done by Karabali, Kim and Nair in [8] and will now be reviewed.

T is essentially the functional Laplacian with respect to the fields A_i^a and as for the usual Laplacian a change of variables produces two terms:

$$
\begin{aligned}
\mathrm{T} &= -\frac{g^2}{2}\int_x \frac{\delta^2}{\delta A_i^a(\boldsymbol{x})^2} = -\frac{g^2}{2}\int_x \frac{\delta^2}{\delta A^a(\boldsymbol{x})\delta \bar{A}^a(\boldsymbol{x})} \\
&= -\frac{g^2}{2}\int_{x,y} \frac{\delta}{\delta A^a(\boldsymbol{x})}\left(\frac{\delta J^b(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})}\frac{\delta}{\delta J^b(\boldsymbol{y})}\right) \\
&= -\frac{g^2}{2}\left(\int_{x,y} \frac{\delta^2 J^b(\boldsymbol{y})}{\delta A^a(\boldsymbol{x})\delta \bar{A}^a(\boldsymbol{x})}\frac{\delta}{\delta J^b(\boldsymbol{y})} + \int_{x,y,z} \frac{\delta J^b(\boldsymbol{y})}{\delta A^a(\boldsymbol{x})}\frac{\delta J^c(\boldsymbol{z})}{\delta \bar{A}^a(\boldsymbol{x})}\frac{\delta^2}{\delta J^b(\boldsymbol{y})\delta J^c(\boldsymbol{z})}\right) \\
&= m\left(\int_y \omega^b(\boldsymbol{y})\frac{\delta}{\delta J^b(\boldsymbol{y})} + \frac{\pi}{N}\int_{y,z}\Omega^{bc}(\boldsymbol{y},\boldsymbol{z})\frac{\delta^2}{\delta J^b(\boldsymbol{y})\delta J^c(\boldsymbol{z})}\right) =: \mathrm{T}_1 + \mathrm{T}_2
\end{aligned} \tag{3.66}
$$

where in the last line the following abbreviations were introduced:

$$
m := \frac{Ng^2}{2\pi} \tag{3.67a}
$$

$$
\omega^b(\boldsymbol{y}) := -\frac{\pi}{N}\int_x \frac{\delta^2 J^b(\boldsymbol{y})}{\delta A^a(\boldsymbol{x})\delta \bar{A}^a(\boldsymbol{x})} \tag{3.67b}
$$

$$
\Omega^{bc}(\boldsymbol{y},\boldsymbol{z}) := -\int_x \frac{\delta J^b(\boldsymbol{y})}{\delta A^a(\boldsymbol{x})}\frac{\delta J^c(\boldsymbol{z})}{\delta \bar{A}^a(\boldsymbol{x})}. \tag{3.67c}
$$

Note that in $2+1$ dimensions the Yang-Mills coupling constant g has mass dimensions $1/2$ and so m (which is – apart from a factor of 2π – the 't Hooft coupling constant [19]) has the dimension of a mass.

The h-transformation behaviour of J^a, viz. $J^a \mapsto h^{ab}J^b + (\partial h h^{-1})^a$, with the adjoint components of h being

$$
h^{ab} = 2\,\mathrm{Tr}(t^a h t^b h^{-1}), \tag{3.68}
$$

shows that ω and Ω transform as

$$
\omega^b(\boldsymbol{y}) \mapsto h^{bc}(y)\omega^c(\boldsymbol{y}) \tag{3.69a}
$$

$$
\Omega^{bc}(\boldsymbol{y},\boldsymbol{z}) \mapsto h^{bd}(y)h^{ce}(z)\Omega^{de}(\boldsymbol{y},\boldsymbol{z}). \tag{3.69b}
$$

Since the functional derivative with respect to J^b transforms as $\delta/\delta J^b \mapsto h^{bc}\delta/\delta J^c$, we see that both T_1 and T_2 are separately h-invariant.

To evaluate ω and Ω one first has to calculate the functional derivative of the current J with respect to A and $\bar{A}$. To this end, one can first rewrite J in terms of A and $M^\dagger$ by using the definitions $J = \partial H H^{-1}$, $H = M^\dagger M$ and $A = -\partial M M^{-1}$:

$$
J = \partial(M^\dagger M)M^{-1}M^{\dagger-1} = \partial M^\dagger M^{\dagger-1} - M^\dagger A M^{\dagger-1}. \tag{3.70}
$$

The expansions in the su(N)-basis $J = J^b t^b$ and $A = -iA^a t^a$ then give

$$
\begin{aligned}
J^b &= 2\,\mathrm{Tr}\left[t^b J\right] = 2\,\mathrm{Tr}\left[t^b\left(\partial M^\dagger M^{\dagger-1} - M^\dagger A M^{\dagger-1}\right)\right] \\
&= 2\,\mathrm{Tr}\left[t^b \partial M^\dagger M^{\dagger-1}\right] + 2i\,\mathrm{Tr}\left[t^b M^\dagger t^a M^{\dagger-1}\right]A^a \\
&= \left(\partial M^\dagger M^{\dagger-1}\right)^b + i(M^\dagger)^{ba}A^a.
\end{aligned} \tag{3.71}
$$

From their definitions it is clear that M depends only on A and $M^\dagger$ only on $\bar{A}$ and so one immediately obtains

$$\frac{\delta J^b(\boldsymbol{y})}{\delta A^a(\boldsymbol{x})} = i\left(M^\dagger(\boldsymbol{y})\right)^{ba} \delta(\boldsymbol{y} - \boldsymbol{x}). \tag{3.72}$$

The derivative with respect to $\bar{A}$ is a bit more involved. Starting again from (3.70) one gets (the arguments $\boldsymbol{x}$ and $\boldsymbol{y}$ are suppressed during the calculation for the sake of clarity)

$$\begin{aligned}
\frac{\delta J(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} &= \partial\left(\frac{\delta M^\dagger}{\delta \bar{A}^a} M\right) M^{-1} M^{\dagger -1} + \partial(M^\dagger M) M^{-1} \frac{\delta M^{\dagger -1}}{\delta \bar{A}^a} \\
&= \partial \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1} - \frac{\delta M^\dagger}{\delta \bar{A}^a} A M^{\dagger -1} + J M^\dagger \frac{\delta M^{\dagger -1}}{\delta \bar{A}^a} \\
&= \partial \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1} - \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1}(\partial M^\dagger M^{\dagger -1} - J) - J \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1} \\
&= \partial \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1} + \frac{\delta M^\dagger}{\delta \bar{A}^a} \partial M^{\dagger -1} + \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1} J - J \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1} \\
&= \partial\left(\frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1}\right) - \left[J, \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1}\right] \\
&= \left[\hat{D}_y, \frac{\delta M^\dagger(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} M^{\dagger -1}(\boldsymbol{y})\right].
\end{aligned} \tag{3.73}$$

One way to evaluate the second term in the commutator is to take the functional derivative of the defining equation of $M^\dagger(\boldsymbol{y})$ (3.4) with respect to $\bar{A}^a(\boldsymbol{x})$:

$$\begin{aligned}
-i\delta(\boldsymbol{y} - \boldsymbol{x})t^a = \frac{\delta(M^{\dagger -1}\bar{\partial}M^\dagger)(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} &= \frac{\delta M^{\dagger -1}}{\delta \bar{A}^a} \bar{\partial} M^\dagger + M^{\dagger -1} \bar{\partial} \frac{\delta M^\dagger}{\delta \bar{A}^a} \\
&= M^{\dagger -1} \frac{\delta M^\dagger}{\delta \bar{A}^a} \bar{\partial} M^{\dagger -1} M^\dagger + M^{\dagger -1} \bar{\partial} \frac{\delta M^\dagger}{\delta \bar{A}^a}.
\end{aligned} \tag{3.74}$$

(Arguments on the right hand side are $\boldsymbol{x}$ for $\bar{A}^a$ and $\boldsymbol{y}$ for everything else.) Multiplying this with $M^\dagger(\boldsymbol{y})$ from the left and with $M^{\dagger -1}(\boldsymbol{y})$ from the right yields

$$\begin{aligned}
-i\delta(\boldsymbol{y} - \boldsymbol{x})M^\dagger(\boldsymbol{y})t^a M^{\dagger -1}(\boldsymbol{y}) &= \frac{\delta M^\dagger}{\delta \bar{A}^a} \bar{\partial} M^{\dagger -1} + \bar{\partial} \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger -1} \\
&= \bar{\partial}_y\left(\frac{\delta M^\dagger(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} M^{\dagger -1}(\boldsymbol{y})\right).
\end{aligned} \tag{3.75}$$

It is now tempting to invert the $\bar{\partial}$ on the right hand side by using the Green's function $\bar{G}$ to obtain

$$\frac{\delta M^\dagger(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} M^{\dagger -1}(\boldsymbol{y}) = -i\bar{G}(\boldsymbol{y}, \boldsymbol{x})M^\dagger(\boldsymbol{x})t^a M^{\dagger -1}(\boldsymbol{x}).$$

The problem with this equation is that the terms on both sides do not transform the same way under a holomorphic transformation, which looks inconsistent. The reason for this is that the Green's function as introduced in section 3.1 were used to invert ∂ and $\bar{\partial}$ in the equations $-AM = \partial M$ and $M^\dagger \bar{A} = \bar{\partial} M^\dagger$. There the derivative operators act on M and $M^\dagger$ transforming as $M \mapsto Mh^\dagger$ and $M^\dagger \mapsto hM^\dagger$, which then dictates the h-transformation behaviour for the Green's functions as given in (3.32). But here the object

on which $\bar{\partial}$ acts transforms as $(\ldots) \mapsto h(\ldots) h^{-1}$, and so one cannot use Green's function transforming as in (3.32) here without losing holomorphic covariance. In order to keep the whole calculation consistent with h-transformations, one can first multiply equation (3.75) with $2\,t^b$ from the left and take the trace which gives

$$-i\delta(\boldsymbol{y} - \boldsymbol{x})\left(M^\dagger(\boldsymbol{y})\right)^{ba} = \bar{\partial}_y \left(\frac{\delta M^\dagger(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} M^{\dagger-1}(\boldsymbol{y})\right)^b. \tag{3.76}$$

Now the object which $\bar{\partial}$ acts on transforms as $(\ldots)^b \mapsto h^{bc}(\ldots)^c$ which is the same transformation as for the adjoint components of $M^\dagger$:

$$(M^\dagger)^{ab} \mapsto h^{ac}(M^\dagger)^{cb} \tag{3.77}$$

and so $\bar{\partial}$ can be inverted using the adjoint Green's function, defined as

$$\bar{G}^{ab}(\boldsymbol{x}, \boldsymbol{y}) := \delta^{ab} \bar{G}(\boldsymbol{x}, \boldsymbol{y}) \tag{3.78}$$

but transforming under h as:

$$\bar{G}^{ab}(\boldsymbol{x}, \boldsymbol{y}) \mapsto h^{ac}(x) \bar{G}^{cd}(\boldsymbol{x}, \boldsymbol{y})(h^{-1})^{db}(y), \tag{3.79}$$

which then yields

$$\left(\frac{\delta M^\dagger(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} M^{\dagger-1}(\boldsymbol{y})\right)^b = -i\bar{G}^{bc}(\boldsymbol{y}, \boldsymbol{x}) M^{\dagger ca}(\boldsymbol{x})$$
$$\equiv -i\left(\bar{G}(\boldsymbol{y}, \boldsymbol{x}) M^\dagger(\boldsymbol{x})\right)^{ba}. \tag{3.80}$$

Note that now both sides have the same h-transformation behaviour, the same as in (3.77). Projecting out the b-component of equation (3.73) finally gives

$$\frac{\delta J^b(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} = -i\left(D_y \bar{G}(\boldsymbol{y}, \boldsymbol{x}) M^\dagger(\boldsymbol{x})\right)^{ba}. \tag{3.81}$$

Now using the results (3.72) and (3.81), Ω can already be evaluated:

$$\Omega^{bc}(\boldsymbol{y}, \boldsymbol{z}) = -\int_x M^{\dagger ba}(\boldsymbol{y}) \delta(\boldsymbol{y} - \boldsymbol{x})\left(D_z \bar{G}(\boldsymbol{z}, \boldsymbol{x}) M^\dagger(\boldsymbol{x})\right)^{ca}$$
$$= -\left(D_z \bar{G}(\boldsymbol{z}, \boldsymbol{y}) M^\dagger(\boldsymbol{y})\right)^{ca} (M^{\dagger-1})^{ab}(\boldsymbol{y}) = -\left(D_z \bar{G}(\boldsymbol{z}, \boldsymbol{y})\right)^{cb}. \tag{3.82}$$

The transformation behaviour of D and $\bar{G}$ in adjoint representation shows that this result is consistent with (3.69b).

Inserting (3.72) into the definition of ω gives:

$$\omega^b(\boldsymbol{y}) = -i\frac{\pi}{N} \int_x \frac{\delta M^{\dagger ba}(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} \delta(\boldsymbol{y} - \boldsymbol{x}) \tag{3.83}$$

The functional derivative of the adjoint components of M can be computed using the result (3.80):

$$
\begin{aligned}
\frac{\delta M^{\dagger ba}(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} &= 2\,\mathrm{Tr}\left\{t^b \frac{\delta M^\dagger(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})} t^a M^{\dagger-1}(\boldsymbol{y}) + t^b M^\dagger(\boldsymbol{y}) t^a \frac{\delta M^{\dagger-1}(\boldsymbol{y})}{\delta \bar{A}^a(\boldsymbol{x})}\right\} \\
&= 2\,\mathrm{Tr}\left\{t^b \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger-1} M^\dagger t^a M^{\dagger-1} - t^b M^\dagger t^a M^{\dagger-1} \frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger-1}\right\} \\
&= \left(\frac{\delta M^\dagger}{\delta \bar{A}^a} M^{\dagger-1}\right)^c 2\,\mathrm{Tr}\left\{t^b t^c M^\dagger t^a M^{\dagger-1} - t^b M^\dagger t^a M^{\dagger-1} t^c\right\} \\
&= -i\left[\bar{G}(\boldsymbol{y},\boldsymbol{x}) M^\dagger(\boldsymbol{x})\right]^{ca} 2\,\mathrm{Tr}\left\{[t^b, t^c] M^\dagger(\boldsymbol{y}) t^a M^{\dagger-1}(\boldsymbol{y})\right\} \\
&= \left[\bar{G}(\boldsymbol{y},\boldsymbol{x}) M^\dagger(\boldsymbol{x})\right]^{ca} f^{bcd} (M^\dagger)^{da}(\boldsymbol{y}) \\
&= f^{bcd}\left[\bar{G}(\boldsymbol{y},\boldsymbol{x}) M^\dagger(\boldsymbol{x}) M^{\dagger-1}(\boldsymbol{y})\right]^{cd}
\end{aligned}
\tag{3.84}
$$

and so we find

$$
\omega^b(\boldsymbol{y}) = -i\frac{\pi}{N} f^{bcd}\left[\bar{G}(\boldsymbol{y},\boldsymbol{x})\right]^{cd}_{\boldsymbol{x}\to\boldsymbol{y}}.
\tag{3.85}
$$

To check the h-transformation property of this result, note that the right hand side transforms as

$$
(\mathrm{rhs})^b(\boldsymbol{y}) := -i\frac{\pi}{N} f^{bcd}\left[\bar{G}(\boldsymbol{y},\boldsymbol{x})\right]^{cd}_{\boldsymbol{x}\to\boldsymbol{y}} \mapsto -i\frac{\pi}{N} f^{bcd} h^{ce}(y) h^{df}(y) \left[\bar{G}(\boldsymbol{y},\boldsymbol{x})\right]^{ef}_{\boldsymbol{x}\to\boldsymbol{y}}
\tag{3.86a}
$$

$$
= -i\frac{\pi}{N} h^{bc}(y) f^{cef} \left[\bar{G}(\boldsymbol{y},\boldsymbol{x})\right]^{ef}_{\boldsymbol{x}\to\boldsymbol{y}}
\tag{3.86b}
$$

$$
= h^{bc}(y)\,(\mathrm{rhs})^c(\boldsymbol{y}),
\tag{3.86c}
$$

which is indeed the expected homogeneous transformation behaviour (3.69a) of $\omega^b(\boldsymbol{y})$. (To arrive at (3.86b), the following identity was used:

$$
\begin{aligned}
f^{bcd} h^{ce} h^{df} &= -2i\,\mathrm{Tr}\left\{[t^d, t^b] h t^e h^{-1}\right\} h^{df} \\
&= -2i\,\mathrm{Tr}\left\{t^b h t^e h^{-1} t^d - h t^e h^{-1} t^b t^d\right\} 2\,\mathrm{Tr}\left\{t^d h t^f h^{-1}\right\} \\
&= -2i\,\mathrm{Tr}\left\{t^b h t^e h^{-1} h t^f h^{-1} - h t^e h^{-1} t^b h t^f h^{-1}\right\} \\
&= -2i\,\mathrm{Tr}\left\{t^b h [t^e, t^f] h^{-1}\right\} \\
&= h^{bc} f^{cef}.\right)
\end{aligned}
\tag{3.87}
$$

As a consistency check for (3.85), one can also derive ω by interchanging the order of functional derivatives, that is, by taking the functional derivative of (3.81) with respect to $A^a(\boldsymbol{x})$ which gives

$$
\begin{aligned}
\omega^b(\boldsymbol{y}) &= i\frac{\pi}{N}\int_x \frac{\delta}{\delta A^a(\boldsymbol{x})}\left[D_y \bar{G}(\boldsymbol{y},\boldsymbol{x}) M^\dagger(\boldsymbol{x})\right]^{ba} \\
&= -\frac{\pi}{N} f^{bcd}\int_x \frac{\delta J^d(\boldsymbol{y})}{\delta A^a(\boldsymbol{x})}\left[\bar{G}(\boldsymbol{y},\boldsymbol{x}) M^\dagger(\boldsymbol{x})\right]^{ca} \\
&= -i\frac{\pi}{N} f^{bcd}\int_x M^{\dagger da}(\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{x})\left[\bar{G}(\boldsymbol{y},\boldsymbol{x}) M^\dagger(\boldsymbol{x})\right]^{ca} \\
&= -i\frac{\pi}{N} f^{bcd}\left[\bar{G}(\boldsymbol{y},\boldsymbol{x})\right]^{cd}_{\boldsymbol{x}\to\boldsymbol{y}},
\end{aligned}
\tag{3.88}
$$

in agreement with (3.85). The problem with this result is that the coincidence limit the Green's function $\bar{G}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\pi(x-y)}$ diverges and therefore one has to introduce some regularisation to make sense of it.

3.7.3. Regularisation

The singular nature of the Green's functions stems from the fact that the source term in their definitions is a delta function. Therefore, the simplest way to regularise them would be to smear out the delta function, i.e. replace it by some sharply peaked, but finite approximation like

$$\delta^\epsilon(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{\pi\epsilon} e^{-|\boldsymbol{x}-\boldsymbol{y}|^2/\epsilon}. \tag{3.89}$$

Here ϵ denotes some small positive real number of mass dimension -2 that should be sent to zero only at the very end of a calculation. This would then lead to the regularised Green's function $\mathcal{G}(\boldsymbol{x}, \boldsymbol{y}) = \int_z G(\boldsymbol{x}, \boldsymbol{z})\delta^\epsilon(\boldsymbol{z}, \boldsymbol{y})$. The problem is that this expression does not transform under h-transformations like the original Green's function $G(\boldsymbol{x}, \boldsymbol{y}) \mapsto h^{\dagger^{-1}}(\bar{x})G(\boldsymbol{x}, \boldsymbol{y})h^\dagger(\bar{y})$, but the regularisation scheme has to preserve holomorphic invariance. Since the field H transforms under h as $H(\boldsymbol{z}) \mapsto h(z)H(\boldsymbol{z})h^\dagger(\bar{z})$, the desired transformation behaviour can be restored by including appropriate factors of H and H^{-1}, serving as a parallel transport:

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{y}) := \int_z G(\boldsymbol{x}, \boldsymbol{z})\delta^\epsilon(\boldsymbol{z}, \boldsymbol{y})H^{-1}(y, \bar{z})H(\boldsymbol{y}) \tag{3.90a}$$

$$\bar{\mathcal{G}}(\boldsymbol{x}, \boldsymbol{y}) := \int_z \bar{G}(\boldsymbol{x}, \boldsymbol{z})\delta^\epsilon(\boldsymbol{z}, \boldsymbol{y})H(z, \bar{y})H^{-1}(\boldsymbol{y}) \tag{3.90b}$$

Defined this way, they have the correct transformation behaviour (3.32) and reproduce the original Green's functions for $\boldsymbol{x} \neq \boldsymbol{y}$ in the limit $\epsilon \to 0$, in which $\delta^\epsilon(\boldsymbol{z}, \boldsymbol{y}) \to \delta(\boldsymbol{z}, \boldsymbol{y})$. The z-integrals in (3.90) can also be carried out explicitly:

$$\begin{aligned}
\mathcal{G}(\boldsymbol{x}, \boldsymbol{y}) &= \int \mathrm{d}^2 z \, \frac{1}{\pi(\bar{x} - \bar{z})} \frac{e^{-|\boldsymbol{z}-\boldsymbol{y}|^2/\epsilon}}{\pi\epsilon} H^{-1}(y, \bar{z})H(\boldsymbol{y}) \\
&= \int \mathrm{d}^2 z \, \frac{1}{\pi(\bar{x} - \bar{y} - \bar{z})} \frac{e^{-z\bar{z}/\epsilon}}{\pi\epsilon} H^{-1}(y, \bar{y} + \bar{z})H(\boldsymbol{y}) \\
&= \int \mathrm{d}^2 z \, \frac{1}{\pi(\bar{x} - \bar{y} - \bar{z})} \left(-\frac{1}{\pi\bar{z}}\right) \partial_z e^{-z\bar{z}/\epsilon} H^{-1}(y, \bar{y} + \bar{z})H(\boldsymbol{y}) \\
&\overset{i.b.p.}{=} -\int \mathrm{d}^2 z \, \partial_z \left[\frac{1}{\pi\bar{z}} \cdot \frac{1}{\pi(\bar{z} - (\bar{x} - \bar{y}))}\right] e^{-z\bar{z}/\epsilon} H^{-1}(y, \bar{y} + \bar{z})H(\boldsymbol{y}) \\
&= -\int \mathrm{d}^2 z \left[\delta^2(\boldsymbol{z}) \frac{1}{\pi(\bar{z} - (\bar{x} - \bar{y}))} + \frac{1}{\pi\bar{z}}\delta^2(\boldsymbol{z} - (\boldsymbol{x} - \boldsymbol{y}))\right] e^{-z\bar{z}/\epsilon} H^{-1}(y, \bar{y} + \bar{z})H(\boldsymbol{y}) \\
&= \frac{1}{\pi(\bar{x} - \bar{y})} - \frac{e^{-|\boldsymbol{x}-\boldsymbol{y}|^2/\epsilon}}{\pi(\bar{x} - \bar{y})} H^{-1}(y, \bar{x})H(\boldsymbol{y}) \\
&= G(\boldsymbol{x}, \boldsymbol{y}) \left[1 - e^{-|\boldsymbol{x}-\boldsymbol{y}|^2/\epsilon} H^{-1}(y, \bar{x})H(\boldsymbol{y})\right] \tag{3.91a}
\end{aligned}$$

$$\bar{\mathcal{G}}(\boldsymbol{x}, \boldsymbol{y}) = \bar{G}(\boldsymbol{x}, \boldsymbol{y}) \left[1 - e^{-|\boldsymbol{x}-\boldsymbol{y}|^2/\epsilon} H(x, \bar{y})H^{-1}(\boldsymbol{y})\right]. \tag{3.91b}$$

In the fourth line the boundary term from the integration by parts vanishes due to the factor of $e^{-z\bar{z}/\epsilon}$. In the fifth line the product rule and the defining property of the Green's function were used. The calculation of $\bar{\mathcal{G}}$ is completely analogous. Note that the matrix factors included in $\mathcal{G}$ and $\bar{\mathcal{G}}$ can be expanded in Taylor series around $\boldsymbol{x} = \boldsymbol{y}$:

$$H^{-1}(y, \bar{x})H(\boldsymbol{y}) = \sum_{n=0}^{\infty} \frac{(\bar{x} - \bar{y})^n}{n!} \left(\bar{\partial}^n H^{-1} H \right)(\boldsymbol{y}) \tag{3.92a}$$

$$H(x, \bar{y})H^{-1}(\boldsymbol{y}) = \sum_{n=0}^{\infty} \frac{(x - y)^n}{n!} \left(\partial^n H H^{-1} \right)(\boldsymbol{y}) \tag{3.92b}$$

Using this as well as the expansion of the exponential, one can check that $\mathcal{G}$ and $\bar{\mathcal{G}}$ indeed still are regulated expressions, i.e. well defined for $\boldsymbol{y} \to \boldsymbol{x}$:

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\pi(\bar{x} - \bar{y})} \left[1 - \sum_{m=0}^{\infty} \frac{[-(x-y)(\bar{x}-\bar{y})]^m}{m!\epsilon^m} \sum_{n=0}^{\infty} \frac{(\bar{x}-\bar{y})^n}{n!} \left(\bar{\partial}^n H^{-1} H \right)(\boldsymbol{y}) \right]$$

$$= \frac{1}{\pi(\bar{x} - \bar{y})} \left[-(\bar{x} - \bar{y})\bar{\partial} H^{-1} H(\boldsymbol{y}) + \frac{(x-y)(\bar{x}-\bar{y})}{\epsilon} + \mathcal{O}\left((x-y), (\bar{x}-\bar{y})^2 \right) \right]$$

$$= \frac{1}{\pi} \left[-\bar{\partial} H^{-1} H(\boldsymbol{y}) + \mathcal{O}\left((x-y), (\bar{x}-\bar{y}) \right) \right] \overset{\boldsymbol{y} \to \boldsymbol{x}}{\to} -\frac{1}{\pi} \bar{\partial} H^{-1} H(\boldsymbol{x}) = -\frac{1}{\pi} J^\dagger(\boldsymbol{x}) \tag{3.93a}$$

$$\bar{\mathcal{G}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\pi} \left[-\partial H H^{-1}(\boldsymbol{y}) + \mathcal{O}\left((x-y), (\bar{x}-\bar{y}) \right) \right] \overset{\boldsymbol{y} \to \boldsymbol{x}}{\to} -\frac{1}{\pi} \partial H H^{-1}(\boldsymbol{x}) = -\frac{1}{\pi} J(\boldsymbol{x}) \tag{3.93b}$$

Plugging this into (3.85) finally gives the result

$$\omega^b(\boldsymbol{y}) = \frac{i}{N} f^{bcd} J^{cd}(\boldsymbol{y}) = J^b(\boldsymbol{y}). \tag{3.94}$$

To summarise, the Hamiltonian expressed in terms of the new variable J takes the form:

$$\mathrm{H} = \mathrm{T} + \mathrm{V} \tag{3.95a}$$

$$\mathrm{T} = \mathrm{T}_1 + \mathrm{T}_2 = m \left(\int_z J^a(\boldsymbol{z}) \frac{\delta}{\delta J^a(\boldsymbol{z})} + \frac{\pi}{N} \int_{z,z'} \Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') \frac{\delta}{\delta J^a(\boldsymbol{z})} \frac{\delta}{\delta J^b(\boldsymbol{z}')} \right) \tag{3.95b}$$

$$\mathrm{V} = \frac{N}{m\pi} \int_z \bar{\partial} J^a(\boldsymbol{z}) \bar{\partial} J^a(\boldsymbol{z}), \tag{3.95c}$$

with

$$m = \frac{g^2 N}{2\pi} \tag{3.96a}$$

$$\Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') = - \left[D_{z'} \bar{G}(\boldsymbol{z}', \boldsymbol{z}) \right]^{ba}. \tag{3.96b}$$

It should be mentioned that the result for T_1 as written in (3.95) is not manifestly holomorphic invariant, because J does not transform homogeneously. The reason why

this could happen, even though the regulators were constructed to preserve holomorphic invariance, is that in the derivation a Green's function was cancelled against a positive power of $\bar{x} - \bar{y}$. However, as noted by Karabali et al. [10, p. 681], T_1 can by an integration by parts be brought to the form

$$T_1 = -m \int_{z,z'} \bar{\partial} J^a(\boldsymbol{z}) \bar{G}^{ab}(\boldsymbol{z}, \boldsymbol{z}') \frac{\delta}{\delta J^b(\boldsymbol{z}')} \tag{3.97}$$

which now has manifest holomorphic invariance and is equivalent to the form in (3.95) up to boundary terms that are assumed to vanish.

When the kinetic energy operator T acts on products of currents at the same point it will produce divergences coming from the coincident limit of $\bar{G}$. Therefore, in those cases, one has to replace the Green's function by its regulated version as given above yielding

$$
\begin{aligned}
\Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') &= -D_{z'}^{bc} \bar{\mathcal{G}}^{ca}(\boldsymbol{z}', \boldsymbol{z}) \\
&= -D_{z'}^{bc} \left\{ \bar{G}(\boldsymbol{z}', \boldsymbol{z}) \left(\delta^{ca} - e^{-|\boldsymbol{z}'-\boldsymbol{z}|^2/\epsilon} \left[H(z', \bar{z}) H^{-1}(\boldsymbol{z}) \right]^{ca} \right) \right\}.
\end{aligned}
\tag{3.98}
$$

In a more recent paper from 1998, Karablai et al. give another derivation for the Hamiltonian in terms of J, treating regularisation issues "in much greater detail" [10, p. 662]. The main difference affecting the derivation of H[J] is that the regulators are introduced at the very beginning of the calculation, which may be a more consistent way to proceed. Here we only cite the result, which has the same form as (3.95). The linear part of the kinetic energy operator, T_1 stays the same. But for the quadratic part there are some subtle differences which will become important later on: Ω is now given by [10, pp. 679 - 681]

$$\Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') = D_{z'}^{bc} \Lambda^{ca}(\boldsymbol{z}', \boldsymbol{z}) \tag{3.99a}$$

$$
\begin{aligned}
\Lambda^{ca}(\boldsymbol{z}', \boldsymbol{z}) &= \int_x \bar{\mathcal{G}}^{dc}(\boldsymbol{x}, \boldsymbol{z}') G(\boldsymbol{x}, \boldsymbol{z}) e^{-|\boldsymbol{x}-\boldsymbol{z}|^2/\epsilon} \left[\frac{\bar{x} - \bar{z}}{\epsilon} H(x, \bar{x}) H^{-1}(z, \bar{x}) \right. \\
&\quad \left. + H(\boldsymbol{x}) \partial_z \left(H^{-1}(z, \bar{x}) H(\boldsymbol{z}) \right) H^{-1}(\boldsymbol{z}) \right]^{da} \tag{3.99b}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\pi(z - z')} \left[1 - e^{-|\boldsymbol{z}-\boldsymbol{z}'|^2/2\epsilon} H(\boldsymbol{z}') H^{-1}(u, \bar{z}') H(\boldsymbol{u}) H^{-1}(z, \bar{u}) \right]^{ca} \\
&\quad + \frac{\epsilon}{\pi} e^{-|\boldsymbol{z}-\boldsymbol{z}'|^2/2\epsilon} [...]^{ca} + \mathcal{O}(\epsilon^2) \tag{3.99c}
\end{aligned}
$$

$$= \bar{G}(\boldsymbol{z}, \boldsymbol{z}') \left(\delta^{ca} - e^{-|\boldsymbol{z}-\boldsymbol{z}'|^2/2\epsilon} \left[H(\boldsymbol{z}') H^{-1}(z, \bar{z}') \right]^{ca} \right) + \dots \tag{3.99d}$$

where $\boldsymbol{u} := (\boldsymbol{z} + \boldsymbol{z}')/2$, [...] in (3.99c) stands for another complicated expression involving H and H^{-1}'s (the exact form is not of interest here) and the ellipsis in (3.99d) refers to terms of higher order in ϵ or $(z - z'), (\bar{z} - \bar{z}')$. Using the last approximation (3.99d) for Λ gives nearly the same Ω as (3.98), but with ϵ replaced by 2ϵ and, more important, the arguments $\bar{z}$ of H and H^{-1} replaced by $\bar{z}'$.

4. The LMY Proposal

In this chapter a detailed review of the paper [12] by Leigh, Minic and Yelnikov (LMY) is given, where the authors want to determine the glueball mass spectrum of pure Yang-Mills theory in $2+1$ dimensions using the KKN formalism. To this end, they try to find the quasi-Gaussian part of the vacuum wave functional by solving the Schrödinger equation up to second order in the KKN variable $\bar{\partial}J$. The hope is that this approximation captures enough non-perturbative information to allow for reliable predictions about the infra-red sector of the theory.

After making a very non-trivial assumption about the spectrum of the kinetic energy operator when acting on a certain class of operators, an analytic solution to the Schrödinger equation in this approximation can be obtained. This solution has the correct high energy limit corresponding to asymptotically free gluons, and — more important — also the correct low energy limit to give rise to confinement, with a string tension that is in good agreement with lattice simulation data. Furthermore, using this vacuum wave functional, the glueball masses can be inferred from the exponential fall-off of vacuum expectation values of appropriate gauge-invariant operators. The agreement of those glueball masses with lattice data is claimed to be "excellent" by the authors, but this statement has to be qualified as discussed here in section 4.2.

The main problem of their calculations is that LMY do not succeed in verifying the aforementioned assumption, and so the proposed vacuum wave functional must be considered conjectural. It would therefore be of great interest to have an independent approach to test their conjecture. One possibility will be explored here in section 6.2.

The other problem is that it is a priori not clear whether (or to what extent, or in which regime) the quasi-Gaussian truncation really approximates the true vacuum state. The main argument in favour of its validity is given a posteriori by the good agreement with lattice simulation data.

4.1. Vacuum Wave Functional

As just explained, the strategy pursued by LMY to obtain the vacuum functional Ψ_0 is to solve the time independent Schrödinger equation

$$H\Psi_0 = E_0 \Psi_0 \tag{4.1}$$

where E_0 denotes the (divergent) vacuum energy. Since it is very hard to find the exact solution, the authors of [12] content themselves with solving (4.1) up to second order in the current $\bar{\partial}J$, i.e. finding the quasi-Gaussian generalization of the QED case[1].

---

[1]See appendix B.

4.1.1. Ansatz

As argued by Feynman in [1], Ψ_0 can be assumed to be real and positive. Furthermore, it should be invariant under gauge- and holomorphic transformations, as well as under spacetime symmetries. Thus, according to the considerations made in sections 3.4 and 3.5, one can make the general ansatz

$$
\begin{aligned}
\Psi_0 &= \exp\left[-\frac{N}{\pi m^2}\int_z \mathrm{Tr}\left(\bar{\partial}J(\boldsymbol{z})\,K\left(\frac{\Delta}{m^2}\right)\bar{\partial}J(\boldsymbol{z}) + \ldots\right)\right] \\
&= \exp\left[-\frac{N}{2\pi m^2}\int_z\left(\bar{\partial}J^a(\boldsymbol{z})\,K^{ab}\left(\frac{\Delta}{m^2}\right)\bar{\partial}J^b(\boldsymbol{z}) + \ldots\right)\right] =: e^P,
\end{aligned}
\tag{4.2}
$$

where the ellipsis refers to quartic and higher terms in $\bar{\partial}J$ (with additional kernels), which will be neglected. Note that since the kernel K is a function of the h-covariant Laplacian Δ which depends on J, this ansatz is not purely quadratic in the current (and therefore called "quasi-Gaussian"). But it is also not the most general ansatz, because only those terms of higher order in J that are necessary for consistency are taken into account.

Now K can be thought of as a formal power series of the dimensionless argument $L := \Delta/m^2$

$$
K(L) = \sum_{n=0}^{\infty} c_n L^n
\tag{4.3}
$$

or in components

$$
K^{ab}(L) = \sum_{n=0}^{\infty}\frac{c_n}{m^{2n}}(\Delta^n)^{ab}, \quad (\Delta^n)^{ab} = \Delta^{ac_1}\Delta^{c_1 c_2}\cdots\Delta^{c_{n-1}b}
\tag{4.4}
$$

and has to be determined by solving the Schrödinger equation (4.1). Even though the ansatz (4.2) looks as if it was local (the currents are evaluated at the same point), the fact that K may be written as an *infinite* power series allows for a non-local solution.

The strategy is now to apply the Hamiltonian (3.95) to Ψ_0 and only keep terms up to second order in J. Terms independent of J will be identified with the vacuum energy and those quadratic in J have to be equal to zero. This will in the end result in a differential equation for K as a function of L.

4.1.2. Calculation

Functional derivatives acting on $\Psi_0 = e^P$ give

$$
\frac{\delta\Psi_0}{\delta J^a(\boldsymbol{z})} = \frac{\delta P}{\delta J^a(\boldsymbol{z})}\Psi_0,
\tag{4.5a}
$$

$$
\frac{\delta^2\Psi_0}{\delta J^a(\boldsymbol{z})\delta J^b(\boldsymbol{z}')} = \left(\frac{\delta^2 P}{\delta J^a(\boldsymbol{z})\delta J^b(\boldsymbol{z}')} + \frac{\delta P}{\delta J^a(\boldsymbol{z})}\frac{\delta P}{\delta J^b(\boldsymbol{z}')}\right)\Psi_0
\tag{4.5b}
$$

and thus

$$
\mathrm{H}\Psi_0 = \left[(\mathrm{T}P) + \underbrace{\frac{\pi m}{N}\int_{z,z'}\Omega^{ab}(\boldsymbol{z},\boldsymbol{z}')\frac{\delta P}{\delta J^a(\boldsymbol{z})}\frac{\delta P}{\delta J^b(\boldsymbol{z}')}}_{=:C} + \frac{N}{\pi m}\int_z \bar{\partial}J^a(\boldsymbol{z})\bar{\partial}J^a(\boldsymbol{z})\right]\Psi_0.
\tag{4.6}
$$

For the evaluation of functional derivatives acting on P, first note that the h-covariant derivative in adjoint representation can be "integrated by parts" just like a usual partial derivative, meaning that for any functions ϕ^a and ψ^a vanishing at infinity we have the identity

$$
\begin{aligned}
\int \phi^a (D\psi)^a &= \int \phi^a D^{ab} \psi^b = \int \phi^a \left(\delta^{ab} \partial - J^{ab} \right) \psi^b \\
&\overset{\text{i.b.p.}}{=} \int \psi^b \left(-\delta^{ab} \partial \phi^a - J^{ab} \phi^a \right) \\
&\overset{a \leftrightarrow b}{=} \int \psi^a \left(-\delta^{ab} \partial \phi^a + J^{ab} \phi^b \right) \\
&= -\int D^{ab} \phi^b \psi^a = -\int (D\phi)^a \psi^a .
\end{aligned}
\tag{4.7}
$$

The minus sign for the derivative coming from the integration by parts in the second line arises for the J-part in the third line because J^{ab} is anti-symmetric in a and b. From this it immediately follows for the h-covariant Laplacian that

$$
\int \phi^a (\Delta\psi)^a = + \int (\Delta\phi)^a \psi^a .
\tag{4.8}
$$

Since K can be expressed as a power series in Δ, the same identity holds with Δ replaced by K. Using this one finds

$$
\frac{\delta P}{\delta J^a(\boldsymbol{z})} = -\frac{N}{2\pi m^2} \left\{ -2\bar{\partial} \left[K^{ab}(L) \bar{\partial} J^b(\boldsymbol{z}) \right] + \int_{z'} \bar{\partial} J^b(\boldsymbol{z}') \frac{\delta K^{bc}(L_{z'})}{\delta J^a(\boldsymbol{z})} \bar{\partial} J^c(\boldsymbol{z}') \right\}
\tag{4.9a}
$$

$$
= \frac{N}{\pi m} \left[\frac{\bar{\partial}}{m} K \left(\frac{\bar{\partial}\partial}{m^2} \right) \right] \bar{\partial} J^a(\boldsymbol{z}) + \mathcal{O}(J^2)
\tag{4.9b}
$$

Only the linear part of this is needed to calculate the term C in (4.6) up to second order in J. Likewise, the h-covariant derivative contained in Ω in C can be replaced by an ordinary derivative and since the currents in C are not evaluated at the same point, we can use the unregulated result (3.96b) and find[2]

$$
\begin{aligned}
C &= \frac{N}{\pi m} \int_{z,z'} \partial_{z'} \bar{G}(\boldsymbol{z}, \boldsymbol{z}') \left[\frac{\bar{\partial}}{m} K \left(\frac{\bar{\partial}\partial}{m^2} \right) \right]_z \bar{\partial} J^a(\boldsymbol{z}) \left[\frac{\bar{\partial}}{m} K \left(\frac{\bar{\partial}\partial}{m^2} \right) \right]_{z'} \bar{\partial} J^a(\boldsymbol{z}') \\
&= \frac{N}{\pi m} \int_{z,z'} \underbrace{\bar{\partial}_z \bar{G}(\boldsymbol{z}, \boldsymbol{z}')}_{=\delta^2(\boldsymbol{z}-\boldsymbol{z}')} \left[K \left(\frac{\bar{\partial}\partial}{m^2} \right) \right]_z \bar{\partial} J^a(\boldsymbol{z}) \left[\frac{\bar{\partial}\partial}{m^2} K \left(\frac{\bar{\partial}\partial}{m^2} \right) \right]_{z'} \bar{\partial} J^a(\boldsymbol{z}') \\
&= \frac{N}{\pi m} \int_z \bar{\partial} J^a(\boldsymbol{z}) \frac{\bar{\partial}\partial}{m^2} K^2 \left(\frac{\bar{\partial}\partial}{m^2} \right) \bar{\partial} J^a(\boldsymbol{z}) \\
&= \frac{N}{\pi m} \int_z \bar{\partial} J^a(\boldsymbol{z}) \left[L\, K^2(L) \right]^{ab} \bar{\partial} J^b(\boldsymbol{z})
\end{aligned}
\tag{4.10}
$$

where all terms of higher order than quadratic in J have been suppressed. So equation (4.6) now becomes

$$
\mathrm{H}\Psi_0 = \left[(\mathrm{T}P) + \frac{N}{\pi m} \int_z \bar{\partial} J^a(\boldsymbol{z}) \left[LK^2(L) + 1 \right]^{ab} \bar{\partial} J^b(\boldsymbol{z}) + \mathcal{O}(J^3) \right] \Psi_0 .
\tag{4.11}
$$

---

[2]The subscripts emphasise which argument the derivatives act on.

The evaluation of TP is more involved, because there are two functional derivatives acting on P which have to be calculated to second order in J. Consequently, one has to keep the second term in (4.9a) which is not easy to compute. Furthermore, there are divergent expressions arising that have to be dealt with by using the regularised version of Ω. Because of those complications, LMY do not succeed in finding TP to second order in J by direct calculation. Instead, they *conjecture* that T applied to the holomorphic invariant operators

$$\mathcal{O}_n := \int \bar{\partial} J^a (\Delta^n)^{ab} \bar{\partial} J^b \tag{4.12}$$

should give

$$\text{T} : \mathcal{O}_n: \overset{?}{=} (2+n)m :\mathcal{O}_n: + \mathcal{O}(J^3) \tag{4.13}$$

where $: \cdot :$ denotes *normal ordering*, which means that divergent terms contributing to E_0 have been subtracted.

Their heuristic argument in favour of this conjecture is that $\text{T}_1 \sim J\delta/\delta J$ basically counts the number of J's in each term of $\mathcal{O}_n$ when expanded in powers of J, which will give something that is not holomorphic invariant. T_2 will then only serve to restore holomorphic invariance. Since $\mathcal{O}_n$ contains a term with $2+n$ J's, they expect the general rule (4.13) to hold.

This rather naive reasoning indeed turns out to work out for $n = 0$ and $n = 1$, at least when one uses one particular approximate form of Ω.[3] But what is puzzling about this argument is that the two parts of T should actually be separately h-invariant (at least up to boundary terms, cf. the remark below equation (3.95)) — this was the essential requirement in the derivation of T$[J]$, and in particular the appearance of the $J\delta/\delta J$ term strongly depends on demanding a h-covariant regulator. So the apparent restoration of h-invariance in the $n = 1$ case might rather be an accident, and the interplay between T_1 and T_2 might be more subtle than argued by LMY.

In fact they find that for $n = 2$ the conjecture fails, but then they claim that this discrepancy might stem from the fact that the $\mathcal{O}_n$ are *local* operators, and so it may be that they "have not taken proper account of the nonlocal character of the theory" [12, p. 19]. Indeed, the calculations were so far done by LMY using an approximate form of Ω which is actually not allowed when T_2 acts on local operators such as $\mathcal{O}_n$. But the problem is, that using a more exact form of Ω as proposed by KKN leads to disagreement with the conjecture even for $n = 0$ and 1.

So in the end LMY admit that they were not able to prove their conjecture (4.13); but they use it anyway with the main justification that in the end it leads to "sensible physical results" [12, p. 7].[4] This is of course rather unsatisfying, and it would be of great interest to get another, independent verification (or maybe falsification) of equation (4.13).

Disregarding all those problems for a moment, and just assuming that (4.13) is true,

---

[3]These calculations are repeated and extended as well as discussed in much detail in chapter 5.

[4]Another, more recent justification for using equation (4.13) is given in the paper by Fukuma et al. [20], where the regularised Hamiltonian is derived in a different way and the same Kernel ODE (4.16) is obtained for the Gaussian part of the vacuum wave functional. But this derivation also relies on an assumption that is put in by hand and can thus not yet be considered rigorous.

one finds

$$
\begin{aligned}
\mathrm{T}P &= -\frac{N}{2\pi m^2}\mathrm{T}\left(\sum_{n=0}^{\infty}\frac{c_n}{m^{2n}}\mathcal{O}_n\right) = -\frac{N}{2\pi m}\left(\sum_{n=0}^{\infty}\frac{c_n}{m^{2n}}(2+n)\mathcal{O}_n\right) + E_0 \\
&= -\frac{N}{2\pi m}\int \bar\partial J^a\left(\sum_{n=0}^{\infty}c_n(2+n)(L^n)^{ab}\right)\bar\partial J^b + E_0 \\
&= \frac{N}{\pi m}\int \bar\partial J^a\left(-K(L) - \frac{L}{2}\frac{\mathrm{d}}{\mathrm{d}L}K(L)\right)^{ab}\bar\partial J^b + E_0
\end{aligned}
\tag{4.14}
$$

where higher order terms in J were again neglected. Plugging the intermediate results (4.6), (4.10) and (4.14) into (4.1) gives:

$$
\int \bar\partial J^a\left(-K - \frac{L}{2}\frac{\mathrm{d}}{\mathrm{d}L}K + LK^2 + 1\right)\bar\partial J^a + \mathcal{O}(J^3) = 0
\tag{4.15}
$$

which finally yields the following differential equation for the kernel:

$$
-K - \frac{L}{2}\frac{\mathrm{d}}{\mathrm{d}L}K + LK^2 + 1 = 0.
\tag{4.16}
$$

4.1.3. Solution

The general solution of the differential equation (4.16) is[5]

$$
K(L) = \frac{1}{\sqrt{L}}\frac{CJ_2(4\sqrt{L}) + Y_2(4\sqrt{L})}{CJ_1(4\sqrt{L}) + Y_1(4\sqrt{L})}
\tag{4.17}
$$

where C is a constant of integration and J_n and Y_n denote the Bessel functions of the first and second kind respectively. The only normalisable vacuum wave functional is obtained for a kernel that is everywhere positive, since otherwise Ψ_0 would grow exponentially for some field configurations. This can only be achieved by taking $C \to \infty$, in which case the kernel becomes

$$
K(L) = \frac{1}{\sqrt{L}}\frac{J_2(4\sqrt{L})}{J_1(4\sqrt{L})} = \frac{1}{\sqrt{|L|}}\frac{I_2(4\sqrt{|L|})}{I_1(4\sqrt{|L|})}.
\tag{4.18}
$$

Here the Bessel functions J_n were replaced by modified Bessel functions I_n using the identities

$$
J_1(ix) = iI_1(x) \qquad J_2(ix) = -I_2(x)
\tag{4.19}
$$

because L (in momentum space) is negative.

Asymptotic Behaviour

There are now two asymptotic regimes of special interest: the high energy (or UV) limit, for which we know from perturbation theory that the vacuum wave functional must approach the QED result, and of course the low energy (or IR) limit, which can give information about the confining properties of the theory. Since

$$
L = \frac{1}{m^2}\partial\bar\partial + \mathcal{O}(\bar\partial, J) = -\frac{1}{4m^2}\boldsymbol{p}^2 + \mathcal{O}(p_i, J),
\tag{4.20}
$$

---

[5]Here we only quote the results; for the detailed derivation see [12].

the UV and IR limit correspond to $L \sim -\boldsymbol{p}^2/4m^2 \to -\infty$ and $L \to 0$ respectively. In these limits the kernel has the following asymptotic behaviour:

$$K(L) \sim \begin{cases} 1/\sqrt{-L} & (L \to -\infty) \\ 1 & (L \to 0) \end{cases} \tag{4.21}$$

So the vacuum wave functional for high energies becomes

$$\Psi_0^{\mathrm{UV}} = \exp\left[-\frac{N}{2\pi m^2} \int_z \bar{\partial} J^a(\boldsymbol{z}) \frac{m}{\sqrt{-\partial\bar{\partial}}} \bar{\partial} J^a(\boldsymbol{z}) \right]$$

$$= \exp\left[-\frac{N}{2\pi m} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{1}{4} B^a(\boldsymbol{p}) \frac{2}{\sqrt{\boldsymbol{p}^2}} B^a(-\boldsymbol{p}) \right] \tag{4.22}$$

$$= \exp\left[-\frac{1}{2g^2} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} B^a(\boldsymbol{p}) \frac{1}{|\boldsymbol{p}|} B^a(-\boldsymbol{p}) \right]$$

and thus correctly reproduces the result for *asymptotically free* gluons (cf. the end of appendix B).

In the IR limit one finds

$$\Psi_0^{\mathrm{IR}} = \exp\left[-\frac{N}{8\pi m^2} \int_x [B^a(\boldsymbol{x})]^2 \right] = \exp\left[-\frac{1}{4g^2 m} \int_x B^a(\boldsymbol{x})^2 \right]. \tag{4.23}$$

As argued by KKN in [9] and explained below, this vacuum wave functional can be used to deduce an area law behaviour of a Wilson loop, corresponding to a *confining* theory, with a string tension

$$\sigma = g^4 \frac{N^2 - 1}{8\pi}, \tag{4.24}$$

which agrees very well ($\sim 3\%$) with lattice simulation data [21].

This result can be derived by means of *dimensional reduction* ([3], [22]): The vacuum expectation value of a Wilson loop [2] $W_C = \mathrm{Tr}\,\mathcal{P}\exp[-\oint_C \mathrm{d}x_i A_i]$ along a curve C is given by the path-integral of W_C weighted with the modulus squared of the vacuum wave functional. Using (4.23) as an approximation will give a factor of $\exp\left[-\frac{1}{2g^2 m}\int_x B^a(\boldsymbol{x})^2\right]$, which is the same as the exponential of the euclidean action of pure Yang-Mills theory in $1+1$ dimensions with coupling $g_2 = mg^2$. So the calculation reduces to that in a $1+1$ dimensional Yang-Mills theory, which can be done exactly: Choosing the gauge $A_2 = 0$ leads to

$$\langle W_C \rangle = \int [dA]\,\mathrm{Tr}\,\mathcal{P}\exp\left[-\oint_C \mathrm{d}x_1 A_1 \right] \exp\left[-\frac{1}{2g^2 m} \int_x \partial_2 A_1 \partial_2 A_1 \right] \tag{4.25a}$$

$$= \mathrm{Tr}\,\mathcal{P}\exp\left[\frac{mg^2}{2} \oint \mathrm{d}x_1 \oint \mathrm{d}y_1 \mathcal{D}(\boldsymbol{x}, \boldsymbol{y}) t^a t^a \right] \tag{4.25b}$$

where $\mathcal{D}$ is the Green's function of ∂_2^2:

$$\partial_2^2 \mathcal{D}(\boldsymbol{x}, \boldsymbol{y}) = \delta^{(2)}(\boldsymbol{x} - \boldsymbol{y}) \tag{4.26a}$$

$$\Rightarrow \mathcal{D}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}|x_2 - y_2|\,\delta^{(1)}(x_1 - y_1) \tag{4.26b}$$

We take the integration contour C to be an rectangle with sides T and R in 1- and 2-direction, respectively. The double line integral in (4.25b) can now easily be computed:

$$\oint \mathrm{d}x_1 \oint \mathrm{d}y_1 \frac{1}{2}|x_2 - y_2| \, \delta^{(1)}(x_1 - y_1) = -RT. \tag{4.27}$$

The factor $t^a t^a$ is just the Casimir operator in fundamental representation, which in our conventions equals $\frac{N^2-1}{2N}$ and so the Wilson loop has an area law behaviour

$$\langle W_C \rangle = N \exp\left[-\frac{mg^2(N^2-1)}{4N}RT\right] = N \exp\left[-\sigma RT\right] \tag{4.28}$$

with the string tension σ given by equation (4.24) as was to be shown.

Since the vacuum expectation value of a Wilson loop along this rectangular curve in the limit $T \gg R$ goes to $\mathrm{e}^{-E(R)T}$, where $E(R)$ denotes the energy of two (static, i.e. infinitely heavy) point sources separated by a distance R (see for example [23, pp. 110 - 111]), the area law shows that such sources feel a confining, linear potential $E(R) = \sigma R$.

Fourier Transform

As explained in [12], one can use standard Bessel function identities to rewrite the inverse kernel as

$$K^{-1}(L) = 1 + 8L \sum_{n=1}^{\infty} \frac{1}{16L - (\gamma_{2,n})^2} \tag{4.29}$$

where $\gamma_{2,n}$ denote the ordered zeros of J_2. Inserting $L \approx \partial\bar{\partial}/m^2 = -\boldsymbol{p}^2/4m^2$ the inverse kernel as a function of momentum $\boldsymbol{p}$ then becomes

$$K^{-1}(\boldsymbol{p}) = 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\boldsymbol{p}^2}{\boldsymbol{p}^2 + M_n^2}, \quad M_n := \frac{m\gamma_{2,n}}{2}. \tag{4.30}$$

The Fourier transform of this is

$$K^{-1}(|\boldsymbol{x}-\boldsymbol{y}|) = \delta(\boldsymbol{x}-\boldsymbol{y}) + \frac{1}{2}\sum_{n=1}^{\infty}\left(\delta(\boldsymbol{x}-\boldsymbol{y}) - \frac{(M_n)^2}{2\pi}K_0(M_n|\boldsymbol{x}-\boldsymbol{y}|)\right). \tag{4.31}$$

and at asymptotically large spatial separation $|\boldsymbol{x}-\boldsymbol{y}| \to \infty$ this becomes

$$K^{-1}(|\boldsymbol{x}-\boldsymbol{y}|) \approx -\frac{1}{4\sqrt{2\pi|\boldsymbol{x}-\boldsymbol{y}|}}\sum_{n=1}^{\infty}(M_n)^{3/2}e^{-M_n|\boldsymbol{x}-\boldsymbol{y}|}. \tag{4.32}$$

4.2. Glueball Spectrum

Using Ψ_0 as motivated in section 4.1 as the vacuum wave functional, the glueball spectrum of the theory can in principle be probed by evaluating vacuum expectation values of operators with the desired quantum numbers J^{PC}. But the actual evaluation of the corresponding path integral with the WZW measure factor would be very difficult. To simplify

things, LMY argue that $\bar{\partial}J$ can be regarded as a free field and so their corresponding two point function should approximately be given by

$$\langle \bar{\partial}J^a(\boldsymbol{x})\bar{\partial}J^b(\boldsymbol{y})\rangle \approx \delta^{ab}\frac{m^2\pi}{2N}K^{-1}(\boldsymbol{x}-\boldsymbol{y}). \tag{4.33}$$

To justify this assertion one should actually make another change of variables in the path integral from H to $\bar{\partial}J$ and show that in doing so the factor $e^{2NS_{WZW}}$ disappears. This argument was indeed brought forward by Leigh et al. in a more recent review article [24]:

As we saw in section 3.6, the change of variables from $(A, \bar{A})$ to H gives rise to a factor of $\det(\nabla\overline{\nabla})$. If we now consider $\bar{\partial}J$ as the variable of integration in the path integral we will get another Jacobian. A calculation analogous to (3.56) yields

$$\mathrm{d}J = [\hat{D}, \mathrm{d}HH^{-1}] \tag{4.34}$$

and consequently

$$\mathrm{d}\mu[H] = \left[\det(\hat{\bar{\partial}}\hat{D})\right]^{-1}\mathrm{d}\mu[\bar{\partial}J]. \tag{4.35}$$

Furthermore, recalling $\hat{\bar{\partial}} = M^\dagger\hat{\overline{\nabla}}M^{\dagger-1}$ from equation (3.63b) as well as $\hat{D} = M^\dagger\hat{\nabla}M^{\dagger-1}$ (which can be shown in the same way), we see that this Jacobian cancels the one we obtained first, as was to be shown. So the correlator can be written as

$$\langle \bar{\partial}J^a(\boldsymbol{x})\bar{\partial}J^b(\boldsymbol{y})\rangle = \int \mathrm{d}\mu[\bar{\partial}J]\,\bar{\partial}J^a(\boldsymbol{x})\bar{\partial}J^b(\boldsymbol{y})\,|\Psi_0|^2 \tag{4.36}$$

which leads to the claimed equality (4.33) in the approximation that interactions coming from the J-terms in K are neglected, i.e. when $K(L)$ is replaced by $K(\boldsymbol{p})$.

4.2.1. Spin 0 States

According to the discussion of quantum numbers in section 3.5, an operator with $J^{PC} = 0^{++}$ is given by $\mathrm{Tr}(\bar{\partial}J\bar{\partial}J)$. Using (4.33), the corresponding correlator can be approximated by

$$\langle \mathrm{Tr}(\bar{\partial}J\bar{\partial}J)_x\,\mathrm{Tr}(\bar{\partial}J\bar{\partial}J)_y\rangle \approx (\text{const.})\left(K^{-1}(|\boldsymbol{x}-\boldsymbol{y}|)\right)^2. \tag{4.37}$$

The approximation neglects interactions encoded in the J dependence of the kernel, but its nonlocal spatial properties are kept. Using (4.32) then gives

$$\langle \mathrm{Tr}(\bar{\partial}J\bar{\partial}J)_x\,\mathrm{Tr}(\bar{\partial}J\bar{\partial}J)_y\rangle \sim \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\sum_{m,n=1}^{\infty}(M_mM_n)^{3/2}\,e^{-(M_m+M_n)|\boldsymbol{x}-\boldsymbol{y}|} \tag{4.38}$$

and the glueball masses can now be inferred from the coefficient in the exponent to be (the * denotes excited states)

$$M(0^{++(*\cdots*)}) = M_m + M_n \quad (m, n \in \mathbb{N}). \tag{4.39}$$

It should be mentioned that Leigh et al. write these masses in a slightly different way, viz.

$$M(0^{++(*\cdots*)}) = M_1 + M_n \quad (n \in \mathbb{N}), \tag{4.40}$$

without any justification. Of course, this makes no difference for the lowest lying states, but they claim the mass of the fourth excited state to be $M(0^{++****}) = M_1 + M_5 = 11.55m$, even though $M_2 + M_3 = 10.02m$ would be smaller than this and perfectly allowed according to the correlator (4.38).

The mass parameter m can be related to the string tension by (4.24) and so all masses can be expressed in units of $\sqrt{\sigma}$. These predictions can then be compared to results from lattice simulations, without adjusting any further parameters. This is a nice feature of LMY's approach to the glueball spectrum, as opposed for example to the supergravity approach [25], which only gives ratios of different glueball masses that need to be fitted to the lattice data.

LMY find excellent agreement (0.8%) of their lowest 0^{++} mass $M(0^{++})_{\text{LMY}} = 4.098\sqrt{\sigma}$ with the lattice value[6] $M(0^{++})_{\text{lattice}} = (4.065 \pm 0.055)\sqrt{\sigma}$. The excited 0^{++} masses differ from the lattice values by $12 - 15\%$, which is a significant deviation since it is about 6 times larger than the given error bars would allow. LMY argue that this discrepancy could arise because the first two excited states might not have been properly resolved in the lattice simulations. If one shifts the labelling of the lattice values for all higher states accordingly, the agreement becomes much better: 0.5%[7] for $M(0^{++***})$ and 2.4% for $M(0^{++****})$, which would be inside the given error bars. But this ad hoc assumption is rather arbitrary, and is not justified by other means than observing that it improves the agreement between their predictions and lattice results.

It should be noted that LMY also give their predictions for 0^{--} states, which they probe with the operator $\text{Tr}(\bar{\partial}J\bar{\partial}J\bar{\partial}J)$. Here the agreement with lattice data is good ($2 - 4\%$, which is within the lattice error bars) without relabelling anything.

4.2.2. Spin 2 States

An operator which is suitable for probing 2^{++} states is given by $\text{Tr}(\bar{\partial}^2 J \bar{\partial}^2 J)$, and the corresponding correlator is

$$
\begin{aligned}
\langle \text{Tr}(\bar{\partial}^2 J \bar{\partial}^2 J)_x \, \text{Tr}(\bar{\partial}^2 J \bar{\partial}^2 J)_y \rangle &\sim \left(\bar{\partial}_x \bar{\partial}_y K^{-1}(|\boldsymbol{x} - \boldsymbol{y}|) \right)^2 \\
&\sim \frac{|\boldsymbol{x} - \boldsymbol{y}|^3}{(\bar{x} - \bar{y})^4} \sum_{m,n=1}^{\infty} (M_m M_n)^{7/2} \, \mathrm{e}^{-(M_m + M_n)|\boldsymbol{x} - \boldsymbol{y}|}.
\end{aligned} \tag{4.41}
$$

The problem is that the 2^{++} masses are thus the same as the ones of 0^{++} states as given in equation (4.39). To resolve this paradox, LMY suggest that the 2^{++} masses should actually be given by

$$
M(2^{++(*\cdots*)}) = M_2 + M_{1+n} \quad (n \in \mathbb{N}), \tag{4.42}
$$

in contrast to (4.40). But again, this assertion is just put in by hand and there is no justification for its validity.

---

[6] The lattice values are taken from [26]. The comparisons by LMY are made for the masses in the limit $N \to \infty$; it should be noted, however, that the agreement for $M(0^{++})$ is in fact very good for any value of N given in [26].

[7] LMY claim it to be 0.05%, but this percentage arises from a wrong number they list in table III in [12], viz. 7.994 instead of the correct value 7.951. Also the value for the third excited mass should actually be 6.685 instead of 6.716. These errors already occur there in table II.

As in the 0^{++} case the agreement with lattice data for the lowest state is good (2.4%), viz. $M(2^{++})_{\text{LMY}} = 6.72\sqrt{\sigma}$ compared to $M(2^{++})_{\text{lattice}} = (6.88 \pm 0.16)\sqrt{\sigma}$, whereas the first two excited masses differ from lattice calculations by 7.6% and 13%. Again, LMY suspect the lattice masses to be labelled in a wrong way, and after relabelling they get much better agreement ($< 1\%$) for $M(2^{++**})$ and $M(2^{++***})$, the deviation of $M(2^{++*})$ remains unchanged at 7.6%. But as before, there is no real justification for relabelling the lattice masses in this peculiar way.

LMY also discuss spin two states with different C and P quantum numbers. Here they also find agreement with lattice values of about $1 - 10\%$, but with the same caveats as before. For instance, they assume that 0^{--} masses are given by $M_1 + M_1 + M_n$ ($n \in \mathbb{N}$) and 2^{--} masses by $M_1 + M_2 + M_{1+n}$ ($n \in \mathbb{N}$), even though in *both* cases their correlators only predict $M_m + M_n + M_k$ ($m, n, k \in \mathbb{N}$).

4.2.3. Summary

To summarise, one can say that the *only* predicted glueball mass that i) comes out of the calculation without further speculative assumptions about identification and labelling of states, and that ii) is in excellent agreement (0.8%) with lattice computations, is the lightest 0^{++} mass. The excited 0^{++} masses as well as the 0^{--} masses also fulfil requirement i), but they differ from the (unrelabelled) lattice predictions by $10 - 15\%$ and $2 - 4\%$, respectively.

In view of these qualifications, the claim of Leigh et al. in their abstract that "the agreement [of the mass spectrum] with available lattice data is excellent" [12, p. 1], seems somewhat exaggerated. Since this "excellent" agreement is one of the justifications for their key conjecture (4.13), its validity becomes more questionable.

5. Testing the LMY Conjecture

Since the conjecture (4.13) made by LMY is the crucial ingredient for the deviation of the approximation to the vacuum wave functional found in section 4.1, it is worthwhile to check it by direct calculation. Since those computations become quite involved for higher n, only the cases $n = 0, 1, 2$ will be considered. Let us for convenience first write down the conjecture (4.13) once again:

$$\text{T} : \mathcal{O}_n : \overset{?}{=} (2+n)m : \mathcal{O}_n : + \mathcal{O}(J^3) \tag{5.1a}$$

$$\mathcal{O}_n = \int \bar{\partial} J^a (\Delta^n)^{ab} \bar{\partial} J^b \tag{5.1b}$$

The kinetic energy operator consists of parts:

$$\text{T} = \text{T}_1 + \text{T}_2 \tag{5.2a}$$

$$\text{T}_1 = m \int_z J_z^a \frac{\delta}{\delta J_z^a} \tag{5.2b}$$

$$\text{T}_2 = m \frac{\pi}{N} \int_{z,z'} \Omega_{z,z'}^{ab} \frac{\delta}{\delta J_z^a} \frac{\delta}{\delta J_{z'}^b} \tag{5.2c}$$

with the compact notation $J_z^a \equiv J^a(\boldsymbol{z})$ etc. The action of T_1 on $\mathcal{O}_n$ up to second order in J is easily evaluated for general n:

$$\frac{\delta \mathcal{O}_n}{\delta J_z^a} = -2\bar{\partial} \left[(\Delta^n)^{ab} \bar{\partial} J_z^b \right] + \int \bar{\partial} J_{z'}^b \frac{\delta (\Delta^n)_{z'}^{bc}}{\delta J_z^a} \bar{\partial} J_{z'}^c \tag{5.3}$$

and therefore

$$\text{T}_1 \mathcal{O}_0 = 2m\mathcal{O}_0 \tag{5.4a}$$

$$\text{T}_1 \mathcal{O}_n = 2m\mathcal{O}_n + \mathcal{O}(J^3) \qquad (n > 0). \tag{5.4b}$$

The action of T_2 is more involved because one has to keep the second term in (5.3) (for $n > 0$), and there are divergent expressions arising which make it necessary to use a regulated version of Ω and to introduce normal ordering. Recall from the end of section

3.7.3 that the carefully regulated form of Ω was found by Karabali et al. in [10] to be

$$\Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') = D^{bc}_{z'}\Lambda^{ca}(\boldsymbol{z}', \boldsymbol{z}) \tag{5.5a}$$

$$\Lambda^{ca}(\boldsymbol{z}', \boldsymbol{z}) = \int_x \bar{\mathcal{G}}^{dc}(\boldsymbol{x}, \boldsymbol{z}')G(\boldsymbol{x}, \boldsymbol{z})\mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{z}|^2/\epsilon}\left[\frac{\bar{x}-\bar{z}}{\epsilon}H(x, \bar{x})H^{-1}(z, \bar{x})\right.$$

$$\left. + H(\boldsymbol{x})\partial_z\left(H^{-1}(z, \bar{x})H(\boldsymbol{z})\right)H^{-1}(\boldsymbol{z})\right]^{da} \tag{5.5b}$$

$$= \frac{1}{\pi(z-z')}\left[1 - \mathrm{e}^{-|\boldsymbol{z}-\boldsymbol{z}'|^2/2\epsilon}H(\boldsymbol{z}')H^{-1}(u, \bar{z}')H(\boldsymbol{u})H^{-1}(z, \bar{u})\right]^{ca} \tag{5.5c}$$

$$+ \frac{\epsilon}{\pi}\mathrm{e}^{-|\boldsymbol{z}-\boldsymbol{z}'|^2/2\epsilon}[...]^{ca}$$

$$+ \mathcal{O}(\epsilon^2)$$

$$= \bar{G}(\boldsymbol{z}, \boldsymbol{z}')\left(\delta^{ca} - \mathrm{e}^{-|\boldsymbol{z}-\boldsymbol{z}'|^2/2\epsilon}\left[H(\boldsymbol{z}')H^{-1}(z, \bar{z}')\right]^{ca}\right) + \ldots \tag{5.5d}$$

For the following calculations, i.e. when T acts on products of $\bar{\partial}J$ at the same point, one should actually use the exact form of Λ displayed in (5.5b). Karabali et al. even explicitly state in their paper that there may be contributions for such calculations from the ... terms in (5.5d) [10, p. 681]. But the approximate form of Λ displayed in (5.5d) is the one LMY choose for their calculations, which we will repeat here first because this demonstrates the subtle dependence of the eigenvalues of $\mathcal{O}_n$ on the exact regularisation.

5.1. First Approximation

In this section we will follow LMY's lead and use as a first approximation

$$\Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') = D^{bc}_{z'}\Lambda^{ca}_{(1)}(\boldsymbol{z}', \boldsymbol{z}), \tag{5.6a}$$

$$\Lambda^{ca}_{(1)}(\boldsymbol{z}, \boldsymbol{z}') := \frac{1}{\pi(z-z')}\left(\delta^{ca} - \mathrm{e}^{-\alpha}\left[H(\boldsymbol{z}')H^{-1}(z, \bar{z}')\right]^{ca}\right), \tag{5.6b}$$

$$\alpha := \frac{|\boldsymbol{z}-\boldsymbol{z}'|^2}{2\epsilon}. \tag{5.6c}$$

5.1.1. n = 0

This is the easiest case, in which the second term in (5.3) is absent and there is only one divergent term. This calculation will also show how those divergent normal ordering terms arise in general which will prove helpful for the $n > 0$ cases.

The second functional derivative of $\mathcal{O}_0$ is

$$\frac{\delta^2\mathcal{O}_0}{\delta J^a_z \delta J^b_{z'}} = -2\delta^{ab}\bar{\partial}^2\delta(\boldsymbol{z}-\boldsymbol{z}') \tag{5.7}$$

and thus

$$\mathrm{T}_2\mathcal{O}_0 = -2m\frac{\pi}{N}\int_z\left[\bar{\partial}^2_z\Omega^{aa}(\boldsymbol{z}, \boldsymbol{z}')\right]_{z'\to z}. \tag{5.8}$$

To evaluate the coincidence limit of $\bar{\partial}^2\Omega$ just expand Ω in powers of $\boldsymbol{z} - \boldsymbol{z}'$ and note that the dangerous $1/(z-z')$ term is effectively absent in the sense that the lowest power of

$z - z'$ appearing in the expansion of Λ is not -1 but 0. Therefore there is no delta function arising when acting with $\bar{\partial}$ on Ω and so $\bar{\partial}$ only acts on $e^{-\alpha}$, giving

$$
\bar{\partial}_z^2 \Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') = D_{z'}^{bc} \left\{ \frac{-1}{\pi(z - z')} \frac{(z - z')^2}{(2\epsilon)^2} e^{-\alpha} \left[H(\boldsymbol{z}')H^{-1}(z, \bar{z}') \right]^{ca} \right\}
$$

$$
= (\delta^{bc}\partial_{z'} - J_{z'}^{bc}) \left\{ \frac{-1}{\pi} \frac{e^{-\alpha}}{(2\epsilon)^2} \sum_{n=0}^{\infty} \frac{(z - z')^{n+1}}{n!} (H\partial^n H^{-1})^{ca}(\boldsymbol{z}') \right\} \qquad (5.9)
$$

$$
= +\frac{1}{4\pi\epsilon^2} \delta^{ba} + \mathcal{O}(z - z')
$$

and therefore

$$
\left[\bar{\partial}_z^2 \Omega^{aa}(\boldsymbol{z}, \boldsymbol{z}') \right]_{z' \to z} = \frac{N^2 - 1}{4\pi\epsilon^2} \qquad (5.10a)
$$

$$
\Rightarrow \quad \mathrm{T}_2 \mathcal{O}_0 = -2m \frac{\pi}{N} \int \frac{N^2 - 1}{4\pi\epsilon^2} = -2m \int \left[\frac{N^2 - 1}{4N\epsilon^2} \right]. \qquad (5.10b)
$$

This divergent term can now be dealt with by defining the normal ordered form of $\mathcal{O}_0$ as

$$
:\mathcal{O}_0: = \int \left(\bar{\partial} J^a \bar{\partial} J^a - \frac{N^2 - 1}{4N\epsilon^2} \right) \qquad (5.11)
$$

which now satisfies

$$
\mathrm{T} :\mathcal{O}_0: = 2m :\mathcal{O}_0: \qquad (5.12)
$$

in agreement with the conjecture for $n = 0$.

For the following calculations, note that the coincident limit of an expressions in which $\bar{\partial}_z$ acts n times on $\Omega(\boldsymbol{z}, \boldsymbol{z}')$ (possibly with further partial derivatives with respect to the other variables) *always* results in a term of order $1/\epsilon^n$ because, as explained above, $\bar{\partial}_z$ only acts on $e^{-\alpha}$ contained in Ω and thus pulls out a factor of $1/\epsilon$. (This is not true for $\bar{\partial}_{z'}$ because this also acts on the $H\partial^n H^{-1}$ part.) Therefore these terms will dictate the normal ordering procedure, but as long as we are not interested in how this looks explicitly, we do not have to calculate them and will only keep the finite terms.

5.1.2. n = 1

For this case it is convenient to simplify the expression for the h-covariant Laplacian acting on $\bar{\partial} J$ as

$$
\Delta^{ab} \bar{\partial} J^b = \delta^{ab} \partial \bar{\partial} J^b - if^{acb} \left(\frac{1}{2} \bar{\partial} J^c \bar{\partial} J^b + J^c \bar{\partial}^2 J^b \right)
$$

$$
= \delta^{ab} \partial \bar{\partial} J^b - if^{acb} J^c \bar{\partial}^2 J^b \qquad (5.13)
$$

$$
= D^{ab} \bar{\partial}^2 J^b,
$$

the first term in brackets dropping out because of the antisymmetry of f^{acb}. So for $n = 1$ equation (5.3) becomes

$$
\frac{\delta \mathcal{O}_1}{\delta J_{z'}^b} = -2\bar{\partial} \left(D^{bc} \bar{\partial}^2 J_{z'}^c \right) + \int \bar{\partial} J_z^c \frac{\delta D_z^{cd}}{\delta J_{z'}^b} \bar{\partial}^2 J_z^d
$$

$$
= -2\bar{\partial} \left(D^{bc} \bar{\partial}^2 J_{z'}^c \right) - if^{cbd} \bar{\partial} J_{z'}^c \bar{\partial}^2 J_{z'}^d \qquad (5.14)
$$

and taking another functional derivative then yields

$$\frac{\delta^2 \mathcal{O}_1}{\delta J_z^a \delta J_{z'}^b} = 2if^{bac}\bar{\partial}_{z'}\left(\delta_{z,z'}\bar{\partial}^2 J_{z'}^c\right) - 2\bar{\partial}_{z'}\left(D_{z'}^{ba}\bar{\partial}_{z'}^2 \delta_{z,z'}\right) - if^{abc}\left(\bar{\partial}_{z'}\delta_{z,z'}\bar{\partial}^2 J_{z'}^c - \bar{\partial} J_{z'}^c \bar{\partial}_{z'}^2 \delta_{z,z'}\right)$$

$$= 2\delta_{z,z'}\bar{\partial}^3 J_{z'}^{ab} + (\bar{\partial}_z\delta\text{-terms})$$

$$(5.15)$$

where $\delta_{z,z'} \equiv \delta(\boldsymbol{z} - \boldsymbol{z}')$ and "$\bar{\partial}_z\delta$-terms" refers to terms in which at least one $\bar{\partial}_z$ acts on a delta function. (Although all derivatives in the first line are with respect to z', one can bring it to the form in the second line by using $\partial_{z'}\delta_{z',z} = -\partial_z\delta_{z',z}$.) When computing $\mathrm{T}\mathcal{O}_1$ all those $\bar{\partial}_z$'s can be integrated by parts and in the end give the coincident limit of $\Omega(\boldsymbol{z} - \boldsymbol{z}')$ with at least one $\bar{\partial}_z$ acting on it. As argued at the end of section 5.1.1 these expressions will only give normal ordering contributions which we will denote by $[...]_{\mathrm{NO}}$ and so we find:

$$\mathrm{T}_2\mathcal{O}_1 = m\frac{2\pi}{N}\int_{\boldsymbol{z}}\left[\Omega^{ab}(\boldsymbol{z},\boldsymbol{z}')\right]_{z'\to z}\bar{\partial}^3 J_z^{ab} + [...]_{\mathrm{NO}}. \qquad (5.16)$$

The coincident limit can again be evaluated by expanding Ω as

$$\Omega^{ab}(\boldsymbol{z},\boldsymbol{z}') = D_{z'}^{bc}\left\{\frac{1}{\pi(z-z')}\left(\delta^{ca} - \sum_{n=0}^{\infty}\frac{(z-z')^n}{n!}(H\partial^n H^{-1})^{ca}(\boldsymbol{z}')\sum_{m=0}^{\infty}\frac{(-\alpha)^m}{m!}\right)\right\}$$

$$= -\frac{1}{\pi}(\delta^{bc}\partial_{z'} - J_{z'}^{bc})\left\{\sum_{n=1}^{\infty}\frac{(z-z')^{n-1}}{n!}(H\partial^n H^{-1})_{z'}^{ca} e^{-\alpha} + \sum_{m=1}^{\infty}\frac{(z-z')^{m-1}(\bar{z}-\bar{z}')^m}{m!(-2\epsilon)^m}\right\}$$

$$= -\frac{1}{\pi}\left\{-\frac{1}{2}\left(H\partial^2 H^{-1}\right)^{ba} + D^{bc}\left(H\partial H^{-1}\right)^{ca}\right\}_{z'} + \dots$$

$$(5.17)$$

where the ellipsis refers to terms of higher order in $(z - z')$ or $(\bar{z} - \bar{z}')$. Using the identities

$$H\partial H^{-1} = -\partial H H^{-1} = -J \qquad (5.18a)$$

$$H\partial^2 H^{-1} = \partial(H\partial H^{-1}) - \partial H(H^{-1}H)\partial H^{-1} = -\partial J + JJ = -DJ \qquad (5.18b)$$

we obtain

$$\Omega^{ab}(\boldsymbol{z},\boldsymbol{z}')_{z\to z'} = \frac{1}{2\pi}\left(DJ\right)^{ba} = \frac{1}{2\pi}(\partial J^{ba} - J^{bc}J^{ca}). \qquad (5.19)$$

The JJ term is symmetric in a and b and will hence drop out when plugging this result into (5.16):

$$\mathrm{T}_2\mathcal{O}_1 = m\frac{2\pi}{N}\int\frac{1}{2\pi}\partial J^{ba}\bar{\partial}^3 J^{ab} + [...]_{\mathrm{NO}}$$

$$= m\int\bar{\partial}J^c(\partial\bar{\partial})\bar{\partial}J^c + [...]_{\mathrm{NO}} = m\mathcal{O}_1 + \mathcal{O}(J^3) + [...]_{\mathrm{NO}}$$

$$(5.20)$$

where the trace identity $J^{ab}J^{ba} = if^{acb}if^{bda}J^c J^d = \underbrace{f^{abc}f^{abd}}_{=N\delta^{cd}}J^c J^d = NJ^c J^c$ was used. So the final result for the kinetic energy operator acting on the normal ordered operator $:\mathcal{O}_1:$ is

$$\mathrm{T}:\mathcal{O}_1: = 3m:\mathcal{O}_1: + \mathcal{O}(J^3), \qquad (5.21)$$

which again agrees with the conjecture.

5.1.3. n = 2

Finally, consider the operator

$$\mathcal{O}_2 = \int \bar{\partial} J^c (\Delta^2)^{cd} \bar{\partial} J^d \overset{\text{p.i.}}{=} \int \Delta^{cd} \bar{\partial} J^d \Delta^{ce} \bar{\partial} J^e \overset{5.13}{=} \int D^{cd} \bar{\partial}^2 J^d D^{ce} \bar{\partial}^2 J^e \tag{5.22}$$

for which

$$\frac{\delta \mathcal{O}_2}{\delta J^b_{z'}} = -2\bar{\partial}^2 \left[D^{bc} \left(D^{cd} \bar{\partial}^2 J^d_{z'} \right) \right] - 2i f^{cbd} \bar{\partial}^2 J^d_{z'} D^{ce} \bar{\partial}^2 J^e_{z'} \tag{5.23}$$

and consequently

$$\frac{\delta^2 \mathcal{O}_2}{\delta J^a_z \delta J^b_{z'}} = -2\Big\{ -i f^{bac} \delta_{z,z'} \bar{\partial}^2 \left(D^{cd} \bar{\partial}^2 J^d_{z'} \right) - i f^{cad} \bar{\partial}^2 \left[D^{bc} \left(\delta_{z,z'} \bar{\partial}^2 J^d_{z'} \right) \right]$$
$$+ f^{cbd} f^{cae} \delta_{z,z'} \bar{\partial}^2 J^d_{z'} \bar{\partial}^2 J^e_{z'} \Big\} + (\bar{\partial}_z \delta\text{-terms}) \tag{5.24a}$$

$$= \delta_{z,z'} 2\Big\{ i f^{bac} \bar{\partial}^2 \left(D^{cd} \bar{\partial}^2 J^d_{z'} \right) + i f^{cad} \bar{\partial}^2 \left(D^{bc} \bar{\partial}^2 J^d_{z'} \right) - \bar{\partial}^2 J^{bc}_{z'} \bar{\partial}^2 J^{ca}_{z'} \Big\}$$
$$+ 2i f^{bac} \partial_{z'} \delta_{z,z'} \bar{\partial}^2 J^c_{z'} + (\bar{\partial}_z \delta\text{-terms}) \tag{5.24b}$$

$$= \delta_{z,z'} 2\Big\{ 2\partial \bar{\partial}^4 J^{ab}_{z'} + \mathcal{O}(J^2) \Big\} + 2\partial_{z'} \delta_{z,z'} \bar{\partial}^4 J^{ab}_{z'} + (\bar{\partial}_z \delta\text{-terms}). \tag{5.24c}$$

Since $\Omega_{z' \to z}$ is of order J, the terms in braces which are quadratic in J will only give $\mathcal{O}(J^3)$ contributions to $\mathrm{T}\mathcal{O}_2$. Repeating the argument from the $n = 1$ case one then finds

$$\mathrm{T}_2 \mathcal{O}_2 = 2m \frac{2\pi}{N} \int \left[\Omega^{ab}_{z,z'} \right]_{z' \to z} \partial \bar{\partial}^4 J^{ab} + m \frac{2\pi}{N} \int \left[\partial_z \Omega^{ab}_{z,z'} \right]_{z' \to z} \bar{\partial}^4 J^{ab} + \mathcal{O}(J^3) + [\ldots]_{\mathrm{NO}} \tag{5.25}$$

The first coincidence term was already computed above, and the second term can be calculated analogously:

$$\partial_z \Omega^{ab}(z, z') = -\frac{1}{\pi} (\delta^{bc} \partial_{z'} - J^{bc}_{z'}) \left\{ \sum_{n=2}^{\infty} (n-1) \frac{(z-z')^{n-2}}{n!} (H\partial^n H^{-1})^{ca}_{z'} \right\} + \ldots$$
$$= -\frac{1}{\pi} \left\{ -\frac{2}{3!} (H\partial^3 H^{-1}) + \frac{1}{2} D \left(H\partial^2 H^{-1} \right) \right\}^{ba}_{z'} + \ldots \tag{5.26}$$

where the ellipsis again refers to terms of higher order in $(z-z')$ or $(\bar{z}-\bar{z}')$. Using equation (5.18b) as well as

$$H\partial^3 H^{-1} = \partial(H\partial^2 H^{-1}) - \partial H(H^{-1}H)\partial^2 H^{-1} = -\partial(DJ) + J(DJ) = -D^2 J \tag{5.27}$$

it follows that

$$\left[\partial_z \Omega^{ab}(z, z') \right]_{z' \to z} = -\frac{1}{\pi} \left\{ \frac{1}{3} \left(D^2 J \right) + \frac{1}{2} D \left(-DJ \right) \right\}^{ba} = \frac{1}{2\pi} \frac{1}{3} \left(D^2 J \right)^{ba} \tag{5.28}$$

and so we obtain

$$\mathrm{T}_2 \mathcal{O}_2 = 2m \int \partial J^a \partial \bar{\partial}^4 J^a + m \int \frac{1}{3} \partial^2 J^a \bar{\partial}^4 J^a + \mathcal{O}(J^3) + [\ldots]_{\mathrm{NO}}$$
$$= 2m \int \bar{\partial} J^a \left(\partial \bar{\partial} \right)^2 \bar{\partial} J^a - \frac{1}{3} m \int \bar{\partial} J^a \left(\partial \bar{\partial} \right)^2 \bar{\partial} J^a + \mathcal{O}(J^3) + [\ldots]_{\mathrm{NO}}. \tag{5.29}$$

This gives the final result

$$\mathrm{T}\mathcal{O}_2 = \left(4 - \frac{1}{3}\right)\mathcal{O}_2 + \mathcal{O}(J^3) + [...]_{\mathrm{NO}} \tag{5.30}$$

and so this time the conjecture is violated.

5.2. Second Approximation

As already mentioned, using the simplified $\Lambda_{(1)}$ instead of Λ is not allowed when T_2 acts on local operators like $\mathcal{O}_n$. So LMY repeat the calculations for $n = 0, 1$ using what they call the "exact regulator" [12, p. 19], which is the one displayed in equation (5.5c):

$$\Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') = D_{z'}^{bc}\Lambda_{(2)}^{ca}(\boldsymbol{z}', \boldsymbol{z}) \tag{5.31a}$$

$$\Lambda_{(2)}^{ca}(\boldsymbol{z}', \boldsymbol{z}) = \frac{1}{\pi(z - z')}\left[1 - \mathrm{e}^{-|\boldsymbol{z} - \boldsymbol{z}'|^2/2\epsilon}H(\boldsymbol{z}')H^{-1}(u, \bar{z}')H(\boldsymbol{u})H^{-1}(z, \bar{u})\right]^{ca}. \tag{5.31b}$$

We will not repeat these calculations here, but the result is that the eigenvalues for $n = 0, 1$ using Λ_2 become

$$\mathrm{T}\mathcal{O}_0 = 2\left(1 - \frac{1}{16}\right)\mathcal{O}_0 \tag{5.32a}$$

$$\mathrm{T}\mathcal{O}_1 = 3\left(1 - \frac{3}{32}\right)\mathcal{O}_1, \tag{5.32b}$$

so the conjecture is in fact violated for all three cases that were investigated by LMY by direct calculation.

5.3. Exact Regulator

Taking a closer look at equation (5.5c) shows that there might be contributions to terms like $\bar{\partial}_z^2\Omega(\boldsymbol{z}, \boldsymbol{z}')_{z'\to z}$ (which appears in the $n = 0$ calculation) from the second summand proportional to $\epsilon\, \mathrm{e}^{-|\boldsymbol{z} - \boldsymbol{z}'|^2/2\epsilon}$, because a derivative acting on the exponent will bring down a factor of ϵ^{-1} and thus give a contribution of order ϵ^0. The same holds true for the $\mathcal{O}(\epsilon^2)$ terms because there are two derivatives acting on Ω, and so a consistent way to calculate $\mathrm{T}_2\mathcal{O}_0$ is to take the *exact* form of Λ, i.e.

$$\Omega^{ab}(\boldsymbol{z}, \boldsymbol{z}') = D_{z'}^{bc}\Lambda^{ca}(\boldsymbol{z}', \boldsymbol{z}) \tag{5.33a}$$

$$\Lambda^{ca}(\boldsymbol{z}', \boldsymbol{z}) = \int_x \bar{\mathcal{G}}^{dc}(\boldsymbol{x}, \boldsymbol{z}')G(\boldsymbol{x}, \boldsymbol{z})\mathrm{e}^{-|\boldsymbol{x} - \boldsymbol{z}|^2/\epsilon}\left[\frac{\bar{x} - \bar{z}}{\epsilon}H(x, \bar{x})H^{-1}(z, \bar{x})\right.$$

$$\left. + H(\boldsymbol{x})\partial_z\left(H^{-1}(z, \bar{x})H(\boldsymbol{z})\right)H^{-1}(\boldsymbol{z})\right]^{da}, \tag{5.33b}$$

$$\bar{\mathcal{G}}^{dc}(\boldsymbol{x}, \boldsymbol{z}') = \frac{1}{\pi(x - z')}\left[1 - \mathrm{e}^{-|\boldsymbol{x} - \boldsymbol{z}'|^2/\epsilon}H(x, \bar{z}')H^{-1}(z', \bar{z}')\right]^{dc}. \tag{5.33c}$$

Because of the complicated form of Λ the calculations become quite involved, so we will only consider the $n = 0$ case[1]. The detailed calculations can be found in appendix C.1 and the result is

$$\mathrm{T} \; :\mathcal{O}_0: \; = 2m \left(1 - \frac{3}{16} \right) \; :\mathcal{O}_0:. \tag{5.34}$$

So once again the conjecture is falsified, this time using — according to KKN — the most accurate (and strictly speaking in this context the only legitimate) regulator.

All the different results show that the result is very sensitive to the exact form of the regulator that is used. As LMY put it: "There are certainly very subtle effects coming from the point-splitting regulator. We have not been able to arrive at a consistent method for dealing with these effects." [12, p. 19]

It would therefore be very desirable to have an independent, maybe more robust way to calculate the kinetic energy eigenvalues of $\mathcal{O}_n$ in order to justify the proposed vacuum wave functional found by LMY. One possible approach will be presented here in section 6.2.

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[1]In fact, the calculation for $n = 0$ also reveals that the verification of the conjecture for higher n by direct calculation using the exact KKN regulator will become an almost insurmountable task.

6. Back to The Original Variables

The approximation to the vacuum wave functional, Ψ_0, was constructed in section 4.1 as a functional of the gauge-invariant variable J. It is now an interesting question if Ψ_0 can be re-expressed in terms of the original Yang Mills connection A, and if so, what it looks like. If one assumes that the use of the new variables was somehow an essential ingredient, one could expect that Ψ_0 as a functional of A takes a rather complicated form. In contrast, it is easy to see that it has exactly the *same* form, just with $\bar{\partial}J$ replaced by ($i/2$ times[1]) the colour-magnetic field B, and the holomorphic covariant Laplacian[2] $\Delta^{(J)} \equiv \frac{1}{2}\{\bar{\partial}, D\}$ replaced by ($1/4$ times[1]) its gauge-covariant counterpart $\Delta^{(A)} \equiv \nabla_i^2 \equiv (\partial_i + A_i)^2$:

$$\Psi_0[J] = \exp\left[-\frac{N}{\pi m^2}\int \mathrm{Tr}\left(\bar{\partial}J\, K\left(\frac{\Delta^{(J)}}{m^2}\right)\bar{\partial}J\right)\right] \tag{6.1a}$$

$$= \exp\left[+\frac{N}{4\pi m^2}\int \mathrm{Tr}\left(B\, K\left(\frac{\Delta^{(A)}}{4m^2}\right)B\right)\right] = \Psi_0[A] \tag{6.1b}$$

(Regarding the $+$ sign, recall that in our conventions $B = -iB^a t^a$ whereas $J = J^a t^a$.) Because of the similarity of $\bar{\partial}J$ and B shown in (3.64), viz. $\bar{\partial}J = \frac{i}{2}M^\dagger B M^{\dagger-1}$, it is enough to show an analogous similarity between $\Delta^{(J)}$ and $\Delta^{(A)}$ in order to verify the assertion (6.1b). To this end, first note that $\Delta^{(A)}$ can be expressed in terms of the complex covariant derivatives as follows:

$$\nabla\overline{\nabla} = \frac{1}{4}(\nabla_1 + i\nabla_2)(\nabla_1 - i\nabla_2) = \frac{1}{4}\left(\nabla_i^2 - i[\nabla_1, \nabla_2]\right)$$

$$= \frac{1}{4}\left(\nabla_i^2 - iB\right) \tag{6.2a}$$

$$\overline{\nabla}\nabla = \frac{1}{4}\left(\nabla_i^2 + iB\right) \tag{6.2b}$$

$$\Rightarrow \{\nabla, \overline{\nabla}\} = \frac{1}{2}\nabla_i^2 = \frac{1}{2}\Delta^{(A)}. \tag{6.2c}$$

Furthermore, it was already shown that $\bar{\partial} = M^\dagger \overline{\nabla} M^{\dagger-1}$ and $D = M^\dagger \nabla M^{\dagger-1}$, and so we have

$$\Delta^{(J)} \equiv \frac{1}{2}\{\bar{\partial}, D\} = \frac{1}{4}M^\dagger \Delta^{(A)} M^{\dagger-1} \tag{6.3}$$

which shows that (6.1b) is correct.

---

[1] These factors could have been avoided if the complex coordinates had been introduced in a normalised way, viz. $z = (x_1 - ix_2)/\sqrt{2}$. We followed the KKN convention $z = x_1 - ix_2$ to avoid too much confusion when comparing the results.

[2] As a technical remark, note that we will from now on adopt the convention that derivative operators act on everything to their right, not only the first object to their right, without using hats.

The fact that Ψ_0 has exactly the same simple quasi-Gaussian form when expressed in terms of A_i immediately raises the question: Why not, instead of going through the painful derivation of the Hamiltonian as a function of J, stick to the original variables and solve the Schrödinger equation there? In other words, why is the transformation to the KKN variables in this context necessary or helpful at all?

The equality of (6.1a) and (6.1b) implies that using the ansatz (6.1b) to solve the Schrödinger equation for the vacuum state in the original variables must lead to the *same* Kernel equation as before — assuming the LMY conjecture holds. Since this conjecture is vital for the derivation but could not be proven in the J-variables, the hope is now that the calculation in the A-variables provides a verification of it, or at least some independent support for its validity.

6.1. Schrödinger Equation

The Schrödinger equation in the Yang-Mills connection A_i basis reads

$$\mathrm{H}[A]\Psi_0[A] = E_0\Psi_0[A], \tag{6.4a}$$

$$\mathrm{H}[A] = \int_x \left(-\frac{g^2}{2}\frac{\delta^2}{\delta A_i^a(\boldsymbol{x})^2} + \frac{1}{2g^2}B^a(\boldsymbol{x})^2 \right) \equiv \mathrm{T}[A] + \mathrm{V}[A]. \tag{6.4b}$$

Using the ansatz $\Psi_0 = \mathrm{e}^P$ leads to

$$\mathrm{H}\Psi_0 = \left(\mathrm{T}P - \frac{g^2}{2}\int_x \frac{\delta P}{\delta A_i^a(\boldsymbol{x})}\frac{\delta P}{\delta A_i^a(\boldsymbol{x})} + \frac{1}{2g^2}\int_x B^a(\boldsymbol{x})^2 \right)\Psi_0. \tag{6.5}$$

Since the calculation in J-variables was carried out only up to second order in J, we will do the same here and only keep terms up to second order in B. The functional derivative of $B^b = \partial_1 A_2^b - \partial_2 A_1^b + f^{bac}A_1^a A_2^c$ is

$$\frac{\delta B^b(\boldsymbol{y})}{\delta A_i^a(\boldsymbol{x})} = \delta^{ba}\left(\delta_{i2}\partial_{1y} - \delta_{i1}\partial_{2y}\right)\delta(\boldsymbol{y}-\boldsymbol{x}) + f^{bac}\left(\delta_{i1}A_2^c - \delta_{i2}A_1^c\right)\delta(\boldsymbol{y}-\boldsymbol{x}) \tag{6.6a}$$

$$= -\epsilon_{ij}\nabla_{jy}^{ba}\delta(\boldsymbol{y}-\boldsymbol{x}), \tag{6.6b}$$

and using the exponent $P = -\frac{N}{8\pi m^2}\int B^a K^{ab} B^b$ as given in equation (6.1b) then gives

$$\frac{\delta P}{\delta A_i^a(\boldsymbol{x})} = -\frac{N}{8\pi m^2}2\epsilon_{ij}\nabla_j^{ab}K^{bc}B^c(\boldsymbol{x}) + \mathcal{O}(B^2) \tag{6.7a}$$

$$\Rightarrow -\frac{g^2}{2}\int_x \frac{\delta P}{\delta A_i^a(\boldsymbol{x})}\frac{\delta P}{\delta A_i^a(\boldsymbol{x})} = -\frac{1}{8g^2m^2}\int_x \left[\nabla_i^{ab}K^{bc}B^c(\boldsymbol{x})\right]\left[\nabla_i^{ad}K^{de}B^e(\boldsymbol{x})\right] + \mathcal{O}(B^3)$$

$$= \frac{1}{8g^2m^2}\int_x B^c(\boldsymbol{x})K^{cb}\Delta^{bd}K^{de}B^e(\boldsymbol{x}) + \mathcal{O}(B^3)$$

$$= \frac{1}{2g^2}\int_x B^a(\boldsymbol{x})\left[LK^2(L)\right]^{ab}B^b(\boldsymbol{x}) + \mathcal{O}(B^3) \tag{6.7b}$$

where now $L \equiv \Delta^{(A)}/(4m^2)$. Note that this is the exact analogue of equation (4.10). Equation (6.5) thus becomes ($\mathcal{O}(B^3)$ terms are suppressed)

$$\mathrm{H}\Psi_0 = \left(\mathrm{T}P + \frac{1}{2g^2}\int_x B^a(\boldsymbol{x})\left[LK^2(L)+1\right]^{ab}B^b(\boldsymbol{x})\right)\Psi_0 \tag{6.8a}$$

$$= \frac{1}{2g^2}\left(-\frac{1}{2m}\sum_{n=0}^{\infty}\frac{c_n}{(4m^2)^n}\mathrm{T}\mathcal{O}_n + \int_x B^a(\boldsymbol{x})\left[LK^2(L)+1\right]^{ab}B^b(\boldsymbol{x})\right)\Psi_0 \tag{6.8b}$$

where the Kernel was again expanded in a power series as

$$K(L) = \sum_{n=0}^{\infty}c_n L^n = \sum_{n=0}^{\infty}\frac{c_n}{(4m^2)^n}\Delta^{(A)n} \tag{6.9}$$

and[3]

$$\mathcal{O}_n[A] = \int B^a(\Delta^{(A)n})^{ab}B^b. \tag{6.10}$$

Now it is clear that the Kernel equation (4.16) (and thus the vacuum wave functional and the glueball spectrum as discussed in sections 4.1.3 and 4.2) will follow if the key conjecture (4.13) holds.

6.2. The LMY Conjecture

So far the calculation was completely analogous to the one using the KKN variables, but now there is a major difference. In the KKN variables, the kinetic energy operator contained the linear term $\mathrm{T}_1 \sim J\delta/\delta J$ which basically counts the number of J's and thus made the conjecture (4.13) a more or less natural guess. (This naive argument of course neglects contributions from the quadratic term T_2, which can nevertheless be important as we saw in chapter 5.) But recall that this term only emerged during the non-trivial regularisation procedure that was needed to express T in terms of J.

Here it is a priori far from obvious that the relation (6.11a) could possibly hold. But T as written in (6.11b) contains two functional derivatives evaluated at the same point which will produce divergent expressions when acting on the local operators $\mathcal{O}_n$. Therefore, some regularisation will have to be introduced here as well, and it becomes evident that the careful regularisation of the kinetic energy operator is the key ingredient to the verification (or falsification) of the conjecture.

In any case, as already discussed, the result must in the end of course be independent of the choice of variables and so here we will have an independent test for the validity of the conjecture, which in terms of the original Yang-Mills variables reads

$$\mathrm{T} :\mathcal{O}_n: \overset{?}{=} (2+n)m :\mathcal{O}_n: + \mathcal{O}(B^3) \tag{6.11a}$$

$$\mathrm{T} = -\frac{g^2}{2}\int_x \frac{\delta^2}{\delta A_i^a(\boldsymbol{x})^2}, \tag{6.11b}$$

---

[3]Note that (6.10) differs from the original definition of $\mathcal{O}_n$ in terms of J by a constant factor of 4^{n+1}. Since this factor would drop out in equation (6.11a) anyway, it is not necessary to keep it in what follows.

where now $\mathcal{O}_n$ as given in (6.10) is a functional of A. Just like in the KKN variable approach we will consider the cases $n = 0, 1, 2$.

6.2.1. n=0

The first and simplest operator reads

$$\mathcal{O}_0 = \int_y B^b(\boldsymbol{y})^2, \tag{6.12}$$

which is (up to a constant) nothing but the potential energy operator V. Using (6.6) one finds

$$\frac{\delta \mathcal{O}_0}{\delta A_i^a(\boldsymbol{x})} = 2\epsilon_{ij} \nabla_j^{ab} B^b(\boldsymbol{x}) \tag{6.13a}$$

$$\Rightarrow \mathrm{T}\mathcal{O}_0 = -\frac{g^2}{2} \int_x \frac{\delta^2 \mathcal{O}_0}{\delta A_i^a(\boldsymbol{x})^2} = g^2 \int_x \nabla_i^{ab} \nabla_i^{ba} \delta(\boldsymbol{x} - \boldsymbol{x}). \tag{6.13b}$$

As expected, this result is divergent and therefore needs to be regularised. Letting tr denote the trace in adjoint representation and using a boldface notation for the spatial 2-vector $\boldsymbol{\nabla} = \nabla_i$, the result (6.13b) can be written as

$$\mathrm{T}\mathcal{O}_0 = g^2 \int_x \mathrm{tr} \left[\boldsymbol{\nabla}_x^2 \delta(\boldsymbol{x} - \boldsymbol{y}) \right]_{\boldsymbol{y} \to \boldsymbol{x}}. \tag{6.14}$$

The delta function now has to be regularised in a gauge-covariant way, which can be done using a method first introduced by Fujikawa for the calculation of the axial anomaly [13][4].

To smear out the delta function one could naively use a properly normalised Gaussian, which can be written as a Fourier integral:

$$\frac{M^2}{4\pi} e^{-M^2(\boldsymbol{x}-\boldsymbol{y})^2/4} = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \, e^{-\boldsymbol{p}^2/M^2} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \, \exp\left(\boldsymbol{\partial}_x^2/M^2\right) e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}. \tag{6.15}$$

The parameter M has mass dimension one and should be taken to infinity at the end of a calculation. The problem with (6.15) is that this naive regularisation will not preserve gauge invariance, as can be seen from the appearance of an ordinary derivative. But in order to get a gauge-covariant regulator, one can simply replace the ordinary by a gauge-covariant derivative. (For the axial anomaly calculation, where quarks are present, one would take $\slashed{\nabla}$. Since we are considering pure Yang-Mills theory without fermions, it is sufficient to take ∇.) The regulated delta function thus becomes:

$$\delta_M(\boldsymbol{x} - \boldsymbol{y}) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \, \exp\left(\boldsymbol{\nabla}_x^2/M^2\right) e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}. \tag{6.16}$$

In the limit $M \to \infty$ this still becomes a delta function as required, but it also has the appropriate gauge transformation properties and is thus a valid regulator.

---

[4]The way this calculation is presented here follows [16], chapter 77.

Now equation (6.14) becomes

$$
\mathrm{T}\mathcal{O}_0 = \lim_{M\to\infty} g^2 \int_x \mathrm{tr}\left\{\int \frac{\mathrm{d}^2 p}{(2\pi)^2} \boldsymbol{\nabla}_x^2 \exp\left[\boldsymbol{\nabla}_x^2/M^2\right] \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right\}_{\boldsymbol{y}\to\boldsymbol{x}} \tag{6.17a}
$$

$$
= m \lim_{M\to\infty} \int_x \frac{1}{N} \mathrm{tr}\left\{\int \frac{\mathrm{d}^2 p}{2\pi} (\boldsymbol{\nabla}+i\boldsymbol{p})^2 \exp\left[(\boldsymbol{\nabla}+i\boldsymbol{p})^2/M^2\right]\right\}_x \tag{6.17b}
$$

where a derivative on the very right now gives a vanishing contribution. Here we used

$$
\boldsymbol{\nabla}_x \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = (\boldsymbol{\partial}_x + \boldsymbol{A}(\boldsymbol{x}))\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}(\boldsymbol{\partial}_x + \boldsymbol{A}(\boldsymbol{x}) + i\boldsymbol{p}) = \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}(\boldsymbol{\nabla}_x + i\boldsymbol{p}) \tag{6.18}
$$

for every covariant derivative in the series expansion, to commute $\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}$ to the very left where it becomes 1 after performing the coincident limit.

The integration variable $\boldsymbol{p}$ can be rescaled to the dimensionless variable $\boldsymbol{p}/M$ (which we will call $\boldsymbol{p}$ again) and then the integrand can be expanded in powers of M. When expanding the exponential and collecting terms one should of course not change the order of individual terms since they do not necessarily commute. But $\boldsymbol{p}$ commutes with everything and so $e^{-\boldsymbol{p}^2}$ can be factored out; also note that any term of odd order in M will also be odd in p and so the integral from $-\infty$ to ∞ will be zero. Using this, the term in braces becomes

$$
\{\ldots\} = M^2 \int \frac{\mathrm{d}^2 p}{2\pi} \left(\boldsymbol{\nabla}^2 + 2iM\boldsymbol{p}\cdot\boldsymbol{\nabla} - M^2\boldsymbol{p}^2\right) \exp\left[\boldsymbol{\nabla}^2/M^2 + 2i\boldsymbol{p}\cdot\boldsymbol{\nabla}/M - \boldsymbol{p}^2\right] \tag{6.19a}
$$

$$
= \int \frac{\mathrm{d}^2 p}{2\pi} \mathrm{e}^{-\boldsymbol{p}^2} \left(I_4 M^4 + I_2 M^2 + I_0\right) + \mathcal{O}(1/M), \tag{6.19b}
$$

with

$$
I_4 := -\boldsymbol{p}^2 = -p_i p_j \delta_{ij} \tag{6.20a}
$$

$$
I_2 := \boldsymbol{\nabla}^2 + (2i)^2(\boldsymbol{p}\cdot\boldsymbol{\nabla})^2 - \boldsymbol{p}^2\left[\boldsymbol{\nabla}^2 + \frac{(2i)^2}{2!}(\boldsymbol{p}\cdot\boldsymbol{\nabla})^2\right]
$$

$$
= \boldsymbol{\nabla}^2 - p_i p_j\left[\delta_{ij}\boldsymbol{\nabla}^2 + 4\nabla_i\nabla_j\right] + p_i p_j p_k p_l\left[2\delta_{ij}\nabla_k\nabla_l\right] \tag{6.20b}
$$

$$
I_0 := \boldsymbol{\nabla}^2\left[\boldsymbol{\nabla}^2 + \frac{(2i)^2}{2!}(\boldsymbol{p}\cdot\boldsymbol{\nabla})^2\right]
$$

$$
+ 2i\boldsymbol{p}\cdot\boldsymbol{\nabla}\left[\frac{2i}{2!}\left(\boldsymbol{\nabla}^2(\boldsymbol{p}\cdot\boldsymbol{\nabla}) + (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}^2\right) + \frac{(2i)^3}{3!}(\boldsymbol{p}\cdot\boldsymbol{\nabla})^3\right]
$$

$$
- \boldsymbol{p}^2\left[\frac{1}{2!}\boldsymbol{\nabla}^4 + \frac{(2i)^2}{3!}\left(\boldsymbol{\nabla}^2(\boldsymbol{p}\cdot\boldsymbol{\nabla})^2 + (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}^2(\boldsymbol{p}\cdot\boldsymbol{\nabla}) + (\boldsymbol{p}\cdot\boldsymbol{\nabla})^2\boldsymbol{\nabla}^2\right)\right.
$$

$$
\left. + \frac{(2i)^4}{4!}(\boldsymbol{p}\cdot\boldsymbol{\nabla})^4\right]
$$

$$
= \left(1 - \boldsymbol{p}^2/2\right)\boldsymbol{\nabla}^4 - 2\left(1 - \boldsymbol{p}^2/3\right)\left[\boldsymbol{\nabla}^2(\boldsymbol{p}\cdot\boldsymbol{\nabla})^2 + (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}^2(\boldsymbol{p}\cdot\boldsymbol{\nabla}) + (\boldsymbol{p}\cdot\boldsymbol{\nabla})^2\boldsymbol{\nabla}^2\right]
$$

$$
+ \frac{8}{3}\left(1 - \boldsymbol{p}^2/4\right)(\boldsymbol{p}\cdot\boldsymbol{\nabla})^4
$$

$$
= \boldsymbol{\nabla}^4 - p_i p_j \left[\frac{1}{2} \delta_{ij} \boldsymbol{\nabla}^4 + 2 \left(\boldsymbol{\nabla}^2 \nabla_i \nabla_j + \nabla_i \boldsymbol{\nabla}^2 \nabla_j + \nabla_i \nabla_j \boldsymbol{\nabla}^2 \right) \right]
$$

$$
+ p_i p_j p_k p_l \left[\frac{2}{3} \delta_{ij} \left(\boldsymbol{\nabla}^2 \nabla_k \nabla_l + \nabla_k \boldsymbol{\nabla}^2 \nabla_l + \nabla_k \nabla_l \boldsymbol{\nabla}^2 \right) + \frac{8}{3} \nabla_i \nabla_j \nabla_k \nabla_l \right]
$$

$$
- p_i p_j p_k p_l p_m p_n \left[\frac{2}{3} \delta_{ij} \nabla_k \nabla_l \nabla_n \nabla_m \right] \tag{6.20c}
$$

All the integrals in (6.19b) are generalised Gaussian integrals which can be computed using the formulas[5]

$$
\int \mathrm{d}^2 p \, \mathrm{e}^{-\boldsymbol{p}^2} = \pi \tag{6.21a}
$$

$$
\int \mathrm{d}^2 p \, \mathrm{e}^{-\boldsymbol{p}^2} p_i p_j = \frac{\pi}{2} \delta_{ij} \tag{6.21b}
$$

$$
\int \mathrm{d}^2 p \, \mathrm{e}^{-\boldsymbol{p}^2} p_i p_j p_k p_l = \frac{\pi}{4} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{6.21c}
$$

$$
\int \mathrm{d}^2 p \, \mathrm{e}^{-\boldsymbol{p}^2} p_i p_j p_k p_l p_m p_n = \frac{\pi}{8} (\delta_{ij}\delta_{kl}\delta_{mn} + \text{permutations}). \tag{6.21d}
$$

This yields

$$
\int \frac{\mathrm{d}^2 p}{2\pi} \, \mathrm{e}^{-\boldsymbol{p}^2} I_4 = -\frac{1}{2} \tag{6.22a}
$$

$$
\int \frac{\mathrm{d}^2 p}{2\pi} \, \mathrm{e}^{-\boldsymbol{p}^2} I_2 = \frac{1}{2}(1 - 3 + 2)\boldsymbol{\nabla}^2 = 0 \tag{6.22b}
$$

$$
\int \frac{\mathrm{d}^2 p}{2\pi} \, \mathrm{e}^{-\boldsymbol{p}^2} I_0 = \frac{1}{2} \left[\left(1 - \frac{5}{2} + \frac{4}{3} + \frac{2}{3} - \frac{1}{2} \right) \boldsymbol{\nabla}^4 \right.
$$

$$
\left. + \left(-1 + \frac{2}{3} + \frac{2}{3} - \frac{1}{2} \right) \nabla_i \boldsymbol{\nabla}^2 \nabla_i + \left(\frac{2}{3} - \frac{1}{2} \right) \nabla_i \nabla_j \nabla_i \nabla_j \right]
$$

$$
= \frac{1}{12} \left(-\nabla_i \nabla_j \nabla_j \nabla_i + \nabla_i \nabla_j \nabla_i \nabla_j \right) = \frac{1}{12} \nabla_i \nabla_j [\nabla_i, \nabla_j]
$$

$$
= \frac{1}{24} (F_{ij})^2 = \frac{1}{12} B^2 \tag{6.22c}
$$

and thus

$$
\mathrm{T}\mathcal{O}_0 = \frac{1}{12} m \frac{1}{N} \int_x B^{ab} B^{ba} - m \int_x \frac{N^2 - 1}{2N} M^4. \tag{6.23}
$$

Using the relations

$$
B^{ab} = f^{acb} B^c \tag{6.24a}
$$

$$
f^{abc} f^{abd} = N \delta^{cd}, \tag{6.24b}
$$

the final result becomes

$$
\mathrm{T}\mathcal{O}_0 = -\frac{1}{12} m \mathcal{O}_0 - m \int_x \frac{N^2 - 1}{2N} M^4. \tag{6.25}
$$

---

[5]The index structure is dictated by symmetry and the prefactors can be found by calculating one particular example.

It is quite remarkable that in the end all ∇'s in (6.22c) conspire to form exactly a B^2 so that the M independent term indeed reproduces $\mathcal{O}_0$. Also the divergent term is the same as the one found by Leigh et al. (see equation (5.10a) and recall that the regulator ϵ there has mass dimension -2) and can be absorbed by defining the normal ordered form of $\mathcal{O}_0$ to be

$$:\mathcal{O}_0: = \int_x \left(B^a B^a + \frac{6(N^2 - 1)}{N} M^4 \right) \tag{6.26}$$

which then satisfies

$$\text{T} :\mathcal{O}_0: = -\frac{1}{12} m :O_0:. \tag{6.27}$$

So the main qualitative feature of the conjecture, namely that $\mathcal{O}_0$ is an eigenvector of T is indeed reproduced in this approach. But unfortunately the quantitative eigenvalue found here is different — not only by magnitude but also by sign. Note however, that this negative eigenvalue is a priori not an unphysical result, since $\mathcal{O}_0$ is not an allowed wave functional because it is not normalisable and so its eigenvalue $-m/12$ cannot be regarded as the energy of a physical state.

6.2.2. n=1

The next operator is

$$\mathcal{O}_1 = \int_y B^b(\boldsymbol{y}) \Delta^{bc} B^c(\boldsymbol{y}) = \int_y B^b(\boldsymbol{y}) \nabla_i^{bd} \nabla_i^{dc} B^c(\boldsymbol{y}). \tag{6.28}$$

Taking the first functional derivative then yields

$$\frac{\delta \mathcal{O}_1}{\delta A_i^a(\boldsymbol{x})} = 2\epsilon_{ij} \nabla_j^{ab} \Delta^{bc} B^c(\boldsymbol{x}) + \int_y B^b(\boldsymbol{y}) \left[f^{bad} \delta(\boldsymbol{y} - \boldsymbol{x}) \nabla_i^{dc} + \nabla_i^{bd} f^{dac} \delta(\boldsymbol{y} - \boldsymbol{x}) \right] B^c(\boldsymbol{y}) \tag{6.29a}$$

$$= 2\epsilon_{ij} \nabla_j^{ab} \Delta^{bc} B^c(\boldsymbol{x}) - 2B^b(\boldsymbol{x}) f^{abd} \nabla_i^{dc} B^c(\boldsymbol{x}) \tag{6.29b}$$

and taking another one gives

$$\frac{\delta^2 \mathcal{O}_1}{\delta A_i^a(\boldsymbol{x})^2} = -2\epsilon_{ij}\epsilon_{ik} \nabla_j^{ab} \Delta^{bc} \nabla_k^{ca} \delta(0) + 2\epsilon_{ij} \nabla_j^{ab} \left[\delta(0) f^{bad} \nabla_i^{dc} + \nabla_i^{bd} \delta(0) f^{dac} \right] B^c(\boldsymbol{x})$$

$$+ 2\epsilon_{ij} \left[\nabla_j^{ba} \delta(0) \right] f^{abd} \nabla_i^{dc} B^c(\boldsymbol{x}) - 2\delta_{ii} B^{ab}(\boldsymbol{x}) \delta(0) f^{bac} B^c(\boldsymbol{x})$$

$$+ 2\epsilon_{ij} B^{ab}(\boldsymbol{x}) \nabla_i^{bc} \nabla_j^{ca} \delta(0) \tag{6.30a}$$

$$= -2 \left[\nabla_i \Delta \nabla_i \delta(0) \right]^{aa} + 4\epsilon_{ij} \left[\nabla_j \delta(0) \right]^{ab} f^{bad} \nabla_i^{dc} B^c(\boldsymbol{x})$$

$$+ 6 \left[B(\boldsymbol{x}) \delta(0) B(\boldsymbol{x}) \right]^{aa} + 2 \left[B(\boldsymbol{x}) B(\boldsymbol{x}) \delta(0) \right]^{aa} \tag{6.30b}$$

where the following identity was used:

$$\epsilon_{ij} \nabla_i^{ab} \nabla_j^{bc} = F_{12}^{ac} = B^{ac}. \tag{6.31}$$

(This can be seen directly from $F_{12}^{\text{adj}} = [\nabla_1^{\text{adj}}, \nabla_2^{\text{adj}}]$, or alternatively by explicitly evaluating the left hand side

$$\epsilon_{ij} \nabla_i^{ab} \nabla_j^{bc} = \epsilon_{ij} \left(\delta^{ab} \partial_i + A_i^{ab} \right) \left(\delta^{bc} \partial_j + A_j^{bc} \right) \tag{6.32a}$$

$$= \epsilon_{ij} \left[\delta^{ac} \partial_i \partial_j + \left(\partial_i A_j^{ac} \right) + A_j^{ac} \partial_i + A_i^{ac} \partial_j + A_i^{ab} A_j^{bc} \right] \tag{6.32b}$$

$$= \epsilon_{ij} \left[\left(\partial_i A_j^{ac} \right) + A_i^{ab} A_j^{bc} \right] \tag{6.32c}$$

$$= (\partial_1 A_2^{ac}) - (\partial_2 A_1^{ac}) + \left(f^{adb} f^{bec} - f^{aeb} f^{bdc} \right) A_1^d A_2^e \tag{6.32d}$$

$$= (\partial_1 A_2^{ac}) - (\partial_2 A_1^{ac}) + f^{abc} f^{bde} A_1^d A_2^e \tag{6.32e}$$

$$= f^{abc} \left[\left(\partial_1 A_2^b \right) - \left(\partial_2 A_1^b \right) + f^{bde} A_1^d A_2^e \right] \tag{6.32f}$$

which agrees with the right hand side; (6.32e) follows from the Jacobi identity

$$f^{adb} f^{bec} + f^{deb} f^{bac} + f^{eab} f^{bdc} = 0.) \tag{6.33}$$

Note that in the last step (6.30b) the delta-functions were pulled inside the adjoint matrix component structure (just like in (6.14)), which is necessary because they become matrices after regularisation. This procedure is again in complete analogy to the calculation of the axial anomaly, where the delta function is also pulled inside the trace to get a non-zero result (see also the remark in [23, p. 53]).

The regularisation can now be performed just like in the $n = 0$ case; the detailed calculation can be found in appendix C.2 and the results are (only the finite, i.e. $\mathcal{O}(M^0)$ terms are shown)

$$\nabla_i \Delta \nabla_i \delta(0) \to \frac{1}{\pi} \frac{1}{60} \left(B[\nabla_i, [\nabla_i, B]] + [\nabla_i, [\nabla_i, B]]B + 4[\nabla_i, B][\nabla_i, B] \right) \tag{6.34a}$$

$$\nabla_j \delta(0) \to \frac{1}{\pi} \frac{1}{24} \epsilon_{jk} [\nabla_k, B] \tag{6.34b}$$

$$\delta(0) \to 0. \tag{6.34c}$$

Using (6.24) and the relations

$$[\nabla_i, B]^{ab} = f^{acb} \nabla_i^{cd} B^d \tag{6.35a}$$

$$[\nabla_i, [\nabla_i, B]]^{ab} = f^{acb} \Delta^{cd} B^d, \tag{6.35b}$$

which can be proved in the sam way as (6.31), one arrives at

$$\text{T}\mathcal{O}_1 = \frac{N g^2}{2\pi} \int \left\{ -\frac{2}{60} \left[B^a \Delta^{ab} B^b + \left(\Delta^{ab} B^b \right) B^a + 4 \left(\nabla_i^{ab} B^b \right) \left(\nabla_i^{ac} B^c \right) \right] \right.$$

$$\left. + \frac{4}{24} \left(\nabla_i^{ab} B^b \right) \left(\nabla_i^{ac} B^c \right) \right\} \tag{6.36a}$$

$$= -\frac{1}{10} m \int B^a \Delta^{ab} B^b. \tag{6.36b}$$

The divergent terms (which are not shown here explicitly) can be absorbed into the definition of the normal ordered form of $\mathcal{O}_1$ similar to (6.26), which then satisfies

$$\text{T} :\mathcal{O}_1: = -\frac{1}{10} m :\mathcal{O}_1:. \tag{6.37}$$

Again it is (a priori) very surprising that $\mathcal{O}_1$ in this calculation turned out to be an eigenvector of T, since this required a very subtle interplay between all the terms arising from the regularisation procedure (cf. appendix C.2), that could not possibly have been guessed. But just like in the $n = 0$ case, the precise eigenvalue found here is negative and contradicts the conjecture (6.11a) also in magnitude.

6.2.3. n=2

Finally, consider the operator

$$\mathcal{O}_2 = \int_y B^b(\boldsymbol{y})\Delta^{bc}\Delta^{cd}B^d(\boldsymbol{y}). \tag{6.38}$$

The first functional derivative is

$$\frac{\delta\mathcal{O}_2}{\delta A_i^a(\boldsymbol{x})} = 2\epsilon_{ij}\nabla_j^{ab}\Delta^{bc}\Delta^{cd}B^d(\boldsymbol{x})$$

$$+ \int_y B^b(\boldsymbol{y})\Big[\big(f^{bae}\delta(\boldsymbol{y}-\boldsymbol{x})\nabla_i^{ec} + \nabla_i^{be}f^{eac}\delta(\boldsymbol{y}-\boldsymbol{x})\big)\Delta^{cd}$$

$$+ \Delta^{bc}\big(f^{cae}\delta(\boldsymbol{y}-\boldsymbol{x})\nabla_i^{ed} + \nabla_i^{ce}f^{ead}\delta(\boldsymbol{y}-\boldsymbol{x})\big)\Big]B^d(\boldsymbol{y}) \tag{6.39a}$$

$$= 2\epsilon_{ij}\nabla_j^{ab}\Delta^{bc}\Delta^{cd}B^d(\boldsymbol{x})$$

$$- 2B^{ae}\nabla_i^{ec}\Delta^{cd}B^d(\boldsymbol{x}) - 2\big(\nabla_i^{eb}B^b(\boldsymbol{x})\big)f^{eac}\Delta^{cd}B^d(\boldsymbol{x}) \tag{6.39b}$$

and so the second one gives

$$\frac{\delta^2\mathcal{O}_2}{\delta A_i^a(\boldsymbol{x})^2} = -2\bigg\{\nabla_i^{ab}\Delta^{bc}\Delta^{cd}\nabla_i^{da}\delta(0) - \epsilon_{ij}\nabla_j^{ab}\Big[\big(f^{bae}\delta(0)\nabla_i^{ec} + \nabla_i^{be}f^{eac}\delta(0)\big)\Delta^{cd}$$

$$+ \Delta^{bc}\big(f^{cae}\delta(0)\nabla_i^{ed} + \nabla_i^{ce}f^{ead}\delta(0)\big)\Big]B^d\bigg\}$$

$$- 2\bigg\{-\epsilon_{ij}f^{abe}\big(\nabla_j^{ba}\delta(0)\big)\nabla_i^{ec}\Delta^{cd}B^d + B^{ae}f^{eac}\delta_{ii}\delta(0)\Delta^{cd}B^d$$

$$+ B^{ae}\nabla_i^{ec}\Big[f^{caf}\delta(0)\nabla_i^{fd} + \nabla_i^{cf}f^{fad}\delta(0)\Big]B^d - \epsilon_{ij}B^{ae}\nabla_i^{ec}\Delta^{cd}\nabla_j^{da}\delta(0)\bigg\}$$

$$- 2\bigg\{f^{eab}\delta_{ii}\delta(0)B^b f^{eac}\Delta^{cd}B^d - \big(\epsilon_{ij}\nabla_i^{eb}\nabla_j^{ba}\delta(0)\big)f^{eac}\Delta^{cd}B^d$$

$$+ \big(\nabla_i^{eb}B^b\big)f^{eac}\Big[f^{caf}\delta(0)\nabla_i^{fd} + \nabla_i^{cf}f^{fad}\delta(0)\Big]B^d$$

$$- \epsilon_{ij}\big(\nabla_i^{eb}B^b\big)f^{eac}\Delta^{cd}\nabla_j^{da}\delta(0)\bigg\} \tag{6.40a}$$

$$= -2\Big[\nabla_i\Delta\Delta\nabla_i\delta(0)\Big]^{aa}$$

$$+ 2\epsilon_{ij}\bigg\{2\Big[\nabla_j\delta(0)\Big]^{ab}f^{bae}\nabla_i^{ec}\Delta^{cd}B^d + \Big[\nabla_j\Delta\delta(0)\Big]^{ac}f^{cae}\nabla_i^{ed}B^d$$

$$- 2\Big[\nabla_j\Delta\nabla_i\delta(0)\Big]^{ae}B^{ea} + \big(\nabla_i^{eb}B^b\big)f^{eac}\Big[\Delta\nabla_j\delta(0)\Big]^{ca}\bigg\}$$

$$- 4NB^c\delta(0)\Delta^{cd}B^d - 2B^{ae}\Big[\nabla_i\delta(0)\Big]^{ec}f^{caf}\nabla_i^{fd}B^d + 2B^{ae}\Big[\Delta\delta(0)\Big]^{ef}B^{fa}$$

$$- 8N\delta(0)B^b\Delta^{bd}B^d + 2N\big(\nabla_i^{eb}B^b\big)\delta(0)\nabla_i^{ed}B^d + 2\big(\nabla_i^{eb}B^b\big)f^{eac}\Big[\nabla_i\delta(0)\Big]^{cf}B^{fa}.$$

$$(6.40b)$$

The regularisation procedure is again the same as above. The detailed calculation can be found in appendix C.2 and the results with up to four derivatives acting on $\delta(0)$ are (only $\mathcal{O}(M^0)$ terms are shown):

$$\nabla_j\Delta\delta(0) \rightarrow \frac{1}{\pi}\frac{1}{240}\Big(\epsilon_{jk}[\nabla_k,\Delta B] - 2B[\nabla_j,B] + 6[\nabla_j,B^2]\Big) \tag{6.41a}$$

$$\Delta\nabla_j\delta(0) \rightarrow \frac{1}{\pi}\frac{1}{240}\Big(\epsilon_{jk}[\nabla_k,\Delta B] + 18B[\nabla_j,B] + 6[\nabla_j,B^2]\Big) \tag{6.41b}$$

$$\epsilon_{ij}\nabla_j\Delta\nabla_i\delta(0) \rightarrow \frac{1}{\pi}\frac{1}{8}\epsilon_{ij}\Big(\nabla_iB^2\nabla_j - \nabla_iB\nabla_jB - B\nabla_iB\nabla_j\Big). \tag{6.41c}$$

The most complicated term is the one in the first line in (6.40b) with six derivatives acting on $\delta(0)$. As shown in appendix C.2, it gets quite difficult to extract a useful result in terms of B and its derivatives from it and therefore it remains unclear at this point whether $\mathcal{O}_2$ is again an eigenvector of T (up to terms $\mathcal{O}(B^3)$). But one can show that *if* $\mathcal{O}_2$ is an eigenvector, then the contribution to $T\mathcal{O}_2$ is:

$$\nabla_i\Delta\Delta\nabla_i\delta(0) \rightarrow -\frac{1}{\pi}\frac{11}{1680}B[\nabla_i,[\nabla_i,[\nabla_j,[\nabla_j,B]]]] + \mathcal{O}(B^3). \tag{6.42}$$

Plugging (6.41), (6.42) as well as (6.34) into (6.40b) and integrating by parts then gives (the divergent normal ordering terms are not shown explicitly)

$$T\mathcal{O}_2 = -\frac{Ng^2}{2\pi}\int_x\left[-\frac{11}{840} + 2\left(\frac{2}{24} + \frac{1}{240} + \frac{1}{240}\right)\right]B^a\Delta^{ab}\Delta^{bc}B^c + \mathcal{O}(B^3) \tag{6.43a}$$

$$= -\frac{143}{840}m\mathcal{O}_2 + \mathcal{O}(B^3). \tag{6.43b}$$

So either $\mathcal{O}_2$ is not an eigenvector of T (cf. the hypothesis discussed in appendix C.2), or its eigenvalue is $-143/840$. Again this (possible) eigenvalue is negative and disagrees with the conjecture. Moreover, together with the first two eigenvalues $-1/12$ and $-1/10$ it contradicts a linear spectrum, which would be crucial for the derivation of a kernel ODE similar to (4.16). The concrete consequences of the eigenvalues found here for the vacuum wave functional will be analysed in 6.4.

6.3. Further Regularisation Issues

The quantitative discrepancy with the LMY results for T acting on the operators $\mathcal{O}_n$ found in section 6.2 is very disturbing. The main difference between the two approaches lies in

the choice of a regularisation scheme, and as we learned in chapter 5 this can have an important impact on the exact eigenvalues. Therefore it is worthwhile to discuss possible issues regarding the regularisation procedure chosen in section 6.2 in a more careful way.

6.3.1. Choosing Different Regulators

The action of T on $\mathcal{O}_n$ generally leads to expressions involving the coincidence limit of covariant derivatives acting on a delta function, as seen in section 6.2. To give a meaning to these divergent results, the delta functions were replaced by a covariantised narrow Gaussian, with a width given by the inverse regularisation scale $1/M$:

$$\delta(\boldsymbol{x} - \boldsymbol{y}) \to \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \, \exp\left(\boldsymbol{\nabla}_x^2/M^2\right) \mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \exp\left(\boldsymbol{\nabla}_x^2/M^2\right) \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{6.44}$$

But to smear out the delta function one could also use other functions with appropriate fall off and normalisation conditions instead of a Gaussian. The corresponding generalisation of (6.44) is

$$\delta(\boldsymbol{x} - \boldsymbol{y}) \to R\left(-\boldsymbol{\nabla}_x^2/M^2\right) \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{6.45}$$

where $R(s)$ is some function that satisfies $R(0) = 1$ and that smoothly goes to 0 as $s \to \infty$.[6]

Now we want to show that the regulated, finite results do not depend on this otherwise arbitrary function $R(s)$. As a definite example, consider the coincidence limit

$$\left[\boldsymbol{\nabla}_x^2 \delta(\boldsymbol{x} - \boldsymbol{y})\right]_{\boldsymbol{y}\to\boldsymbol{x}} \tag{6.46}$$

which appeared in the evaluation of $\mathrm{T}\mathcal{O}_0$ in section 6.2.1. Using the general regulator (6.45) this can again be evaluated in momentum space:

$$(6.46) \to \left[\boldsymbol{\nabla}_x^2 R\left(-\boldsymbol{\nabla}_x^2/M^2\right) \delta(\boldsymbol{x} - \boldsymbol{y})\right]_{\boldsymbol{y}\to\boldsymbol{x}} \tag{6.47a}$$

$$= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \, (\boldsymbol{\nabla} + i\boldsymbol{p})^2 \, R\left[-(\boldsymbol{\nabla} + i\boldsymbol{p})^2/M^2\right] \tag{6.47b}$$

$$= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \, M^2 \left(-M^2 \boldsymbol{p}^2 + 2iM\boldsymbol{p}\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}^2\right) R\left(\boldsymbol{p}^2 - 2i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{M} - \frac{\boldsymbol{\nabla}^2}{M^2}\right). \tag{6.47c}$$

Now the function R can be expanded in inverse powers of M as

$$R\left(\boldsymbol{p}^2 - (\dots)\right) = R(\boldsymbol{p}^2) - R'(\boldsymbol{p}^2)(\dots) + \frac{1}{2}R''(\boldsymbol{p}^2)(\dots)^2 + \frac{1}{3!}R'''(\boldsymbol{p}^2)(\dots)^3$$
$$+ \frac{1}{4!}R''''(\boldsymbol{p}^2)(\dots)^4 + \mathcal{O}(1/M^5), \tag{6.48}$$

$$(\dots) := 2i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{M} + \frac{\boldsymbol{\nabla}^2}{M^2}.$$

---

[6]These asymptotic values also have to be approached fast enough as will be seen below, cf. (6.52).

Plugging this into (6.47c) and keeping only finite, i.e. $\mathcal{O}(M^0)$ terms gives, after some rearranging

$$(6.46) \rightarrow - \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \left\{ \frac{1}{2} \left(\boldsymbol{p}^2 R'' + 2R' \right) \boldsymbol{\nabla}^4 \right.$$

$$+ \frac{2}{3} \left(\boldsymbol{p}^2 R''' + 3R'' \right) p_i p_j \left(\boldsymbol{\nabla}^2 \nabla_i \nabla_j + \nabla_i \boldsymbol{\nabla}^2 \nabla_j + \nabla_i \nabla_j \boldsymbol{\nabla}^2 \right) \quad (6.49)$$

$$\left. + \frac{2}{3} \left(\boldsymbol{p}^2 R'''' + 4R''' \right) p_i p_j p_k p_l \nabla_i \nabla_j \nabla_k \nabla_l \right\}$$

where the argument $\boldsymbol{p}^2$ of R was omitted for brevity. The momentum integrals can be carried out using repeated integration by parts:

$$\int \mathrm{d}^2 p \left(\boldsymbol{p}^2 R'' + 2R' \right) = \pi \int_0^\infty \mathrm{d}s \, (sR''(s) + 2R'(s))$$

$$= \pi \int_0^\infty \mathrm{d}s \, R'(s) = -\pi \qquad (6.50a)$$

$$\int \mathrm{d}^2 p \left(\boldsymbol{p}^2 R''' + 3R'' \right) p_i p_j = \pi \int_0^\infty \mathrm{d}s \, (sR'''(s) + 3R''(s)) \frac{\delta_{ij}}{2} s^2$$

$$= \frac{\pi}{2} \delta_{ij} \int_0^\infty \mathrm{d}s \, R''(s) s = \frac{\pi}{2} \delta_{ij} \qquad (6.50b)$$

$$\int \mathrm{d}^2 p \left(\boldsymbol{p}^2 R'''' + 4R''' \right) p_i p_j p_k p_l = \pi \int_0^\infty \mathrm{d}s \, (sR''''(s) + 4R'''(s)) \frac{\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{8} s^2$$

$$= \frac{\pi}{8} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) \int_0^\infty \mathrm{d}s \, R'''(s) s^2$$

$$= -\frac{\pi}{4} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right), \qquad (6.50c)$$

where the index structure and the combinatorial factors follow from symmetry considerations. This leads to the result

$$(6.46) \rightarrow - \frac{1}{4\pi} \left[\left(-\frac{1}{2} + \frac{2}{3} - \frac{1}{6} \right) \boldsymbol{\nabla}^4 + \left(\frac{1}{3} - \frac{1}{6} \right) \nabla_i \boldsymbol{\nabla}^2 \nabla_i - \frac{1}{6} \nabla_i \nabla_j \nabla_i \nabla_j \right] \qquad (6.51a)$$

$$= \frac{1}{2\pi} \frac{1}{12} B^2, \qquad (6.51b)$$

which indeed agrees with the result found in section 6.2.1.

Note that for the integrations by parts in (6.50) it was implicitly assumed that the regulator function R fulfils the following conditions:

$$s^m R^{(m)}(s) \big|_{s=0,\infty} = 0 \qquad (m = 1, 2, 3). \qquad (6.52)$$

Those are not only met by the original (Fujikawa) regulator function $R(s) = \mathrm{e}^{-s}$, but also by many other examples including e^{-s^2}, $1/(1+s^\alpha)$ for $\alpha > 0$, $1/\ln(\mathrm{e}+s)$ or $1 - \frac{2}{\pi} \arctan(s)$. If, on the other hand those conditions are not fulfilled, then the integrals in (6.50) would typically diverge and R would not be an admissible regulator. So in conclusion *any regulator* that can be used to extract a finite result for $\mathrm{T}\mathcal{O}_0$ will give the *same result*.

Even though this was only checked for $\mathcal{O}_0$ here, one can expect that the calculations for $\mathcal{O}_n$ with $n > 0$ will go analogously, with the only difference that the condition (6.52) would have to be fulfilled also for higher values of m, according to the number of derivatives acting on the delta function. But this is not a problem since the examples above show that they are typically fulfilled for any $m \in \mathbb{N}$.

6.3.2. A More Careful Analysis

So far the regularisation procedure was to replace the delta function by a covariant regulator whenever its coincidence limit appeared during a calculation. As mentioned earlier, this is essentially the same as the standard textbook method that is used to calculate the axial anomaly[7], where it is widely believed to give the correct result. The main difference is that instead of $\overleftrightarrow{\nabla}$ here ∇ was used in the argument of the regulating function, since there are no fermions in the theory. Nevertheless, it is not quite obvious that this method really provides a consistent regularisation scheme in this context, and in this section this issue will be reviewed in a bit more rigorous way.

One possible consistency problem with the regularisation as described above might be the following: When T acts on $\mathcal{O}_n$, the first functional derivative already produces a delta function. But since its argument is not identically zero, there was no need to use the regulator at this point. It was only introduced when the second functional derivative yielded a $\delta(0)$. So in other words, the limit $M \to \infty$ was performed for one regulator before it was done for the other one. It would certainly be more consistent to treat all delta functions on an equal footing right from the beginning and take $M \to \infty$ simultaneously at the end.

One can also view this regularisation scheme in the following way: The origin of the delta function clearly lies in the fundamental quantum commutation relations (2.21)

$$[A_i^a(\boldsymbol{x}), E_j^b(\boldsymbol{y})] = -i\delta^{ab}\delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{6.53}$$

In order to avoid UV divergences, the right hand side can be smeared out in a gauge-covariant way:

$$[A_i^a(\boldsymbol{x}), E_j^b(\boldsymbol{y})] = -i\delta_{ij}R^{ab}(-\boldsymbol{\nabla}_x^2/M^2)\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{6.54}$$

where R is some regulator function as discussed in section 6.3.1 and ab denote the adjoint indices of this operator valued function. The asymptotic $R(0) = 1$, or $R^{ab}(0) = \delta^{ab}$, ensures that the original local canonical commutation relations (6.53) are reproduced in the limit $M \to \infty$. Under a gauge transformation

$$A_i^a(\boldsymbol{x}) \mapsto g^{ab}(\boldsymbol{x})A_i^b(\boldsymbol{x}) + \left(g(\boldsymbol{x})\partial_j g^{-1}(\boldsymbol{x})\right)^b, \tag{6.55}$$

where $g^{ab} = 2\,\mathrm{Tr}\left(t^a g\, t^b g^{-1}\right)$ are the adjoint components of the SU(N) matrix g, the left hand side of (6.54) transforms as

$$[A_i^a(\boldsymbol{x}), E_j^b(\boldsymbol{y})] \mapsto g^{ac}(\boldsymbol{x})g^{bd}(\boldsymbol{y})[A_i^c(\boldsymbol{x}), E_j^d(\boldsymbol{y})]. \tag{6.56}$$

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[7]See, for instance [15, chapter 22], [16, chapter 77], [23, chpater 3] or [27, chapter 19.2].

The gauge-covariant derivative in the argument of R also transforms homogeneously and thus makes sure that the right hand side has the same transformation property, so that gauge invariance of the theory is preserved in this regularisation scheme. (Note that on the other hand, the axiom of locality is violated, which is an inevitable consequence of any point splitting regularisation.)

In A-representation, the regularised commutation relation (6.54) implies that the colour-electric field operator becomes

$$E_j^b(\boldsymbol{y}) = i \int_z R^{bc}(\boldsymbol{y}, \boldsymbol{z}) \frac{\delta}{\delta A_j^c(\boldsymbol{z})} \tag{6.57}$$

where the short hand notation

$$R^{bc}(\boldsymbol{y}, \boldsymbol{z}) := R^{bc}(-\boldsymbol{\nabla}_y^2/M^2)\delta(\boldsymbol{y} - \boldsymbol{z}) \tag{6.58a}$$

$$= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} R^{bc}(-\boldsymbol{\nabla}_y^2/M^2)\mathrm{e}^{i\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{z})} \tag{6.58b}$$

was introduced. (The momentum space representation will be most convenient for doing actual calculations.) Equation (6.57) can easily be verified by acting with it on $A_i^a(\boldsymbol{x})$ and observing that this reproduces minus the right hand side of (6.54).

Next, we need to work out the regularised version of the kinetic energy operator. Using (6.57) to replace the E's in $\mathrm{T} = \frac{g^2}{2} \int_x E_i^a(\boldsymbol{x})^2$ would lead to

$$\mathrm{T} = -\frac{g^2}{2} \int_x \int_y R^{ac}(\boldsymbol{x}, \boldsymbol{y}) \frac{\delta}{\delta A_i^c(\boldsymbol{y})} \int_z R^{ad}(\boldsymbol{x}, \boldsymbol{z}) \frac{\delta}{\delta A_i^d(\boldsymbol{z})}, \tag{6.59}$$

but a potential problem is that in this form T is not manifestly self-adjoint. To make sure that the regularised kinetic energy operator is still self-adjoint, one can proceed as follows (cf. also the discussion in [7, p. 143]). The matrix element of T between two wave functionals Ψ_1 and Ψ_2 can be written as ([$\mathrm{d}A$] denotes the path-integral measure)

$$\langle \Psi_1 | \mathrm{T} | \Psi_2 \rangle = \frac{g^2}{2} \int [\mathrm{d}A] \, \Psi_1^* \int_x E_i^a(\boldsymbol{x})^2 \, \Psi_2 \tag{6.60a}$$

$$= \frac{g^2}{2} \int [\mathrm{d}A] \int_x \left(E_i^a(\boldsymbol{x}) \Psi_1 \right)^* \left(E_i^a(\boldsymbol{x}) \, \Psi_2 \right) \tag{6.60b}$$

$$= \frac{g^2}{2} \int [\mathrm{d}A] \int_x \left(\int_y R^{ac}(\boldsymbol{x}, \boldsymbol{y}) \frac{\delta}{\delta A_i^c(\boldsymbol{y})} \Psi_1 \right)^* \left(\int_z R^{ad}(\boldsymbol{x}, \boldsymbol{z}) \frac{\delta}{\delta A_i^d(\boldsymbol{z})} \, \Psi_2 \right) \tag{6.60c}$$

$$= -\frac{g^2}{2} \int [\mathrm{d}A] \, \Psi_1^* \int_{x,y,z} \frac{\delta}{\delta A_i^c(\boldsymbol{y})} R^{ac}(\boldsymbol{x}, \boldsymbol{y}) R^{ad}(\boldsymbol{x}, \boldsymbol{z}) \frac{\delta}{\delta A_i^d(\boldsymbol{z})} \, \Psi_2 \tag{6.60d}$$

$$\tag{6.60e}$$

and so T is given by

$$\mathrm{T} = -\frac{g^2}{2} \int_{x,y,z} \frac{\delta}{\delta A_i^c(\boldsymbol{y})} R^{ac}(\boldsymbol{x}, \boldsymbol{y}) R^{ad}(\boldsymbol{x}, \boldsymbol{z}) \frac{\delta}{\delta A_i^d(\boldsymbol{z})}. \tag{6.61}$$

This construction makes sure that T remains manifestly self-adjoint upon regularisation. To simplify things, note that the two regulators can be combined as

$$\int_x R^{ac}(\boldsymbol{x}, \boldsymbol{y}) R^{ad}(\boldsymbol{x}, \boldsymbol{z}) = \int_x \left[R^{ac}(-\boldsymbol{\nabla}_x^2/M^2)\delta(\boldsymbol{x} - \boldsymbol{y}) \right] \left[R^{ad}(-\boldsymbol{\nabla}_x^2/M^2)\delta(\boldsymbol{x} - \boldsymbol{z}) \right] \quad \text{(6.62a)}$$

$$= \int_x \delta(\boldsymbol{x} - \boldsymbol{y}) R^{ca}(-\boldsymbol{\nabla}_x^2/M^2) R^{ad}(-\boldsymbol{\nabla}_x^2/M^2)\delta(\boldsymbol{x} - \boldsymbol{z}) \quad \text{(6.62b)}$$

$$= \int_x \delta(\boldsymbol{x} - \boldsymbol{y}) \left[R^2(-\boldsymbol{\nabla}_x^2/M^2) \right]^{cd} \delta(\boldsymbol{x} - \boldsymbol{z}) \quad \text{(6.62c)}$$

$$= \left[R^2 \right]^{cd}(\boldsymbol{y}, \boldsymbol{z}) \quad \text{(6.62d)}$$

where the two R's could be combined to one R^2 because $\boldsymbol{\nabla}^2$ commutes with itself. Now if R is a valid regularisation function as discussed in section 6.3.1, then R^2 will also have this property. But as seen in section 6.3.1, the physical, i.e. M-independent results do not depend on the choice of R, so the two R's in (6.63a) can effectively be replaced by one, and so the regularised kinetic energy operator is

$$\text{T} = -\frac{g^2}{2} \int_{x,y} \frac{\delta}{\delta A_i^a(\boldsymbol{x})} R^{ab}(\boldsymbol{x}, \boldsymbol{y}) \frac{\delta}{\delta A_i^b(\boldsymbol{y})} \quad \text{(6.63a)}$$

$$= -\frac{g^2}{2} \int_{x,y} \frac{\delta R^{ab}(\boldsymbol{x}, \boldsymbol{y})}{\delta A_i^a(\boldsymbol{x})} \frac{\delta}{\delta A_i^b(\boldsymbol{y})} - \frac{g^2}{2} \int_{x,y} R^{ab}(\boldsymbol{x}, \boldsymbol{y}) \frac{\delta}{\delta A_i^b(\boldsymbol{y})} \frac{\delta}{\delta A_i^a(\boldsymbol{x})}. \quad \text{(6.63b)}$$

Note that the first term, in which one of the functional derivatives hits the regulator, would effectively be absent in the more naive regularisation procedure used in section 6.2. Also, this term is reminiscent of the linear part $\text{T}_1[J]$ of the KKN kinetic energy operator, cf. equation (3.95). Recalling from equation (5.4) that this term was the one that gave a contribution of $2m$ to the eigenvalues of $\mathcal{O}_n$, one could imagine that the disagreement with the LMY results found in section 6.2 stems from neglecting this term there. But even without explicitly calculating the functional derivative of R, one can easily see that this term must in fact vanish. To this end, one can write the first part of T as

$$\text{T}_1 = -\frac{g^2}{2} \int_y X_i^b(\boldsymbol{y}) \frac{\delta}{\delta A_i^b(\boldsymbol{y})}, \qquad X_i^b(\boldsymbol{y}) := \int_x \frac{\delta R^{ab}(\boldsymbol{x}, \boldsymbol{y})}{\delta A_i^a(\boldsymbol{x})}. \quad \text{(6.64)}$$

Now without evaluating X, it is clear that for dimensional reasons X must have mass dimension $+1$, and since the argument of the regulator is $\boldsymbol{\nabla}^2/M^2$, the only candidate that survives the limit $M \to \infty$ is ∇. Furthermore, X has only one free colour index, and so the covariant derivative, which carries two colour indices, has to be contracted with a structure constant. So X has to be either of the form

$$X_i^b \propto f^{cdd} \nabla_i^{cb} = 0 \quad \text{(6.65)}$$

which vanishes due to the antisymmetry of the structure constants, or of the form

$$X_i^b \propto f^{bcd} \nabla_i^{dc} = N A_i^b, \quad \text{(6.66)}$$

where only the gauge field part of ∇ is left. Since A does not transform homogeneously under a gauge transformation, such a term would imply that T_1 is not gauge-invariant[8]. But the regularisation scheme certainly respects gauge invariance, and so the conclusion is that the constant of proportionality in (6.66) must turn out to be zero, which means that T_1 vanishes altogether. So finally we are only left with

$$\mathrm{T} = -\frac{g^2}{2} \int_{x,y} R^{ab}(\boldsymbol{x}, \boldsymbol{y}) \frac{\delta}{\delta A_i^b(\boldsymbol{y})} \frac{\delta}{\delta A_i^a(\boldsymbol{x})}. \tag{6.67}$$

Note that one could also have started directly from this expression by demanding a gauge-invariantly point-split version of the kinetic energy operator, where $R(\boldsymbol{x}, \boldsymbol{y})$ would be any regulator that also serves as a parallel transport from $\boldsymbol{y}$ to $\boldsymbol{x}$. The argument in favour of the vanishing of the linear term would also have applied in this case, implying that the regularised kinetic energy operator is properly self-adjoint. In fact, this also shows that T is self-adjoint for any ordering of R and the functional derivatives.

Now that the regularisation has been implemented at a more fundamental level, let us check whether it reproduces the result from section 6.2. Using (6.6) and integrating by parts leads to

$$\mathrm{T}_2 \mathcal{O}_0 = g^2 \int_{x,y} R^{ab}(\boldsymbol{x}, \boldsymbol{y}) \frac{\delta}{\delta A_i^b(\boldsymbol{y})} \epsilon_{ij} \nabla_{jx}^{ac} B^c(\boldsymbol{x}) \tag{6.68a}$$

$$= g^2 \int_{x,y} R^{ab}(\boldsymbol{x}, \boldsymbol{y}) \nabla_{ix}^{ac} \nabla_{ix}^{cb} \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{6.68b}$$

$$= g^2 \int_{x,y} \delta(\boldsymbol{x} - \boldsymbol{y}) \nabla_{ix}^{bc} \nabla_{ix}^{ca} R^{ab}(\boldsymbol{x}, \boldsymbol{y}) \tag{6.68c}$$

$$= g^2 \int_x \left[\boldsymbol{\nabla}_x^2 R(-\boldsymbol{\nabla}_x^2/M^2) \delta(\boldsymbol{x} - \boldsymbol{y}) \right]_{\boldsymbol{y} \to \boldsymbol{x}}^{bb}, \tag{6.68d}$$

which is indeed exactly the same expression that was used previously, i.e. the one in (6.17) or (6.47a).

For the operators $\mathcal{O}_n$ with $n > 0$ it is sufficient to observe that the calculations in section 6.2 will be reproduced with the only difference that instead of $\delta(0)$ the regulator $R(\boldsymbol{x}, \boldsymbol{y})$ (with appropriate indices) will appear, of which the coincident limit has to be taken in the end. This leads exactly to all the regulated expressions that were used in section 6.2; in particular it gives a more rigorous justification for the rule that $\delta(0)$ has to be pulled inside the adjoint index structure before regularising it.

So the more careful investigation of possible regularisation issues only confirmed the results that were found in section 6.2, and it remains unclear why they disagree with the ones obtained using the KKN variables.

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[8]A potential loophole to this argument would be that it might be gauge-invariant only up to boundary terms, like $\mathrm{T}_1[J]$, cf. the remark below equation (3.95). But there this was only possible because the corresponding symmetry transformations were holomorphic functions h, for which $\bar{\partial} h = 0$. Here the gauge transformations do not have a similar property and so this possibility is ruled out.

6.4. A Modified Kernel

In section 6.2 it was argued that the negative kinetic energy eigenvalues found there for the operators $\mathcal{O}_n$ are a priori not unphysical, because they do not correspond to observable energies of physical states. But of course there remains the question of how they would alter the result for the vacuum wave functional, which does have physical consequences.

6.4.1. Linear Spectrum

To answer this question, let us first assume a linear spectrum taking into account only the eigenvalues for the first two operators. In section 6.2 it was found that

$$\mathrm{T} \; :\mathcal{O}_0: \; = -\frac{1}{12}m \; :\mathcal{O}_0: \tag{6.69a}$$

$$\mathrm{T} \; :\mathcal{O}_1: \; = -\frac{1}{10}m \; :\mathcal{O}_1:, \tag{6.69b}$$

which suggests a modified conjecture

$$\mathrm{T} \; :\mathcal{O}_n: \; = -\left(\frac{1}{12} + \frac{n}{60}\right)m \; :\mathcal{O}_n:. \tag{6.70}$$

In doing so the result for $\mathcal{O}_2$, viz. $-\frac{143}{840}m$, which is not compatible with a linear spectrum, is disregarded. We will nevertheless use it as a first approximation, just to illustrate the effect of negative eigenvalues on the form of the vacuum wave functional, in a scenario where the kernel can be computed analytically. It is also instructive to see the comparison to the LMY proposal, where agreement with the conjecture was also found only for the first two eigenvalues, if at all.

So assuming for the moment that (6.70) is true, then (6.8b) becomes ($\mathcal{O}(B^3)$ terms are again suppressed)

$$\mathrm{H}\Psi_0 = \frac{1}{2g^2}\left(\frac{1}{2}\sum_{n=0}^{\infty}\frac{c_n}{(4m^2)^n}\left(\frac{1}{12}+\frac{n}{60}\right)\mathcal{O}_n + \int_x B^a(\boldsymbol{x})\left[LK^2(L)+1\right]^{ab}B^b(\boldsymbol{x}) + E_0\right)\Psi_0 \tag{6.71a}$$

$$= \frac{1}{2g^2}\int_x B^a(\boldsymbol{x})\left(\frac{1}{2}\sum_{n=0}^{\infty}c_n\left(\frac{1}{12}+\frac{n}{60}\right)L^n + LK^2(L)+1\right)^{ab}B^b(\boldsymbol{x})\Psi_0 + E_0\Psi_0 \tag{6.71b}$$

$$= \frac{1}{2g^2}\int_x B^a(\boldsymbol{x})\left(\frac{1}{24}K(L) + \frac{1}{120}L\frac{\mathrm{d}}{\mathrm{d}L}K(L) + LK^2(L)+1\right)^{ab}B^b(\boldsymbol{x})\Psi_0 + E_0\Psi_0 \tag{6.71c}$$

and so in order to solve the Schrödinger equation to quadratic order in B, the kernel has to satisfy the ordinary differential equation

$$\frac{1}{24}K(L) + \frac{1}{120}L\frac{\mathrm{d}}{\mathrm{d}L}K(L) + LK^2(L) + 1 = 0. \tag{6.72}$$

This ODE has the same form as the one derived by LMY, because the $\mathcal{O}_n$ spectrum is still assumed to be linear. Only the coefficients of the first two terms changed due to the modified conjecture (6.70), in particular they now have the opposite sign.

The general solution of (6.72) can be found in the same way as before (see [12] for details) and reads

$$K(L) = -\frac{1}{\sqrt{L}} \frac{CY_5(240\sqrt{L}) + J_5(240\sqrt{L})}{CY_4(240\sqrt{L}) + J_4(240\sqrt{L})} \tag{6.73}$$

where again C is a constant of integration and J_n and Y_n are Bessel functions of the first and second kind, respectively. So the modification of the conjecture changed the general solution for the kernel in two ways: First, the order of the Bessel functions changed from 2 and 1 to 5 and 4, according to the ratio of the numerical coefficients entering the conjecture. And second, the overall sign changed owing to the negative sings of the eigenvalues.

This overall minus sign might at first glance look disturbing, because in order to obtain a normalisable wave functional, K has to be positive everywhere. But luckily there is still the freedom to choose the constant of integration in a suitable way. To see how it should be chosen, one can again proceed as in [12]. First note that in momentum space $L = -\boldsymbol{p}^2/4m^2$ is negative and so the solution should be rewritten in terms of modified Bessel functions using the relations[9]

$$J_4(ix) = I_4(x) \qquad\qquad Y_4(ix) = -\frac{2}{\pi}K_4(x) + iI_4(x) \tag{6.74a}$$

$$J_5(ix) = iI_5(x) \qquad\qquad Y_5(ix) = \frac{2}{\pi}iK_5(x) - I_5(x), \tag{6.74b}$$

giving

$$K(-|L|) = +\frac{1}{\sqrt{|L|}} \frac{K_5(240\sqrt{|L|}) + C'I_5(240\sqrt{|L|})}{K_4(240\sqrt{|L|}) - C'I_4(240\sqrt{|L|}} \tag{6.75}$$

with

$$C' := \frac{i - C}{C} \frac{\pi}{2i} . \tag{6.76}$$

Now the only choice that leads to a kernel which is positive for all momenta is $C' = 0$ (or equivalently $C = i$) and the corresponding kernel reads

$$K(-|L|) = \frac{1}{\sqrt{|L|}} \frac{K_5(240\sqrt{|L|})}{K_4(240\sqrt{|L|})}. \tag{6.77}$$

Even though the modified conjecture (6.70) on which this derivation relies, was not proved (and even seems to be violated for $n = 2$ and presumably also for $n > 2$), this result at least shows that negative eigenvalues for $\mathcal{O}_n$ can still lead to physical sensible results and do not have to be discarded a priori because of potential inconsistencies.

Let us now investigate the asymptotic behaviour of this solution. In the high energy limit we find

$$K(L) \sim \frac{1}{\sqrt{|L|}} \qquad (|L| \to \infty) \tag{6.78}$$

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[9]See, for example [28] and [29].

which agrees with the LMY result and indeed leads to a wave functional corresponding to asymptotically free gluons (see appendix B). But in the infra-red the asymptotic behaviour of the kernel is

$$K(L) \sim \frac{1}{30|L|} \qquad (|L| \to 0) \tag{6.79}$$

which is qualitatively different from the LMY kernel that approached a constant in this limit. Recall from section 4.1.3 that a constant kernel can be used to derive an area law for a Wilson loop, corresponding to a confining vacuum. So the solution (6.77) does not share this important property, which is of course very unsatisfying. Also note that the limit (6.79) shows that the expansion of the kernel implicitly changed from the Taylor series $K(L) = \sum_{n=0}^{\infty} c_n L^n$ to the more general ansatz of a Laurent series $K(L) = \sum_{n=-1}^{\infty} c_n L^n$. This corresponds to the choice for the constant of integration made above, which was necessary in order to obtain a normalisable wave functional.

6.4.2. General IR Behaviour

Since the derivation of the solution (6.77) relied on the conjecture (6.70) that presumably does not hold for all n, one could imagine that a deviation from a linear spectrum for higher n could alter the infra-red behaviour of the kernel. Unfortunately, this is not the case and the IR limit is in fact independent of the eigenvalues for higher n. This can be seen as follows: Instead of transforming (6.71b) to the ODE form (6.71c), one can alternatively compute the c_n in the expansion $K(L) = \sum_{n=0}^{\infty} c_n L^n$ directly from (6.71b) by comparing coefficients of equal powers of L. To order L^0 this yields

$$\frac{1}{24} c_0 + 1 = 0 \quad \Rightarrow c_0 = -24 \tag{6.80}$$

where only the $n = 0$ eigenvalue $-1/12$ was used. In the IR limit higher order contributions in L can be neglected, and so the vacuum wave functional would approach $\Psi_0^{\mathrm{IR}} = \exp\left[+\frac{3N}{\pi m^2} \int (B^a)^2\right]$, which has to be rejected because it is not normalisable. The only way out is to allow the series expansion of K to start with a negative power of L. Note that since only the $n = 0$ eigenvalue was used in this argument, the inclusion of at least one negative power of L in the kernel is an inevitable consequence of the negative sign of the $n = 0$ eigenvalue.

Now using $K = \sum_{n=-1}^{\infty} c_n L^n$ leads to the $\mathcal{O}(L^{-1})$ equation

$$-\frac{1}{2} c_{-1} t_{-1} + (c_{-1})^2 = 0 \qquad \overset{c_0 \neq 0}{\Rightarrow} \quad c_{-1} = \frac{1}{2} t_{-1} \tag{6.81}$$

where t_{-1} denotes the eigenvalue of $\mathcal{O}_{-1}$, i.e.

$$\mathrm{T} : \mathcal{O}_{-1} := t_{-1}\, m\, :\mathcal{O}_{-1}:. \tag{6.82}$$

In order to obtain a normalisable vacuum wave functional this time, c_{-1} has to be negative (since L itself is negative), which will be the case if and only if $t_{-1} < 0$. This was implicitly assumed when the eigenvalues were linearly interpolated, so that $t_{-1} = -\frac{1}{15}$, but of course (6.82) would have to be checked (it is not clear that $\mathcal{O}_{-1}$ is an eigenvector at all) and the value of t_{-1} would have to be derived by direct calculation to make this argument sound.

This is not an easy task because the covariant Laplacian can not simply be inverted, so the IR limit $K \sim 1/|L|$ remains speculative at this point.

It should be mentioned that there is in principle still a possibility to obtain the desired asymptotic form $K \sim$ const., even though the $n = 0$ eigenvalue is negative: The above argument only showed that in this case the kernel expansion must include *at least* one negative power of L, but it is a priori not clear that it cannot contain infinitely many of them. In other words, if one allows the kernel to have an infinite principal part of the Laurent series, it could still approach a constant as the argument goes to zero along the negative real axis (as for example the function $\exp(1/L)$). Since the exact form of the eigenvalues of $\mathcal{O}_n$ for general n is not known, it could as well be that such a function would ultimately be the solution to the *correct* kernel ODE. But the technical difficulty mentioned above to handle $\mathcal{O}_n$ with $n < 0$ makes it difficult to test this assertion.

7. Conclusion

In this thesis an analytic approach to determine the glueball mass spectrum of pure Yang-Mills theory in 2+1 dimensions, proposed by Leigh et al. [12] was carefully analysed. Their proposal is based on a gauge-invariant reformulation of the theory in the Hamiltonian framework, that was developed by Karabali et al. ([7], [8], [9], [10]) and which was also reviewed here in detail.

The most important outcome of the KKN parametrisation for the calculations by LMY is the form of the Hamiltonian, which, as was shown here, depends in a very subtle way on the correct regularisation procedure. "Correct" in this case means in particular that it must respect the new holomorphic symmetry that was obtained as a consequence of the construction of the gauge-invariant variables.

LMY made an ansatz for the vacuum wave functional that is quasi Gaussian in the KKN variable $\bar{\partial}J$, but with a non-trivial kernel that is implicitly J-dependent in order to respect the holomorphic symmetry. Then their strategy was to determine this kernel by solving the Schrödinger equation explicitly up to second order in J. However, in order to be able to solve the Schrödinger equation analytically they had to use a conjecture about the kinetic energy spectrum of the operators $\mathcal{O}_n = \int \bar{\partial}J^a \, (\Delta^n)^{ab} \, \bar{\partial}J^b$, which turned out to be violated. As was discussed here, the only calculations that gave results in agreement with the conjecture were done by LMY using an approximation to the KKN regulator that is inadequate for the class of local operators which were investigated.

The specific form of the conjecture, in particular the linear increase of the eigenvalues with n, is of vital importance for the derivation of the vacuum wave functional, and thus for the determination of the glueball masses. But the linearity was violated in all calculations, even when the simplified regulator was used which at least gave agreement for $n = 0, 1$. Therefore, strictly speaking, LMY can not make any valid statement about the glueball spectrum. Moreover, as was discussed here, even if the conjecture is assumed to hold, the resulting glueball masses are not in such excellent agreement with lattice data as claimed by the authors.

On the other hand, the low energy limit of the proposed vacuum wave functional, which can be used to deduce an area law for a Wilson loop operator and thus implies confinement, does not depend on the whole conjecture but only on the lowest eigenvalue. So the quantitative result of confinement is a solid result of the LMY analysis. The quantitative prediction for the string tension, however, is sensitive to the exact eigenvalue for $n = 0$, and the good agreement with lattice data was only found assuming that it equals 2. As the careful calculation in section 5.3 showed, this value changes by $3/16 \approx 20\%$ when the most accurate form of the KKN regulator is used. So the qualitative success of this prediction is also partially lost.

Regarding the contradiction with their conjecture, Leigh et al. take the following point of view: "We believe, however, that this result is not complete and the correct spectrum should be equidistant [...]. Thus the calculation that we have outlined above needs to be

reconsidered carefully. We do not at this time have a consistent understanding of all these issues as the required calculations are tedious" [12, p. 19], but that "at least in principle, it should be possible to prove (or disprove) it [the conjecture] by direct computation." [12, p. 2].

This was a major motivation to repeat the calculations using a different approach here in chapter 6. It was shown there that the vacuum wave functional proposed by LMY can easily be re-expressed in terms of the original Yang-Mills gauge field variables and has *exactly the same* quasi-Gaussian form. Furthermore, the calculation to solve the Schrödinger equation in the original variables is completely analogous to the one done by LMY, and in particular the only key ingredient is again the *same* conjecture about the kinetic energy spectrum. First of all, this means that the use of the KKN variables is in fact of no relevance for the LMY proposal[1], in sharp contrast to their attitude: "we believe that our results demonstrate the importance of 'corner variables' [...] in the pure Yang-Mills sector." [12, p. 16] Second, this alternative approach can thus provide an independent test of the LMY conjecture and is therefore very valuable.

The major difficulty to overcome is to find an appropriate regularisation for the kinetic energy operator. In the KKN approach this was achieved by a point-splitting procedure that respects the holomorphic invariance, and basically the same was done here, with the corresponding symmetry that had to be preserved being gauge invariance. This regularisation scheme is essentially the same that is widely used in the calculation of the axial anomaly, and it was also argued that here in the Hamiltonian formalism it leads to a covariantly point-split kinetic energy operator that remains self-adjoint.

Even though the qualitative feature of the conjecture, namely that the $\mathcal{O}_n$ are eigenvectors of the kinetic energy operator, was reproduced for $n = 0, 1$ (and possibly for $n = 2$), the quantitative results, i.e. the concrete eigenvalues, were quite different. The most striking difference is that the eigenvalues turned out to be negative. It was shown that this is a priori not a physical contradiction, since these negative eigenvalues are not observable, and because they can still lead to a normalisable vacuum wave functional; but the effect is that this change of sign alters the low energy limit, so that the vacuum wave functional probably does not imply confinement in this regime.

The discrepancy with the LMY results is of course very disturbing. To be more precise, the problem is *not* that the results disagree with the conjecture, because this was already found to be violated in the LMY approach. The contradiction lies in the disagreement with the actual values that were calculated using the KKN variables, even when using the most accurate form of the regulator as proposed by KKN. It should be emphasised that since the kinetic energy spectrum of the $\mathcal{O}_n$ directly affects the physical predictions (like confinement and the glueball masses), it *must be independent* of the choice of variables and regularisation scheme.

This implies that (at least) one of the two calculations/regularisation methods is wrong, and as long as there are no fundamental reasons to prefer one of them (which would certainly be the ultimate goal), one can only rely on the physical consequences that can be drawn. Since an equally spaced kinetic energy spectrum is inconsistent with either of

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[1]It should be noted that this does not apply to the KKN formalism in itself, since for example KKN themselves pursued a different approximation scheme, where the vacuum wave functional can not be re-expressed as a local functional of the original field strength. [9]

them, no predictions about the glueball spectrum can be made so far, and so no physical hints can be found this way. The most reliable prediction is the infra-red behaviour of the vacuum wave functional, which in the LMY proposal leads to confinement, whereas in the new approach it does not, or at least this cannot be shown for the time being. This might be interpreted as a hint in favour of the LMY procedure, but it could as well just indicate that the quasi-Gaussian approximation is simply insufficient to capture the most important non-perturbative features of the theory. After all, the only justification for using this quasi-Gaussian approximation by Leigh et al. was that in their approach it leads to physical sensible predictions. When this feature is lost, the whole ansatz has to be questioned.

To conclude, there are many questions that remain unanswered and that have to be left for future research, which is not surprising in this challenging field of research. The most urgent one regarding the LMY approach would be, how the discrepancies between the two regularisation schemes can be resolved, and what solid statements can be made about the form of the kinetic energy spectrum. One could also regard the numerous problems of the LMY approach that became evident in this thesis as a reason to focus on other methods, like for example the approximation scheme proposed by KKN ([9], [30]).

Appendix

A. Green's Functions for ∂ and $\bar{\partial}$

We want to calculate the Green's function G for the operator ∂, i.e. the function satisfying

$$\partial G(\boldsymbol{x}) \equiv \frac{1}{2}(\partial_1 + i\partial_2)G(\boldsymbol{x}) \stackrel{!}{=} \delta^2(\boldsymbol{x}). \tag{A.1}$$

The Fourier transform $\tilde{G}(\boldsymbol{k}) = \int \mathrm{d}^2x\, G(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ then satisfies

$$\frac{1}{2}(-ik_1 + k_2)\tilde{G}(\boldsymbol{k}) = 1 \quad \Rightarrow \tilde{G}(\boldsymbol{k}) = \frac{2}{-ik_1 + k_2} \tag{A.2}$$

and transforming back yields

$$G(\boldsymbol{x}) = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{2}{-ik_1 + k_2} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{A.3a}$$

$$= \begin{cases} \int_{-\infty}^{0} \frac{\mathrm{d}k_1}{2\pi} e^{-ik_1x_1}\left(-2ie^{k_1x_2}\right) & (x_2 > 0) \\ \int_{0}^{\infty} \frac{\mathrm{d}k_1}{2\pi} e^{-ik_1x_1}\left(2ie^{k_1x_2}\right) & (x_2 < 0) \end{cases} \tag{A.3b}$$

$$= \frac{-i}{\pi}\frac{1}{-ix_1 + x_2} = \frac{1}{\pi}\frac{1}{x_1 + ix_2} = \frac{1}{\pi\bar{z}} \tag{A.3c}$$

where first the k_2 integral was performed by applying the residue theorem to a contour closing the real line with a semicircle in the lower or upper half-plane for $x_2 > 0$ or $x_2 < 0$ respectively. Using the translational invariance of ∂ we have the final result

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{\pi(\bar{z} - \bar{z}')}. \tag{A.4}$$

The calculation for $\bar{G}$ goes analogously giving

$$\bar{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{\pi(z - z')}. \tag{A.5}$$

B. Vacuum Wave Functional for Pure QED in 2+1 Dimensions

Here we want to find the vacuum wave functional for pure QED, i.e. photons without any matter fields, in 2+1 dimensions. Like in the Yang-Mills case we include the coupling constant g in the field A_μ and adopt the temporal gauge $A_0 = 0$. Then the Lagrangian density for this theory is

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2g^2} \left[(\partial_0 A_i)^2 - B^2 \right] \tag{B.1}$$

with the magnetic field $B = F_{12} = \partial_1 A_2 - \partial_2 A_1 = \epsilon_{ij} \partial_i A_j$. The canonical momenta are (minus) the components of the electric field

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = \frac{1}{g^2} \partial_0 A_i = -E_i \tag{B.2}$$

and the Hamiltonian density is

$$\mathcal{H} = \Pi_i \partial_0 A_i - \mathcal{L} = \frac{g^2}{2} E_i^2 + \frac{1}{2g^2} B^2. \tag{B.3}$$

The commutation relation for A and E is

$$- [A_i(\boldsymbol{x}), E_j(\boldsymbol{y})] = i\delta_{ij}\delta^2(\boldsymbol{x} - \boldsymbol{y}), \tag{B.4}$$

so in A-representation $E_i = i\frac{\delta}{\delta A_i}$ and

$$\mathcal{H}(\boldsymbol{x}) = -\frac{g^2}{2} \frac{\delta^2}{\delta A_i(\boldsymbol{x})^2} + \frac{1}{2g^2} B(\boldsymbol{x})^2. \tag{B.5}$$

Like in the non-abelian case we have to impose the Gauss law constraint on physical states Ψ:

$$\partial_i E_i(\boldsymbol{x})\Psi = 0, \tag{B.6}$$

which again is the same as requiring physical states to be invariant under residual (i.e. time independent) gauge transformations. To solve the Schrödinger equation for the vacuum wave functional $\Psi_0[A]$,

$$\mathrm{H}\Psi_0 \equiv \int \mathcal{H}(\boldsymbol{x})\Psi_0 \, \mathrm{d}^2 x = E_0 \Psi_0, \tag{B.7}$$

where E_0 denotes the vacuum energy, we make the ansatz

$$\Psi_0[A] = \exp\left(-\frac{1}{2g^2} \int_{y_1, y_2} B(\boldsymbol{y}_1) f(\boldsymbol{y}_1 - \boldsymbol{y}_2) B(\boldsymbol{y}_2) \right) \tag{B.8}$$

where $f(\boldsymbol{x}) = f(-\boldsymbol{x})$ is some symmetric kernel function we would like to determine. Note that here in the abelian case this ansatz is manifestly gauge invariant (as it should be) since $B = F_{12}$ is gauge invariant, unlike in the non-abelian case[1]. The functional derivative acting on B gives

$$\frac{\delta B(\boldsymbol{y})}{\delta A_i(\boldsymbol{x})} = -\epsilon_{ij}\partial_j^y \delta^2(\boldsymbol{y} - \boldsymbol{x}) \tag{B.9}$$

and therefore (∂_i is with respect to $\boldsymbol{x}$)

$$\frac{\delta \Psi_0}{\delta A_i(\boldsymbol{x})} = -\frac{1}{g^2}\epsilon_{ij}\int_y \partial_j f(\boldsymbol{x} - \boldsymbol{y})B(\boldsymbol{y})\Psi_0 \tag{B.10a}$$

$$\frac{\delta^2 \Psi_0}{\delta A_i(\boldsymbol{x})^2} = \left[-\frac{1}{g^2}\epsilon_{ij}\epsilon_{ik}\partial_j\partial_k f(0) + \frac{1}{g^4}\left(\epsilon_{ij}\int_y \partial_j f(\boldsymbol{x} - \boldsymbol{y})B(\boldsymbol{y})\right)^2\right]\Psi_0 \tag{B.10b}$$

$$= \left[-\frac{1}{g^2}\Delta f(0) + \frac{1}{g^4}\left(\int_y \partial_i f(\boldsymbol{x} - \boldsymbol{y})B(\boldsymbol{y})\right)^2\right]\Psi_0. \tag{B.10c}$$

Inserting this in (B.7) then yields

$$\int_x\left[\frac{1}{2}\Delta f(0) - \frac{1}{2g^2}\left(\int_y \partial_i f(\boldsymbol{x} - \boldsymbol{y})B(\boldsymbol{y})\right)^2 + \frac{1}{2g^2}B(\boldsymbol{x})^2\right] = E_0. \tag{B.11}$$

The first term gives the divergent vacuum energy and for the other two terms to cancel we must have:

$$\int_x \int_y \partial_i f(\boldsymbol{x} - \boldsymbol{y})B(\boldsymbol{y})\int_{y'} \partial_i f(\boldsymbol{x} - \boldsymbol{y}')B(\boldsymbol{y}') = \int_x B(\boldsymbol{x})^2 \tag{B.12}$$

In Fourier space this becomes

$$\int_p (-ip_i)(+ip_i)f(\boldsymbol{p})f(-\boldsymbol{p})B(\boldsymbol{p})B(-\boldsymbol{p}) = \int_p \boldsymbol{p}^2 f(\boldsymbol{p})^2 B(\boldsymbol{p})B(-\boldsymbol{p}) \overset{!}{=} \int_p B(\boldsymbol{p})B(-\boldsymbol{p}) \tag{B.13}$$

so we find

$$f(\boldsymbol{p}) = \frac{1}{|\boldsymbol{p}|}, \tag{B.14a}$$

$$\Psi_0 = \exp\left(-\frac{1}{2g^2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\, B(\boldsymbol{p})\frac{1}{|\boldsymbol{p}|}B(-\boldsymbol{p})\right). \tag{B.14b}$$

Now the difference in the Yang-Mills case is the interaction part in the magnetic field

$$B^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc}A_1^b A_2^c. \tag{B.15}$$

But in the UV limit this term vanishes due to asymptotic freedom, so we expect the same asymptotic form for the vacuum wave functional there (except for an additional sum over group indices, of course).

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[1]Even though in the non-abelian case B transforms homogeneously, the analogue of (B.8) would not be gauge invariant because the two B-fields are taken at different space points. To restore gauge invariance one would have to include for example two Wilson lines serving as parallel transport between the B's. This approach can be found in [31].

C. Regulated Calculations

C.1. In KKN Variables

In this section $\mathrm{T}\mathcal{O}_0$ will be evaluated using the the exact KKN regulator. Recall from chapter 5 that

$$\mathrm{T}_1\mathcal{O}_0 = 2m\mathcal{O}_0, \tag{C.1.1a}$$

$$\mathrm{T}_2\mathcal{O}_0 = -2m\frac{\pi}{N}\int_z \left[\bar{\partial}_z^2\Omega(\boldsymbol{z},\boldsymbol{z}')\right]^{aa}_{\boldsymbol{z}'\to\boldsymbol{z}}. \tag{C.1.1b}$$

The coincidence limit

$$\bar{\partial}_z^2\Omega^{\mathrm{T}}(\boldsymbol{z},\boldsymbol{z}')_{\boldsymbol{z}'\to\boldsymbol{z}} = \bar{\partial}_z^2\partial_{z'}\Lambda(\boldsymbol{z}',\boldsymbol{z})_{\boldsymbol{z}'\to\boldsymbol{z}} - J(\boldsymbol{z}')\bar{\partial}_z^2\Lambda(\boldsymbol{z}',\boldsymbol{z})_{\boldsymbol{z}'\to\boldsymbol{z}} \tag{C.1.2}$$

will be performed using (5.33):

$$\Lambda(\boldsymbol{z}',\boldsymbol{z}) = \int_x \bar{\mathcal{G}}^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{z}')G(\boldsymbol{x},\boldsymbol{z})\mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{z}|^2/\epsilon}\left[\frac{\bar{x}-\bar{z}}{\epsilon}H(\boldsymbol{x})H^{-1}(z,\bar{x})\right.$$

$$\left. + H(\boldsymbol{x})\partial_z\left(H^{-1}(z,\bar{x})H(\boldsymbol{z})\right)H^{-1}(\boldsymbol{z})\right] \tag{C.1.3a}$$

$$= \int_x \frac{1}{\pi^2(x-z')}\left[1 - \mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{z}'|^2/\epsilon}H(x,\bar{z}')H^{-1}(\boldsymbol{z}')\right]^{\mathrm{T}}\mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{z}|^2/\epsilon}$$

$$\cdot\left[\frac{1}{\epsilon}H(\boldsymbol{x})H^{-1}(z,\bar{x}) + \frac{1}{\bar{x}-\bar{z}}\left(H(\boldsymbol{x})\partial_zH^{-1}(z,\bar{x}) + H(\boldsymbol{x})H^{-1}(z,\bar{x})J(\boldsymbol{z})\right)\right]. \tag{C.1.3b}$$

Here the adjoint indices were dropped in favour of an implicit matrix notation where $()^{\mathrm{T}}$ denotes the transposed matrix.

First consider the term with two derivatives acting on Λ:

$$\bar{\partial}_z^2\Lambda(\boldsymbol{z}',\boldsymbol{z}) = \int_x \frac{1}{\pi^2(x-z')}\left[1 - \mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{z}'|^2/\epsilon}H(x,\bar{z}')H^{-1}(\boldsymbol{z}')\right]^{\mathrm{T}}\mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{z}|^2/\epsilon}$$

$$\cdot\left\{\left(\frac{x-z}{\epsilon}\right)^2\left[\frac{1}{\epsilon}H(\boldsymbol{x})H^{-1}(z,\bar{x}) + \frac{1}{\bar{x}-\bar{z}}(\ldots)\right]\right.$$

$$+ 2\frac{x-z}{\epsilon}\left[\frac{1}{(\bar{x}-\bar{z})^2}(\ldots) + \frac{1}{\bar{x}-\bar{z}}H(\boldsymbol{x})H^{-1}(z,\bar{x})\bar{\partial}J(\boldsymbol{z})\right] \tag{C.1.4}$$

$$+ \frac{2}{(\bar{x}-\bar{z})^3}(\ldots) + \frac{2}{(\bar{x}-\bar{z})^2}H(\boldsymbol{x})H^{-1}(z,\bar{x})\bar{\partial}J(\boldsymbol{z})$$

$$\left. + \frac{1}{\bar{x}-\bar{z}}H(\boldsymbol{x})H^{-1}(z,\bar{x})\bar{\partial}^2J(\boldsymbol{z})\right\}$$

where $(\dots) = H(\boldsymbol{x})\partial_z H^{-1}(z,\bar{x}) + H(\boldsymbol{x})H^{-1}(z,\bar{x})J(z)$. Now the coincidence limit $\boldsymbol{z}' \to \boldsymbol{z}$ can be performed and after shifting and rescaling the integration variable to $\boldsymbol{x} \to \boldsymbol{z} + \sqrt{\epsilon}\boldsymbol{x}$ one gets (note that the integration measure gets an additional factor of ϵ):

$$\bar{\partial}_z^2 \Lambda(\boldsymbol{z}', \boldsymbol{z})_{\boldsymbol{z}' \to \boldsymbol{z}} = \int_x \frac{\mathrm{e}^{-\boldsymbol{x}^2}}{\pi^2} \frac{\sqrt{\epsilon}}{x} \left[1 - \mathrm{e}^{-\boldsymbol{x}^2} I_0\right]^{\mathrm{T}}$$

$$\cdot \left\{ \frac{x^2}{\epsilon^2} I_2 + \left(\frac{x^2}{\bar{x}} + \frac{2x}{\bar{x}^2} + \frac{2}{\bar{x}^3} \right) \frac{1}{\epsilon^{3/2}} I_1 \right. \tag{C.1.5}$$

$$\left. + \left(\frac{2x}{\bar{x}} + \frac{2}{\bar{x}^2} \right) \frac{1}{\epsilon} I_2 \bar{\partial} J(\boldsymbol{z}) + \frac{1}{\bar{x}} \frac{1}{\epsilon^{1/2}} I_2 \bar{\partial}^2 J(\boldsymbol{z}) \right\}$$

with

$$I_0 := H(z + \sqrt{\epsilon}x, \bar{z})H^{-1}(\boldsymbol{z}) \tag{C.1.6a}$$

$$I_1 := (\dots)|_{\boldsymbol{x} \to \boldsymbol{z} + \sqrt{\epsilon}\boldsymbol{x}}$$
$$= H(\boldsymbol{z} + \sqrt{\epsilon}\boldsymbol{x})\partial_z H^{-1}(z, \bar{z} + \sqrt{\epsilon}\bar{x}) + H(\boldsymbol{z} + \sqrt{\epsilon}\boldsymbol{x})H^{-1}(z, \bar{z} + \sqrt{\epsilon}\bar{x})J(\boldsymbol{z}) \tag{C.1.6b}$$

$$I_2 := H(\boldsymbol{z} + \sqrt{\epsilon}\boldsymbol{x})H^{-1}(z, \bar{z} + \sqrt{\epsilon}\bar{x}). \tag{C.1.6c}$$

To find the finite, i.e. $\mathcal{O}(\epsilon^0)$ terms in (C.1.5), all expressions appearing there have to be expanded in powers of $\sqrt{\epsilon}$. Rewriting all terms involving ∂'s and $\bar{\partial}$'s acting on H's and H^{-1}'s in terms of $J = \partial H H^{-1}$ and its derivatives, one arrives at (all arguments are $\boldsymbol{z}$ and are being suppressed):

$$I_0 = 1 + \epsilon^{1/2}xJ + \epsilon\frac{x^2}{2}(\partial J + J^2) + \dots \tag{C.1.7a}$$

$$I_1 = 0 - \epsilon^{1/2}\bar{x}\bar{\partial}J - \epsilon\left[x\bar{x}J\bar{\partial}J + \frac{\bar{x}^2}{2}\bar{\partial}^2 J\right]$$
$$- \epsilon^{3/2}\left[\frac{x^2\bar{x}}{2}(\partial J\bar{\partial}J + J^2\bar{\partial}J) + \frac{x\bar{x}^2}{2}(2\bar{\partial}J\bar{\partial}J + J\bar{\partial}^2 J) + \frac{\bar{x}^3}{6}\bar{\partial}^3 J\right] + \dots \tag{C.1.7b}$$

$$I_2 = 1 + \epsilon^{1/2}xJ + \epsilon\left[\frac{x^2}{2}(\partial J + J^2) + x\bar{x}\bar{\partial}J\right]$$
$$+ \epsilon^{3/2}\left[\frac{x^3}{6}(\partial^2 J + 2\partial JJ + J\partial J + J^3) + \frac{x^2\bar{x}}{2}\bar{\partial}(\partial J + J^2) + \frac{x\bar{x}^2}{2}\bar{\partial}^2 J\right] + \dots \tag{C.1.7c}$$

Inserting this into (C.1.5), using that $J^{\mathrm{T}} = -J$ in adjoint representation, multiplying out and keeping only terms up to $\mathcal{O}(\epsilon^0)$ then leads to:

$$\bar{\partial}_z^2 \Lambda(\boldsymbol{z}', \boldsymbol{z})_{\boldsymbol{z}' \to \boldsymbol{z}} = \int_x \frac{\mathrm{e}^{-\boldsymbol{x}^2}}{\pi^2} \left\{ \left(1 - \mathrm{e}^{-\boldsymbol{x}^2}\right) \left[\frac{x^2\bar{x}^2}{2}\bar{\partial}^2 J \right.\right.$$

$$\left. - \left(\frac{x}{\bar{x}} + \frac{2}{\bar{x}^2} + \frac{2}{x\bar{x}^3}\right)\left(x\bar{x}J\bar{\partial}J + \frac{\bar{x}^2}{2}\bar{\partial}^2 J\right) + \frac{2}{\bar{x}^2}J + \frac{1}{x\bar{x}}\bar{\partial}^2 J\right]$$

$$\left. + \mathrm{e}^{-\boldsymbol{x}^2}xJ\left[-\frac{2}{x\bar{x}^2}\bar{\partial}J + \frac{2}{x\bar{x}^2}\bar{\partial}J\right]\right\} \tag{C.1.8a}$$

$$= \int_x \frac{e^{-\boldsymbol{x}^2}}{\pi^2} \left(1 - e^{-\boldsymbol{x}^2}\right) \left[\frac{\boldsymbol{x}^4}{2} - \frac{\boldsymbol{x}^2}{2} - 1\right] \bar{\partial}^2 J \tag{C.1.8b}$$

$$= \frac{1}{\pi} \left[\frac{7}{8} - \frac{3}{8} - \frac{1}{2}\right] \bar{\partial}^2 J = 0. \tag{C.1.8c}$$

Note that only terms involving equal powers of x and $\bar{x}$ were kept, since only those give a non-vanishing contribution after performing the integration. (In addition, terms with at least two negative powers of x and $\bar{x}$ were kept as a consistency check, because those are potentially dangerous terms for which the integral would diverge. But all those terms cancel each other, so the final result is finite — as required for a regulated calculation.)

So there is no contribution from the second term in (C.1.2). The first term can be calculated in the same way:

$$\begin{aligned}
\bar{\partial}_z^2 \partial_{z'} \Lambda(\boldsymbol{z}', \boldsymbol{z}) = \int_x \frac{1}{\pi^2} &\left\{ \frac{1}{(x-z')^2} \left[1 - e^{-|\boldsymbol{x}-\boldsymbol{z}'|^2/\epsilon} H(x, \bar{z}') H^{-1}(\boldsymbol{z}')\right]^{\mathrm{T}} \right. \\
&\left. - \frac{e^{-|\boldsymbol{x}-\boldsymbol{z}'|^2/\epsilon}}{x-z'} \left[\frac{\bar{x} - \bar{z}'}{\epsilon} H(x, \bar{z}') H^{-1}(\boldsymbol{z}') + H(x, \bar{z}') \partial H^{-1}(\boldsymbol{z}')\right]^{\mathrm{T}} \right\} \\
\cdot e^{-|\boldsymbol{x}-\boldsymbol{z}|^2/\epsilon} &\left\{ \left(\frac{x-z}{\epsilon}\right)^2 \left[\frac{1}{\epsilon} H(\boldsymbol{x}) H^{-1}(z, \bar{x}) + \frac{1}{\bar{x} - \bar{z}}(\ldots)\right] \right. \\
&+ 2\frac{x-z}{\epsilon} \left[\frac{1}{(\bar{x} - \bar{z})^2}(\ldots) + \frac{1}{\bar{x} - \bar{z}} H(\boldsymbol{x}) H^{-1}(z, \bar{x}) \bar{\partial} J(\boldsymbol{z})\right] \\
&+ \frac{2}{(\bar{x} - \bar{z})^3}(\ldots) + \frac{2}{(\bar{x} - \bar{z})^2} H(\boldsymbol{x}) H^{-1}(z, \bar{x}) \bar{\partial} J(\boldsymbol{z}) \\
&\left. + \frac{1}{\bar{x} - \bar{z}} H(\boldsymbol{x}) H^{-1}(z, \bar{x}) \bar{\partial}^2 J(\boldsymbol{z}) \right\}
\end{aligned} \tag{C.1.9}$$

Proceeding as before, one then finds

$$\begin{aligned}
\bar{\partial}_z^2 \partial_{z'} \Lambda(\boldsymbol{z}', \boldsymbol{z})_{\boldsymbol{z}' \to \boldsymbol{z}} = \int_x \frac{1}{\pi^2} &\left\{ \frac{1}{x^2} \left[1 - e^{-\boldsymbol{x}^2} I_0\right]^{\mathrm{T}} - \frac{e^{-\boldsymbol{x}^2}}{x} \left[\bar{x} I_0 + \epsilon^{1/2} I_0'\right]^{\mathrm{T}} \right\} \\
\cdot e^{-\boldsymbol{x}^2} &\left\{ \frac{x^2}{\epsilon^2} I_2 + \left(\frac{x^2}{\bar{x}} + \frac{2x}{\bar{x}^2} + \frac{2}{\bar{x}^3}\right) \frac{1}{\epsilon^{3/2}} I_1 \right. \\
&\left. + \left(\frac{2x}{\bar{x}} + \frac{2}{\bar{x}^2}\right) \frac{1}{\epsilon} I_2 \bar{\partial} J + \frac{1}{\bar{x}} \frac{1}{\epsilon^{1/2}} I_2 \bar{\partial}^2 J \right\}
\end{aligned} \tag{C.1.10}$$

where

$$I_0' := H(z + \sqrt{\epsilon}x, \bar{z}) \partial H^{-1}(\boldsymbol{z}) \tag{C.1.11a}$$

$$= -J - \epsilon^{1/2} x J^2 - \epsilon \frac{x^2}{2}(\partial J + J^2)J + \ldots \tag{C.1.11b}$$

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Inserting this as well as the expansions (C.1.7) and collecting terms $\mathcal{O}(\epsilon^0)$ finally yields

$$\bar{\partial}_z^2 \partial_{z'} \Lambda(\boldsymbol{z}', \boldsymbol{z})_{z' \to z} = \int_x \frac{\mathrm{e}^{-\boldsymbol{x}^2}}{\pi^2} \Bigg\{ \Big[1 - (1 + \boldsymbol{x}^2)\mathrm{e}^{-\boldsymbol{x}^2} \Big]$$
$$\cdot \Big[\frac{\boldsymbol{x}^4}{4} \bar{\partial}^2 (\partial J + J^2) - \frac{\boldsymbol{x}^2}{2} (2\bar{\partial}J\bar{\partial}J + J\bar{\partial}^2 J) - J\bar{\partial}^2 J \Big]$$
$$+ \mathrm{e}^{-\boldsymbol{x}^2} \Big[\frac{\boldsymbol{x}^6}{2} - \frac{\boldsymbol{x}^4}{2} - \boldsymbol{x}^2 \Big] J\bar{\partial}^2 J \Bigg\} \tag{C.1.12a}$$

$$\hat{=} \int_x \frac{\mathrm{e}^{-\boldsymbol{x}^2}}{\pi^2} \Bigg\{ \Big[1 - (1 + \boldsymbol{x}^2)\mathrm{e}^{-\boldsymbol{x}^2} \Big] \Big(-\frac{\boldsymbol{x}^2}{2} + 1 \Big)$$
$$- \mathrm{e}^{-\boldsymbol{x}^2} \Big[\frac{\boldsymbol{x}^6}{2} - \frac{\boldsymbol{x}^4}{2} - \boldsymbol{x}^2 \Big] \Bigg\} \bar{\partial}J\bar{\partial}J \tag{C.1.12b}$$

$$= \frac{1}{\pi} \frac{3}{16} \bar{\partial}J\bar{\partial}J. \tag{C.1.12c}$$

The $\hat{=}$ means that this is an equality up to total derivatives with respect to $\bar{z}$, which do not contribute in (C.1.1b):

$$\mathrm{T}_2 \mathcal{O}_0 = -2m \frac{\pi}{N} \frac{1}{\pi} \frac{3}{16} \int_z \bar{\partial} J^{ab} \bar{\partial} J^{ba} = -2m \frac{3}{16} \mathcal{O}_0 \tag{C.1.13}$$

and thus

$$\mathrm{T}\mathcal{O}_0 = 2m \Big(1 - \frac{3}{16} \Big) \mathcal{O}_0 \tag{C.1.14}$$

where divergent terms $\mathcal{O}(\epsilon^{-1})$ are not shown explicitly. We only want to note here, without showing the explicit calculation, that the only divergent term contributing is indeed $1/4\pi\epsilon^2$, confirming the result from section 5.1.1.

C.2. In Original Variables

Here we show the regularised evaluations of the terms that appear during the calculation of $\mathrm{T}\mathcal{O}_1[A]$ in equation (6.30b):

$$(\mathrm{I}) := \delta(0) \tag{C.2.1a}$$
$$(\mathrm{II}) := \nabla_j \delta(0) \tag{C.2.1b}$$
$$(\mathrm{III}) := \nabla_i \Delta \nabla_i \delta(0) \tag{C.2.1c}$$

The regularisation procedure was explained in section 6.2.1 and leads to

$$(\mathrm{I}) = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} M^2 \exp \big[\boldsymbol{\nabla}^2/M^2 + 2i\boldsymbol{p} \cdot \boldsymbol{\nabla}/M \big] \tag{C.2.2a}$$

$$(\mathrm{II}) = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} M^2 (\nabla_j + iMp_j) \exp \big[\boldsymbol{\nabla}^2/M^2 + 2i\boldsymbol{p} \cdot \boldsymbol{\nabla}/M \big] \tag{C.2.2b}$$

$$(\mathrm{III}) = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} M^2 (\nabla_i + iMp_i)(\boldsymbol{\nabla} + iM\boldsymbol{p})^2 (\nabla_i + iMp_i) \exp \big[\boldsymbol{\nabla}^2/M^2 + 2i\boldsymbol{p} \cdot \boldsymbol{\nabla}/M \big]$$
$$\tag{C.2.2c}$$

with a mass parameter M that should be sent to ∞ in the end. The integrands can now be expanded in powers of M and the integrations can be performed. The easiest one is (I), which gives

$$\text{(I)} = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} \left[M^2 + \boldsymbol{\nabla}^2 + \frac{1}{2}(2i)^2(\boldsymbol{p} \cdot \boldsymbol{\nabla})^2 + \mathcal{O}(1/M^2) \right] \tag{C.2.3a}$$

$$= \frac{1}{\pi}\frac{1}{4} \left[M^2 + \boldsymbol{\nabla}^2 - \boldsymbol{\nabla}^2 + \mathcal{O}(1/M^2) \right] \tag{C.2.3b}$$

$$= \frac{1}{4\pi} M^2 + \mathcal{O}(1/M^2) \tag{C.2.3c}$$

where the formulas (6.21) were used to evaluate the Gaussian integrals. Also note that odd powers of M always vanish because each such term also has an odd number of p's and thus gives zero when integrated from $-\infty$ to ∞.

The next more complicated term is

$$\text{(II)} = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} \left[M^2 \left(\nabla_j - 2p_j\, \boldsymbol{p} \cdot \boldsymbol{\nabla} \right) \right.$$
$$\left. + \nabla_j (\boldsymbol{\nabla}^2 - 2(\boldsymbol{p} \cdot \boldsymbol{\nabla})^2) - p_j \left((\boldsymbol{p} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}^2 + \boldsymbol{\nabla}^2(\boldsymbol{p} \cdot \boldsymbol{\nabla}) - \frac{8}{6}(\boldsymbol{p} \cdot \boldsymbol{\nabla})^3 \right) \right] \tag{C.2.4a}$$

$$= \frac{1}{\pi}\frac{1}{4} \left[M^2 (\nabla_j - \nabla_j) \right.$$
$$\left. + \nabla_j (\boldsymbol{\nabla}^2 - \boldsymbol{\nabla}^2) - \frac{1}{2}(\nabla_j \boldsymbol{\nabla}^2 + \boldsymbol{\nabla}^2 \nabla_j) + \frac{1}{3}(\nabla_j \boldsymbol{\nabla}^2 + \nabla_i \nabla_j \nabla_i + \boldsymbol{\nabla}^2 \nabla_j) \right] \tag{C.2.4b}$$

$$= \frac{1}{\pi}\frac{1}{24} \left(-\nabla_j \boldsymbol{\nabla}^2 - \boldsymbol{\nabla}^2 \nabla_j + 2\nabla_k \nabla_j \nabla_k \right) \tag{C.2.4c}$$

$$= \frac{1}{\pi}\frac{1}{24} \left(-[\nabla_j, \nabla_k]\nabla_k - \nabla_k[\nabla_k, \nabla_j] \right) \tag{C.2.4d}$$

$$= \frac{1}{\pi}\frac{1}{24} \epsilon_{jk} \left(-[\nabla_1, \nabla_2]\nabla_k - \nabla_k[\nabla_2, \nabla_1] \right) \tag{C.2.4e}$$

$$= \frac{1}{\pi}\frac{1}{24} \epsilon_{jk} [\nabla_k, B] \tag{C.2.4f}$$

where (6.21) was used again, and $\mathcal{O}(1/M^2)$ terms are not shown explicitly.

One could in principle proceed with the last term (III) in the same way. But since there are four derivatives acting on the delta-function, one has to expand the exponent to sixth order and the integrand becomes quite complicated which makes the calculation very tedious. Moreover, in the case $n = 2$ which we will also investigate, the situation becomes even worse. Therefore it will prove helpful to continue doing the calculations using a computer algebra system like Maple.

In Maple there is a package called `Physics` which provides the possibility to introduce non-commuting variables by setting them up as `quantumoperators`. In this way the covariant derivative operators $\nabla_1 \to$ `n1` and $\nabla_2 \to$ `n2` can be handled. The finite contribution to the integral in (III) (excluding the factor of $1/\pi$ for convenience) can then be computed as follows:

```
    |\^/|       Maple 14 (X86 64 LINUX)
._|\|   |/|_.  Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2010
 \  MAPLE  /   All rights reserved. Maple is a trademark of
 <____ ____>   Waterloo Maple Inc.
      |        Type ? for help.
>
> with(Physics):
> exp6 := x -> 1 + x + x^2/2 + x^3/3! + x^4/4! + x^5/5! + x^6/6!:
> Setup(quantumoperators={n1, n2, B}):
> nn := n1^2 + n2^2:
> m1 := n1 + I*M*p1:
> m2 := n2 + I*M*p2:
> mm := m1^2 + m2^2:
> pn := p1*n1 + p2*n2:
> pp :=p1^2 + p2^2:
> integrand := M^2*(m1*mm*m1 + m2*mm*m2)*exp6(nn/M^2 + 2*I*pn/M)*exp(-pp):
> integrand0 := limit(convert(series(integrand, M=infinity, 7), polynom), M=0):
> integral := value(Intc(integrand0/(4*Pi), p1, p2));
```

Maple Code C.1: Calculation of (III)

Since there are four derivatives, each containing one M, and one additional factor of M^2 from the momentum integral measure, the exponential has to be expanded to sixth order to get all terms contributing to the M^0 result. The corresponding function `exp6` is defined after starting the `Physics` environment, and therefore it uses the non-commutative multiplication operator, which is essential for the calculation. The terms of order M^0 are extracted from the integrand by expanding it to seventh order at $M = \infty$, i.e. down to order M^0 and then sending M to zero to remove the divergent normal ordering terms.

The result is (a common factor of 60 is pulled out for convenience):

```
> 60*integral;
   3                  2     2                3                  3        2        2
n1  n2 n1 n2 + n1  n2 n1  n2 - 4 n1 n2 n1  n2 + n1 n2 n1 n2  + n1 n2  n1 n2

         2     2              3              2     2              3
     + n2  n1 n2  n1 - 4 n2 n1  n2 n1 + n2 n1  n2 n1  + n2 n1 n2 n1

               3        3              3  2        3  2                  4
     - 4 n2 n1 n2  n1 + n2  n1 n2 n1 - n2  n1  n2 - n1  n2  n1 + 2 n1 n2  n1

         2  2  2        2  2  2        2  3              4              2  3
     + 4 n1  n2  n1  + 4 n2  n1  n2  - n1 n2  n1  + 2 n2 n1  n2  - n2 n1  n2

             3              2                        2
     - 4 n1 n2  n1  n2 - 5 n1  n2 n1 n2 n1 + 10 n1 n2 n1  n2 n1

                   2        2                              2
     - 5 n1 n2 n1 n2 n1  - 5 n2  n1 n2 n1 n2 - 5 n2 n1 n2 n1 n2

               2
     + 10 n2 n1 n2  n1 n2
```

Maple Code C.2: Result for (III)

Written in this way, the result is obviously not very useful and one needs a strategy to extract from it the terms that are supposed to be contained. According to the conjecture, it should contain $B\Delta B = B[\nabla_i, [\nabla_i, B]]$ where $B = [\nabla_2, \nabla_1]$. But since for the final result of $T\mathcal{O}_1$ one has to take the spatial integral and the trace (in adjoint representation) of this expression, there are two other terms possible which will be proportional to $B\Delta B$

after an integration by parts: $[\nabla_i, B][\nabla_i, B]$ and $[\nabla_i, [\nabla_i, B]]B$.

Now each of these three terms, when written down entirely in terms of ∇'s, contains some particular sequences of ∇'s which are characteristic of it, i.e. which are not contained in any of the other two, for example:

$$B[\nabla_i, [\nabla_i, B]] \supset \nabla_2 \nabla_1 \nabla_2 \nabla_1 \nabla_1 \nabla_1 \tag{C.2.5a}$$

$$[\nabla_i, [\nabla_i, B]]B \supset \nabla_1 \nabla_1 \nabla_1 \nabla_2 \nabla_1 \nabla_2 \tag{C.2.5b}$$

$$[\nabla_i, B][\nabla_i, B] \supset \nabla_1 \nabla_1 \nabla_2 \nabla_2 \nabla_1 \nabla_1 \tag{C.2.5c}$$

By looking at the result obtained above one can now conclude that *if* those three terms are contained, then they will come with the prefactors $1, 1$ and 4, respectively.

Now one can subtract those terms from the result and see what is left over:

```
> B := n1*n2 - n2*n1:
> n1B := n1*B - B*n1:
> n2B := n2*B - B*n2:
> LB := n1*n1B - n1B*n1 + n2*n2B - n2B*n2:
> simplify(60*integral - B*LB - LB*B);
         2       2        2       2            4                   3
4 n1 n2  n1 n2 + 4 n2 n1  n2 n1  + 4 n1 n2  n1 - 8 n1 n2 n1  n2

                 2          3                  2
    - 8 n1 n2 n1 n2 n1  - 8 n1 n2  n1 n2 + 16 n1 n2 n1  n2 n1

          2  2  2         2                  2       2
    + 4 n2  n1  n2  - 8 n2  n1 n2 n1 n2 + 4 n2  n1 n2  n1

            4          3                  2
    + 4 n2 n1  n2 - 8 n2 n1  n2 n1 - 8 n2 n1 n2 n1 n2

                2          3         2       2
    + 16 n2 n1 n2  n1 n2 - 8 n2 n1 n2  n1 + 4 n1  n2 n1  n2

        2              2  2  2
    - 8 n1  n2 n1 n2 n1 + 4 n1  n2  n1

> simplify(% - 4*(n1B^2 + n2B^2));
                              0
```

Maple Code C.3: Subtractions from the result for (III)

As the 0 at the bottom shows there is nothing left, meaning that the last result is indeed

$$(III) = \frac{1}{\pi}\frac{1}{60}\Big(B[\nabla_i, [\nabla_i, B]] + [\nabla_i, [\nabla_i, B]]B + 4[\nabla_i, B][\nabla_i, B] \Big) \tag{C.2.6}$$

apart from divergent $\mathcal{O}(M^2)$ terms which are not of interest here.

For the $T\mathcal{O}_2[A]$ evaluation one also needs (see equation (6.40b))

$$(IV) := \nabla_j \Delta \delta(0) \tag{C.2.7a}$$

$$(V) := \Delta \nabla_j \delta(0) \tag{C.2.7b}$$

$$(VI) := \epsilon_{ij} \nabla_j \Delta \nabla_i \delta(0) \tag{C.2.7c}$$

$$(VII) := \nabla_i \Delta \Delta \nabla_i \delta(0). \tag{C.2.7d}$$

Proceeding as before leads to

$$(\text{IV}) = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} M^2 (\nabla_j + iMp_j)(\boldsymbol{\nabla} + iM\boldsymbol{p})^2 \exp\left[\boldsymbol{\nabla}^2/M^2 + 2i\boldsymbol{p}\cdot\boldsymbol{\nabla}/M\right] \quad (\text{C.2.8a})$$

$$(\text{V}) = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} M^2 (\boldsymbol{\nabla} + iM\boldsymbol{p})^2 (\nabla_j + iMp_j) \exp\left[\boldsymbol{\nabla}^2/M^2 + 2i\boldsymbol{p}\cdot\boldsymbol{\nabla}/M\right] \quad (\text{C.2.8b})$$

$$(\text{VI}) = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} M^2 \epsilon_{ij} (\nabla_j + iMp_j)(\boldsymbol{\nabla} + iM\boldsymbol{p})^2 (\nabla_i + iMp_i)$$
$$\cdot \exp\left[\boldsymbol{\nabla}^2/M^2 + 2i\boldsymbol{p}\cdot\boldsymbol{\nabla}/M\right]. \quad (\text{C.2.8c})$$

$$(\text{VII}) = \frac{1}{\pi} \int \frac{\mathrm{d}^2 p}{4\pi} \mathrm{e}^{-\boldsymbol{p}^2} M^2 (\nabla_i + iMp_i)(\boldsymbol{\nabla} + iM\boldsymbol{p})^2 (\boldsymbol{\nabla} + iM\boldsymbol{p})^2 (\nabla_i + iMp_i)$$
$$\cdot \exp\left[\boldsymbol{\nabla}^2/M^2 + 2i\boldsymbol{p}\cdot\boldsymbol{\nabla}/M\right]. \quad (\text{C.2.8d})$$

The integrals can again be carried out using Maple. First consider the computation of (IV) (excluding the factor of $1/\pi$) for $j = 1$:

```
> exp5 := x -> 1 + x + x^2/2 + x^3/3! + x^4/4! + x^5/5!:
> integrand := M^2*(m1*mm)*exp5(nn/M^2 + 2*I*pn/M)*exp(-pp):
> integrand0 := limit(convert(series(integrand, M=infinity, 6), polynom), M=0):
> integral := value(Intc(integrand0/(4*Pi), p1, p2)):
> 240*integral;
              2      4             3         2   3           2    2
-n2 n1 n2 n1   - n2   n1 + 4 n2 n1   n2 - n2   n1  + 4 n1 n2   n1  + 4 n1 n2 n1 n2 n1

            2             3           2                     3      4      3    2
   - n2 n1   n2 n1 + 4 n2   n1 n2 + 9 n1   n2 n1 n2 + 4 n2 n1 n2   - n1 n2   - n1   n2

          2    2            2           2       2
   - 6 n1   n2   n1 - 11 n1 n2 n1   n2 - 6 n2   n1 n2
```

Maple Code C.4: Calculation of (IV)

This time the integrand contains at most M^5 and so it suffices to expand the exponential to fifth order. In the result a common factor of 240 was pulled out to make it look nicer. It is now again possible to figure out what terms can be contained in this result. To be brief, let us just subtract the terms shown in (6.41a) (for $j = 1$),

$$+[\nabla_2, [\nabla_i, [\nabla_i, B]]] \to \quad (\text{n2n1n1B + n2n2n2B}) \quad (\text{C.2.9a})$$
$$-2B[\nabla_1, B] \to \quad - \text{2*B*n1B} \quad (\text{C.2.9b})$$
$$+6[\nabla_1, B^2] \to \quad + \text{6*(n1*B*B - B*B*n1)} \quad (\text{C.2.9c})$$

to prove the assertion:

```
> B := n1*n2 - n2*n1:
> n1B := n1*B - B*n1:
> n2B := n2*B - B*n2:
> n1n1B := n1*n1B - n1B*n1:
> n2n2B := n2*n2B - n2B*n2:
> n1n1n1B := n1*n1n1B - n1n1B*n1:
> n2n1n1B := n2*n1n1B - n1n1B*n2:
> n1n2n2B := n1*n2n2B - n2n2B*n1:
> n2n2n2B := n2*n2n2B - n2n2B*n2:
>
> simplify(240*integral - n2n1n1B - n2n2n2B);
          3                2                    2   2          2   2
2 n2 n1  n2 + 2 n2 n1   n2 n1 + 4 n1 n2   n1  - 6 n1   n2   n1

                                  2                  2                      2
      + 4 n1 n2 n1 n2 n1 + 6 n1   n2 n1 n2 - 8 n1 n2 n1   n2 - 4 n2 n1 n2 n1

> simplify(% + 2*B*n1B);
          2                2   2          2   2              2
6 n2 n1   n2 n1 + 6 n1 n2   n1  - 6 n1   n2   n1 + 6 n1   n2 n1 n2

                   2                  2
      - 6 n1 n2 n1   n2 - 6 n2 n1 n2 n1

> simplify(% - 6*(n1*B*B - B*B*n1));
                                  0
```

Maple Code C.5: Subtractions from the result of (IV)

The same calculation can be done for $j = 2$ and gives the same result, thus confirming (6.41a).

The computation for (V) is almost identical, so there is no need to show it here. The only difference is that in the result the term $B[\nabla_j, B] \rightarrow$ B*n1B comes with a factor of 18 instead of -2.

For (VI) the analogous computation that proves (6.41c) is:

```
> integrand := M^2*(m2*mm*m1 - m1*mm*m2)*exp6(nn/M^2 + 2*I*pn/M)*exp(-pp):
> integrand0 := limit(convert(series(integrand, M=infinity, 7), polynom), M=0):
> integral := value(Intc(integrand0/(4*Pi), p1, p2)):
> 8*integral;
    3    2         3    2        2              2   3        2              2
n2 n1   n2  - n1 n2   n1  - n2 n1   n2 n1 n2 - n2   n1   n2 - n2   n1 n2 n1

            2   2             2   3        2             2              2
      + 2 n2   n1   n2 n1 + n1   n2   n1 + n1   n2 n1 n2  + n1 n2   n1 n2 n1

            2   2                                            2   2
      - 2 n1   n2   n1 n2 - 3 n2 n1 n2 n1 n2 n1 + 2 n2 n1 n2   n1

                                      2   2                  2
      + 3 n1 n2 n1 n2 n1 n2 - 2 n1 n2 n1   n2  + n2 n1 n2 n1   n2

                  2
      - n1 n2 n1 n2   n1
>
> simplify(% - (n1*B*B*n2 - n2*B*B*n1 + n2*B*n1*B - n1*B*n2*B - B*n1*B*n2 + B*n2*B*n1) );
                                        0
```

Maple Code C.6: Calculation of (VI)

Now for the last term (VII), which contains six derivatives, the exponential has to be

expanded to eighth order and the result gets quite messy:

```
> exp8 := x -> 1 + x + x^2/2 + x^3/3! + x^4/4! + x^5/5! + x^6/6! + x^7/7! + x^8/8!:
> integrand := M^2*(m1*mm*mm*m1 + m2*mm*mm*m2)*exp8(nn/M^2 + 2*I*pn/M)*exp(-pp):
> integrand0 := limit(convert(series(integrand, M=infinity, 9), polynom), M=0):
> 1680*integral;
              4                      2   3                       3   3           2       4
35 n2 n1 n2  n1 n2 - 97 n1 n2 n1  n2  n1 - 221 n2 n1 n2  n1  - 5 n2 n1  n2 n1

               3   3              2   2   2   2              4
     - 249 n2 n1  n2  n1 - 368 n2  n1  n2  n1 + 35 n1 n2 n1  n2 n1

              2       4              2   3   2              5           5
     + 205 n2  n1 n2  n1 + 129 n1 n2  n1  n2 - 89 n1 n2 n1  n2 + 23 n1 n2  n1 n2

              3   2              5          2   3   2
     + 505 n2 n1  n2  n1 n2 + 9 n2 n1 n2 n1  + 619 n1 n2  n1  n2

              2   2       2              3   3           2       4
     - 475 n1  n2  n1 n2  n1 - 249 n1 n2  n1  n2 - 5 n1 n2  n1 n2

              3   2       2              3       3              2   3   2       5
     + 255 n1  n2  n1 n2  + 37 n2 n1  n2  n1 + 619 n2 n1  n2  n1  + 9 n1  n2 n1 n2

              3   3              3       3              5           4       2
     - 333 n1  n2  n1 n2 - 53 n1  n2 n1 n2 + 9 n1 n2 n1 n2  - 75 n2 n1  n2 n1

              3       3          2   3   2              5               3           3
     + 37 n1 n2  n1 n2  + 129 n2  n1  n2  n1 + 9 n2  n1 n2 n1 - 53 n2  n1 n2 n1

              2       3           3       2          3       2
     + 161 n2  n1 n2 n1 n2  + 189 n1 n2 n1  n2 n1 + 357 n1  n2 n1  n2 n1

             3       2          2   4   2              2       3
     - 63 n1  n2 n1 n2 n1  - 184 n2  n1  n2  - 91 n2 n1 n2  n1 n2

                     4              2       2   2              2
     - 35 n1 n2 n1 n2 n1  - 147 n1  n2 n1  n2 n1  - 1179 n1 n2  n1 n2 n1 n2 n1

                   2                              2
     - 1179 n2 n1  n2 n1 n2 n1 n2 + 249 n2 n1 n2 n1 n2 n1  n2

                       2              2          2              2   2       2
     + 249 n1 n2 n1 n2 n1 n2  n1 + 433 n2 n1  n2 n1 n2  n1 - 147 n2  n1 n2  n1 n2

             3       2              2   2   2              2   3
     - 63 n2  n1 n2 n1 n2  + 155 n2 n1  n2  n1  n2 - 97 n2 n1 n2  n1  n2

                       4              3       2              2       3
     - 35 n2 n1 n2 n1 n2  - 335 n2 n1  n2 n1 n2  - 91 n1 n2 n1  n2 n1

                 3       2       2       3              3   2
     - 335 n1 n2  n1 n2 n1  - 259 n2  n1 n2  n1 n2 - 307 n2 n1 n2 n1  n2

              2   2   2              2       3          4
     + 155 n1 n2  n1  n2  n1 + 155 n1  n2 n1 n2  n1 - 147 n1  n2 n1 n2 n1

               3   2       2       3              3   2
     - 307 n1 n2  n1 n2  n1  + 155 n2  n1 n2 n1  n2 + 505 n1 n2  n1  n2 n1

              2   2       2          2       3              3   2
     - 475 n2  n1  n2 n1  n2 + 161 n1  n2 n1 n2 n1  + 189 n2 n1 n2  n1 n2

              2   3              2       2   2          2   3
     + 295 n1  n2  n1 n2 n1 + 785 n2 n1  n2 n1  n2  + 295 n2  n1  n2 n1 n2

              2       3          2       2   2          2   2
     - 259 n1  n2 n1  n2 n1 + 785 n1 n2  n1 n2  n1  - 449 n2 n1 n2 n1  n2  n1
```

```
                2                    2                    2   2                                        2             2
  - 197 n2  n1 n2 n1 n2 n1  - 827 n2 n1   n2   n1 n2 n1 + 433 n1 n2   n1 n2 n1   n2

              2   2                            2             2           2   2
  + 643 n1   n2   n1 n2 n1 n2 + 181 n2 n1 n2   n1 n2 n1  + 643 n2   n1   n2 n1 n2 n1

                    2   2                 2   2                         2             2
  - 449 n1 n2 n1 n2 n1   n2 + 181 n1 n2 n1   n2   n1 n2 + 181 n1 n2 n1   n2 n1 n2

              2             2               2   2                   2             2
  - 197 n1   n2 n1 n2   n1 n2 + 181 n2 n1 n2   n1   n2 n1 - 197 n2   n1 n2 n1   n2 n1

              2             2               2   2             3   4
  - 197 n1   n2 n1 n2 n1 n2  - 449 n1 n2 n1 n2   n1   n2 + 131 n2   n1  n2

            4   3                   2                 4   2   2         4   2   2
  + 75 n1 n2   n1  + 249 n1 n2 n1 n2   n1 n2 n1 + 96 n1   n2   n1  + 96 n2   n1   n2

                2   5           2   2   4           6             2   2   4           6
  - 9 n1 n2   n1  + 40 n1   n2   n1  + 11 n2 n1   n2  + 40 n2   n1   n2  + 11 n1 n2   n1

              3   4            2               3       2
  + 131 n1   n2   n1 + 249 n2 n1 n2 n1   n2 n1 n2 + 357 n2   n1 n2   n1 n2

              3       3           3       3           4     2           4     2
  - 187 n2   n1 n2   n1 - 187 n1   n2 n1   n2 + 51 n2   n1 n2   n1 + 51 n1   n2 n1   n2

                    5           4     2         3   2       2             2         2   3
  - 89 n2 n1 n2   n1 - 75 n1 n2   n1 n2  + 255 n2   n1   n2 n1  + 199 n1   n2 n1   n2

              2   4   2           3   2   3           4                 2   5
  - 184 n1   n2   n1  - 107 n2   n1   n2  - 147 n2   n1 n2 n1 n2 - 9 n2 n1   n2

              2       2   3         3   3
  + 199 n2   n1 n2   n1  - 333 n2   n1   n2 n1 + 681 n2 n1 n2 n1 n2 n1 n2 n1

          5   2                                       2       4
  - 9 n2   n1   n2 + 681 n1 n2 n1 n2 n1 n2 n1 n2 + 205 n1   n2 n1   n2

            3   2   3             3   3           2   2   2   2               5
  - 107 n1   n2   n1  - 221 n1 n2 n1   n2  - 368 n1   n2   n1   n2 + 23 n2 n1   n2 n1

            4   3         5   2                   2   2
  + 75 n2 n1   n2  - 9 n1   n2   n1 - 449 n2 n1 n2 n1 n2   n1

              2   2
  - 827 n1 n2   n1   n2 n1 n2
```

Maple Code C.7: Calculation of (VII)

To extract a useful result, one can now proceed as in the evaluation of (III). First note that the contribution to $T\mathcal{O}_2$ is proportional to $\int \operatorname{tr}(\text{VII})$. Since the conjecture states that this should again be proportional to $\mathcal{O}_2$, one may expect that the result contains the following terms:

$$B[\nabla_i, [\nabla_i, [\nabla_j, [\nabla_j, B]]]] \to \texttt{B*LLB} \qquad (\text{C.2.10a})$$

$$[\nabla_i, B][\nabla_i, [\nabla_j, [\nabla_j, B]]] \to \texttt{nBnLB} \qquad (\text{C.2.10b})$$

$$[\nabla_i, [\nabla_i, B]][\nabla_j, [\nabla_j, B]] \to \texttt{LB*LB} \qquad (\text{C.2.10c})$$

$$[\nabla_j, [\nabla_i, [\nabla_i, B]]][\nabla_j, B] \to \texttt{nLBnB} \qquad (\text{C.2.10d})$$

$$[\nabla_j, [\nabla_j, [\nabla_i, [\nabla_i, B]]]]B \to \texttt{LLB*B} \qquad (\text{C.2.10e})$$

The first one is proportional to the integrand of $\mathcal{O}_2$ in matrix notation and the last four

are equivalent to it (up to sign) upon integration by parts. Assuming that *only* those five terms are contained in the result, it is again possible to find a characteristic sequence of ∇'s for three of them:

$$B[\nabla_i, [\nabla_i, [\nabla_j, [\nabla_j, B]]]] \supset \nabla_2\nabla_1\nabla_2\nabla_1\nabla_1\nabla_1\nabla_1\nabla_1 \tag{C.2.11a}$$

$$[\nabla_i, [\nabla_i, B]][\nabla_j, [\nabla_j, B]] \supset -\nabla_1\nabla_1\nabla_1\nabla_2\nabla_2\nabla_1\nabla_1\nabla_1 \tag{C.2.11b}$$

$$[\nabla_j, [\nabla_j, [\nabla_i, [\nabla_i, B]]]]B \supset \nabla_1\nabla_1\nabla_1\nabla_1\nabla_1\nabla_2\nabla_1\nabla_2 \tag{C.2.11c}$$

Looking at the result above then reveals that they must come with the factors $+9, +107$ and $+9$, respectively. Therefore we can subtract them from the result:

```
> B := n1*n2 - n2*n1:
> n1B := n1*B - B*n1:
> n2B := n2*B - B*n2:
> LB := n1*n1B - n1B*n1 + n2*n2B - n2B*n2:
> n1LB := n1*LB - LB*n1:
> n2LB := n2*LB - LB*n2:
> LLB := n1*n1LB - n1LB*n1 + n2*n2LB - n2LB*n2:
> nBnLB := n1B*n1LB + n2B*n2LB:
> nLBnB := n1LB*n1B + n2LB*n2B:
>
> simplify(1680*integral - 9*B*LLB - 107*LB*LB - 9*LLB*B);
          2                 2         2        2        2                   2    2
-1160 n2   n1 n2 n1 n2 n1   + 816 n2   n1 n2   n1 n2 - 503 n2 n1 n2 n1   n2   n1

                 2   2                             2    2
    - 836 n1 n2 n1   n2   n1 n2 - 476 n2 n1 n2 n1 n2   n1

                   2   2               2        2   2
    - 503 n1 n2 n1 n2   n1   n2 + 812 n1 n2   n1 n2   n1

             2                 2        2             2
    - 1160 n1   n2 n1 n2 n1 n2   + 820 n2   n1 n2 n1   n2 n1

               2   2               2           2
    - 881 n1 n2   n1   n2 n1 n2 + 820 n1   n2 n1 n2   n1 n2

                 3         2         4                       2    2
    - 710 n2 n1   n2 n1 n2   - 192 n1   n2 n1 n2 n1 - 881 n2 n1   n2   n1 n2 n1

                   2   2               2    2
    - 476 n1 n2 n1 n2 n1   n2   - 836 n2 n1 n2   n1   n2 n1

           2   2                       2           2
    + 616 n2   n1   n2 n1 n2 n1 + 1198 n1 n2 n1   n2 n1 n2

                   2         2         2        2                    5
    + 1198 n2 n1 n2   n1 n2 n1   + 541 n1 n2   n1 n2 n1   n2 - 352 n1 n2   n1 n2

                 2   2     2                 4                   3    2
    - 448 n2   n1   n2 n1   n2 + 1088 n1 n2 n1   n2 n1 - 334 n2 n1 n2 n1   n2

                 3         3         3   2     2             4         2
    - 160 n2 n1   n2 n1   + 576 n1   n2   n1 n2   + 336 n2 n1   n2 n1

                 2         3                 3   2         2   3   2
    - 1312 n1   n2 n1   n2 n1 - 334 n1 n2 n1 n2   n1   + 156 n1   n2   n1   n2

                                                 2   2
    + 681 n1 n2 n1 n2 n1 n2 n1 n2 + 616 n1   n2   n1 n2 n1 n2

                 2   3         2         3                 3         2
    + 278 n1 n2 n1   n2   n1 - 384 n2   n1 n2 n1 n2   - 864 n1 n2 n1   n2 n1
```

$$+ 320\, n_2 n_1^2 n_2^3 n_1 n_2 - 448\, n_1^2 n_2^2 n_1 n_2^2 n_1 + 268\, n_1^2 n_2^3 n_1 n_2 n_1$$

$$- 864\, n_2 n_1^3 n_2^2 n_1 n_2 + 880\, n_1^3 n_2^2 n_1 n_2 n_1 - 80\, n_2 n_1 n_2 n_1 n_2^4$$

$$- 220\, n_1^2 n_2^3 n_1 n_2 n_1 + 268\, n_2^2 n_1 n_2^3 n_1 n_2 + 816\, n_1^2 n_2^2 n_1 n_2^2 n_1$$

$$- 384\, n_1^3 n_2 n_1 n_2^2 n_1 + 357\, n_2 n_1 n_2 n_1^2 n_2 n_1 n_2$$

$$+ 195\, n_1 n_2 n_1 n_2^2 n_1 n_2 n_1 - 1233\, n_1 n_2^2 n_1 n_2 n_1 n_2 n_1$$

$$- 1233\, n_2 n_1^2 n_2 n_1 n_2 n_1 n_2 + 195\, n_2 n_1 n_2 n_1 n_2 n_1^2 n_2$$

$$+ 357\, n_1 n_2 n_1 n_2^2 n_1 n_2 n_1 + 320\, n_1 n_2 n_1^2 n_2 n_1^3 - 80\, n_1 n_2 n_1 n_2 n_1^4$$

$$- 160\, n_1^2 n_2 n_1 n_2^3 n_1 + 768\, n_1 n_2 n_1^3 n_2 n_1^2 - 368\, n_2^2 n_1 n_2^2 n_1^2$$

$$+ 520\, n_2^2 n_1 n_2^3 n_1 + 616\, n_2 n_1 n_2^4 n_1 - 160\, n_2^2 n_1 n_2 n_1 n_2^3$$

$$+ 40\, n_1 n_2^2 n_1 n_2^4 + 155\, n_1 n_2^2 n_1 n_2^2 n_1 + 278\, n_2 n_1 n_2^3 n_1 n_2$$

$$+ 40\, n_2^2 n_1^2 n_2^4 - 184\, n_2^2 n_1 n_2^4 + 96\, n_1^2 n_2^2 n_1 + 136\, n_1 n_2^6$$

$$+ 256\, n_2^3 n_1^4 n_2 + 200\, n_1 n_2^4 n_1^3 + 40\, n_1^2 n_2^2 n_1^4 - 184\, n_1^2 n_2^4 n_1^2$$

$$+ 768\, n_2^3 n_1^2 n_2 n_1 n_2 + 136\, n_2 n_1 n_2^6 + 200\, n_2 n_1^4 n_2^3$$

$$+ 256\, n_1^3 n_2^4 n_1 - 220\, n_2^2 n_1 n_2 n_1^3 n_2 - 160\, n_1^3 n_2 n_1 n_2^3$$

$$- 410\, n_2 n_1^3 n_2^3 n_1 - 464\, n_2 n_1 n_2^5 n_1 - 672\, n_2^3 n_1 n_2^3 n_1$$

$$- 352\, n_2 n_1^5 n_2 n_1 - 160\, n_2^3 n_1 n_2 n_1^3 + 646\, n_2^2 n_1 n_2^3 n_1^2$$

$$+ 616\, n_1^2 n_2^4 n_1 n_2 + 96\, n_1^4 n_2 n_1^2 n_2 + 336\, n_1^4 n_2 n_1^2 n_2$$

$$+ 40\, n_2^2 n_1 n_2^4 n_1 + 681\, n_2 n_1 n_2 n_1 n_2 n_1 n_2 n_1 - 368\, n_1^2 n_2^2 n_1^2 n_2$$

$$- 160\, n_1^3 n_2 n_1^3 n_2 + 96\, n_2^4 n_1 n_2^2 n_1 - 560\, n_1^3 n_2 n_1 n_2^3$$

$$- 672\, n_1^3 n_2^3 n_1 n_2 - 464\, n_1 n_2^5 n_1 n_2 + 646\, n_1^2 n_2^3 n_1 n_2^2$$

$$- 410\, n_1^3 n_2^3 n_1 n_2 + 96\, n_2^4 n_1^2 n_2^2 - 384\, n_2^3 n_1 n_2^3 n_1$$

$$+ 520\, n_1^2 n_2^2 n_1^3 n_2 + 156\, n_2 n_1^2 n_2^3 n_1 - 560\, n_2^3 n_1 n_2^3 n_1$$

$$+ 576\, n_2^3 n_1^2 n_2^2 n_1 + 155\, n_2 n_1^2 n_2^2 n_1 n_2 + 812\, n_2^2 n_1^2 n_2 n_1^2 n_2^2$$

```
          3      3                         4                   3          2
   - 384 n1   n2 n1   n2 + 1088 n2 n1 n2   n1 n2 - 710 n1 n2   n1 n2 n1


            2       3              2              2
   - 1312 n2   n1 n2   n1 n2 + 541 n2 n1   n2 n1 n2   n1


              3   2                  4
   + 880 n2 n1   n2   n1 n2 - 192 n2   n1 n2 n1 n2
```

Maple Code C.8: First subtractions from the result of (VII)

Note (as a consistency check) that the indicative terms shown in (C.2.11) are indeed gone. The two other terms in (C.2.10) do not contain a distinguishing sequence of ∇'s. For example they both contain

$$(\mathrm{i}) := [\nabla_i, B][\nabla_i, [\nabla_j, [\nabla_j, B]]] \supset +\nabla_2\nabla_1\nabla_1\nabla_1\nabla_1\nabla_1\nabla_2 \tag{C.2.12a}$$

$$(\mathrm{ii}) := [\nabla_j, [\nabla_i, [\nabla_i, B]]][\nabla_j, B] \supset +\nabla_2\nabla_1\nabla_1\nabla_1\nabla_1\nabla_1\nabla_2, \tag{C.2.12b}$$

which appears in the last result with a factor of $+136$. If it contains, say, $a(\mathrm{i}) + b(\mathrm{ii})$, then one can only conclude that

$$a + b = 136. \tag{C.2.13}$$

To determine the ratio of a and b one has to be more clever and find a sequence which comes in (i) and (ii) with a different factor. For instance,

$$[\nabla_i, B][\nabla_i, [\nabla_j, [\nabla_j, B]]] \supset +\nabla_1\nabla_1\nabla_2\nabla_1\nabla_1\nabla_1\nabla_1\nabla_2 \tag{C.2.14a}$$

$$[\nabla_j, [\nabla_i, [\nabla_i, B]]][\nabla_j, B] \supset +6 \cdot \nabla_1\nabla_1\nabla_2\nabla_1\nabla_1\nabla_1\nabla_1\nabla_2, \tag{C.2.14b}$$

because the $\nabla_1\nabla_2\nabla_1$ subsequence can only come from the constellation $B\nabla_1$ in the first case, whereas in the second case it can also stem from $-\nabla_1 B$ (and both possibilities are realised three times). So by looking at the last result one can conclude that

$$a + 6b = 616. \tag{C.2.15}$$

Note that this conclusion could not have been drawn before subtracting the first three terms, because they also contain the sequences shown in (C.2.12) and (C.2.14). From (C.2.13) and (C.2.15) it now follows that

$$a = 40 \qquad\qquad b = 96 \tag{C.2.16}$$

and subtracting the corresponding terms from the last result then gives:

```
> remainder := simplify(% - (40*nBnLB + 96*nLBnB));
                    2         3              2   3              4   3
remainder := 148 n2   n1 n2 n1   n2 - 170 n1 n2 n1   n2   n1 + 160 n1 n2   n1


              3   2                  2   2                   2   2
   + 376 n2 n1   n2   n1 n2 - 215 n1 n2 n1 n2   n1   n2 - 20 n1 n2 n1   n2   n1 n2


            2           2              2   2                     2   2
   + 124 n1   n2 n1 n2   n1 n2 + 616 n1   n2   n1 n2 n1 n2 - 761 n1 n2   n1   n2 n1 n2


              2   2                  2           2
   - 761 n2 n1   n2   n1 n2 n1 + 133 n1 n2   n1 n2 n1   n2 + 681 n1 n2 n1 n2 n1 n2 n1 n2
```

```
                    2          2        2        2                  2 2
+ 670 n1 n2 n1  n2 n1 n2  + 670 n2 n1 n2  n1 n2 n1  - 476 n2 n1 n2 n1 n2  n1

                 2                    2          2                    2
- 459 n2 n1 n2 n1  n2 n1 n2 + 435 n2 n1 n2 n1 n2 n1  n2 - 752 n2  n1 n2 n1 n2 n1

             2             2             2 2
- 752 n1  n2 n1 n2 n1 n2  - 476 n1 n2 n1 n2 n1  n2  + 681 n2 n1 n2 n1 n2 n1 n2 n1

                      2           2 4 2              3 3
+ 435 n1 n2 n1 n2 n1 n2  n1 - 320 n1  n2  n1  - 138 n2 n1  n2  n1

            2 2             2                           2           2 2
+ 616 n2  n1  n2 n1 n2 n1 - 459 n1 n2 n1 n2  n1 n2 n1 - 215 n2 n1 n2 n1  n2  n1

               4 3         3 4              2 2
+ 160 n2 n1  n2  + 160 n1  n2  n1 - 20 n2 n1 n2  n1  n2 n1

              2           2           2         2                 2 3
+ 133 n2 n1  n2 n1 n2  n1 + 124 n2  n1 n2 n1  n2 n1 - 170 n2 n1 n2  n1  n2

         2 3             2       2 2         2 4 2
+ 580 n2  n1  n2 n1 n2 + 524 n1 n2  n1 n2  n1  - 320 n2  n1  n2

         2 2     2         2 3                   3       2
- 568 n2  n1  n2 n1  n2 + 580 n1  n2  n1 n2 n1 - 398 n2 n1  n2 n1 n2

         3 2             3 2         3       2
+ 376 n1 n2  n1  n2 n1 + 34 n2 n1 n2 n1  n2  - 398 n1 n2  n1 n2 n1

         2 2     2         2 2 2         2       3
- 568 n1  n2  n1 n2  n1 + 155 n2 n1  n2  n1 n2 + 148 n1  n2 n1 n2  n1

           2 2 2             3 2       2       2 2
+ 155 n1 n2  n1  n2  n1 + 34 n1 n2 n1 n2  n1  + 524 n2 n1  n2 n1  n2

         3 4             2                   2
+ 160 n2  n1  n2 - 657 n1 n2  n1 n2 n1 n2 n1 - 657 n2 n1  n2 n1 n2 n1 n2

           2 3 2             3 3             3 3         3         3
+ 60 n1  n2  n1  n2 - 138 n1 n2  n1  n2 - 480 n1 n2 n1  n2  - 160 n1  n2 n1 n2

         3         3       2 2 2 2         2 3 2
- 160 n2  n1 n2 n1  - 368 n2  n1  n2  n1 + 606 n2 n1  n2  n1

         3 2     2         2 3 2         3 3
+ 480 n1  n2  n1 n2  + 606 n1 n2  n1  n2  - 480 n1  n2  n1 n2

         2 2 2 2         2       2 3         3 3         3 3
- 368 n1  n2  n1  n2  + 480 n1  n2 n1  n2  - 480 n2 n1 n2  n1  - 480 n2  n1  n2 n1

         2 3 2             2       2 3         3 2       2
+ 60 n2  n1  n2  n1 + 480 n2  n1 n2  n1  + 480 n2  n1  n2 n1
```

Maple Code C.9: Second subtractions from the result of (VII)

Note that again the sequences that were used to identify the last two terms are gone, which means that the values obtained for a and b were correct. But unfortunately the remainder is not zero, and so the assumption that the result for (VII) contains *only* the five terms in (C.2.10) is certainly wrong.

But recall that the eigenvalue relation was only conjectured to hold apart from $\mathcal{O}(B^3)$ terms, and so it could still be satisfied if the remainder only contained terms that are cubic in B. In particular, the indicator sequences that were chosen are all at most quadratic in B, so the reasoning above would not be spoiled by potential cubic terms and the coefficients

for the five terms (C.2.10) that were found would still be correct. But unfortunately the remainder contains terms like $\nabla_1\nabla_2\nabla_2\nabla_2\nabla_2\nabla_1\nabla_1\nabla_1$, which also can be at most quadratic in B and so this possibility is again ruled out.

Now the last loophole is that the remainder could contain (apart from $\mathcal{O}(B^3)$ terms) total derivative terms like

$$[\nabla_i, B[\nabla_i, [\nabla_j, [\nabla_j, B]]]] \tag{C.2.17a}$$

$$[\nabla_i, [\nabla_i, [\nabla_j, [\nabla_j, B^2]]]] \quad \text{etc.,} \tag{C.2.17b}$$

which would not contribute to the final result for $T\mathcal{O}_2$. But then there are two problems: First, the sequences used to extract the coefficients of the expected terms would no longer be characteristic for them, so it is not sufficient to investigate the remainder (i.e. the result shown in Maple Code C.9), but one has to start from the first result (Maple Code C.7). And second, there are a lot of possible total derivative terms like those shown in (C.2.17), and it is not clear with which coefficients each of them should come. These problems make it very difficult to show that *the result only contains the five terms shown in* (C.2.10) *plus higher order and total derivative terms (*)*, and so it remains unclear at this point whether the contributions from (VII) to $T\mathcal{O}_2$ are indeed proportional to $\mathcal{O}_2$. It might still be true, however.

Even though the attempt to verify the aforementioned hypothesis *(*)* was not successful, it is at least possible to show what the contribution to the eigenvalue of $\mathcal{O}_2$ would be, assuming the hypothesis is true. To this end, note that there is a particular sequence of ∇'s, viz.

$$\nabla_2\nabla_1\nabla_1\nabla_1\nabla_1\nabla_1\nabla_1\nabla_2, \tag{C.2.18}$$

which *must* come from the sequence $-B\nabla_1\nabla_1\nabla_1\nabla_1 B$. This sequence, on the other hand, *cannot* come from a total derivative since those must all have at least one ∇ at the very left or at the very right. Now observe that the five terms in (C.2.10) contain the sequence (C.2.18) with factors of $-1, +1, -1, +1$ and -1, respectively. But they contribute to the eigenvalue of $\mathcal{O}_2$ with factors of $+1, -1, +1, -1$ and $+1$, respectively. This means that no matter what higher order and total derivative terms might be contained in the result for (VII), its contribution to the eigenvalue will be given by minus the factor of the term (C.2.18) in the result shown in Maple Code C.7. In other words, under the hypothesis *(*)*, the contribution from (VII) to $T\mathcal{O}_2$ is

$$-\frac{1}{\pi}\frac{11}{1680}B[\nabla_i, [\nabla_i, [\nabla_j, [\nabla_j, B]]]] + \mathcal{O}(B^3). \tag{C.2.19}$$

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Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

München, den 15. Juni 2011

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