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Elitenetzwerk Bayern

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Jordan Wigner Transformations and Quantum Spin Systems on Graphs

Tobias Ried

Thesis Adviser: Prof. Dr. Simone Warzel
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Abstract

The Jordan Wigner transformation is an important tool in the study of one-dimensional quantum spin systems. After reviewing the one-dimensional Jordan Wigner transformation, generalisations to graphs are discussed. As a first result it is proved that the occurrence of statistical gauge fields in the transformed Hamiltonian is related to the structure of the graph. To avoid the difficulties caused by statistical gauge fields, an indirect transformation due to Wosiek and Szczerba is introduced and applied to a particular quantum spin system on a square lattice – the constrained Gamma matrix model. In a random exterior magnetic field, a locality estimate on the Heisenberg evolution – a zero-velocity Lieb Robinson type bound in disorder average – is proved for the constrained Gamma matrix model, using localisation results for the two-dimensional Anderson model.

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Contents

Chapter 1

Introduction

QUANTUM spin systems are an important playground for studying many-body quantum phenomena. Due to the fact that the Hilbert space at each lattice site is finite-dimensional they are relatively simple models, yet still capture some important physical features that would otherwise be too difficult to treat in the general framework. This includes the study of quantum dynamics, thus focussing on many-body scattering effects.

In the study of one-dimensional quantum spin systems, a transformation between spins (or hard-core bosons) and fermions first introduced by JORDAN and WIGNER [WJ28] has proven to be a valuable tool. It can be used to transfer results from (non-interacting) fermionic systems to spin systems with nearest neighbour interactions (Heisenberg model, XY-model, XXZ-model et cetera).

Recent developments in the theory of random Schrödinger operators, in particular the Anderson model, make it possible to analyse random quantum spin systems and look for analogues of dynamical localisation (the notion of spectral or eigenfunction localisation does not make sense any longer in many-body systems).

This thesis has the following objectives:

(i) Review the one-dimensional Jordan Wigner transformation and discuss generalisations to general (finite, connected) simple graphs. In particular it will be proven that the occurrence of statistical gauge fields in the transformed Hamiltonian is related to the structure of the graph.

(ii) Introduce an indirect Jordan Wigner transformation due to WOSIEK and SZCZERBA [Wos82, Szc85] and apply it to a particular quantum spin system – the constrained Gamma Matrix model.

(iii) Present zero-velocity Lieb Robinson bounds (in disorder average), whose validity can be used to define localisation of quantum spin systems in random magnetic fields similar to strong dynamical localisation in the Anderson model. These can be used to prove exponential decay of (averaged) ground state correlations [HSS12], and therefore provide a simplified version of the mobility gap concept introduced by HASTINGS [Has10].

(iv) Prove a zero-velocity Lieb Robinson type bound for the constrained Gamma Matrix Model, thereby providing another example (next to the isotropic random XY chain discussed by HAMZA *et al.* [HSS12]) of the above concepts.

In the remainder of this introductory chapter, some basic notions of quantum mechanics on combinatorial graphs and quantum spin systems are reviewed.

1.1 Quantum mechanics on combinatorial graphs

Combinatorial graphs are the underlying lattices in tight binding models of electrons in solid state physics, hard core bosons in optical lattices, and spin systems (just to name a few applications).

A combinatorial graph $\mathbb{G} = (V, E)$ consists of a set of vertices $V = V(\mathbb{G})$ and edges (or links) $E = E(\mathbb{G}) \subset \{\{x, y\} : x, y \in V\}$ connecting them. Two vertices x and y are said to be neighbouring, $x \sim y$, if $\{x, y\} \in E$. The graph G is assumed to be simple, i.e. two vertices are at most connected by one edge and vertices are not connected to themselves. Let $\mathrm{Adj}(\mathbb{G})$ be the adjacency matrix of $\mathbb{G}$, i.e. the symmetric $|V| \times |V|$-matrix with elements

$$\mathrm{Adj}(\mathbb{G})(x, y) = \begin{cases} 1, & \text{if } x \sim y \\ 0, & \text{otherwise} \end{cases}$$

A simple path $\mathscr{P}_N$ is a graph with vertex set $V = \{v_0, \dots, v_N\}$ and edges $E = \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{N-1}, v_N\}\}$ such that $v_k \neq v_j$ for all $k, j \in \{1, \dots, N\}$. There is a natural ordering on $\mathscr{P}_N$, namely $v_j \leq v_k$ if and only if $j \leq k$, i.e. v_j lies on the path from v_0 to v_k.

If additionally the first and the last vertex v_0 and v_N are connected by an edge, the resulting graph is called a cycle or simple loop $\mathscr{C}_{N+1}$ of length $N + 1$.

Bethe lattice and Cayley tree

The Bethe lattice $\mathbb{B}_z$ with coordination number $z \geq 2$ is the (infinite) connected, cycle-free graph with the property that each vertex has exactly z neighbours. Fixing one particular vertex as root o, the collection of its z neighbours is referred to as the *first shell*. Each vertex in the first shell has another $(z-1)$ child vertices,

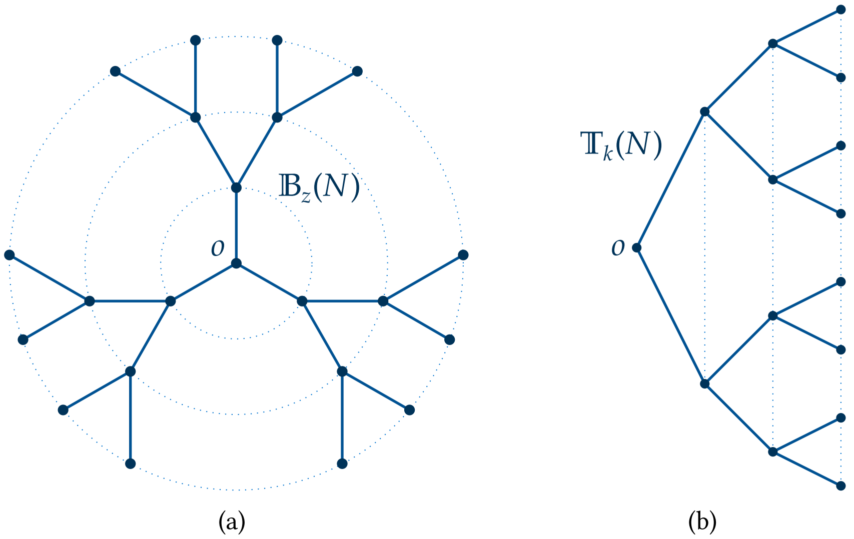


Figure 1.1: (a) Cayley tree $\mathbb{B}_z(N)$ and (b) its forward part $\mathbb{T}_k(N)$.

combined forming the second shell. Proceeding along up to N shells, the resulting tree $\mathbb{B}_z(N)$ is called *Cayley tree* (of size N). The nth shell has $\mathcal{N}_n = z(z-1)^{n-1}$ vertices and the Cayley tree $\mathbb{B}_z(N)$ has a total number of

$$\mathcal{N} = 1 + \sum_{n=1}^{N} \mathcal{N}_n = 1 + z\frac{(z-1)^N - 1}{z-2} = \frac{z(z-1)^N - 2}{z-2}$$

vertices.

There is a natural (partial) ordering on $\mathbb{B}_z$, defined by

$$x \preceq y \quad :\Leftrightarrow \quad \text{x lies on the unique path from } o \text{ to } y.$$

Given two vertices $x, y \in \mathbb{B}_z$ their distance $\operatorname{dist}(x, y)$ is given by the length of the unique path connecting them. In particular, the nth shell is the set of all vertices x with $\operatorname{dist}(x, o) = n$, and $\mathbb{B}_z(N) = \{x \in \mathbb{B}_z : \operatorname{dist}(x, o) \leq N\}$.

Sometimes it is customary to consider only the forward part $\mathbb{T}_k(N) = \mathbb{B}_z^+(N)$ of a Cayley tree, i.e. the regular rooted tree graph with branching number $k = z - 1$.

1.2 Quantum spin systems

This section is intended to describe the usual mathematical framework when discussing quantum spin systems [BR02, Nac06]. Let $\mathbb{G}$ be a connected (finite or infinite) graph, e.g. the d-dimensional lattice $\mathbb{Z}^d$, $d \geq 1$, or the Bethe lattice $\mathbb{B}_z$ with coordination number $z \geq 2$ (see section 1.1).

For each $x \in \mathbb{G}$ let $\mathfrak{H}_x = \mathfrak{H}_{\{x\}}$ be a finite dimensional Hilbert space[1]. For simplicity it is assumed that all the Hilbert spaces $\mathfrak{H}_x$ have the same dimension $n \geq 2$, such that $\mathfrak{H}_x \cong \mathbb{C}^n \; \forall x \in \mathbb{G}$. Denote by $\mathfrak{P}_0(\mathbb{G})$ the set of *finite* subsets of $\mathbb{G}$. For $\Lambda \in \mathfrak{P}_0(\mathbb{G})$ define $\mathfrak{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathfrak{H}_x$.

Observables of the system at site x are elements of the C^*-algebra $\mathfrak{A}_{\{x\}} = \mathfrak{B}(\mathfrak{H}_x)$ of bounded linear operators on $\mathfrak{H}_x$, which is isomorphic to the algebra of $n \times n$ complex matrices $\mathfrak{M}_n(\mathbb{C})$. The algebra of observables for a system in a finite set Λ is $\mathfrak{A}_\Lambda = \mathfrak{B}(\mathfrak{H}_\Lambda) = \bigotimes_{x\in\Lambda} \mathfrak{A}_{\{x\}}$.

Since for two disjoint finite sets $\Lambda_1 \cap \Lambda_2 = \emptyset$ one has $\mathfrak{H}_{\Lambda_1\cup\Lambda_2} = \mathfrak{H}_{\Lambda_1} \otimes \mathfrak{H}_{\Lambda_2}$, there is a natural embedding of $\mathfrak{A}_{\Lambda_1}$ into $\mathfrak{A}_{\Lambda_1\cup\Lambda_2}$ given by

$$\mathfrak{A}_{\Lambda_1} \ni A \mapsto A \otimes \mathbb{1}_{\Lambda_2} \in \mathfrak{A}_{\Lambda_1\cup\Lambda_2},$$

i.e. $\mathfrak{A}_{\Lambda_1} \cong \mathfrak{A}_{\Lambda_1} \otimes \mathbb{1}_{\Lambda_2} \subseteq \mathfrak{A}_{\Lambda_1\cup\Lambda_2}$. The C^*-algebra of *local observables* is defined as the union of the increasing family $(\mathfrak{A}_\Lambda)_{\Lambda\in\mathfrak{P}_0(\mathbb{G})}$,

$$\mathfrak{A}_{\mathrm{loc}} = \bigcup_{\Lambda\in\mathfrak{P}_0(\mathbb{G})} \mathfrak{A}_\Lambda,$$

and its norm closure is the C^*-algebra of *quasi-local observables* $\mathfrak{A} = \overline{\mathfrak{A}_{\mathrm{loc}}}$.

An *interaction* (or potential) on the quantum spin system, defining a quantum spin model, is a map $\Phi : \mathfrak{P}_0(\mathbb{G}) \to \mathfrak{A}$ with the properties that $\Phi(X) \in \mathfrak{A}_X$ and $\Phi(X)^* = \Phi(X)$ for each $X \in \mathfrak{P}_0(\mathbb{G})$. It is called *finite range* if there is an $R > 0$ such that for all finite sets X with $\mathrm{diam}(X) > R$ one has $\Phi(X) = 0$. The *Hamiltonian* with a finite set Λ and an interaction Φ is given by

$$H_\Lambda = H_\Lambda^\Phi = \sum_{X\subseteq\Lambda} \Phi(X).$$

It is a self-adjoint element of $\mathfrak{A}_\Lambda$.

Time evolution

The time evolution (Heisenberg dynamics) of a quantum spin system corresponding to an interaction Φ (respectively, the Hamiltonian H_Λ) is given by the one-parameter group $(\alpha_t^\Lambda)_{t\in\mathbb{R}}$ of automorphisms of $\mathfrak{A}_\Lambda$,

$$\alpha_t^\Lambda(A) = e^{\mathrm{i}tH_\Lambda} A e^{-\mathrm{i}tH_\Lambda}, \quad A \in \mathfrak{A}_\Lambda.$$

It is well-defined by the spectral theorem (H_Λ being self-adjoint).

[1] The notation $x \in \mathbb{G}$ is used throughout this thesis instead of $x \in V(\mathbb{G})$ if there is no room for confusion.

Example 1.1 (The XY model). Let $\mathfrak{H}_x = \mathbb{C}^2$ for each $x \in \mathbb{G}$. The anisotropic XY model in an external field is defined by the one- and two-body interaction

$$\Phi(X) = \begin{cases} \mu_x[(1+\gamma_x)\sigma_x^1\sigma_y^1 + (1-\gamma_x)\sigma_x^2\sigma_y^2] & \text{if } X = \{x,y\} \text{ and } \operatorname{dist}(x,y) = 1 \\ \nu_x\sigma_x^3 & \text{if } X = \{x\} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices, and the real-valued sequences $\{\mu_x\}$, $\{\gamma_x\}$ and $\{\nu_x\}$ describe the coupling strength, anisotropy and external magnetic field strength, respectively.

The Hamiltonian for a finite subset Λ of $\mathbb{G}$ is then given by

$$H_\Lambda^{XY} = \sum_{\substack{x,y\in\Lambda \\ \text{dist(x,y)=1}}} \mu_x[(1+\gamma_x)\sigma_x^1\sigma_y^1 + (1-\gamma_x)\sigma_x^2\sigma_y^2] + \sum_{x\in\Lambda}\nu_x\sigma_x^3 \tag{1.1}$$

and, formally, the Hamiltonian of the full system may be written as

$$H^{XY} = \sum_{\substack{x,y\in\mathbb{G} \\ \text{dist(x,y)=1}}} \mu_x[(1+\gamma_x)\sigma_x^1\sigma_y^1 + (1-\gamma_x)\sigma_x^2\sigma_y^2] + \sum_{x\in\mathbb{G}}\nu_x\sigma_x^3$$

Elements in $\mathfrak{A}_\Lambda$, $\Lambda \in \mathfrak{P}_0(\mathbb{G})$, are $2^{|\Lambda|}\times 2^{|\Lambda|}$-matrices corresponding to polynomials in the Pauli matrices and the 2×2 identity matrix. Quasi-local observables are given by uniform limits of such polynomials.

For completeness and later convenience, the most important properties of the Pauli matrices are stated below.

(i) The commutator between two Pauli matrices is given by

$$\left[\sigma^\alpha, \sigma^\beta\right] = 2\mathrm{i}\epsilon_{\alpha\beta\gamma}\sigma^\gamma$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi Cività tensor.

(ii) The anti-commutator between two Pauli matrices is $\left\{\sigma^\alpha, \sigma^\beta\right\} = 2\delta_{\alpha\beta}\mathbb{1}$.

(iii) $(\sigma^\alpha)^2 = \mathbb{1}$, $\operatorname{Tr}(\sigma^\alpha) = 0$

(iv) $\sigma^\alpha\sigma^\beta = \delta_{\alpha\beta}\mathbb{1} + \mathrm{i}\epsilon_{\alpha\beta\gamma}\sigma^\gamma$

Chapter 2

Jordan-Wigner transformation

THE Jordan Wigner transformation, introduced by JORDAN and WIGNER in [WJ28], has proven to be a valuable tool in studying spin chains and their (quantum) phase transitions in one dimensional systems, most prominently in the application to the quantum XY model in one dimension by LIEB *et al.* [LSM61].

2.1 The Jordan Wigner transformation in one dimension

Consider the XY model on $\mathbb{Z}$ as defined above. In particular, the following section will be concerned with the Hamiltonians $H^{XY}_{[-N,N]}$ for $N \in \mathbb{N}$ and make use of an equivalence between the Pauli matrices and fermionic operators.

This method, known as *Jordan Wigner transformation*, dates back to JORDAN and WIGNER [WJ28] and was used by LIEB, SCHULZ and MATTIS [LSM61] in the exact solution of the anisotropic XY chain. The infinite model was studied by ARAKI and MATSUI [AM85] after the correspondence between the Pauli and CAR algebras had been given by ARAKI [Ara84].

2.1.1 Jordan Wigner transformation

For $N \in \mathbb{N}$ let

$$H_N := H^{XY}_{[-N,N]} = \sum_{x=-N}^{N-1} \mu_x[(1+\gamma_x)\sigma^1_x\sigma^1_{x+1} + (1-\gamma_x)\sigma^2_x\sigma^2_{x+1}] + \sum_{x=-N}^{N} \nu_x\sigma^3_x \tag{2.1}$$

As a first step towards establishing the Jordan Wigner transformation, one introduces the set of *raising and lowering operators* (hard core bosons)

$$b^*_x = \sigma^+_x = \frac{1}{2}\left(\sigma^1_x + \mathrm{i}\sigma^2_x\right), \quad b_x = \sigma^-_x = \frac{1}{2}\left(\sigma^1_x - \mathrm{i}\sigma^2_x\right) \tag{2.2}$$

for each $x \in [-N, N] \cap \mathbb{Z}$.

They are *locally anticommuting*, $\{b_x^*, b_x\} = \mathbb{1}$, *commuting* on different lattice sites, $[b_x^*, b_y] = [b_x^*, b_y^*] = [b_x, b_y] = 0$ for $x \neq y$, and satisfy a *hard core condition*, $b_x^2 = (b_x^*)^2 = 0$ (*mixed algebra of raising/lowering operators*).

Furthermore, they have the properties

$$\begin{aligned} &[\sigma_x^3, b_x^*] = 2b_x^*, \quad [\sigma_x^3, b_x] = -2b_x \\ &\sigma_x^3 b_x = -b_x, \quad b_x\sigma_x^3 = b_x, \quad \sigma_x^3 b_x^* = b_x^*, \quad b_x^*\sigma_x^3 = -b_x^* \end{aligned} \tag{2.3}$$

By definition, the raising/lowering operators preserve the local structure, and satisfy the relations

$$\sigma_x^1 = b_x + b_x^*, \quad \sigma_x^2 = \mathrm{i}(b_x - b_x^*), \quad \sigma_x^3 = 2b_x^*b_x - \mathbb{1} = b_x^*b_x - b_xb_x^* \tag{2.4}$$

In particular, one has for $y \neq x$

$$\begin{aligned} \sigma_x^1\sigma_y^1 + \sigma_x^2\sigma_y^2 &= 2(b_x^*b_y + b_y^*b_x) \\ \sigma_x^1\sigma_y^1 - \sigma_x^2\sigma_y^2 &= 2(b_xb_y + b_y^*b_x^*) \end{aligned} \tag{2.5}$$

These can be used to express H_N in terms of the b operators.

$$H_N = \sum_{x=-N}^{N-1} 2\mu_x[b_x^*b_{x+1} + b_{x+1}^*b_x + \gamma_x(b_xb_{x+1} + b_{x+1}^*b_x^*)] + \sum_{x=-N}^{N} \nu_x(2b_x^*b_x - \mathbb{1}) \tag{2.6}$$

In order to construct fermionic creation and annihilation operators c_j^* and c_j, respectively, obeying the canonical anti-commutation relations (CARs)

$$\{c_x, c_y^*\} = \delta_{xy}\mathbb{1}, \quad \{c_x, c_y\} = \{c_x^*, c_y^*\} = 0, \quad \text{for all } x, y \in [-N, N] \cap \mathbb{Z} \tag{2.7}$$

one needs to break the local structure of the bosonic operators b defined above. One possible choice is to define $c_{-N}^{(*)} := b_{-N}^{(*)}$ and

$$c_x := \left(\prod_{y=-N}^{x-1} \sigma_y^3\right) b_x, \quad c_x^* = b_x^* \left(\prod_{y=-N}^{x-1} \sigma_y^3\right), \quad y = -N+1, \dots, N \tag{2.8}$$

The operator $S_x = \prod_{y=-N}^{x-1} \sigma_y^3$ is usually referred to as *string*, *kink* or *soliton operator*[1] (especially in the physics literature) [Fra13].

[1]Let $|\pm\rangle$ be the two eigenstates of σ^1. Then S_x creates a *kink* at lattice site x,

$$S_x\left(|+\rangle \otimes \cdots \otimes |+\rangle\right) = |-\rangle \otimes \cdots \otimes |-\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$$

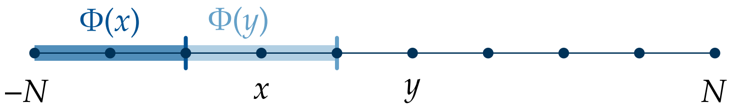


Figure 2.1: Graphical representation of the lattice sites contributing to the phases $\Phi(x)$ and $\Phi(y)$ in the one dimensional Jordan Wigner transformation. They are directly related to the string operator (see remark).

Lemma 2.1 (Jordan Wigner Transformation). The operators defined in (2.8) satisfy the canonical anticommutator relations (2.7).

Proof. Without loss of generality let $x < y$. Then by relation 2.3 and the commutativity of the b operators on different lattice sites one has (see also figure 2.1 for a graphical representation of which sites contribute to the product in the Jordan Wigner transformation)

$$c_x c_y^* = \left(\prod_{z=-N}^{x-1} \sigma_z^3 \right) b_x b_y^* \left(\prod_{z=-N}^{y-1} \sigma_z^3 \right) = \left(b_y^* \prod_{z=-N}^{y-1} \sigma_z^3 \right) \underbrace{\sigma_x^3 b_x \sigma_x^3}_{=-b_x} \left(\prod_{z=-N}^{x-1} \sigma_z^3 \right) = -c_y^* c_x$$

Since $c_x^* c_x = b_x^* \left(\prod_{z=-N}^{x-1} \sigma_z^3\right) \left(\prod_{z=-N}^{x-1} \sigma_z^3\right) b_x = b_x^* b_x$ and the bs are locally anti commuting the c operators inherit this property. By the hard core condition and essentially the same calculation as above one proves that $\{c_x^*, c_y^*\} = \{c_x, c_y\} = 0$ for all $x, y \in \{-N, \dots, N\}$.

∎

Remark. Another, equivalent way of constructing fermionic operators c from the hard core bosons b is to define (c.f. [LSM61] and the discussion in section 2.2)

$$c_x := e^{\mathrm{i}\Phi(x)} b_x, \quad c_x^* = b_x^* e^{-\mathrm{i}\Phi(x)} \tag{2.9}$$

with $\Phi(x) = \pi \sum_{y=-N}^{x-1} b_y^* b_y$ and inverse transformation

$$b_x = e^{-\mathrm{i}\Phi(x)} c_x, \quad b_x^* = c_x^* e^{\mathrm{i}\Phi(x)}. \tag{2.10}$$

This is due to the fact that the operators $(b_y^* b_y)_{y=-N,\dots,N}$ are mutually commuting and thus

$$\exp\left(\mathrm{i}\pi \sum_{y=-N}^{x-1} b_y^* b_y \right) = \prod_{y=-N}^{x-1} \exp\left(\mathrm{i}\pi b_y^* b_y\right) = \prod_{y=-N}^{x-1} \exp\left(\tfrac{1}{2}\mathrm{i}\pi(\sigma_y^3 + \mathbb{1})\right) = \prod_{y=-N}^{x-1} (-\sigma_y^3)$$

which is unitarily equivalent to the string operator S_x. ♦

Using the relations $c_x^* c_x = b_x^* b_x$, $c_x^* c_{x+1} = -b_x^* b_{x+1}$ and $c_x c_{x+1} = b_x b_{x+1}$, one gets

$$H_N = \sum_{x=-N}^{N-1} 2\mu_x[-c_x^* c_{x+1} - c_{x+1}^* c_x + \gamma_x(c_x c_{x+1} + c_{x+1}^* c_x^*)] + \sum_{x=-N}^{N} \nu_x(2c_x^* c_x - \mathbb{1}). \tag{2.11}$$

2.1.2 Extension to the infinite chain

The extension of the Jordan Wigner transformation to the (two-sided) infinite chain $\mathbb{Z}$ reveals some subtleties that were studied by ARAKI and MATSUI in their discussion of ground states of the XY-model on $\mathbb{Z}$ [Ara84, AM85]. It turns out that the C^*-algebra $\mathfrak{A}^{CAR}$ generated by $\{c_x, c_x^* : x \in \mathbb{Z}\}$, and the C^*-algebra $\mathfrak{A}^P$ generated by the Pauli spin matrices $\{\sigma_x^\alpha : \alpha = 1,2,3, x \in \mathbb{Z}\}$ are different C^*-subalgebras of an enlarged C^*-algebra $\widehat{\mathfrak{A}}$. Only their even parts with respect to a certain automorphism of $\widehat{\mathfrak{A}}$ coincide. The problem is that the infinite product $\prod_{x=0}^{-\infty} \sigma_x^3$ is not a quasi-local observable[2].

Define the automorphism $\Theta_- : \mathfrak{A}^P \mapsto \mathfrak{A}^P$,

$$\Theta_-(A) := \lim_{N\to\infty} \left(\prod_{x=0}^{-N} \sigma_x^3\right) A \left(\prod_{x=0}^{-N} \sigma_x^3\right),$$

which corresponds to a rotation of all the spins on the left half of the lattice $x \le 0$ by π around the σ^3-axis. Indeed, a simple application of the (anti-)commutation relations of the Pauli matrices shows that for $\alpha = 1,2$,

$$\Theta_-(\sigma_x^\alpha) = \lim_{N\to\infty} \left(\prod_{y=0}^{-N} \sigma_y^3\right) \sigma_x^\alpha \left(\prod_{y=0}^{-N} \sigma_y^3\right) = \sigma_x^\alpha \lim_{N\to\infty} \underbrace{\left(\prod_{x=0}^{-N} \sigma_y^3\right)^2}_{=\mathbb{1}} = \sigma_x^\alpha \quad \text{for } x > 0,$$

$$\Theta_-(\sigma_x^\alpha) = \lim_{N\to\infty} \left(\prod_{y=0}^{-N} \sigma_y^3\right) \sigma_x^\alpha \left(\prod_{y=0}^{-N} \sigma_y^3\right) = -\sigma_x^\alpha \quad \text{for } x \le 0,$$

and $\Theta_-(\sigma_x^3) = \sigma_x^3$ for all $x \in \mathbb{Z}$. Further, one can easily see that Θ_- is an involution, $\Theta_-^2 = \mathbb{1}$.

Now let T be such that

$$T^2 = \mathbb{1}, \quad T^* = T, \quad TAT = \Theta_-(A) \quad \forall A \in \mathfrak{A}^P$$

[2] Indeed, one has $\left\|\prod_{x=0}^{-N} \sigma_x^3\right\| = 1$ for all $n \in \mathbb{N}$, thus it cannot converge in norm.

and define $\widehat{\mathfrak{A}}$ as the C^*-algebra generated by $\mathfrak{A}^P$ and T. It can be decomposed into a direct sum $\widehat{\mathfrak{A}} = \mathfrak{A}^P T^0 + \mathfrak{A}^P T^1 = \mathfrak{A}^P + \mathfrak{A}^P T$ and with this decomposition at hand, the automorphism Θ_- can be extended to $\widehat{\mathfrak{A}}$ by setting

$$\Theta_-(A_1 + A_2 T) := \Theta_-(A_1) + \Theta_-(A_2)T, \quad A_1, A_2 \in \mathfrak{A}^P.$$

Remark. Introducing T and the enlarged C^*-algebra $\widehat{\mathfrak{A}}$ was necessary to adapt the Jordan Wigner transformation 2.8 to the case of an infinite chain.

If $\Lambda = [-N, N] \cap \mathbb{Z}$, then $\Theta_-(A) = \left(\prod_{x=0}^{-N} \sigma_x^3\right) A \left(\prod_{x=0}^{-N} \sigma_x^3\right) = TAT$ for all $A \in \mathfrak{A}^P$. In particular $T = \prod_{x=0}^{-N} \sigma_x^3 \in \mathfrak{A}^P$, so $\widehat{\mathfrak{A}} = \mathfrak{A}^P$, and the above reduces to the discussion in section 2.1.1. ♦

Having introduced the enlarged algebra $\widehat{\mathfrak{A}}$ it is possible to define the creation and annihilation operators $c_x^*, c_x \in \widehat{\mathfrak{A}}$ by

$$\begin{aligned} c_x^* &:= TS_x\sigma_x^+ = TS_x\left(\frac{\sigma_x^1 + \mathrm{i}\sigma_x^2}{2}\right) \\ c_x &:= TS_x\sigma_x^- = TS_x\left(\frac{\sigma_x^1 - \mathrm{i}\sigma_x^2}{2}\right) \end{aligned} \qquad S_x = \begin{cases} \prod_{y=1}^{x-1} \sigma_y^3 & \text{, if } x \geq 2 \\ \mathbb{1} & \text{, if } x = 1 \\ \prod_{y=0}^{x} \sigma_y^3 & \text{, if } x \leq 0 \end{cases}.$$

Here one can see that T was used to substitute $\prod_{x=-\infty}^{0} \sigma_x^3$. In particular T has all the necessary properties to make the family of operators c_x, c_x^*, $x \in \mathbb{Z}$, fulfil canonical anticommutation relations. This follows from $[S_x, S_y] = 0$, $S_x^2 = \mathbb{1}$ and $\Theta_-(S_x) = S_x$ for all $x, y \in \mathbb{Z}$. Also notice that for $x \leq y$ one has

$$S_x\sigma_y^{\pm} = \begin{Bmatrix} \left(\prod_{z=1}^{x-1} \sigma_z^3\right)\sigma_y^{\pm}, & x \geq 2 \\ \sigma_y^{\pm}, & x = 1 \\ \left(\prod_{z=0}^{x} \sigma_z^3\right)\sigma_y^{\pm}, & x \leq 0 \end{Bmatrix} = \begin{Bmatrix} \sigma_y^{\pm}\left(\prod_{z=1}^{x-1} \sigma_z^3\right), & x \geq 2 \\ \sigma_y^{\pm}, & x = 1 \\ \Theta_-(\sigma_y^{\pm})\left(\prod_{z=0}^{x} \sigma_z^3\right), & x \leq 0 \end{Bmatrix} = \Theta_-(\sigma_y^{\pm})S_x$$

and for $x > y$

$$S_x\sigma_y^{\pm} = \begin{Bmatrix} \left(\prod_{z=1}^{x-1} \sigma_z^3\right)\sigma_y^{\pm}, & x \geq 2 \\ \sigma_y^{\pm}, & x = 1 \\ \left(\prod_{z=0}^{x} \sigma_z^3\right)\sigma_y^{\pm}, & x \leq 0 \end{Bmatrix} = \begin{Bmatrix} -\Theta_-(\sigma_y^{\pm})\left(\prod_{z=1}^{x-1} \sigma_z^3\right), & x \geq 2 \\ \sigma_y^{\pm}, & x = 1 \\ \sigma_y^{\pm}\left(\prod_{z=0}^{x} \sigma_z^3\right), & x \leq 0 \end{Bmatrix} = -\Theta_-(\sigma_y^{\pm})S_x$$

Hence,

$$\begin{aligned} c_x c_x^* &= (TS_x\sigma_x^-)(TS_x\sigma_x^+) = S_x\Theta_-(\sigma_x^-)T^2 S_x\sigma_x^+ = S_x\Theta_-(\sigma_x^-)S_x\sigma_x^+ = S_x\Theta_-(\sigma_x^-)\Theta_-(\sigma_x^+)S_x \\ &= S_x\left(\mathbb{1} - \Theta_-(\sigma_x^+)\Theta_-(\sigma_x^-)\right)S_x = \mathbb{1} - S_x\Theta_-(\sigma_x^+)S_x\sigma_x^- = \mathbb{1} - S_x\Theta_-(\sigma_x^+)T^2 S_x\sigma_x^- \\ &= \mathbb{1} - TS_x\sigma_x^+ TS_x\sigma_x^- = \mathbb{1} - c_x^* c_x, \end{aligned}$$

and for $x < y$ one gets

$$\begin{aligned} c_x c_y^* &= T S_x \sigma_x^- T S_y \sigma_y^+ = S_x \Theta_-(\sigma_x^-) S_y \sigma_y^+ = -S_x S_y \sigma_x^- \sigma_y^+ = -S_y S_x \sigma_y^+ \sigma_x^- \\ &= -S_y \Theta_-(\sigma_y^+) S_x \sigma_x^- = -S_y \Theta_-(\sigma_y^+) T^2 S_x \sigma_x^- = -T S_y \sigma_y^+ T S_x \sigma_x^- = -c_y^* c_x. \end{aligned}$$

Similarly it can be proven that $\{c_x, c_y\} = \{c_x^*, c_y^*\} = 0$.

Let $\mathfrak{A}^{CAR}$ denote the C^*-subalgebra of $\widehat{\mathfrak{A}}$ generated by those operators. It follows immediately, that

$$\Theta_-(c_x^*) = \begin{cases} c_x^* & \text{, if } x \geq 1 \\ -c_x^* & \text{, if } x \leq 0 \end{cases} \quad \Theta_-(c_x) = \begin{cases} c_x & \text{, if } x \geq 1 \\ -c_x^* & \text{, if } x \leq 0 \end{cases}$$

Next, the relation between $\mathfrak{A}^P$ and $\mathfrak{A}^{CAR}$ is described. To do so, one has to introduce another involutive automorphism Θ of $\widehat{\mathfrak{A}}$, describing the rotation of all spins around the σ^3-axis by π. By the construction of $\widehat{\mathfrak{A}}$ it suffices to define

$$\begin{aligned} \Theta(\sigma_x^\alpha) &= -\sigma_x^\alpha \quad (\alpha = 1, 2) \\ \Theta(\sigma_x^3) &= \sigma_x^3 \\ \Theta(T) &= T \end{aligned}$$

for $x \in \mathbb{Z}$ and extend it to an automorphism on $\widehat{\mathfrak{A}}$. Accordingly, the action on the fermionic operators is

$$\Theta(c_x^*) = -c_x^*, \quad \Theta(c_x) = -c_x \quad (x \in \mathbb{Z}).$$

Clearly, $\Theta^2 = \mathbb{1}$, and by definition Θ leaves $\mathfrak{A}^P$ and $\mathfrak{A}^{CAR}$ invariant, i.e. for $A \in \mathfrak{A}^{P/CAR}$ also $\Theta(A) \in \mathfrak{A}^{P/CAR}$.

Θ can now be used to decompose both subalgebras into an even ($\Theta = 1$) and an odd part ($\Theta = -1$). For an operator $A \in \mathfrak{A}^{P/CAR}$ one has $A = A_+ + A_-$ where $A_\pm = \frac{1}{2}(A \pm \Theta(A))$, and $\mathfrak{A}^{P/CAR} = \mathfrak{A}_+^{P/CAR} + \mathfrak{A}_-^{P/CAR}$ with $\mathfrak{A}_\pm^{P/CAR} = \{A \in \mathfrak{A}^{P/CAR} : \Theta(A) = \pm A\}$. The relation between the spin and CAR algebras is given by

$$\mathfrak{A}_+^P = \mathfrak{A}_+^{CAR}, \quad \mathfrak{A}_-^P = T\mathfrak{A}_-^{CAR}. \tag{2.12}$$

Define another involutive automorphism $\widetilde{\Theta}_- : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ by

$$\widetilde{\Theta}_-(A_1 + TA_2) = A_1 - TA_2, \quad A_{1,2} \in \mathfrak{A}^P$$

Then

$$\begin{aligned} \Theta\left(\widetilde{\Theta}_-(A_1 + TA_2)\right) &= \Theta\left(A_1 - TA_2\right) = \Theta(A_1) - T\Theta(A_2) \\ &= \widetilde{\Theta}_-\left(\Theta(A_1) + T\Theta(A_2)\right) = \widetilde{\Theta}_-\left(\Theta(A_1 + TA_2)\right) \end{aligned}$$

i.e. $\Theta\widetilde{\Theta}_- = \widetilde{\Theta}_-\Theta$ on $\widehat{\mathfrak{A}}$.

Since $\widehat{\mathfrak{A}} = \mathfrak{A}^P + T\mathfrak{A}^P$ the extended algebra $\widehat{\mathfrak{A}}$ can be decomposed into

$$\widehat{\mathfrak{A}} = \mathfrak{A}_+^{CAR} + \mathfrak{A}_-^{CAR} + T\mathfrak{A}_+^{CAR} + T\mathfrak{A}_-^{CAR}$$
$$\mathfrak{A}^P = \mathfrak{A}_+^{CAR} + T\mathfrak{A}_-^{CAR}$$

The local Hamiltonian H_N (2.1) is quadratic in the spin operators, and therefore lies in the positive subalgebra of $\mathfrak{A}^P$. But the positive subalgebras of the spin and CAR algebras coincide, which concludes the discussion of the Jordan Wigner transformation in the case of an infinite chain.

2.2 Generalisations to graphs

The LIEB-SCHULTZ-MATTIS ansatz 2.9 can be used to define a Jordan Wigner transformation on arbitrary (connected, finite) graphs. In this setting the Hamiltonian transforms to a more difficult one including statistical gauge fields which in general cannot be removed by gauge transformations.

Let $\mathbb{G} = (V, E)$ be a finite, connected graph and $\{c_x\}_{x\in\mathbb{G}}$ a fermionic field on $\mathbb{G}$. Define the new operators

$$a_x = e^{\mathrm{i}\Phi(x)}c_x, \quad a_x^* = c_x^* e^{-\mathrm{i}\Phi(x)} \tag{2.13}$$

with $\Phi(x) = \pi\sum_{z\in\mathbb{G}}\varphi(x,z)c_z^*c_z = \pi\sum_{z\neq x}\varphi(x,z)c_z^*c_z$, $\varphi(x,z)\in\mathbb{R}$ for all $x, z\in\mathbb{G}$. The latter equality uses that, without loss of generality, one can assume $\varphi(x,x) = 0$, since $e^{\mathrm{i}\phi(x,x)c_x^*c_x}c_x = c_x$.

Indeed, since $c_x^*c_x$ is bounded, one has the norm-convergent series representation

$$e^{\mathrm{i}\alpha c_x^*c_x}c_x = \left(\mathbb{1} + \sum_{k\geq 1}\frac{(\mathrm{i}\alpha)^k}{k!}(c_x^*c_x)^k\right)c_x = c_x \quad ,\alpha\in\mathbb{R}$$

due to the property $c_x^2 = 0$. The same applies to $c_x^* e^{\mathrm{i}\alpha c_x^*c_x} = c_x^*$.

By the canonical anti commutation relations the a operators satisfy

$$\begin{aligned}
a_x a_y^* &= e^{\mathrm{i}\Phi(x)}c_x c_y^* e^{-\mathrm{i}\Phi(y)} = \delta_{xy}\mathbb{1} - e^{\mathrm{i}\Phi(x)}c_y^*c_x e^{-\mathrm{i}\Phi(y)} \\
&= \delta_{xy}\mathbb{1} - \underbrace{e^{\mathrm{i}\pi\varphi(x,y)c_y^*c_y}c_y^*e^{-\mathrm{i}\pi\varphi(x,y)c_y^*c_y}}_{=e^{\mathrm{i}\pi\varphi(x,y)}c_y^*}\, e^{\mathrm{i}\Phi(x)}e^{-\mathrm{i}\Phi(y)}\, \underbrace{e^{\mathrm{i}\pi\varphi(y,x)c_x^*c_x}c_x e^{-\mathrm{i}\pi\varphi(y,x)c_x^*c_x}}_{=e^{-\mathrm{i}\pi\varphi(y,x)}c_x} \\
&= \delta_{xy}\mathbb{1} - e^{\mathrm{i}\pi(\varphi(x,y)-\varphi(y,x))}c_y^*e^{-\mathrm{i}\Phi(y)}e^{\mathrm{i}\Phi(x)}c_x \\
&= \delta_{xy}\mathbb{1} - e^{\mathrm{i}\pi(\varphi(x,y)-\varphi(y,x))}a_y^*a_x
\end{aligned} \tag{2.14}$$

for $x, y \in \mathbb{G}$. If

$$\varphi(x,y) - \varphi(y,x) = \vartheta \mod 2 \quad \forall x \neq y, \tag{2.15}$$

then the a operators are said to satisfy hard core *anyonic* statistics, i.e.

$$a_x a_y^* = \delta_{xy}\mathbb{1} - e^{i\pi\vartheta} a_y^* a_x \tag{2.16}$$

The parameter $\vartheta \in [0,1]$ interpolates between fermionic ($\vartheta = 0$) and hard-core bosonic ($\vartheta = 1$) statistics, and the hard-core property $(a_x^{(*)})^2 = 0$ follows from the same property of the fermionic c operators.

In the calculation above the equalities

$$\begin{aligned} e^{i\pi\varphi(x,y)c_y^*c_y} c_y^* e^{-i\pi\varphi(x,y)c_y^*c_y} &= e^{i\pi\varphi(x,y)} c_y^* \\ e^{i\pi\varphi(y,x)c_x^*c_x} c_x e^{-i\pi\varphi(y,x)c_x^*c_x} &= e^{-i\pi\varphi(y,x)} c_x \end{aligned}$$

have been used. They follow from the following observation: in a basis where $c_x^* c_x$ is diagonal, $c_x^* c_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_x$, $c_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_x$, and $c_x^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_x$. Then

$$\begin{aligned} e^{i\pi\varphi(x,y)c_y^*c_y} c_y^* e^{-i\pi\varphi(x,y)c_y^*c_y} &= e^{i\pi\varphi(x,y)c_y^*c_y} c_y^* = \begin{pmatrix} e^{i\pi\varphi(x,y)} & 0 \\ 0 & 1 \end{pmatrix}_y \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_y = \begin{pmatrix} 0 & e^{i\pi\varphi(x,y)} \\ 0 & 0 \end{pmatrix}_y \\ &= e^{i\pi\varphi(x,y)} c_y^*. \end{aligned}$$

An analogous calculation yields the second equality.

Unless stated otherwise it shall from now on be assumed that $\vartheta = 1$, so that the a operators describe hard-core bosons. Let H be the Hamiltonian of free fermions on $\mathbb{G}$ with nearest neighbour hopping,

$$H = \sum_{x\sim y} (c_x^* c_y + c_y^* c_x).$$

Then the transformation 2.13 yields the unitarily equivalent Hamiltonian

$$H = \sum_{x\sim y} (a_x^* e^{i\Phi(x)} e^{-i\Phi(y)} a_y + a_y^* e^{i\Phi(y)} e^{-i\Phi(x)} a_x) = \sum_{x\sim y} (a_x^* e^{-iA(x,y)} a_y + a_y^* e^{iA(x,y)} a_x) \tag{2.17}$$

with a statistical gauge field

$$A(x,y) = \pi \sum_{z\neq x,y} \big[\varphi(y,z) - \varphi(x,z)\big] c_z^* c_z = \pi \sum_{z\neq x,y} \big[\varphi(y,z) - \varphi(x,z)\big] a_z^* a_z, \tag{2.18}$$

where it has been taken into account that $a_x^* e^{i\beta a_x^* a_x} = a_x^*$ and $e^{i\beta a_x^* a_x} a_x = a_x$ for all $\beta \in \mathbb{R}$ due to the hard-core property $(a_x^{(*)})^2 = 0$.

Statistical gauge transformations

From calculation 2.14 it follows immediately that transformations of the form $\tilde{a}_x = e^{\mathrm{i}\Lambda(x)} a_x$ with $\Lambda(x) = \pi \sum_{z \neq x} \lambda(x,z) a_z^* a_z$ leave the particle statistics unchanged iff

$$\lambda(x,y) = \lambda(y,x) \quad \text{mod } 2 \quad \forall x \neq y. \tag{2.19}$$

They can be used to modify the statistical gauge field $A(x,y)$ (2.18) while preserving particle statistics. Explicitly, the Hamiltonian 2.17 is unitarily equivalent to

$$\begin{aligned} H &= \sum_{x \sim y} (\tilde{a}_x^* e^{\mathrm{i}\Lambda(x)} e^{-\mathrm{i}A(x,y)} e^{-\mathrm{i}\Lambda(y)} \tilde{a}_y + \tilde{a}_y^* e^{\mathrm{i}\Lambda(y)} e^{\mathrm{i}A(x,y)} e^{-\mathrm{i}\Lambda(x)} \tilde{a}_x) \\ &= \sum_{x \sim y} (\tilde{a}_x^* e^{-\mathrm{i}\tilde{A}(x,y)} \tilde{a}_y + \tilde{a}_y^* e^{\mathrm{i}\tilde{A}(x,y)} \tilde{a}_x) \end{aligned} \tag{2.20}$$

with

$$\begin{aligned} \tilde{A}(x,y) &= \pi \sum_{z \neq x,y} \Big[(\varphi(y,z) - \varphi(x,z)) + (\lambda(y,z) - \lambda(x,z)) \Big] a_z^* a_z \\ &= \pi \sum_{z \neq x,y} \Big[(\varphi(y,z) - \varphi(x,z)) + (\lambda(y,z) - \lambda(x,z)) \Big] \tilde{a}_z^* \tilde{a}_z \end{aligned} \tag{2.21}$$

In principle one would like to use such *statistical gauge transformations* to get rid of the statistical gauge field in 2.17 altogether. This could be achieved by letting $\lambda(w,z) = -\varphi(w,z)$ for all $w \neq z$. But by condition 2.19 one would have

$$-\varphi(w,z) = -\varphi(z,w) \quad \text{mod } 2 \quad \forall w \neq z$$

in contradiction to 2.15,

$$\varphi(w,z) - \varphi(z,w) = 1 \quad \text{mod } 2 \quad \forall w \neq z.$$

In the following section it will be shown that the occurrence of statistical gauge fields in Jordan Wigner transformations has to do with the structure of the underlying graph $\mathbb{G}$.

2.2.1 Examples and the existence of special Jordan Wigner transformations

It is an interesting question whether there are Jordan Wigner transformations on a given graph $\mathbb{G}$ which map fermions to hard-core bosons without introducing a statistical gauge field A. Such fields correspond to higher order non-local interactions,

$$
\begin{aligned}
a_x^* e^{-\mathrm{i}A(x,y)} a_y &= a_x^* \left(\prod_{z \neq x,y} e^{-\mathrm{i}\pi(\varphi(y,z)-\varphi(x,z)) a_z^* a_z} \right) a_y \\
&= a_x^* \prod_{z \neq x,y} \left(\mathbb{1} + (e^{-\mathrm{i}\pi(\varphi(y,z)-\varphi(x,z))} - 1) a_z^* a_z \right) a_y
\end{aligned}
$$

since $e^{\mathrm{i}\alpha a_z^* a_z} = e^{\mathrm{i}\alpha} a_z^* a_z + (\mathbb{1} - a_z^* a_z)$, $\alpha \in \mathbb{R}$.

Equation 2.18 shows that $A(x,y) = 0$ if and only if

$$
\varphi(x,z) = \varphi(y,z) \mod 2 \quad \forall x \sim y, z \neq x, y.
$$

Together with the statistics condition 2.15, this yields the following system for the phases $\varphi(x,y)$, $x \neq y \in \mathbb{G}$:

$$
\begin{aligned}
\mathrm{Adj}(\mathbb{G})(x,y)(\varphi(x,z) - \varphi(y,z)) &= 0 \mod 2 \quad \forall z \neq x, y \\
\varphi(x,y) - \varphi(y,x) &= 1 \mod 2,
\end{aligned}
\tag{2.22}
$$

with the adjacency matrix $\mathrm{Adj}(\mathbb{G})$ of $\mathbb{G}$. Letting e denote the number of edges in $\mathbb{G}$ and v the number of vertices, there are in total $v(v-1)$ variables $\varphi(x,y)$ and $\frac{v(v-1)}{2}$ equations defining the particle statistics together with $e(v-2)$ equations for the non-occurrence of a statistical gauge field. For the graph to be minimally connected, one has $e \geq v-1$, thus the system will in general be overdetermined for $v > 4$, since

$$
\frac{v(v-1)}{2} - e(v-2) \leq \frac{v(v-1)}{2} - (v-1)(v-2) < 0 \quad \text{if } v > 4.
$$

Definition 2.2 (Special Jordan Wigner transformations). A transformation of the above type satisfying the conditions 2.22 is henceforth called special Jordan Wigner transformation in this thesis.

Example 2.3 (Simple Paths). Let $\mathbb{G} = \mathscr{P}_N$ be a simple path of length N with vertices $v_0, \dots, v_N$. Then, as in the one dimensional Jordan Wigner transformation,

$$
\varphi(x,z) = \Theta(x,z) := \begin{cases} 1, & \text{if } z < x \\ 0, & \text{if } z \geq x \end{cases}
$$

is a solution to 2.22. This can be seen by noting that

$$
\Theta(z,x) = \left\{ \begin{matrix} 1, & \text{if } x < z \\ 0, & \text{if } x \geq z \end{matrix} \right\} = \Theta(x,z) + 1 - \delta_{xz} \mod 2 \tag{2.23}
$$

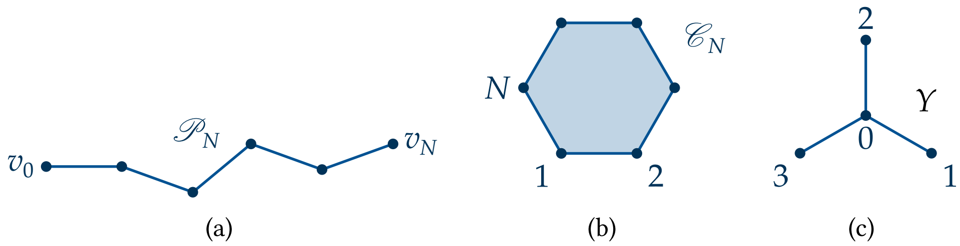


Figure 2.2: The graphs considered in examples 2–4: (a) Simple path $\mathscr{P}_N$ (b) Cycle graph $\mathscr{C}_N$ (c) Y-graph.

so $\Theta(z,x) - \Theta(x,z) = 1 - \delta_{xz} = 1 \mod 2$ for $x \neq z$. Also, if $x \sim y$, one has

$$\Theta(x,z) - \Theta(y,z) = \begin{Bmatrix} -\delta_{zy}, & \text{if } x \prec y \\ \delta_{zx}, & \text{if } y \prec x \end{Bmatrix} = 0 \quad \text{for } z \neq x, y.$$

Hence there exists a special Jordan Wigner transformation on simple paths $\mathscr{P}_N$, which is of course not surprising, since this is the exact same case as in one dimension. ♦

Example 2.4 (Cycle Graph). Let $\mathbb{G}$ be a cycle graph $\mathscr{C}_N$, $N \geq 3$, then the system of equations 2.22 determining the phases $\varphi(x,y)$ is given by

$$\begin{aligned} \varphi(1,z) &= \varphi(2,z) \mod 2 \quad \forall z \geq 3 \\ \varphi(2,z) &= \varphi(3,z) \mod 2 \quad \forall z \geq 4, z \leq 1 \\ &\vdots \\ \varphi(N-1,z) &= \varphi(N,z) \mod 2 \quad \forall z \leq N-2 \\ \varphi(N,z) &= \varphi(1,z) \mod 2 \quad \forall 2 \leq z \leq N-1 \end{aligned}$$

together with the statistics conditions

$$\varphi(x,y) = \varphi(y,x) + 1 \mod 2 \quad \forall x \neq y \in \{1, \dots, N\}$$

In particular, one has

$$\begin{aligned} \varphi(1,3) &= \varphi(2,3) = \varphi(3,2) + 1 = \cdots = \varphi(N,2) + 1 \\ &= \varphi(1,2) + 1 = \varphi(2,1) = \varphi(3,1) = \varphi(1,3) + 1 \end{aligned}$$

meaning that one can construct the contradiction $0 = 1$ out of part of the above conditions, so there exists no special Jordan Wigner transformation on $\mathscr{C}_N$ for any $N \geq 3$. ♦

Example 2.5 (Y-graph)**.** Let $\mathbb{G}$ be a Y-graph (sometimes also called 3-legged star graph or claw), i.e. the graph with vertex set $V = \{0,1,2,3\}$ and edge set $E = \{\{0,j\} : j = 1,2,3\}$. The system of equations 2.22 reads (everything mod 2)

$$\begin{aligned} &\varphi(0,2) = \varphi(1,2) \qquad &&\varphi(0,1) = \varphi(2,1) \qquad &&\varphi(0,1) = \varphi(3,1) \\ &\varphi(0,3) = \varphi(1,3) \qquad &&\varphi(0,3) = \varphi(2,3) \qquad &&\varphi(0,2) = \varphi(3,2) \end{aligned}$$

$$\varphi(x,y) = \varphi(y,x) + 1 \quad \forall x \neq y$$

But then

$$\begin{aligned} \varphi(0,2) &= \varphi(1,2) = \varphi(2,1) + 1 = \varphi(0,1) + 1 = \varphi(3,1) + 1 \\ &= \varphi(1,3) = \varphi(0,3) = \varphi(2,3) = \varphi(3,2) + 1 = \varphi(0,2) + 1 \end{aligned}$$

So $0 = 1$, contradiction.

This shows that there cannot be a special Jordan Wigner transformation on the Y-graph. ♦

Theorem 2.6. *Let $\mathbb{G} = (V,E)$ be a finite, connected graph. Then there exists a special Jordan Wigner transformation in the above sense if and only if the graph is a simple path, $\mathbb{G} = \mathscr{P}_{|V|-1}$.*

Proof. If $\mathbb{G} = \mathscr{P}_{|V|-1}$, then the special Jordan Wigner transformation is given in example 2.3.

For the converse assume that $\mathbb{G}$ is not a simple path. Then there are two cases:

(i) $\mathbb{G}$ is a cycle graph $\mathscr{C}_{|V|}$ (ii) $\mathbb{G}$ contains a Y-graph

Assume $\mathbb{G}$ is a cycle graph. It was proved in example 2.4 that the system 2.22 has no solution, so there exists no special Jordan Wigner transformation.

Therefore assume that $\mathbb{G} \neq \mathscr{P}_{|V|+1}, \mathscr{C}_{|V|}$. In this case it contains a Y-subgraph $\mathbb{G}_Y = (\{v_0,v_1,v_2,v_3\}, \{\{v_0,v_j\} : j = 1,2,3\})$.

Assuming further that the system of equations 2.22 admits a solution on $\mathbb{G}$, the equations in particular have to be true for $\mathbb{G}_Y$. But as shown in example 2.5 these equations are enough to produce the contradiction

$$\begin{aligned} \varphi(v_0,v_2) &= \varphi(v_1,v_2) = \varphi(v_2,v_1) + 1 = \varphi(v_0,v_1) + 1 = \varphi(v_3,v_1) + 1 \\ &= \varphi(v_1,v_3) = \varphi(v_0,v_3) = \varphi(v_2,v_3) = \varphi(v_3,v_2) + 1 = \varphi(v_0,v_2) + 1, \end{aligned}$$

i.e. $0 = 1$.

Hence the system 2.22 cannot have solutions on $\mathbb{G}$. ■

2.2.2 The two-dimensional Jordan Wigner transformation

Let $\Lambda = [-L, L]^2 \cap \mathbb{Z}^2$ be a square in $\mathbb{Z}^2$ centred at 0, and $\{c_x, c_x^* : x \in \Lambda\}$ be a family of fermionic annihilation/creation operators on Λ. The canonical basis vectors are denoted by e_1 and e_2.

As an immediate consequence of theorem 2.6 there cannot exist a special Jordan Wigner transformation on the two-dimensional lattice Λ. Nevertheless, there do exist Jordan Wigner transformations leading to statistical gauge fields. They are solutions to the condition 2.15 (for $\vartheta = 1$),

$$\varphi(x, y) - \varphi(y, x) = 1 \mod 2 \quad \forall x \neq y, \tag{2.24}$$

and shall be discussed in the following two subsections.

The FRADKIN-WANG solution

Let $\operatorname{Arg}(z) = \operatorname{Im} \operatorname{Log} z \in [-\pi, \pi)$ be the relative angle between z (identifying $z = z_1 + \mathrm{i}z_2$ with $z = (z_1, z_2) \in \Lambda$) and an arbitrary reference axis. Then by the property

$$\operatorname{Arg}(z) = \operatorname{Arg}(-z) + \pi \mod 2\pi$$

one can see that $\varphi(x, z) = \frac{1}{\pi}\operatorname{Arg}(z - x)$ satisfies condition 2.24. The resulting Jordan Wigner transformation goes back to FRADKIN and WANG [Fra89, Wan92].

Under this transformation a Hamiltonian of hard core bosons with nearest-neighbour hopping

$$H = \sum_{x\in\Lambda} \sum_{j=1,2} \left(b_x^* b_{x+e_j} + b_{x+e_j}^* b_x \right)$$

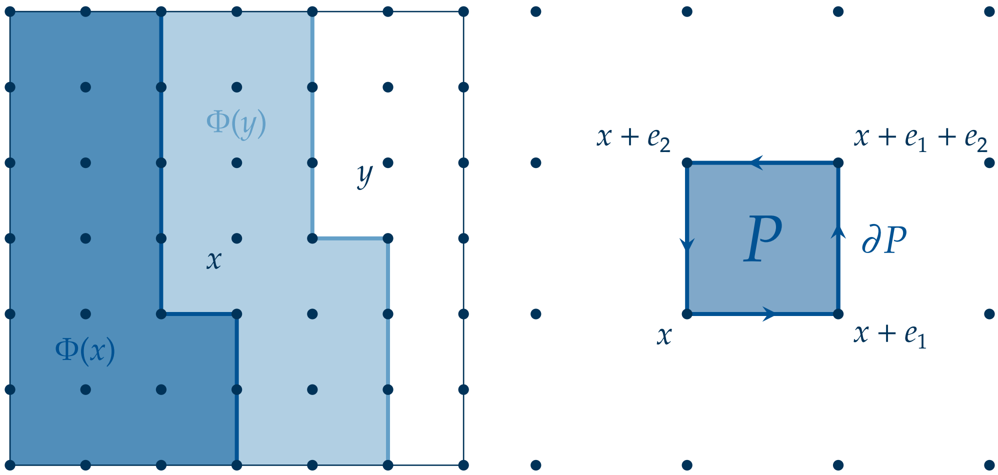
transforms to the Hamiltonian

$$H = \sum_{x\in\Lambda} \sum_{j=1,2} \left(c_x^* e^{\mathrm{i}A(x,x+e_j)} c_{x+e_j} + c_{x+e_j}^* e^{-\mathrm{i}A(x,x+e_j)} c_x \right)$$

of fermions with nearest-neighbour hopping coupled to a statistical gauge field $A(x, y)$ defined on the edges (x, y) of the lattice, with

$$A(x, x + e_j) = \pi \sum_{z\neq x, x+e_j} \left[\varphi(x + e_j, z) - \varphi(x, z) \right] c_z^* c_z.$$

The statistical gauge field A generates a flux through elementary plaquettes P of the lattice Λ. Let P be such an elementary plaquette with corners $x, x+e_1, x+$



(a) (b)

Figure 2.3: Jordan Wigner transformation in two-dimensional lattices: (a) The lattice sites contributing to the phases in the AZZOUZ solution (b) Elementary plaquette in Λ.

$e_1 + e_2$, and $x + e_2$ (see figure 2.3). Then the statistical flux through P is given by $B_P = \sum_{\ell \in \partial P} A(\ell)$, explicitly

$$\begin{aligned}
B_P &= A(x, x+e_1) + A(x+e_1, x+e_1+e_2) - A(x+e_2, x+e_1+e_2) - A(x, x+e_2) \\
&= \pi \left[\varphi(x+e_2, x) - \varphi(x+e_1, x)\right] c_x^* c_x \\
&\quad + \pi \left[\varphi(x, x+e_1) - \varphi(x+e_1+e_2, x+e_1)\right] c_{x+e_1}^* c_{x+e_1} \\
&\quad + \pi \left[\varphi(x+e_1, x+e_1+e_2) - \varphi(x+e_2, x+e_1+e_2)\right] c_{x+e_1+e_2}^* c_{x+e+1+e_2} \\
&\quad + \pi \left[\varphi(x+e_1+e_2, x+e_2) - \varphi(x, x+e_2)\right] c_{x+e_2}^* c_{x+e_2} \\
&= \tfrac{\pi}{2} c_x^* c_x + \tfrac{\pi}{2} c_{x+e_1}^* c_{x+e_1} + \tfrac{\pi}{2} c_{x+e_1+e_2}^* c_{x+e_1+e_2} - \tfrac{3\pi}{2} c_{x+e_2}^* c_{x+e_2}
\end{aligned}$$

The AZZOUZ solution

In his Ph.D. thesis M. AZZOUZ proposed another solution to the two dimensional Jordan Wigner transformation[3] [Azz93]. It uses a more natural generalisation of the one-dimensional solution, with a phase function taking only the values 0 or

[3]Apparently unaware of this result, SHAOFENG published the same solution a few years later [Sha95]. His conclusions were wrong in several points, though [BA01].

1,

$$\varphi(x,z) = \Theta(x_1 - z_1)(1 - \delta_{x_1 z_1}) + \Theta(x_2 - z_2)\delta_{x_1 z_1}. \tag{2.25}$$

Here, $x = (x_1, x_2), z = (z_1, z_2)$ and Θ is the step function

$$\Theta(x_j - z_j) = \begin{cases} 1 & \text{if} \quad -L \leq z_j < x_j \\ 0 & \text{if} \quad x_j \leq z_j \leq L \end{cases}.$$

To check that condition 2.24 is indeed satisfied one only needs to recall the property 2.23 of the step function to see that

$$\begin{aligned}
\varphi(z,x) &= (\Theta(x_1 - z_1) + 1 - \delta_{x_1 z_1})(1 - \delta_{x_1 z_1}) + (\Theta(x_2 - z_2) + 1 - \delta_{x_2 z_2})\delta_{x_1 z_1} \\
&= \Theta(x_1 - z_1)(1 - \delta_{x_1 z_1}) + (1 - \delta_{x_1 z_1})^2 + \Theta(x_2 - z_2)\delta_{x_1 z_1} + (1 - \delta_{x_2 z_2})\delta_{x_1 z_1} \\
&= \varphi(x,z) + (1 - \delta_{x_1 z_1}) + (1 - \delta_{x_2 z_2})\delta_{x_1 z_1} \\
&= \varphi(x,z) + 1 - \delta_{x_1 z_1}\delta_{x_2 z_2} = \varphi(x,z) + 1 \quad \text{if } x \neq z
\end{aligned}$$

Then one has

$$\Phi(x) = \pi \sum_{z \neq x} \varphi(x,z) c_z^* c_z = \pi \left(\sum_{z_1=-L}^{x_1-1} \sum_{z_2=-L}^{L} c^*_{(z_1,z_2)} c_{(z_1,z_2)} + \sum_{z_2=-L}^{x_2-1} c^*_{(x_1,z_2)} c_{(x_1,z_2)} \right)$$

which results in a statistical gauge field of the form

$$\begin{aligned}
A(x, x + e_1) &= \pi \left(\sum_{z_2=x_2+1}^{L} c^*_{(x_1,z_2)} c_{(x_1,z_2)} + \sum_{z_2=-L}^{x_2-1} c^*_{(x_1+1,z_2)} c_{(x_1+1,z_2)} \right) \\
A(x, x + e_2) &= 0
\end{aligned}$$

in the transformed fermionic Hamiltonian, and thus the statistical flux

$$B_P = \pi \left(c^*_{x+e_2} c_{x+e_2} - c^*_{x+e_1} c_{x+e_1} \right).$$

It is interesting to note that even though the two Jordan Wigner transformations produce a different flux B_P, the two transformed Hamiltonians are still unitarily equivalent and the corresponding statistical gauge transformation (in the sense of 2.19) is given by

$$\lambda(x,y) = -\tfrac{1}{\pi}\mathrm{Arg}(z - x) + \Theta(x_1 - z_1)(1 - \delta_{x_1 z_1}) + \Theta(x_2 - z_2)\delta_{x_1 z_1}$$

Remark. AZZOUZ's solution 2.25 has an immediate generalisation to higher dimensional lattices, for instance in three dimensions, $\Lambda = [-L, L]^3 \cap \mathbb{Z}^3$, the function $\varphi(x,z)$ would read [BA01, Koc95, HZ93]

$$\begin{aligned}
\varphi(x,z) = \Theta(x_3 - z_3)(1 - \delta_{x_3 z_3}) &+ \Theta(x_1 - z_1)(1 - \delta_{x_1 z_1})\delta_{x_3 z_3} \\
&+ \Theta(x_2 - z_2)(1 - \delta_{x_2 z_2})\delta_{x_1 z_1}\delta_{x_3 z_3}
\end{aligned}$$

♦

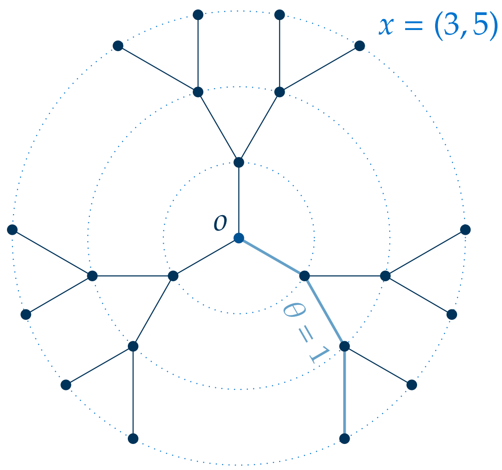


Figure 2.4: Polar coordinates on the Cayley tree.

2.2.3 A Jordan Wigner transformation for tree graphs

Having discussed the Jordan Wigner transformation for two-dimensional lattices, it is straightforward to construct such a transformation for Cayley trees $B_z(N)$ by representing each vertex in terms of suitable *polar coordinates* and using a modification of the AZZOUZ solution. Invoking theorem 2.6, also in this case there will be a statistical gauge field in the transformed Hamiltonian.

To this aim one assigns each vertex x of the tree the coordinates $x = (r_x, \theta_x)$, where $r_x = \operatorname{dist}(x, o) \in \{0, \dots, N\}$ denotes the shell in which x is situated and $\theta_x \in \{1, \dots, \mathcal{N}_{r_x}\}$ is the "angle" measured from some reference path (see figure 2.4). Then, given a family of hard-core bosonic operators b on $\mathbb{B}_z(N)$, it is possible to define Jordan-Wigner fermions via $c_x = e^{\mathrm{i}\Phi(x)} b_x$ with

$$\Phi(x) = \pi \sum_{z \neq x} \varphi(x, z) b_z^* b_z \tag{2.26}$$

$$= \pi \left(b_o^* b_o + \sum_{r_z=1}^{r_x - 1} \sum_{\theta_z = 1}^{\mathcal{N}_{r_x}} b^*_{(r_z,\theta_z)} b_{(r_z,\theta_z)} + \sum_{\theta_z = \theta_{x+1}}^{\mathcal{N}_{r_x}} b^*_{(r_x,\theta_z)} b_{(r_x,\theta_z)} \right), \tag{2.27}$$

i.e. $\varphi(x, z) = \Theta(r_x - r_z)(1 - \delta_{r_x r_z}) + \Theta(\theta_x - \theta_z)\delta_{r_x r_z}$. The verification of condition 2.24 is done by a calculation analogous to the one in the two-dimensional case.

The Hamiltonian of nearest neighbour hopping hard-core bosons is then unitarily equivalent to

$$H = \sum_{x \sim y} \left(c_x^* e^{\mathrm{i}A(x,y)} c_y + c_y^* e^{-\mathrm{i}A(x,y)} c_x \right)$$

with statistical gauge field (assuming $x \sim y$, $x \prec y$)

$$A(x,y) = \pi \sum_{\theta_z=1}^{\theta_x-1} a^*_{(r_x,\theta_z)} a_{(r_x,\theta_z)} + \pi \sum_{\theta_z=\theta_y+1}^{\mathcal{N}_{r_y}} a^*_{(r_y,\theta_z)} a_{(r_y,\theta_z)}.$$

2.3 The Wosiek-Szczerba approach

An alternative approach to a generalisation of the one-dimensional Jordan Wigner transformation[4] is based upon the observation that the Pauli matrices σ^1 and σ^2 are generators of a 2×2 matrix representation of the Clifford algebra $\mathfrak{Cl}(2)$, since $\{\sigma^\alpha, \sigma^\beta\} = 2\delta_{\alpha\beta}\mathbb{1}$.

Clifford algebras in a nutshell

Let $d \in \mathbb{N}$. The Clifford algebra $\mathfrak{Cl}(2d)$ is the algebra generated by $2d$ elements satisfying the Clifford algebra relation

$$\{\Gamma^a, \Gamma^b\} = \Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\delta_{ab}\mathbb{1}, \quad a, b \in \{1, \dots, 2d\} \tag{2.28}$$

Thus the Clifford algebra consists of

$$\begin{aligned}
&\mathbb{1}, \\
&\Gamma^a, \quad a = 1, \dots, 2d \\
&\Gamma^a\Gamma^b, \quad 1 \le a < b \le 2d \\
&\Gamma^a\Gamma^b\Gamma^c, \quad 1 \le a < b < c \le 2d \\
&\vdots \\
&\Gamma^1 \cdots \Gamma^{2d}
\end{aligned}$$

The special element $\Gamma^{2d+1} = (-\mathrm{i})^d \Gamma^1 \cdots \Gamma^{2d}$ with the property $(\Gamma^{2d+1})^2 = \mathbb{1}$ anticommutes with all the other generators of the Clifford algebra, $\{\Gamma^a, \Gamma^{2d+1}\} = 0$ for all $a = 1, \dots, 2d$.

One can show that the antisymmetric products of two Γ's,

$$\Gamma^{ab} = \frac{1}{2\mathrm{i}}[\Gamma^a, \Gamma^b] = -\mathrm{i}\Gamma^a\Gamma^b, \quad 1 \le a < b \le 2d \tag{2.29}$$

generate a representation of SO($2d$), and for $1 \le a < b \le 2d+1$ a representation of SO($2d+1$) [Geo99].

[4]Nambu [Nam50] used a similar argument in the one dimensional case, which is why Wosiek and Szczerba [Wos82, Szc85] refer to this method as the "Nambu trick".

The Clifford algebra $\mathfrak{Cl}(2d)$ can be represented by the algebra $\mathfrak{M}_{2^d}(\mathbb{C})$ of $2^d \times 2^d$ matrices, and generators satisfying the above relation for their anticommutator. For $d = 1$ these are usually represented by the two Pauli matrices σ^1 and σ^2 (in this case, the element Γ^3 is represented by σ^3, since $\sigma^3 = -\mathrm{i}\sigma^1\sigma^2$), whereas in the case $d = 2$ a representation of the $\mathfrak{Cl}(4)$ generators is given by the Dirac Gamma matrices $\Gamma^1, \dots, \Gamma^4$.

For later use, some elementary properties of the elements Γ^{ab} are stated here:

$$[\Gamma^a, \Gamma^{bc}] = 0 \text{ for } a \neq b, c \tag{2.30}$$

$$\{\Gamma^a, \Gamma^{ab}\} = 0 \text{ for } a \neq b \tag{2.31}$$

$$[\Gamma^{ab}, \Gamma^{cd}] = 0 \text{ for } (a, b) \neq (c, d) \tag{2.32}$$

$$\{\Gamma^{ab}, \Gamma^{ac}\} = 0 \text{ for } b \neq c \text{ and } a \neq b, c \tag{2.33}$$

2.3.1 Link operators and their algebra

Let H be the Hamiltonian of free fermions, described by fermionic creation and annihilation operators c_x, c_x^* on a finite, symmetric digraph[5] $\mathbb{L} = (\Lambda, E)$,

$$H = 2\mu \sum_{xy\in E} \left(c_x^* c_y + c_y^* c_x\right) + \sum_{x\in\Lambda} \nu_x (2c_x^* c_x - \mathbb{1}). \tag{2.34}$$

The parameter $\mu \in \mathbb{R}$ models the hopping strength to the nearest neighbour lattice site, and $\{\nu_x\} \subset \mathbb{R}$ describe an on-site external potential in which the fermions move. It will later be assumed that the external potential is given by i.i.d. random variables (see chapter 3.2).

The fermionic operators can be expressed in terms of two self-adjoint *Majorana operators* for each site,

$$\xi_x = c_x^* + c_x \quad \text{and} \quad \eta_x = -\mathrm{i}(c_x - c_x^*), \tag{2.35}$$

with the properties

$$\begin{aligned} \zeta_x^* &= \zeta_x \\ \{\zeta_x, \varsigma_y\} &= 2\delta_{xy}\delta_{\zeta\varsigma} \quad \text{with} \quad \zeta, \varsigma \in \{\xi, \eta\} \end{aligned} \tag{2.36}$$

that is, the operators generate a representation of $\mathfrak{Cl}(2|\Lambda|)$. In particular, $\zeta_x^2 = \mathbb{1}$ for all $x \in \Lambda$ and $\zeta = \xi, \eta$.

Then, substituting $c_x^* = \frac{1}{2}(\xi_x - \mathrm{i}\eta_x)$ and $c_x = \frac{1}{2}(\xi_x + \mathrm{i}\eta_x)$ in the Hamiltonian H yields

$$H = \mu \sum_{xy\in E} \left(\mathrm{i}\xi_x\eta_y - \mathrm{i}\eta_x\xi_y\right) + \sum_{x\in\Lambda} \nu_x \mathrm{i}\xi_x\eta_x$$

[5] That is, a directed graph $\mathbb{L} = (\Lambda, E)$ with the property that if $(x, y) \in E$, then also $(y, x) \in E$

ξ

η

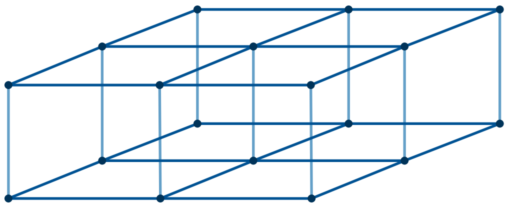


Figure 2.5: Example of the double graph for a two-dimensional square lattice

To allow a more compact description of the products of two Majorana fermions on neighbouring vertices and an easier framework for the proof of his theorem, SZCZERBA used the notion of a double lattice (see figure 2.5) in his paper [Szc85].

Definition 2.7. The (directed) double lattice/graph $\widetilde{\mathbb{L}}$ is the directed graph with vertex set $\widetilde{\Lambda} = \Lambda \times \{\xi, \eta\}$ and edge set $\tilde{E}$, where $(x_\zeta, y_\varsigma) = ((x, \zeta), (y, \varsigma)) \in \tilde{E}$ if one of the following holds: (i) $xy \in E$, or (ii) $x = y$ and $\zeta \neq \varsigma$.

Operators defined on the edges of the (double) graph are usually called *link operators*. Given a family of link operators $\{S(\ell) : \ell \in \tilde{E}\}$, and a path $\gamma = \ell_1 \circ \cdots \circ \ell_n$ of length $n \in \mathbb{N}$, one defines the associated *path operator* as

$$S(\gamma) = (-\mathrm{i})^{n-1} S(\ell_1) \cdots S(\ell_n)$$

(for a degenerate path consisting of only one vertex v one sets $S(v) = \mathrm{i}\mathbb{1}$).

Definition 2.8 (Link algebra). Let $\{S(\ell) : \ell \in \tilde{E}\}$ be a family of operators defined on the edges of the double lattice (*link operators*). They satisfy the *link algebra* if they have the following properties:

(i) $S(\ell)^* = S(\ell)$, $(S(\ell))^2 = \mathbb{1}$ for all $\ell \in \tilde{E}$

(ii) $\{S(\ell), S(\ell')\} = 0$ if the edges $\ell, \ell' \in \tilde{E}$ have one common vertex, and $[S(\ell), S(\ell')] = 0$ otherwise

(iii) $\mathrm{Tr}\left(\prod_{x \in \Lambda} S(x_\xi, x_\eta)\right) = 0$

(iv) If γ is closed, then $S(\gamma) = \mathrm{i}\mathbb{1}$.

Example 2.9. The operators on $\widetilde{\mathbb{L}}$ defined by $S(x_\zeta, y_\varsigma) = \mathrm{i}\zeta_x \varsigma_y$, where $\zeta, \varsigma \in \{\xi, \eta\}$ and ξ_x, η_x being a family of Majorana fermions on Λ as above, satisfy the link algebra.

The Hamiltonian H, expressed in terms of these link operators, is given by

$$H = \mu \sum_{xy \in E} \Big(S(x_\xi, y_\eta) - S(x_\eta, y_\xi) \Big) + \sum_{x \in \Lambda} \nu_x S(x_\xi, x_\eta).$$

The first two defining properties (i)-(ii) follow immediately from the Majorana algebra 2.36. To see that (iv) holds, let $\gamma = \ell_1 \circ \cdots \circ \ell_N$ be a closed path of length N in $\widetilde{\mathbb{L}}$, $\ell_i = ((x_i, \zeta^i), (x_{i+1}, \zeta^{i+1}))$, $i = 1, \dots, N-1$, $\ell_N = ((x_N, \zeta^N), (x_1, \zeta^1))$. Then

$$\begin{aligned} S(\gamma) &= (-\mathrm{i})^{N-1} (\mathrm{i}\zeta^1_{x_1} \zeta^2_{x_2})(\mathrm{i}\zeta^2_{x_2} \zeta^3_{x_4}) \cdots (\mathrm{i}\zeta^{N-1}_{x_{N-1}} \zeta^N_{x_N})(\mathrm{i}\zeta^N_{x_N} \zeta^1_{x_1}) \\ &= (-\mathrm{i})^{N-1} \mathrm{i}^N \zeta^1_{x_1} (\zeta^2_{x_2})^2 \cdots (\zeta^N_{x_N})^2 \zeta^1_{x_1} = \mathrm{i}\mathbb{1} \end{aligned}$$

Finally, one has

$$\mathrm{Tr} \prod_{x \in \Lambda} S(x_\xi, x_\eta) = \mathrm{Tr} \prod_{x \in \Lambda} (\mathrm{i}\xi_x \eta_x) = \mathrm{Tr} \prod_{x \in \Lambda} (2c^*_x c_x - \mathbb{1}) = \prod_{x \in \Lambda} \mathrm{Tr}_{\mathfrak{H}_x} (2c^*_x c_x - \mathbb{1}) = 0$$

since the operators $\{2c^*_x c_x - \mathbb{1}\}_{x \in \Lambda}$ are mutually commuting with eigenvalues ± 1 of the same multiplicity. ♦

SZCZERBA now proved in [Szc85] that the properties of the link operators (definition 2.8) determine the link algebra uniquely up to unitary isomorphisms. This result can be used to prove an implicit version of the Jordan Wigner transformation by finding a family of operators satisfying the link algebra. The main result is a representation of the link algebra in terms of higher-dimensional gamma matrices with appropriate constraints that reduce the dimensionality to the required two per lattice site describing fermionic degrees of freedom. The precise statement of the theorem and its proof are presented in the following. The next chapter will then deal with an application of this method to a square lattice.

Theorem 2.10 (Szczerba [Szc85]). *Let $\mathbb{L} = (\Lambda, E)$ be a directed graph and $\widetilde{\mathbb{L}} = (\tilde{\Lambda}, \tilde{E})$ its associated double graph. Then, given a set $\{S'(\ell) : \ell \in \tilde{E}\}$ of link operators on a finite-dimensional Hilbert space $\mathfrak{H}$ satisfying the link algebra, there exists a family of Majorana operators $\{\xi_x, \eta_x : x \in \Lambda\}$ on $\mathfrak{H}$ obeying the relations 2.36, such that*

$$S'(\ell) = \mathrm{i}\zeta_x \varsigma_y \quad , \ell = (x_\zeta, y_\varsigma) \in \tilde{E}, \quad \zeta, \varsigma \in \{\xi, \eta\} \tag{2.37}$$

If $\{S(\ell) : \ell \in \tilde{E}\}$ denotes the link operators from example 2.9, there is a unitary transformation U with

$$U S(\ell) U^* = S'(\ell) \quad \textit{for all } \ell \in \tilde{E} \tag{2.38}$$

The proof of theorem 2.10 proceeds in two steps by reduction of the problem to a rooted spanning tree of $\mathbb{L}$, where the natural ordering on rooted trees can be used to construct the required operators ξ and η, and later extended to the whole graph.

Lemma 2.11. Let $\mathbb{T} = (\tilde{\Lambda}, \tilde{E}_{\mathbb{T}})$ be a directed tree on $\tilde{\Lambda}$ with an arbitrary vertex fixed as root $\tilde{o} = o_\xi$, and let $\{S'(\ell) : \ell \in \tilde{E}_{\mathbb{T}}\}$ be a family of link operators on a finite-dimensional Hilbert space $\mathfrak{H}$ with properties (i)-(ii) and (iii) replaced by the condition

$$\mathrm{Tr}\Big(S'(\gamma_{o_\eta}) \prod_{x\neq o} S'(\gamma_{x_\xi}) S'(\gamma_{x_\eta})\Big) = 0, \tag{2.39}$$

where γ_{x_ζ} denotes the unique path from the root of $\widetilde{\mathbb{T}}$ to the vertex x_ζ (property (iv) holds automatically, since a tree does not contain any cycles). Then there exists a family of operators $\{\xi_x, \eta_x : x \in \Lambda\}$ obeying the Clifford relations 2.36, such that

$$S'(\ell) = \mathrm{i}\zeta_x \varsigma_y \quad , \ell = (x_\zeta, y_\varsigma) \in \tilde{E}_{\mathbb{T}}, \quad \zeta, \varsigma \in \{\xi, \eta\} \tag{2.40}$$

Proof [Szc85]. Consider the family of *path operators* $\{S'(\gamma_{\tilde{x}}) : \tilde{x} \neq \tilde{o} \in \tilde{E}_{\mathbb{T}}\}$. They have the properties

$$\begin{aligned} S'(\gamma_{\tilde{x}})^* &= S'(\gamma_{\tilde{x}}) \\ \{S'(\gamma_{\tilde{x}}), S'(\gamma_{\tilde{y}})\} &= 2\delta_{\tilde{x}\tilde{y}} \quad \text{for } \tilde{x}, \tilde{y} \neq \tilde{o} \end{aligned} \tag{2.41}$$

A proof of this and some other useful properties of the path operators are presented in appendix A. Defining $S = \mathrm{i}^{|\Lambda|-1} S'(\gamma_{o_\eta}) \prod_{x\neq o} S'(\gamma_{x_\xi}) S'(\gamma_{x_\eta})$, by means of 2.41 one has

$$\begin{aligned} S^* &= (-\mathrm{i})^{|\Lambda|-1} \Big(\prod_{x\neq o} S'(\gamma_{x_\eta}) S'(\gamma_{x_\xi})\Big) S'(\gamma_{o_\eta}) = (-\mathrm{i})^{|\Lambda|-1} S'(\gamma_{o_\eta}) \Big(\prod_{x\neq o} S'(\gamma_{x_\eta}) S'(\gamma_{x_\xi})\Big) \\ &= (-\mathrm{i})^{|\Lambda|-1} S'(\gamma_{o_\eta}) (-1)^{|\Lambda|-1} \Big(\prod_{x\neq o} S'(\gamma_{x_\xi}) S'(\gamma_{x_\eta})\Big) = S, \end{aligned}$$

and

$$\begin{aligned} S^2 &= \mathrm{i}^{|\Lambda|-1} S'(\gamma_{o_\eta}) \Big(\prod_{x\neq o} S'(\gamma_{x_\xi}) S'(\gamma_{x_\eta})\Big) \mathrm{i}^{|\Lambda|-1} S'(\gamma_{o_\eta}) \Big(\prod_{x\neq o} S'(\gamma_{x_\xi}) S'(\gamma_{x_\eta})\Big) \\ &= (-1)^{|\Lambda|-1} S'(\gamma_{o_\eta})^2 \Big(\prod_{x\neq o} S'(\gamma_{x_\xi}) S'(\gamma_{x_\eta})\Big)^2 \\ &= (-1)^{|\Lambda|-1} \prod_{x\neq o} S'(\gamma_{x_\xi}) S'(\gamma_{x_\eta}) S'(\gamma_{x_\xi}) S'(\gamma_{x_\eta}) \\ &= (-1)^{|\Lambda|-1} (-1)^{|\Lambda|-1} \prod_{x\neq o} S'(\gamma_{x_\xi})^2 S'(\gamma_{x_\eta})^2 = \mathbb{1} \end{aligned}$$

Further, by assumption 2.39, $\mathrm{Tr} S = 0$. Therefore there exists a unitary automorphism W of $\mathfrak{H}$ that diagonalises S,

$$WSW^* = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

The corresponding eigenspaces are denoted by $\mathfrak{H}^{\pm}$. Now for each $\tilde{x} \neq \tilde{o}$ the path operators $S'(\gamma_{\tilde{x}})$ commute with S, $[S, S'(\gamma_{\tilde{x}})] = 0$, so that in the ONB where S is diagonal the path operators are block diagonal,

$$WS'(\gamma_{\tilde{x}})W^* = \begin{pmatrix} T^-_{\tilde{x}} & 0 \\ 0 & T^+_{\tilde{x}} \end{pmatrix}, \quad \tilde{x} \neq \tilde{o}.$$

By the properties 2.41 it follows immediately that for all $\tilde{x}, \tilde{y} \neq \tilde{o}$ one has

$$\begin{aligned} (T^{\#}_{\tilde{x}})^* &= T^{\#}_{\tilde{x}} \\ \{T^{\#}_{\tilde{x}}, T^{\#}_{\tilde{y}}\} &= 2\delta_{\tilde{x}\tilde{y}} \end{aligned} \quad , \quad \# \in \{+,-\}. \tag{2.42}$$

It is a basic result in the theory of operator algebras (cf. Theorem 5.2.5 in [BR02]) that the operators $T^{\#}_{\tilde{x}}$ are uniquely determined up to unitary transformations. Hence, there exists a unitary isomorphism $V : \mathfrak{H}^- \to \mathfrak{H}^+$ such that

$$VT^-_{\tilde{x}}V^* = T^+_{\tilde{x}} \quad \text{for all } \tilde{x} \neq o_\xi, o_\eta \tag{2.43}$$

Here it is important that from 2.42 it does not follow the existence of such a unitary transformation for $\tilde{x} = o_\eta$. One rather gets from

$$\begin{aligned} \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = WSW^* &= \mathrm{i}^{|\Lambda|-1} \begin{pmatrix} T^-_{o_\eta} \prod_{x \neq o} T^-_{x_\xi} T^-_{x_\eta} & 0 \\ 0 & T^+_{o_\eta} \prod_{x \neq o} T^+_{x_\xi} T^+_{x_\eta} \end{pmatrix} \\ &= \mathrm{i}^{|\Lambda|-1} \begin{pmatrix} T^-_{o_\eta} \prod_{x \neq o} T^-_{x_\xi} T^-_{x_\eta} & 0 \\ 0 & T^+_{o_\eta} V \prod_{x \neq o} T^+_{x_\xi} T^+_{x_\eta} V^* \end{pmatrix} \end{aligned}$$

the relations

$$\begin{aligned} \mathrm{i}^{|\Lambda|-1} \Big(T^-_{o_\eta} \prod_{x \neq o} T^-_{x_\xi} T^-_{x_\eta} \Big) &= -\mathbb{1} \\ \mathrm{i}^{|\Lambda|-1} \Big(T^+_{o_\eta} V \prod_{x \neq o} T^+_{x_\xi} T^+_{x_\eta} V^* \Big) &= \mathbb{1} \end{aligned}$$

and thus by $(T^{\#}_{o_\eta})^2 = \mathbb{1}$,

$$T^+_{o_\eta} = -VT^-_{o_\eta}V^*. \tag{2.44}$$

This is the key observation in the proof of this lemma, since this unitary isomorphism can now be used to construct the desired operators. Also, the above step reveals one drawback of the WOSIEK-SZCZERBA method: it is only used that such a V exists, but not how it is constructed (which will be quite difficult if the number of lattice sites $|\Lambda|$ is very high).

Now define the operator

$$\eta_o = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}, \tag{2.45}$$

with the properties

$$\begin{aligned} (\eta_o)^* &= \eta_o' \\ (\eta_o)^2 &= \mathbb{1} \\ \{\eta_o, WS'(\gamma_{o_\eta})W^*\} &= 0 \\ [\eta_o, WS'(\gamma_{\tilde{x}})W^*] &= 0 \quad \text{for all } \tilde{x} \neq o_\xi, o_\eta \end{aligned} \tag{2.46}$$

The first two equalities follow immediately from the definition of η_o, for the third equality one uses

$$\begin{aligned} \eta_o WS'(\gamma_{o_\eta})W^* &= \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix} \begin{pmatrix} T^-_{o_\eta} & 0 \\ 0 & T^+_{o_\eta} \end{pmatrix} = \begin{pmatrix} 0 & V^*T^+_{o_\eta} \\ VT^-_{o_\eta} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -T^-_{o_\eta}V^* \\ -T^+_{o_\eta}V & 0 \end{pmatrix} = \begin{pmatrix} -T^-_{o_\eta} & 0 \\ 0 & -T^+_{o_\eta} \end{pmatrix} \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix} = -WS'(\gamma_{o_\eta})W^*\eta_o \end{aligned}$$

by means of 2.44. The same calculation for $S'(\gamma_{\tilde{x}})$ ($\tilde{x} \neq o_\xi, o_\eta$) and 2.43 yields the fourth equality.

Finally, define for $x \neq o$ the operators

$$\begin{aligned} \xi_o &= -\mathrm{i}WS'(\gamma_{o_\eta})W^*\eta_o = \mathrm{i}\eta_o WS'(\gamma_{o_\eta})W^* \\ \xi_x &= -\mathrm{i}\xi_o WS'(\gamma_{x_\xi})W^* \\ \eta_x &= -\mathrm{i}\xi_o WS'(\gamma_{x_\eta})W^* \end{aligned} \tag{2.47}$$

Since $\xi_o^2 = -\mathrm{i}WS'(\gamma_{o_\eta})W^*\eta_o \mathrm{i}\eta_o WS'(\gamma_{o_\eta})W^* = \mathbb{1}$, the latter two equalities can be used to represent a path operator as $WS'(\gamma_{x_\zeta})W^* = \mathrm{i}\xi_x\zeta_x$ for $\zeta = \xi, \eta$, and, collecting all the previous properties, one gets

$$\begin{aligned} (\zeta_x)^* &= \zeta_x \\ \{\zeta_x, \varsigma_y\} &= 2\delta_{xy}\delta_{\zeta\varsigma} \quad \text{with} \quad \zeta, \varsigma \in \{\xi, \eta\} \end{aligned}$$

It remains to show that each link operators can be represented as product of two of these operators. Thereto let $\ell = (x_\zeta, y_\varsigma) \in \tilde{E}$ and assume first that

$x_\zeta = o_\xi$. Then $S'(\ell) = S'(\gamma_{y_\varsigma}) = W^* \mathrm{i}\xi_o \varsigma_y W$. For all other edges with $x_\zeta \neq o_\xi$ one has $\gamma_{x_\zeta} \circ (x_\zeta, y_\varsigma) = \gamma_{y_\varsigma}$ and therefore, by definition of the path operators, $S'(\gamma_{y_\varsigma}) = -\mathrm{i}S'(\gamma_{x_\zeta})S'(x_\zeta, y_\varsigma)$. Since $S'(\gamma_{x_\zeta})^2 = \mathbb{1}$, it follows that

$$\begin{aligned} S'(x_\zeta, y_\varsigma) &= \mathrm{i}S'(\gamma_{x_\zeta})S'(\gamma_{y_\varsigma}) = \mathrm{i}W^* \mathrm{i}\xi_o \zeta_x W\, W^* \mathrm{i}\xi_o \varsigma_y W \\ &= W^*(-\mathrm{i}\xi_o \zeta_x \xi_o \varsigma_y) W = W^* \mathrm{i}\xi_o^2 \zeta_x \varsigma_y W = W^* \mathrm{i}\zeta_x \varsigma_y W \end{aligned}$$

This concludes the proof. ■

With lemma 2.11 at hand, it is straightforward to prove the theorem for general graphs.

Proof of SZCZERBA's theorem 2.10 [Szc85]. Fix an arbitrary point $x_\xi \equiv o$, and let $\widetilde{\mathbb{T}}$ be a directed spanning tree of $\widetilde{\mathbb{L}}$ with root o. By lemma 2.11, applied to the family $\{S'(\ell) : \ell \in \tilde{E}_{\mathbb{T}}\}$ of link operators on the spanning tree, there exists a family of Majorana operators defined on *all* the vertices of $\widetilde{\mathbb{L}}$ such that $WS'(\ell)W^* = \mathrm{i}\zeta_x \varsigma_y$ for any $\ell = (x_\zeta, y_\varsigma) \in \tilde{E}_{\mathbb{T}}$. Indeed, the only property that needs to be checked in order to be able to apply this lemma is the trace equality 2.39. This follows directly from property (iii) (definition 2.8) of operators satisfying the link algebra since $S'(x_\xi, x_\eta) = \mathrm{i}S'(\gamma_{x_\xi})S'(\gamma_{x_\eta})$ as argued at the end of the proof of lemma 2.11. This reduces point (iii) of the link algebra to 2.39.

Using these Majorana fermions, one can define new link operators on the whole double lattice $\tilde{L}$ by

$$\tilde{S}(\ell) = \mathrm{i}\zeta_x \varsigma_y, \quad \ell = (x_\zeta, y_\varsigma) \in \tilde{E}.$$

These operators are unitarily equivalent to the link operators $S'(\ell)$ for $\ell \in \tilde{E}_{\mathbb{T}}$. Furthermore, for *any* edge in $\mathbb{L}$ one has

$$\begin{aligned} S'(x_\zeta, y_\varsigma) &= \mathrm{i}S'(\gamma_{x_\zeta})S'(\gamma_{y_\varsigma}) \\ \tilde{S}(x_\zeta, y_\varsigma) &= \mathrm{i}\tilde{S}(\gamma_{x_\zeta})\tilde{S}(\gamma_{y_\varsigma}) \end{aligned} \tag{2.48}$$

where γ_{x_ζ} and γ_{y_ς} denote the unique paths from the root o to x_ζ (y_ς respectively) along the directed spanning tree $\mathbb{T}$. Therefore the corresponding path operators are products of link operators to edges in $\tilde{E}_{\mathbb{T}}$ only, which implies

$$WS'(\gamma_{x_\zeta})W^* = \tilde{S}(\gamma_{x_\zeta}).$$

By means of 2.48 it then follows that

$$\begin{aligned} S'(\ell) = \mathrm{i}S'(\gamma_{x_\zeta})S'(\gamma_{y_\varsigma}) &= \mathrm{i}W^*\tilde{S}(\gamma_{x_\zeta})WW^*\tilde{S}(\gamma_{y_\varsigma})W = W^*\mathrm{i}\tilde{S}(\gamma_{x_\zeta})\tilde{S}(\gamma_{y_\varsigma})W \\ &= W^*\tilde{S}(\ell)W \end{aligned}$$

for all edges $\ell \in \tilde{E}$. This concludes the proof that the link algebra (definition 2.8) defines the link operators uniquely up to unitary transformations ($U = W^*$). ■

The reason why this result can be thought of as a generalisation of the usual Jordan Wigner transformation is explained in context of the "Gamma Matrix Model" in the following chapter.

Chapter 3

The Gamma matrix model (GMM)

In order to demonstrate the applicability of the Wosiek-Szczerba method, the quantum spin system on a square lattice $\Lambda = [1, L]^2 \cap \mathbb{Z}^2$ with periodic boundary conditions and Hamiltonian (Gamma matrix model)

$$H^\Gamma = \mu \sum_{x \in \Lambda} \left(\Gamma_x^1 \Gamma_{x+e_1}^{25} + \Gamma_x^3 \Gamma_{x+e_2}^{45} - \Gamma_x^{15} \Gamma_{x+e_1}^{2} - \Gamma_x^{35} \Gamma_{x+e_2}^{4} \right) + \sum_{x \in \Lambda} \nu_x \Gamma_x^5, \tag{3.1}$$

with nearest-neighbour interaction strength $\mu \in \mathbb{R}$ and in an external magnetic field described by $\{\nu_x\}_{x\in\Lambda} \subset \mathbb{R}$, is discussed. The Hamiltonian acts on $\mathfrak{H} = \bigotimes_{x\in\Lambda} \mathbb{C}^4$.

H^Γ can be interpreted physically as the Hamiltonian of spin-3/2 particles fixed at the sites of the lattice Λ with short-range quadrupole-octupole interactions. A similar Hamiltonian (Gamma matrix model) has been discussed recently by Yao *et al.* [YZK09] and Whitsitt *et al.* [WCF12] in the context of a spin-$\frac{3}{2}$ generalisation of the Kitaev model on a honeycomb lattice [Kit06] to lattices with connectivity greater than three. It is argued in [YZK09] that the ground state of their Hamiltonian is an algebraic spin liquid, i.e. a spin liquid whose fermionic excitations (spinons) are gapless.

If one defines the matrices ([MNZ04])

$$S^1 = \begin{pmatrix} & \frac{\sqrt{3}}{2} & & \\ \frac{\sqrt{3}}{2} & & 1 & \\ & 1 & & \frac{\sqrt{3}}{2} \\ & & \frac{\sqrt{3}}{2} & \end{pmatrix}, \quad S^2 = \begin{pmatrix} & -\mathrm{i}\frac{\sqrt{3}}{2} & & \\ \mathrm{i}\frac{\sqrt{3}}{2} & & -\mathrm{i} & \\ & \mathrm{i} & & -\mathrm{i}\frac{\sqrt{3}}{2} \\ & & \mathrm{i}\frac{\sqrt{3}}{2} & \end{pmatrix}, \quad S^3 = \begin{pmatrix} \frac{3}{2} & & & \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & -\frac{3}{2} \end{pmatrix}$$

the Gamma matrices can be represented by symmetric bilinear combinations of the SU(2) spin-$\frac{3}{2}$ matrices, namely

$$\Gamma^1 = Q^{23} = \tfrac{1}{\sqrt{3}}\{S^2, S^3\} = \sigma^3 \otimes \sigma^2 \qquad \Gamma^3 = Q^{12} = \tfrac{1}{\sqrt{3}}\{S^1, S^2\} = \sigma^2 \otimes \mathbb{1}$$
$$\Gamma^2 = Q^{13} = \tfrac{1}{\sqrt{3}}\{S^1, S^3\} = \sigma^3 \otimes \sigma^1 \qquad \Gamma^4 = Q^{(2)} = \tfrac{1}{\sqrt{3}}\left((S^1)^2 - (S^2)^2\right) = \sigma^1 \otimes \mathbb{1}$$
$$\Gamma^5 = Q^{(0)} = (S^3)^2 - \tfrac{5}{4}\mathbb{1} = \sigma^3 \otimes \sigma^3$$

Hence the Γ^a matrices may be interpreted as spin-$\frac{3}{2}$ quadrupole (or nematic) operators Q (the Γ^{a5} operators corresponding to spin octupole operators) [MNZ04, Wu06, TZY06, WCF12] or, alternatively, as two-orbit spin-$\frac{1}{2}$ operators [Wen03, WCF12].

3.1 Jordan Wigner transformation for the constrained GMM

In the discussion of the Gamma matrix model it is useful to introduce a family of (elementary) plaquette (flux) operators. For a vertex $x \in \Lambda$ the elementary plaquette $P \equiv P_x$ is defined as the square with corners $x, x+e_1, x+e_1+e_2, x+e_2$. Its boundary ∂P will always be supposed to be positively oriented. Given such a plaquette, one can define

$$\begin{aligned} W_P &= (\Gamma^1_x \Gamma^2_{x+e_1})(\Gamma^3_{x+e_1}\Gamma^4_{x+e_1+e_2})(\Gamma^2_{x+e_1+e_2}\Gamma^1_{x+e_2})(\Gamma^4_{x+e_2}\Gamma^3_x) \\ &= -\Gamma^{13}_x \Gamma^{32}_{x+e_1}\Gamma^{24}_{x+e_1+e_2}\Gamma^{41}_{x+e_2}. \end{aligned} \tag{3.2}$$

Additionally, for a *fixed* value of x_2 (respectively x_1), one can define two global (flux) operators

$$\begin{aligned} W_X \equiv W_X(x_2) &= \prod_{z_1=1}^{L} \Gamma^1_{(z_1,x_2)}\Gamma^2_{(z_1+1,x_2)} \\ W_Y \equiv W_Y(x_1) &= \prod_{z_2=1}^{L} \Gamma^3_{(x_1,z_2)}\Gamma^4_{(x_1,z_2+1)} \end{aligned} \tag{3.3}$$

The collection of the plaquette and the global (flux) operators will be denoted by $\{W_\alpha : \alpha \in \{X, Y, P : P \text{ elementary plaquette}\}\}$. They have the following important properties:

Lemma 3.1. The family $\{W_\alpha\}_{\alpha\in\{X,Y,P:P \text{ elementary plaquette}\}}$ satisfies the algebra

$$\begin{aligned} &W_\alpha^* = W_\alpha, \quad W_\alpha^2 = \mathbb{1} \\ &[W_\alpha, W_\beta] = 0, \quad [W_\alpha, H^\Gamma] = 0 \\ &\mathrm{Tr}(W_{\alpha_1} \cdots W_{\alpha_k}) = 0 \end{aligned}$$

for all α, β and $\alpha_i \neq \alpha_j$ whenever $i \neq j$.

In physics language, the operators W_α correspond to $\mathbb{Z}_2$ fluxes through the elementary plaquettes P and two global $\mathbb{Z}_2$ fluxes [Wen03, YZK09]. To make this more apparent, one may write $W_\alpha = -\exp(\mathrm{i}\Phi_\alpha)$ with Φ_α taking the values 0 and π.

Proof. The self-adjointness of the plaquette and global flux operators follows immediately from the self-adjointness of Γ^a and Γ^{ab} for $a, b = 1, \dots, 5$, the identity $W_\alpha^2 = \mathbb{1}$ from $(\Gamma^a)^2 = (\Gamma^{ab})^2 = \mathbb{1}$ for $a, b = 1, \dots, 5$.

Given two elementary plaquettes P and Q that have no vertex in common, the property $[W_P, W_Q] = 0$ is trivial. If they have one common vertex, the commutator is zero as a consequence of equation 2.32, e.g. if P and Q have their lower left respectively upper right corner x in common, then

$$\begin{aligned} W_P W_Q &= (-\Gamma^{13}_x \Gamma^{32}_{x+e_1} \Gamma^{24}_{x+e_1+e_2} \Gamma^{41}_{x+e_2})(-\Gamma^{13}_{x-e_1-e_2} \Gamma^{32}_{x-e_2} \Gamma^{24}_x \Gamma^{41}_{x-e_1}) \\ &= (-\Gamma^{13}_{x-e_1-e_2} \Gamma^{32}_{x-e_2} \Gamma^{24}_x \Gamma^{41}_{x-e_1})(-\Gamma^{13}_x \Gamma^{32}_{x+e_1} \Gamma^{24}_{x+e_1+e_2} \Gamma^{41}_{x+e_2}) = W_Q W_P. \end{aligned}$$

In case they have two vertices in common, then the commutation property follows from equation 2.33. For example, if the bottom two vertices of P, x and $x + e_1$, and the top two vertices of Q coincide, then

$$\begin{aligned} W_P W_Q &= (-\Gamma^{13}_x \Gamma^{32}_{x+e_1} \Gamma^{24}_{x+e_1+e_2} \Gamma^{41}_{x+e_2})(-\Gamma^{13}_{x-e_2} \Gamma^{32}_{x+e_1-e_2} \Gamma^{24}_{x+e_1} \Gamma^{41}_x) \\ &= (-1)^2 (-\Gamma^{13}_{x-e_2} \Gamma^{32}_{x+e_1-e_2} \Gamma^{24}_{x+e_1} \Gamma^{41}_x)(-\Gamma^{13}_x \Gamma^{32}_{x+e_1} \Gamma^{24}_{x+e_1+e_2} \Gamma^{41}_{x+e_2}) = W_Q W_P. \end{aligned}$$

As for the global flux operators, one has

$$W_X W_Y = \prod_{z_1=1}^{L} \Gamma^1_{(z_1,x_2)} \Gamma^2_{(z_1+1,x_2)} \prod_{z_2=1}^{L} \Gamma^3_{(x_1,z_2)} \Gamma^4_{(x_1,z_2+1)} = W_Y W_X$$

since the only non-trivial commutators are those between Gamma matrices at the intersection point $x = (x_1, x_2)$ of the two global loops

$$\begin{aligned} \gamma_X &= \gamma_X(x_2) = (1, x_2) \to (2, x_2) \to \cdots \to (L, x_2) \to (1, x_2) \\ \gamma_Y &= \gamma_Y(x_1) = (x_1, 1) \to (x_1, 2) \to \cdots \to (x_1, L) \to (x_1, 1), \end{aligned}$$

where the commutation relation follows from the short calculation

$$\Gamma^2_x \Gamma^1_x \Gamma^4_x \Gamma^3_x = \Gamma^4_x \Gamma^3_x \Gamma^2_x \Gamma^1_x.$$

Given W_X and a plaquette operator W_P, there are three possible situations:

(i) P and the loop γ_X have no common vertex, in which case $[W_X, W_P] = 0$.

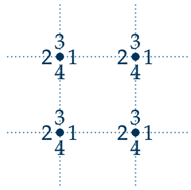


Figure 3.1: Sketch of the assignment of the Gamma matrices to the edges of the square lattice Λ.

(ii) The bottom edge of P, $(x, x+e_1)$, lies in γ_X. Then $[W_X, W_P] = 0$ follows from $[\Gamma^2_x\Gamma^1_x\Gamma^2_{x+e_1}\Gamma^1_{x+e_1}, \Gamma^{13}_x\Gamma^{32}_{x+e_1}] = -[\Gamma^{12}_x\Gamma^{12}_{x+e_1}, \Gamma^{13}_x\Gamma^{32}_{x+e_1}] = 0$, these terms being the only operators in W_X and W_P that do not commute trivially. Indeed, one has by (2.33)

$$\begin{aligned}\Gamma^{12}_x\Gamma^{12}_{x+e_1}\Gamma^{13}_x\Gamma^{32}_{x+e_1} &= \Gamma^{12}_x\Gamma^{13}_x\Gamma^{12}_{x+e_1}\Gamma^{32}_{x+e_1} = \left(-\Gamma^{13}_x\Gamma^{12}_x\right)\left(-\Gamma^{32}_{x+e_1}\Gamma^{12}_{x+e_1}\right)\\ &= \Gamma^{13}_x\Gamma^{32}_{x+e_1}\Gamma^{12}_x\Gamma^{12}_{x+e_1}\end{aligned}$$

(iii) In case the top edge of P lies in γ_X, then $[W_X, W_P] = 0$ can be shown by a similar calculation to the one in (ii).

Analogously, it follows that $[W_Y, W_P] = 0$ for all elementary plaquettes P.

In the same manner, using (2.30)-(2.33), one can prove that the flux operators commute with the Hamiltonian H^Γ, by showing that the commutators of each summand in H^Γ with W_α are zero, distinguishing the cases where they have none, one or two vertices in common.

The proof of the trace property is based on the identities $\mathrm{Tr}\Gamma^a = 0$ and $\mathrm{Tr}\Gamma^{ab} = 0$ (Γ^{ab} being antisymmetric), as well as the tensor product structure of the flux operators. Since the trace property is not essential in the remainder of this thesis, its rather lengthy proof is omitted here. ■

In the lattice Λ there are $|\Lambda| = L^2$ elementary plaquettes P, which give rise to $|\Lambda| - 1$ *independent* plaquette operators W_P, since there is the constraint

$$\prod_{P \text{ elem. plaq.}} W_P = \mathbb{1}$$

which follows from the assumption of periodic boundary conditions and the fact that $(\Gamma^a)^2 = \mathbb{1}$. As for the global flux operators, defining $W_X \equiv W_X(x_2)$ and $W_Y \equiv W_Y(x_1)$ for *fixed* x_2 and x_1 determines $W_{X/Y}(y_{2/1})$ for any other y_2 and y_1. This follows from $W_X(y_2) = W_X \prod_{P \in \mathscr{S}(x_2,y_2)} W_P$ where $\mathscr{S}(x_2, y_2)$ denotes the set of all elementary plaquettes in the strip $\{z \in \Lambda : z_2 \in [\min(x_2,y_2), \max(x_2,y_2)]\}$.

Similarly, $W_Y(y_1) = W_Y \prod_{P\in\mathscr{S}(x_1,y_1)} W_P$. Therefore, there are altogether $|\Lambda| + 1$ independent elements in $\{W_\alpha\}$.

Since they all commute with the Hamiltonian, eigenstates of H^Γ can be chosen to be eigenstates of the $\{W_\alpha\}$ as well, such that the Hilbert space $\mathfrak{H}$ splits into sectors $\mathfrak{H}_{\{w_\alpha\}}$ of fixed flux configurations, i.e. $W_\alpha \psi_{\{w_\alpha\}} = w_\alpha \psi_{\{w_\alpha\}}$ for $\psi_{\{w_\alpha\}} \in \mathfrak{H}_{\{w_\alpha\}}$ and for each α,

$$\mathfrak{H} = \bigoplus_{\{w_\alpha\}\in\{\pm1\}^{|\Lambda|+1}} \mathfrak{H}_{\{w_\alpha\}} \tag{3.4}$$

The orthogonal projections onto the respective subsectors are given by

$$\Xi_{\{w_\alpha\}} = \left(\frac{\mathbb{1} + w_x W_X}{2}\right)\left(\frac{\mathbb{1} + w_y W_Y}{2}\right) \prod_{\substack{P\subset\mathbb{L} \\ P \text{ elem. plaq.}}} \left(\frac{\mathbb{1} + w_P W_P}{2}\right)$$

The projection onto the sector with $w_\alpha = 1\ \forall\alpha$ will be of particular importance and is henceforth just denoted by $\Xi = \Xi_{\{1\}}$. In particular, one has

$$\mathfrak{H} = \Xi\mathfrak{H} \oplus (\mathbb{1} - \Xi)\mathfrak{H}.$$

In view of the previous chapter, the connection to link operators is described next. Let $\mathbb{L} = (\Lambda, E)$ be the symmetric digraph corresponding to the square lattice, and let $\widetilde{\mathbb{L}}$ be its directed double lattice. For an edge $\ell \in \tilde{E}$ one defines the family

$$\begin{aligned}
S^\Gamma(x_\xi, (x+e_1)_\xi) &= -S^\Gamma((x+e_1)_\xi, x_\xi) = \Gamma_x^1\Gamma_{x+e_1}^2 \\
S^\Gamma(x_\xi, (x+e_2)_\xi) &= -S^\Gamma((x+e_2)_\xi, x_\xi) = \Gamma_x^3\Gamma_{x+e_2}^4 \\
S^\Gamma(x_\eta, (x+e_1)_\eta) &= -S^\Gamma((x+e_1)_\eta, x_\eta) = \Gamma_x^{15}\Gamma_{x+e_1}^{25} \\
S^\Gamma(x_\eta, (x+e_2)_\eta) &= -S^\Gamma((x+e_2)_\eta, x_\eta) = \Gamma_x^{35}\Gamma_{x+e_2}^{45} \\
S^\Gamma(x_\xi, x_\eta) &= -S^\Gamma(x_\eta, x_\xi) = \Gamma_x^5
\end{aligned} \tag{3.5}$$

of link operators.

In terms of these operators, the Hamiltonian can be written as

$$\begin{aligned}
H^\Gamma = \mu \sum_{x\in\Lambda} \Big(& S^\Gamma(x_\xi, (x+e_1)_\eta) + S^\Gamma(x_\xi, (x+e_2)_\eta) \\
& - S^\Gamma(x_\eta, (x+e_1)_\xi) - S^\Gamma(x_\eta, (x+e_2)_\xi)\Big) + \sum_{x\in\Lambda} \nu_x S^\Gamma(x_\xi, x_\eta)
\end{aligned} \tag{3.6}$$

The link operators 3.5 clearly satisfy properties (i) and (ii) of the link algebra (see definition 2.8). Also, $S^\Gamma(\ell^{-1}) = -S^\Gamma(\ell)$ for all $\ell \in \tilde{E}$ by definition.

For an elementary plaquette P in $\Lambda \times \{\xi\}$ one has

$$\begin{aligned} S^\Gamma(\partial P) &= (-\mathrm{i})^3 S^\Gamma(x_\xi, (x+e_1)_\xi) S^\Gamma((x+e_1)_\xi, (x+e_1+e_2)_\xi) \cdot \\ &\quad \cdot S^\Gamma((x+e_1+e_2)_\xi, (x+e_2)_\xi) S^\Gamma((x+e_2)_\xi, x_\xi) \\ &= (-\mathrm{i})^3 W_P = \mathrm{i} W_P. \end{aligned}$$

Furthermore, for the loop

$$\gamma_x = (x_\xi, (x+e_1)_\xi) \to ((x+e_1)_\xi, (x+e_1)_\eta) \to ((x+e_1)_\eta, x_\eta) \to (x_\eta, x_\xi)$$

with arbitrary corner $x \in \Lambda$,

$$\begin{aligned} S^\Gamma(\gamma_x) &= (-\mathrm{i})^3 S^\Gamma(x_\xi, (x+e_1)_\xi) S^\Gamma((x+e_1)_\xi, (x+e_1)_\eta) S^\Gamma((x+e_1)_\eta, x_\eta) S^\Gamma(x_\eta, x_\xi) \\ &= (-\mathrm{i})^3 \Gamma^1_x \Gamma^2_{x+e_1} \Gamma^5_{x+e_1} (-\Gamma^{15}_x \Gamma^{25}_{x+e_1})(-\Gamma^5_x) = (-\mathrm{i})^3 \Gamma^1_x \Gamma^{15}_x \Gamma^5_x \mathrm{i} \Gamma^{25}_{x+e_1} \Gamma^{25}_{x+e_1} \\ &= (-\mathrm{i})^2 (-\Gamma^1_x \Gamma^5_x \Gamma^{15}_x) \mathbb{1}_{x+e_1} = \mathrm{i} (\Gamma^{15}_x)^2 \mathbb{1}_{x+e_1} = \mathrm{i}\mathbb{1} \end{aligned}$$

Substituting $(1 \leftrightarrow 3)$ and $(2 \leftrightarrow 4)$ proves the identity $S^\Gamma(\tilde{\gamma}_x) = \mathrm{i}\mathbb{1}$ for the loop

$$\tilde{\gamma}_x = (x_\xi, (x+e_2)_\xi) \to ((x+e_2)_\xi, (x+e_2)_\eta) \to ((x+e_2)_\eta, x_\eta) \to (x_\eta, x_\xi).$$

For the global loop $\gamma_X = (1, x_2)_\xi \to (2, x_2)_\xi \to \cdots \to (L, x_2)_\xi \to (1, x_2)_\xi$ (with x_2 fixed) one has

$$\begin{aligned} S^\Gamma(\gamma_X) &= (-\mathrm{i})^{L-1} S^\Gamma((1,x_2)_\xi, (2,x_2)_\xi) \cdots S^\Gamma((L-1,x_2)_\xi, (L,x_2)_\xi) S^\Gamma((L,x_2)_\xi, (1,x_2)_\xi) \\ &= (-\mathrm{i})^{L-1} \Gamma^1_{(1,x_2)} \Gamma^2_{(2,x_2)} \cdots \Gamma^1_{(L-1,x_2)} \Gamma^2_{(L,x_2)} \Gamma^1_{(L,x_2)} \Gamma^2_{(1,x_2)} = (-\mathrm{i})^{L-1} W_X \end{aligned}$$

Analogously, $S^\Gamma(\gamma_Y) = (-\mathrm{i})^{L-1} W_Y$ for a global loop in x_2-direction at fixed x_1.

Therefore, the link operators $S^\Gamma(\ell)$ satisfy property (iv) of the link algebra if $W_P = \mathbb{1}$ for all elementary plaquettes P, $W_X = W_Y = \mathbb{1}$, and $L \in 4\mathbb{N}$. In this case, $S^\Gamma(\gamma) = \mathrm{i}\mathbb{1}$ for all closed loops in the double lattice $\widetilde{\mathbb{L}}$, since any such loop operator can be decomposed into a product of loop operators corresponding to elementary plaquettes or either of the two global loop operators $W_{X/Y}$.

The identities $W_\alpha = \mathbb{1}\ \forall \alpha \in \{X, Y, P\}$ hold exactly in the sector $\mathfrak{H}_{\{1\}}$ of the Hilbert space.

Lemma 3.2. The projection operator Ξ commutes with all link operators, $[\Xi, S^\Gamma(\ell)] = 0$ for all $\ell \in \tilde{E}$.

Proof. By the very same argumentation as in lemma 3.1 it follows that $[W_\alpha, S^\Gamma(\ell)] = 0$, and thus $[\frac{\mathbb{1}+W_\alpha}{2}, S^\Gamma(\ell)] = 0$ for all α and $\ell \in \tilde{E}$. This implies $[\Xi, S^\Gamma(\ell)] = 0$ for all $\ell \in \tilde{E}$. ∎

By lemma 3.2 the link operators leave the space $\mathfrak{H}_{\{1\}} = \Xi\mathfrak{H} = \operatorname{ran}\Xi$ invariant. It follows that the family of constrained link operators

$$S^{\Gamma}_{\Xi}(\ell) = \Xi S^{\Gamma}(\ell)\Xi\big|_{\operatorname{ran}\Xi} \quad , \ell \in \tilde{E} \tag{3.7}$$

inherits properties (i), (ii) and (iv) from the corresponding ones of the operators $S^{\Gamma}(\ell)$.

Any other closed path in $\widetilde{\mathbb{L}}$ can be decomposed into a product of the above elementary loops. Therefore, property (iv) holds for the family of constrained operators $\{S^{\Gamma}_{\Xi}(\ell) : \ell \in \tilde{E}\}$. The constrained Hamiltonian is given by

$$\begin{aligned} H^{\Gamma}_{\Xi} = \mu \sum_{x\in\Lambda} \Big(& S^{\Gamma}_{\Xi}(x_{\xi}, (x+e_1)_{\eta}) + S^{\Gamma}_{\Xi}(x_{\xi}, (x+e_2)_{\eta}) \\ & - S^{\Gamma}_{\Xi}(x_{\eta}, (x+e_1)_{\xi}) - S^{\Gamma}_{\Xi}(x_{\eta}, (x+e_2)_{\xi}) \Big) + \sum_{x\in\Lambda} \nu_x S^{\Gamma}_{\Xi}(x_{\xi}, x_{\eta}). \end{aligned} \tag{3.8}$$

It still remains to check the trace property (iii). By the identity

$$\begin{aligned} \prod_{x\in\Lambda} \Gamma^5_x &= \prod_{x\in\Lambda} (-\mathrm{i})^2 \Gamma^1_x\Gamma^2_x\Gamma^3_x\Gamma^4_x = (-1)^{|\Lambda|} \Big(\prod_{x\in\Lambda} \Gamma^1_x\Gamma^2_x \Big)\Big(\prod_{x\in\Lambda} \Gamma^3_x\Gamma^4_x \Big) \\ &= (-1)^{|\Lambda|}(-1)^{2(L-1)} \Big(\prod_{x_1=1}^{L}\prod_{x_2=1}^{L} \Gamma^1_{(x_1,x_2)}\Gamma^2_{(x_1+1,x_2)} \Big)\Big(\prod_{x_2=1}^{L}\prod_{x_1=1}^{L} \Gamma^3_{(x_1,x_2)}\Gamma^4_{(x_1,x_2+1)} \Big) \\ &= \Big(\prod_{x_1=1}^{L} W_Y(x_1) \Big)\Big(\prod_{x_2=1}^{L} W_X(x_2) \Big) \end{aligned}$$

it follows that, on the subspace $\Xi\mathfrak{H}$, one has $\Xi\big(\prod_{x\in\Lambda} \Gamma^5_x \big)\Xi\big|_{\operatorname{ran}\Xi} = \mathbb{1}$ and thus

$$\operatorname{Tr}_{\Xi\mathfrak{H}} \left(\prod_{x\in\Lambda} S^{\Gamma}_{\Xi}(x_{\xi}, x_{\eta}) \right) = \operatorname{Tr}_{\Xi\mathfrak{H}} \left(\Xi\Big(\prod_{x\in\Lambda} \Gamma^5_x \Big)\Xi\Big|_{\Xi\mathfrak{H}} \right) = \operatorname{Tr}_{\Xi\mathfrak{H}} \left(\mathbb{1}_{\Xi\mathfrak{H}} \right) = \operatorname{rk}\Xi = 2^{|\Lambda|-1}$$

Hence, the family of constrained link operators does not fulfil the link algebra and SZCZERBA's theorem is not directly applicable. Therefore an auxiliary Hilbert space $\mathfrak{H}_0 \cong \mathbb{C}^2$ and an operator Γ_0 on $\mathfrak{H}_0$ with the properties

$$(\Gamma_0)^* = \Gamma_0, \quad (\Gamma_0)^2 = \mathbb{1}, \quad \operatorname{Tr}_{\mathfrak{H}_0}\Gamma_0 = 0$$

needs to be introduced (e.g. $\Gamma_0 = \sigma^3_0$), and the definition of the link operators has to be slightly modified.

Let $\tilde{\mathfrak{H}} = \mathfrak{H}_0 \otimes \mathfrak{H}$ and fix a vertex v_ξ in the double lattice $\widetilde{\mathbb{L}}$. For concreteness, the choice $v_\xi = (1,1)_\xi$ is made here. Then the modified link operators are defined as

$$\tilde{S}^\Gamma(\ell) = \begin{cases} \mathbb{1} \otimes S^\Gamma(\ell) & \text{if } v_\xi \notin \ell \\ \Gamma_0 \otimes S^\Gamma(\ell) & \text{if } v_\xi \in \ell \end{cases} \tag{3.9}$$

By the identities $\mathrm{i}W_P = S^\Gamma(\partial P)$ and $\mathrm{i}W_{X/Y} = S^\Gamma(\gamma_{X/Y})$ the plaquette and global flux operators can be naturally extended to $\tilde{\mathfrak{H}}$. Since in a closed loop containing o_ξ there are exactly two adjoining edges, and $\Gamma_0^2 = \mathbb{1}$, this extension is trivial, $\widetilde{W}_\alpha = \mathbb{1} \otimes W_\alpha$.

Further, the restrictions $\tilde{S}^\Gamma_\Xi(\ell)$ to $\tilde{\Xi}\tilde{\mathfrak{H}} = (\mathbb{1}\otimes\Xi)\tilde{\mathfrak{H}} = \mathfrak{H}_0 \otimes \Xi\mathfrak{H}$ are link operators satisfying (i), (ii) and (iv) of the link algebra. Property (iii) now does hold due to the introduction of the Γ_0 factor. Indeed,

$$\mathrm{Tr}_{\tilde{\Xi}\tilde{\mathfrak{H}}}\left(\prod_{x\in\Lambda} \tilde{S}^\Gamma_\Xi(x_\xi, x_\eta)\right) = \mathrm{Tr}_{\mathfrak{H}_0}\Gamma_0 \cdot \mathrm{Tr}_{\Xi\mathfrak{H}}\left(\prod_{x\in\Lambda} \tilde{S}^\Gamma_\Xi(x_\xi, x_\eta)\right) = 0.$$

SZCZERBA's theorem 2.10 hence applies to the Hamiltonian

$$\begin{aligned}
\widetilde{H}^\Gamma = \mu \sum_{x\in\Lambda} \Big(& \tilde{S}^\Gamma(x_\xi, (x+e_1)_\eta) + \tilde{S}^\Gamma(x_\xi, (x+e_2)_\eta) \\
& - \tilde{S}^\Gamma(x_\eta, (x+e_1)_\xi) - \tilde{S}^\Gamma(x_\eta, (x+e_2)_\xi)\Big) + \sum_{x\in\Lambda} \nu_x \tilde{S}^\Gamma(x_\xi, x_\eta) \\
= \mathbb{1} \otimes H^\Gamma + \mu(\Gamma_0 - \mathbb{1}) \otimes \Big(& \Gamma^1_v \Gamma^{25}_{v+e_1} + \Gamma^1_{v-e_1}\Gamma^{25}_v + \Gamma^3_v\Gamma^{45}_{v+e_2} + \Gamma^3_{v-e_2}\Gamma^{45}_v \\
& - \Gamma^{15}_v\Gamma^2_{v+e_1} - \Gamma^{15}_{v-e_1}\Gamma^5_v + \Gamma^{35}_v\Gamma^4_{v+e_2} + \Gamma^{35}_{v-e_2}\Gamma^4_v\Big) + \nu_v(\Gamma_0 - \mathbb{1}) \otimes \Gamma^5_v
\end{aligned}$$

constrained to the flux sector $\mathfrak{H}_0 \otimes \mathfrak{H}_{\{1\}}$.

Since the operator $\Gamma_0 \otimes \mathbb{1}$ commutes with $\mathbb{1} \otimes H^\Gamma$ as well as with the other terms in $\widetilde{H}^\Gamma$, eigenstates ψ of $\widetilde{H}^\Gamma$ can be classified by the property $(\Gamma_0 \otimes \mathbb{1})\psi = \pm\psi$. In particular, in the sector of the Hilbert space where $\Gamma_0 \otimes \mathbb{1} = \mathbb{1} \otimes \mathbb{1}$ one has $\widetilde{H}^\Gamma = \mathbb{1} \otimes H^\Gamma$.

This may help in the understanding of the physics of the extended Gamma matrix model, is however not essential in the further discussion, which is based on SZCZERBA's theorem and therefore relies crucially on the validity of the link algebra.

Corollary 3.3 (Jordan Wigner transformation for the Gamma matrix model). *The (extended) Gamma matrix model $\widetilde{H}^\Gamma$ with constraints* *(i)* $W_P = \mathbb{1}$ *for all elementary plaquettes* P *and*

(ii) $W_X = W_Y = \mathbb{1}$

is unitarily equivalent to free fermions on the square lattice $\mathbb{L} = (\Lambda, E)$ *(with periodic boundary conditions),*

$$H = 2\mu \sum_{xy \in E} (c_x^* c_y + c_y^* c_x) + \sum_{x \in \Lambda} \nu_x (2c_x^* c_x - \mathbb{1}).$$

In particular, there exists a unitary transformation U *such that*

$$\widetilde{H}_\Xi^\Gamma = UHU^* \tag{3.10}$$

where $\widetilde{H}_\Xi^\Gamma = \Xi \widetilde{H}^\Gamma \Xi\big|_{\operatorname{ran}\Xi}$.

Remark. As illustrated by Szczerba in his paper [Szc85], the above construction with link operators expressed as products of Dirac Gamma matrices, also works on a lattice with different vertex degrees, at least if they are all even. In this case, the Dirac Gamma matrices have to be replaced by generalised Gamma matrices (generators of $\mathfrak{Cl}(\deg(x))$). ♦

3.2 Dynamical Localisation and Lieb Robinson Bounds for the GMM

The correspondence between the constrained Gamma matrix model and free fermions on the lattice Λ can be used to get insights in the behaviour of the system when the on-site magnetic field is not deterministic, but given by i.i.d. random variables. The theory of free fermions in a random on-site potential on a planar lattice is well-understood in terms of localisation properties, which imply certain locality bounds for the constrained Gamma matrix model.

3.2.1 The Anderson model on $\mathbb{Z}^d$

Let $\{v_x\}_{x \in \mathbb{Z}^d}$ $(d \geq 1)$ be a family of real-valued independent and identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with common distribution P_0

$$\mathbb{P}\{v_x \in A\} = P(A) \quad \text{for all Borel sets } A \subset \mathbb{R} \text{ and } x \in \mathbb{Z}^d.$$

It is further assumed that P is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ with bounded density and compact support, $P(\mathrm{d}\nu) = \rho(\nu)\,\mathrm{d}\nu$ with $\rho \in L_0^\infty(\mathbb{R})$.

The Anderson model on $\mathbb{Z}^d$ is the random Hamiltonian

$$H_\omega = K + \lambda V_\omega \quad \text{on} \quad \ell^2(\mathbb{Z}^d) \tag{3.11}$$

where $K = -\Delta - 2d$ is up to a constant shift the graph Laplacian on $\mathbb{Z}^d$, explicitly for $f \in \ell^2(\mathbb{Z}^d)$,

$$(Kf)(x) = -\sum_{\substack{y\in\mathbb{Z}^d \\ \text{dist}(x,y)=1}} f(y),$$

the random potential V_ω is the multiplication operator by the i.i.d. random variables,

$$(V_\omega f)(x) = v_x f(x),$$

and $\lambda \geq 0$ models the disorder strength.

It is a classical result in the theory of random Schrödinger operators, that the spectrum of H_ω (as well as its spectral components) is almost surely deterministic, $\sigma(H_\omega) = \Sigma$, $\sigma^\#(H_\omega) = \Sigma^\#$ ($\# \in \{pp, ac, sc\}$). This follows from the ergodicity of H_ω with respect to lattice translations. In the i.i.d. case considered here, one has $\sigma(H_\omega) = [-2d, 2d] + \text{supp}P$ [Pas80, KS80, KM82].

Definition 3.4 (Localisation Types). One says that the Anderson model exhibits

(i) *spectral localisation* in the energy regime $I \subset \mathbb{R}$ if there is $\mathbb{P}$-almost surely only pure point spectrum in I, i.e.

$$I \cap \Sigma^{ac} = I \cap \Sigma^{sc} = \emptyset \quad \text{a.s.}$$

(ii) *exponential eigenfunction localisation* in I away from some random centres of localisation ξ_ω, if for eigenfunctions ψ_ω of H_ω

$$|\psi_\omega(x)| \leq C_\omega e^{-\eta|x-\xi_\omega|} \quad \mathbb{P}-\text{almost surely}$$

with inverse localisation length $\eta = \eta(I) > 0$

(iii) *(strong) dynamical localisation* in the energy interval I if

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left|\langle \delta_y, e^{-itH_\omega} P_I(H_\omega)\delta_x\rangle\right|^2\right] \leq Ce^{-\eta|x-y|} \tag{3.12}$$

with $C = C(I) < \infty$ and inverse localisation length $\eta = \eta(I) > 0$. Here, $\{\delta_x\}_{x\in\mathbb{Z}^d}$ denotes the canonical orthonormal basis of $\ell^2(\mathbb{Z}^d)$, $\delta_x(y) = \delta_{xy}$ for all $x, y \in \mathbb{Z}^d$, and $P_I(H_\omega)$ is the spectral projection of H_ω onto I.

In this definition, dynamical localisation is the strongest notion of localisation, and if it holds in some energy interval I, then the spectrum in I is a.s. pure point. Dynamical localisation also implies that for energies in I there is no quantum transport in the sense that all moments of the position operator[1] $|X|$

[1] $(|X|\psi)(x) := |x|\psi(x)$ for $x \in \mathbb{Z}^d$

are finite for all times,

$$\sup_{t\in\mathbb{R}} \| |X|^p e^{-itH_\omega} P_I(H_\omega)\psi \|^2 < \infty \quad \mathbb{P}-\text{almost surely}$$

for all $p > 0$ and $\psi \in \ell^2(\mathbb{Z}^d)$ compactly supported [Sto11].

One way of proving strong dynamical localisation in the Anderson model on $\mathbb{Z}^d$ is via the *fractional moments method* (or AIZENMAN-MOLCHANOV method [AM93]), which proceeds via exponential bounds on the fractional moments of the *Green's function* G_ω of the random Hamiltonian H_ω,

$$G_\omega(x,y;z) = \langle \delta_x, \frac{1}{H_\omega - z}\delta_y \rangle, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

Restrictions of the Hamiltonian or the Green's function to subsets $\Lambda \subset \mathbb{Z}^d$ are denoted by H_ω^Λ and $G_\omega^\Lambda(x,y;z)$.

By a rank-two perturbation argument using Kreĭn's formula one can prove for all $s \in (0,1)$ the a priori bound[2]

$$\mathbb{E}_{x,y} \left| G_\omega(x,y;z) \right|^s \leq \frac{C_s(\rho)}{\lambda^s} \tag{3.13}$$

for some constant $C_s(\rho) < \infty$ and all $x,y \in \mathbb{Z}^d$, $z \in \mathbb{C} \setminus \mathbb{R}$, $\lambda > 0$ [Sto11, AW13].

Making use of the resolvent identity

$$G_\omega^\Lambda(x,y;z) = G_\omega^{\{x\}}(x,x;z)\delta_{xy} + G_\omega^\Lambda(x,x;z) \sum_{u:d(u,x)=1} G_\omega^{\Lambda\setminus\{x\}}(u,y;z)$$

and noting that the Green's function $G_\omega^{\Lambda\setminus\{x\}}$ does not depend on $V(x)$, one can show that for $x \neq y$

$$\begin{aligned}
\mathbb{E} \left| G_\omega^\Lambda(x,y;z) \right|^s &\leq \sum_{u:d(u,x)=1} \mathbb{E} \left[\left| G_\omega^\Lambda(x,x;z) \right|^s \left| G_\omega^{\Lambda\setminus\{x\}}(u,y;z) \right|^s \right] \\
&\leq \frac{C_s}{\lambda^s} \sum_{u:d(u,x)=1} \mathbb{E} \left| G_\omega^{\Lambda\setminus\{x\}}(u,y;z) \right|^s .
\end{aligned}$$

The first inequality is a straightforward application of Jensen's inequality, whereas in the second step first a disorder average over $V(x)$ was taken, using the a priori bound $\mathbb{E}_x \left| G_\omega^\Lambda(x,x;z) \right|^s \leq C_s \lambda^{-s}$. Iterating this expansion, one can prove the following theorem [AW13] due to AIZENMAN and MOLCHANOV [AM93]

[2] $\mathbb{E}_{x,y}[\cdot] = \mathbb{E}\left[\cdot \,\middle|\, \{V(u)\}_{u\notin x,y} \right]$ denotes the conditional expectation with $\{V(u)\}_{u\notin x,y}$ fixed

Theorem 3.5 (Complete localisation for high disorder). *For any disorder strength λ satisfying*

$$\lambda > (2dC_s)^{1/s} \tag{3.14}$$

for some $s \in (0,1)$ there is a constant $C < \infty$ such that for all $\Lambda \subset \mathbb{Z}^d$ and $x, y \in \Lambda$

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \mathbb{E} \left| G^{\Lambda}_{\omega}(x, y; z) \right|^s \leq C e^{-\eta d(x,y)} \tag{3.15}$$

uniformly in Λ with a localisation length $\eta < \log(\frac{|\lambda|^s}{2dC_s})$.

The connection to the definition of dynamical localisation is established by the *eigenfunction correlator* $Q_\omega(x, y; \cdot)$ of H_ω, which is defined as the total variation of the spectral measure associated with two sites $x, y \in \mathbb{Z}^d$. That is, for any Borel set $I \subset \mathbb{R}$,

$$Q_\omega(x, y; I) = \sup_{\substack{F \in \mathscr{C}(\mathbb{R}) \\ \|F\|_\infty \leq 1}} \left| \langle \delta_x, P_I(H_\omega) F(H_\omega) \delta_y \rangle \right| .$$

It has the important property that

$$\sup_{t \in \mathbb{R}} \left| \langle \delta_y, e^{-\mathrm{i}tH_\omega} P_I(H_\omega) \delta_x \rangle \right|^2 \leq \sup_{t \in \mathbb{R}} \left| \langle \delta_y, e^{-\mathrm{i}tH_\omega} P_I(H_\omega) \delta_x \rangle \right| \leq Q_\omega(x, y; I)$$

where the first inequality simply follows from $\left| \langle \delta_y, e^{-\mathrm{i}tH_\omega} P_I(H_\omega) \delta_x \rangle \right| \leq 1$ and the second one by the particular choice $F(H_\omega) = e^{-\mathrm{i}tH_\omega}$.

Under the present assumptions on the single-site distribution, for each $s \in (0, 1)$ there is a constant $c_s(\rho) < \infty$ such that for any bounded open set $I \subset \mathbb{R}$ the eigenfunction correlator, averaged over the disorder, satisfies [AW13]

$$\mathbb{E} Q_\omega(x, y; I) \leq c_s(\rho) \sup_{|\eta| > 0} \int_I \mathbb{E} \left| G_\omega(x, y; E + \mathrm{i}\eta) \right|^s \mathrm{d}E. \tag{3.16}$$

Putting everything together, it follows that for large disorder $\lambda > (2dC_s)^{1/s}$ the Anderson model exhibits strong dynamical localisation throughout the entire spectrum.

Of course, the above is just a sketch of the most basic localisation result in the discrete Anderson model. Much more could be said about different disorder regimes, and far more general methods than the one used here are nowadays at hand. A more detailed account on the topic can be found in the notes by KIRSCH [Kir07] and STOLZ [Sto11], as well as in the book by AIZENMAN and WARZEL [AW13], which is still in preparation at the time of writing of this thesis.

3.2.2 Lieb Robinson bounds and exponential clustering

Consider a quantum spin system on a finite vertex set Λ as described in section 1.2. Then for two local observables (cf. the definition in section 1.2) $A \in \mathfrak{A}_{\Omega_1}$, $B \in \mathfrak{A}_{\Omega_2}$ (Ω_1 and Ω_2 finite, disjoint, $\Omega_1 \cup \Omega_2 \subset \Lambda$) one has

$$[A, B] := [A \otimes \mathbb{1}_{\Lambda \setminus \Omega_1}, B \otimes \mathbb{1}_{\Lambda \setminus \Omega_2}] = 0$$

Lieb-Robinson bounds are bounds on the commutator after time evolution of one of the operators of the form

$$\left\| \left[\alpha_t^{\Lambda}(A), B \right] \right\| \leq C_{\Omega_1, \Omega_2} \|A\| \, \|B\| \, e^{-\eta(\mathrm{dist}(\Omega_1, \Omega_2) - v_{LR}|t|)} \tag{3.17}$$

The velocity v_{LR} depends on the specific interaction Φ and is finite for a fairly general class of interactions, which are essentially finite-range [LR72]. Physically, v_{LR} is the group velocity with which information or excitations in Ω_1 can propagate under the Heisenberg evolution to Ω_2.

Lieb Robinson bounds have been successfully used to prove exponential clustering in presence of a spectral gap, i.e. a spectral gap implies exponential decay of ground state correlations [Has04, NS06]. Since the converse implication is in general false, and the assumption of a spectral gap quite strong, one would like to find weaker requirements on quantum spin systems that allow one to infer exponentially decaying correlation functions.

An interesting step in this direction is a result due to HAMZA, SIMS and STOLZ [HSS12]. In their paper they prove that *zero-velocity* Lieb Robinson bounds imply exponentially decaying ground state correlations up to a logarithmic correction in the gap size. The validity of a zero-velocity Lieb Robinson bound has been proposed by the above authors as a simplified version of HASTINGS' definition of a *mobility gap*, i.e. a gap to propagating excitations [Has10].

Let H_Λ be the Hamiltonian

$$H_\Lambda = 2\mu \sum_{x \in \Lambda} \sum_{k=1}^{d} \left(c_x^* c_{x+e_k} + c_{x+e_k}^* c_x \right) + \sum_{x \in \Lambda} \nu_x (2 c_x^* c_x - \mathbb{1}), \quad \mu, \nu_x \in \mathbb{R}$$

on $\mathfrak{H} = \bigotimes_{x \in \Lambda} \mathbb{C}^2$, $\Lambda \subset \mathbb{Z}^d$ finite and connected. Using the anticommutation relations it can be brought into a symmetric form,

$$H_\Lambda = \mu \sum_{x \in \Lambda} \sum_{k=1}^{d} \left(c_x^* c_{x+e_k} - c_x c_{x+e_k}^* + c_{x+e_k}^* c_x - c_{x+e_k} c_x^* \right) + \sum_{x \in \Lambda} \nu_x (c_x^* c_x - c_x c_x^*),$$

and introducing the vector $C = (c_1, \dots, c_{|\Lambda|}, c_1^*, \dots, c_{|\Lambda|}^*)^t$ this can be written compactly as

$$H_\Lambda = C^* \mathrm{M}^{(\Lambda)} C \quad \text{with} \quad \mathrm{M}^{(\Lambda)} = \begin{pmatrix} \mathrm{A}^{(\Lambda)} & 0 \\ 0 & -\mathrm{A}^{(\Lambda)} \end{pmatrix} \in \mathbb{C}^{2|\Lambda| \times 2|\Lambda|}$$

and $\mathrm{A}^{(\Lambda)} \in \mathbb{C}^{|\Lambda| \times |\Lambda|}$ given by

$$\mathrm{A}^{(\Lambda)} = \mu \mathrm{Adj}(\Lambda) + \mathrm{diag}(\nu_1, \dots, \nu_{|\Lambda|})$$

Here, $\mathrm{Adj}(\Lambda)$ stands for the adjacency matrix of $\Lambda \subset \mathbb{Z}^d$ considered as (undirected) combinatorial graph. Since $\mathrm{Adj}(\Lambda)$ is symmetric, $\mathrm{A}^{(\Lambda)}$ is self-adjoint, thus also $(\mathrm{M}^{(\Lambda)})^* = \mathrm{M}^{(\Lambda)}$. In particular, by the spectral theorem for self-adjoint matrices, there is a unitary matrix U that diagonalises $\mathrm{A}^{(\Lambda)}$, $\mathrm{U}\mathrm{A}^{(\Lambda)}\mathrm{U}^* = \mathrm{D} = \mathrm{diag}(\lambda_1, \dots, \lambda_{|V|})$. Then the unitary matrix $\mathrm{W} = \mathrm{U} \oplus \mathrm{U}$ diagonalises $\mathrm{M}^{(\Lambda)}$,

$$\mathrm{W}\mathrm{M}^{(\Lambda)}\mathrm{W}^* = \begin{pmatrix} \mathrm{D} & 0 \\ 0 & -\mathrm{D} \end{pmatrix}$$

This fact can be used to introduce creation and annihilation operators for new fermions $B = (b_1, \dots, b_{|\Lambda|}, b_1^*, \dots, b_{|\Lambda|})$, namely

$$B = \mathrm{W}C$$

It follows from a straightforward computation that the canonical anticommutation relations are in this vector-valued formalism equivalent to

$$CC^* + \mathbb{J}(CC^*)^t \mathbb{J} = \mathbb{1}, \quad \mathbb{J} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Since $\mathbb{J}$ commutes with W, it follows that

$$BB^* + \mathbb{J}(BB^*)^t \mathbb{J} = \mathrm{W}\left(CC^* + \mathbb{J}(CC^*)^t \mathbb{J}\right)\mathrm{W}^* = \mathbb{1}$$

hence the b-operators indeed are fermionic operators. In terms of these operators the Hamiltonian H_Λ can be written as

$$\begin{aligned} H_\Lambda &= C^* \mathrm{M} C = B^* W \mathrm{M} W^* B = B^* \begin{pmatrix} \mathrm{D} & 0 \\ 0 & -\mathrm{D} \end{pmatrix} B \\ &= \sum_{x \in \Lambda} \lambda_x (b_x^* b_x - b_x b_x^*) = \sum_{x \in \Lambda} 2\lambda_x b_x^* b_x - E^{(\Lambda)} \mathbb{1} \end{aligned}$$

where $E^{(\Lambda)} = \sum_{x \in \Lambda} \lambda_x$.

In this form it is particularly easy to calculate the time evolution of the fermionic operators. For any $x \in \Lambda$ one has

$$[H_\Lambda, b_x] = -2\lambda_x b_x$$

and thus, by definition of the Heisenberg dynamics,

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_t^\Lambda(b_x) = \mathrm{i}\alpha_t^\Lambda([H_\Lambda, b_x]) = -2\mathrm{i}\lambda_x b_x.$$

Together with $\alpha_0^\Lambda(b_x) = b_x$ it follows that $\alpha_t^\Lambda(b_x) = e^{-2\mathrm{i}\lambda_x t} b_x$, and consequently

$$\alpha_t^\Lambda(B) = \begin{pmatrix} e^{-2\mathrm{i}\mathrm{D}t} & 0 \\ 0 & e^{2\mathrm{i}\mathrm{D}t} \end{pmatrix} B$$

This result can be used to obtain the time evolution of C, for

$$\begin{aligned} \alpha_t^\Lambda(C) &= \alpha_t^\Lambda(\mathrm{W}^* B) = \mathrm{W}^* \alpha_t^\Lambda(B) = \mathrm{W}^* \begin{pmatrix} e^{-2\mathrm{i}\mathrm{D}t} & 0 \\ 0 & e^{2\mathrm{i}\mathrm{D}t} \end{pmatrix} B = \mathrm{W}^* \begin{pmatrix} e^{-2\mathrm{i}\mathrm{D}t} & 0 \\ 0 & e^{2\mathrm{i}\mathrm{D}t} \end{pmatrix} \mathrm{W} C \\ &= e^{-2\mathrm{i}\mathrm{M}^{(\Lambda)}t} C = \begin{pmatrix} e^{-2\mathrm{i}\mathrm{A}^{(\Lambda)}t} & 0 \\ 0 & e^{2\mathrm{i}\mathrm{A}^{(\Lambda)}t} \end{pmatrix} C. \end{aligned}$$

Introducing the short-hand notation $\mathrm{A}_{xy}^{(\Lambda)}(t) = \left(e^{-\mathrm{i}\mathrm{A}^{(\Lambda)}t}\right)_{xy}$, one can immediately write down the Heisenberg evolution of the c-operators,

$$\alpha_t^\Lambda(c_x) = \sum_{y\in\Lambda} \mathrm{A}_{xy}^{(\Lambda)}(-2t) c_y + \sum_{y\in\Lambda} \mathrm{A}_{xy}^{(\Lambda)}(2t) c_y^*. \tag{3.18}$$

It shall from now on be assumed that the family $\{\nu_x\}_{x\in\Lambda}$, describing the on-site potential in the fermionic Hamiltonian, is a family of i.i.d. random variables as in section 3.2.1. Then H_Λ is essentially the second quantisation of the Anderson model on $\ell^2(\Lambda)$ with disorder strength $\lambda = \frac{1}{2\mu}$. In particular it is reasonable to define

Definition 3.6. The matrices $\mathrm{M}^{(\Lambda)}$ are dynamically localised if there exist $C, \eta > 0$ such that for all $\Lambda \subset \mathbb{Z}^d$ and $x, y \in \Lambda$

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left|\mathrm{A}_{x,y}^{(\Lambda)}(t)\right|\right] \leq C e^{-\eta d(x,y)}. \tag{3.19}$$

It has been proved by Hamza *et al.* [HSS12], that dynamical localisation of the matrices $\mathrm{M}^{(\Lambda)}$ implies zero-velocity Lieb-Robinson bounds after disorder average in one dimension.

Example 3.7 (Dynamical localisation implies zero-velocity Lieb-Robinson bounds after disorder average in one dimension). Let $\mathfrak{H} = \bigotimes_{x=1}^{N} \mathbb{C}^2$ and consider the XY-model on $\mathfrak{H}$. Looking back at section 2.1.1 a short calculation shows that in this case

$$\mathrm{A}^{(N)} = \begin{pmatrix} \nu_1 & -\mu & & & \\ -\mu & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\mu \\ & & & -\mu & \nu_N \end{pmatrix}$$

The following holds:

Theorem 3.8 (Hamza, Sims, Stolz [HSS12]). $\mathrm{M}^{(N)}$ *is dynamically localised and there exist* $C', \eta > 0$ *such that for all* $1 \leq x < y$*, and any* $N \geq y$ *one has*

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\|[\alpha_t^N(X), Y]\right\|\right] \leq C'\|X\|\,\|Y\|e^{-\eta d(x,y)} \tag{3.20}$$

for all $X \in \mathfrak{A}_{\{x\}}$ *and* $Y \in \mathfrak{A}_{[y,N]}$.[3]

Furthermore, letting ψ_0 *be the almost sure (normalised) ground state of* H_N*, there is a constant* $C < \infty$ *and a inverse correlation length* $\eta' > 0$ *such that*

$$\mathbb{E}\left[\left|\langle\psi_0, XY\psi_0\rangle - \langle\psi_0, X\psi_0\rangle\langle\psi_0, Y\psi_0\rangle\right|\right] \leq C\|X\|\,\|Y\|Ne^{-\eta' d(x,y)}. \tag{3.21}$$

♦

If one instead considers the constrained Gamma matrix model on $\mathfrak{H}_0 \otimes \Xi\mathfrak{H}$ in a random exterior magnetic field $\{\nu_x\}_{x\in\Lambda}$, which is assumed to satisfy the assumptions of subsection 3.2.1, it would be interesting to see if similar results also hold.

Using a Jordan Wigner transformation in the sense of Szczerba (theorem 2.10) it has been proven in corollary 3.3 that the GMM-Hamiltonian is unitarily equivalent to the second quantised version of the Anderson model on $\ell^2(\Lambda)$. In a regime of high disorder $\lambda \gg 1$ (cf. condition 3.14) or, equivalently, low kinetic energy $\mu \sim 1/\lambda \ll 1$, one has complete dynamical localisation, which corresponds to dynamical localisation of the matrix $\mathrm{A}^{(\Lambda)} = \mu\mathrm{Adj}(\Lambda) + \mathrm{diag}(\nu_1, \ldots, \nu_{|\Lambda|})$.

Definition 3.9. For $\Omega \subset \Lambda$ let $\mathfrak{S}_\Omega$ be the C^* algebra generated by the set $\{\tilde{S}_\Xi^\Gamma(\ell) : \ell \in \widetilde{\Omega}\}$ of link operators corresponding to edges ℓ in the (doubled) set $\widetilde{\Omega}$[4].

[3] In the context of this theorem, $\mathfrak{A}_\Omega$ refers to the local algebra generated by the Pauli spin matrices in Ω.

[4] Ω being identified in a natural way as subgraph of $\mathbb{L}$.

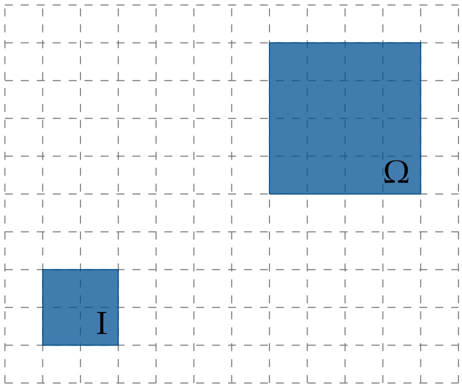


Figure 3.2: On the definition of the relation $\mathrm{I} < \Omega$.

Lemma 3.10. For fixed $x \in \Lambda$ let c_x be a fermionic annihilation operator and $Y \in \mathfrak{S}_\Omega$ such that $x \notin \Omega$ (assuming for simplicity that Ω is a box subset of Λ with the property $x_i < y_i$ for all $y \in \Omega$, $i = 1, 2$)[5]. Then dynamical localisation of the matrix $A^{(\Lambda)}$ implies

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\| \left[\alpha_t^\Lambda(c_x), U^*YU\right] \right\| \right] \leq \underbrace{4C\left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^2}_{=:C'} \|Y\| e^{-\eta \operatorname{dist}(x,\Omega)} \tag{3.22}$$

with C being the constant from definition 3.6.

Proof. Fix the vertex $(1,1)$ as root o of the (directed) spanning tree of Λ depicted in figure 3.3 (directed away from the root) and complete it to a spanning tree of $\widetilde{\mathbb{L}}$ by adding the (directed) edges (x_ξ, x_η) for $x \in \Lambda$. The assumptions on Ω imply that $o \notin \Omega$.

By equation 3.18 one has

$$[\alpha_t^\Lambda(c_x), U^*YU] = \sum_{z\in\Lambda} \mathrm{A}_{xz}^{(\Lambda)}(2t)[c_z, U^*YU] + \sum_{z\in\Lambda} \mathrm{A}_{xz}^{(\Lambda)}(-2t)[c_z^*, U^*YU]. \tag{3.23}$$

Since

$$c_z = \tfrac{1}{2}(\xi_z + \mathrm{i}\eta_z) = \tfrac{1}{2}(-\mathrm{i}\xi_o)U^*(\tilde{S}_\Xi^\Gamma(\gamma_{z_\xi}) + \mathrm{i}\tilde{S}_\Xi^\Gamma(\gamma_{z_\eta}))U \quad \text{for } z \neq o, \tag{3.24}$$

$$c_o = \tfrac{1}{2}(\xi_o + \mathrm{i}\eta_o) = \tfrac{1}{2}(-\mathrm{i}U^*\tilde{S}_\Xi^\Gamma(\gamma_{o_\eta})U\eta_o + \mathrm{i}\eta_o) \tag{3.25}$$

[5]To shorten notation, this shall be written as $\{x\} < \Omega$. More generally, for sets $\mathrm{I}, \Omega \subset \mathbb{Z}^d$ the relation $\mathrm{I} < \Omega$ is defined by $x_1 < y_1$ and $x_2 < y_2$ for all $x \in \mathrm{I}$ and $y \in \Omega$. See figure 3.2.

by the construction 2.47 in SZCZERBA's theorem ($U = W^*$ here as remarked at the end of the proof of theorem 2.10), the commutators in 3.23 can be examined more closely.

Firstly, the properties of η_o (2.46) imply that for any $v, w \neq o$ the commutator $[\eta_o, U^* \tilde{S}^\Gamma_\Xi(v_\zeta, w_\varsigma) U]$ is zero. This follows from the observation that

$$\begin{aligned}
\left[\eta_o, U^* \tilde{S}^\Gamma_\Xi(v_\zeta, w_\varsigma) U\right] &= U^* \left[U\eta_o U^*, \tilde{S}^\Gamma_\Xi(v_\zeta, w_\varsigma)\right] U \\
&= U^* \left[U\eta_o U^*, \mathrm{i}\tilde{S}^\Gamma_\Xi(\gamma_{v_\zeta}) \tilde{S}^\Gamma_\Xi(\gamma_{w_\varsigma})\right] U \\
&= \mathrm{i}U^* \left[U\eta_o U^*, \tilde{S}^\Gamma_\Xi(\gamma_{v_\zeta})\right] \tilde{S}^\Gamma_\Xi(\gamma_{w_\varsigma}) U + \mathrm{i}U^* \tilde{S}^\Gamma_\Xi(\gamma_{v_\zeta}) \left[U\eta_o U^*, \tilde{S}^\Gamma_\Xi(\gamma_{w_\varsigma})\right] U \\
&= \mathrm{i} \underbrace{\left[\eta_o, U^* \tilde{S}^\Gamma_\Xi(\gamma_{v_\zeta}) U\right]}_{=0 \text{ by } (2.46)} U^* \tilde{S}^\Gamma_\Xi(\gamma_{w_\varsigma}) U + \mathrm{i}U^* \tilde{S}^\Gamma_\Xi(\gamma_{v_\zeta}) U \underbrace{\left[\eta_o, U^* \tilde{S}^\Gamma_\Xi(\gamma_{w_\varsigma}) U\right]}_{=0 \text{ by } (2.46)} \\
&= 0.
\end{aligned}$$

The identity in the second line was proved at the end of the proof of lemma 2.11, and third line is a simple application of Leibniz's rule for commutators. Therefore, Y being in the local algebra generated by the link operators corresponding to edges in $\widetilde{\Omega}$, it follows that $[\eta_o, U^* Y U] = 0$ (by repeated application of Leibniz's rule if necessary).

Further, one has

$$\begin{aligned}
[\xi_o, U^* Y U] &= \left[-\mathrm{i}U^* \tilde{S}^\Gamma_\Xi(\gamma_{o_\eta}) U \eta_o, U^* Y U\right] \\
&= -\mathrm{i}\left[U^* \tilde{S}^\Gamma_\Xi(\gamma_{o_\eta}) U, U^* Y U\right] \eta_o - \mathrm{i}U^* \tilde{S}^\Gamma_\Xi(\gamma_{o_\eta}) U \left[\eta_o, U^* Y U\right] \\
&= -\mathrm{i}U^* \underbrace{\left[\tilde{S}^\Gamma_\Xi(\gamma_{o_\eta}), Y\right]}_{=0 \text{ since } o \notin \Omega} U\eta_o - \mathrm{i}U^* \tilde{S}^\Gamma_\Xi(\gamma_{o_\eta}) U \underbrace{\left[\eta_o, U^* Y U\right]}_{=0} \\
&= 0,
\end{aligned}$$

concluding the prove that $[c_o, U^* Y U] = 0$.

For $z \neq o$, the representation 3.24 yields

$$\begin{aligned}
[c_z, U^* Y U] &= -\tfrac{\mathrm{i}}{2}\xi_o \left[U^* \left(\tilde{S}^\Gamma_\Xi(\gamma_{z_\xi}) + \mathrm{i}\tilde{S}^\Gamma_\Xi(\gamma_{z_\eta})\right) U, U^* Y U\right] \\
&\quad - \tfrac{\mathrm{i}}{2} \underbrace{[\xi_o, U^* Y U]}_{=0} U^* \left(\tilde{S}^\Gamma_\Xi(\gamma_{z_\xi}) + \mathrm{i}\tilde{S}^\Gamma_\Xi(\gamma_{z_\eta})\right) U \\
&= -\tfrac{\mathrm{i}}{2}\xi_o U^* \left[\tilde{S}^\Gamma_\Xi(\gamma_{z_\xi}) + \mathrm{i}\tilde{S}^\Gamma_\Xi(\gamma_{z_\eta}), Y\right] U.
\end{aligned}$$

Denote by $\overline{\Omega}$ the set of all vertices in the strip to the right of Ω and of the same height as Ω (not including Ω, see figure 3.3). Then by definition of the path

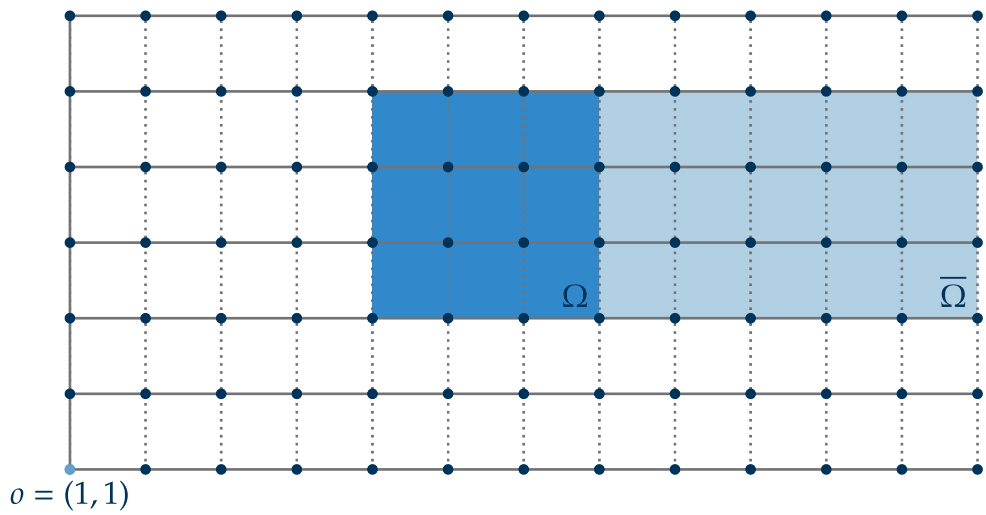


Figure 3.3: The spanning tree with root of $\mathbb{L}$ is shown, together with the definition of the set $\overline{\Omega}$.

operators, the assumed spanning tree and $Y \in \mathfrak{S}_\Omega$, it follows that

$$[\tilde{S}^\Gamma_\Xi(\gamma_{z_\xi}) + \mathrm{i}\tilde{S}^\Gamma_\Xi(\gamma_{z_\eta}), Y] = 0 \quad \text{for all } z \notin \Omega \cup \overline{\Omega},$$

and therefore

$$[c_z, U^*YU] = 0 \quad \text{for all } z \notin \Omega \cup \overline{\Omega}.$$

Since $c^*_y = \frac{1}{2}(\xi_z - \mathrm{i}\eta_z)$ for all $z \in \Lambda$, analogous considerations show that $[c^*_z, U^*YU] = 0$ for all $z \notin \Omega \cup \overline{\Omega}$.

Taking this into account, equation 3.23 becomes

$$[\alpha^\Lambda_t(c_x), U^*YU] = \sum_{z\in\Omega\cup\overline{\Omega}} \mathrm{A}^{(\Lambda)}_{xz}(2t)[c_z, U^*YU] + \sum_{z\in\Omega\cup\overline{\Omega}} \mathrm{A}^{(\Lambda)}_{xz}(-2t)[c^*_z, U^*YU].$$

By dynamical localisation and the bound $\|\,[A, B]\,\| \leq 2\|A\|\,\|B\|$ on the commutator of two operators, one gets

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in\mathbb{R}}\left\|\left[\alpha_t^\Lambda(c_x), U^*YU\right]\right\|\right] \\
&\quad\le \sum_{z\in\Omega\cup\overline{\Omega}} \mathbb{E}\left[\sup_{t\in\mathbb{R}}\left|\mathrm{A}^{(\Lambda)}_{xz}(2t)\right|\right]\left\|[c_z, U^*YU]\right\| + \sum_{z\in\Omega\cup\overline{\Omega}} \mathbb{E}\left[\sup_{t\in\mathbb{R}}\left|\mathrm{A}^{(\Lambda)}_{xz}(-2t)\right|\right]\left\|[c_z^*, U^*YU]\right\| \\
&\quad\le 2\|U^*YU\| \sum_{z\in\Omega\cup\overline{\Omega}} \mathbb{E}\left[\sup_{t\in\mathbb{R}}\left|\mathrm{A}^{(\Lambda)}_{xz}(2t)\right|\right] + 2\|U^*YU\| \sum_{z\in\Omega\cup\overline{\Omega}} \mathbb{E}\left[\sup_{t\in\mathbb{R}}\left|\mathrm{A}^{(\Lambda)}_{xz}(-2t)\right|\right] \\
&\quad\le 4\|Y\| \sum_{z\in\Omega\cup\overline{\Omega}} Ce^{-\eta d(x,z)} \le C'\|Y\|e^{-\eta\,\mathrm{dist}(x,\Omega)}
\end{aligned}$$

with $C' = 4C\left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^2$. ■

Clearly, the same estimate holds for $\alpha_t^\Lambda(c_x^*)$. By the homomorphism property of α_t^Λ and Leibniz's rule,

$$\begin{aligned}
[\alpha_t^\Lambda(c_xc_y), U^*YU] &= [\alpha_t^\Lambda(c_x)\alpha_t^\Lambda(c_y), U^*YU] \\
&= \alpha_t^\Lambda(c_x)[\alpha_t^\Lambda(c_y), U^*YU] + [\alpha_t^\Lambda(c_x), U^*YU]\alpha_t^\Lambda(c_y)
\end{aligned}$$

it follows that (assuming $x, y \notin \Omega$)

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in\mathbb{R}}\left\|[\alpha_t^\Lambda(c_xc_y), Y]\right\|\right] &\le C'\|Y\|e^{-\eta\,\mathrm{dist}(x,\Omega)} + C'\|Y\|e^{-\eta\,\mathrm{dist}(y,\Omega)} \\
&\le C'\|Y\|e^{-\eta\,\mathrm{dist}(\{x,y\},\Omega)}
\end{aligned}$$

The inequality remains valid if c_xc_y is exchanged by $c_x^*c_y$, $c_xc_y^*$, $c_x^*c_y^*$ etc. Thus, since the link operators $S(x_\zeta, y_\varsigma)$ can be expressed in terms of no more than four of the above terms, e.g.

$$\begin{aligned}
S(x_\xi, y_\xi) &= \mathrm{i}\xi_x\xi_y = \mathrm{i}(c_x^* + c_x)(c_y^* + c_y) = \mathrm{i}(c_x^*c_y^* + c_xc_y^* + c_x^*c_y + c_xc_y) \\
S(x_\xi, x_\eta) &= \mathrm{i}\xi_x\eta_x = \mathrm{i}(c_x^* + c_x)\mathrm{i}(c_x^* - c_x) = c_x^*c_x - c_xc_x^*
\end{aligned}$$

one has

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}}\left\|[\alpha_t^\Lambda(S(x_\zeta, y_\varsigma)), Y]\right\|\right] \le 4C'\|Y\|e^{-\eta\,\mathrm{dist}(\{x,y\},\Omega)} \le \underbrace{4C'(1+e^\eta)}_{=:C''}\|Y\|e^{-\eta\,\mathrm{dist}(x,\Omega)}$$

and

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\| [\alpha_t^\Lambda(S(x_\xi, x_\eta)), Y] \right\| \right] \le 2C' \|Y\| e^{-\eta \mathrm{dist}(x,\Omega)},$$

respectively.

Let β_t^Λ be the Heisenberg evolution associated with the Hamiltonian $\widetilde{H}_\Xi^\Gamma(\Lambda)$,

$$\beta_t^\Lambda(A) = e^{\mathrm{i}t\widetilde{H}_\Xi^\Gamma} A e^{-\mathrm{i}t\widetilde{H}_\Xi^\Gamma}.$$

Lemma 3.11. Let $x, y \in \Lambda$, and $Y \in \mathfrak{S}_\Omega$ such that $x, y \notin \Omega$, $x, y < \Omega$. Assume further that the matrix $A^{(\Lambda)}$ is dynamically localised in the sense of definition 3.6. Then

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\| [\beta_t^\Lambda(\tilde{S}_\Xi^\Gamma(x_\zeta, y_\varsigma)), Y] \right\| \right] \le C'' \|Y\| e^{-\eta \mathrm{dist}(x,\Omega)} \tag{3.26}$$

and

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\| [\beta_t^\Lambda(\tilde{S}_\Xi^\Gamma(x_\xi, x_\eta)), Y] \right\| \right] \le 2C' \|Y\| e^{-\eta \mathrm{dist}(x,\Omega)}. \tag{3.27}$$

Proof. By Szczerba's theorem, $\tilde{S}_\Xi^\Gamma(\ell) = US(\ell)U^*$ for all (directed) edges $\ell \in \widetilde{\mathbb{L}}$. Since

$$\begin{aligned}
&\left\| \left[\beta_t^\Lambda(\tilde{S}_\Xi^\Gamma(\ell)), Y\right] \right\| = \left\| U^* \left[\beta_t^\Lambda(\tilde{S}_\Xi^\Gamma(\ell)), Y\right] U \right\| \\
&= \left\| U^* e^{\mathrm{i}t\widetilde{H}_\Xi^\Gamma} U\, U^* \tilde{S}_\Xi^\Gamma(\ell) U\, U^* e^{-\mathrm{i}t\widetilde{H}_\Xi^\Gamma} U\, U^* Y U - U^* Y U\, U^* e^{\mathrm{i}t\widetilde{H}_\Xi^\Gamma} U\, U^* \tilde{S}_\Xi^\Gamma(\ell) U U^* e^{-\mathrm{i}t\widetilde{H}_\Xi^\Gamma} U \right\| \\
&= \left\| e^{\mathrm{i}tH_\Lambda} S(\ell) e^{-\mathrm{i}tH_\Lambda}\, U^* Y U - U^* Y U\, e^{\mathrm{i}tH_\Lambda} S(\ell) e^{-\mathrm{i}tH_\Lambda} \right\| \\
&= \left\| \left[\alpha_t^\Lambda(S(\ell)), U^* Y U\right] \right\|
\end{aligned}$$

it follows by lemma 3.10 that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\| [\beta_t^\Lambda(\tilde{S}_\Xi^\Gamma(x_\zeta, y_\varsigma)), Y] \right\| \right] &= \mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\| [\alpha_t^\Lambda(S(x_\zeta, y_\varsigma)), U^* Y U] \right\| \right] \\
&\le C'' \|Y\| e^{-\eta \mathrm{dist}(x,\Omega)}
\end{aligned} \tag{3.28}$$

and analogously for $\tilde{S}_\Xi^\Gamma(x_\xi, x_\eta)$. ∎

Having lemmata 3.10 and 3.11 at hand, the following zero-velocity Lieb Robinson bound in disorder average can be proved:

Proposition 3.12. *Assume that the constrained Gamma Matrix Model $\widetilde{H}_{\Xi}^{\Gamma}$ is dynamically localised in the sense of definition 3.6. Then there exist constants $c, \eta > 0$ such that for any two sets $\mathrm{I}, \Omega \subset \Lambda$ with $\mathrm{I} < \Omega$ one has*

$$\mathbb{E}\left[\left\| [\beta_t^\Lambda(X), Y] \right\|\right] \leq c \min(1, |t|)\, \|X\|\, \|Y\|\, e^{-\eta \operatorname{dist}(\mathrm{I}, \Omega)} \tag{3.29}$$

for all $X \in \mathfrak{S}_{\mathrm{I}}$ and $Y \in \mathfrak{S}_{\Omega}$.

Proof. **Part 1:** $\mathbb{E}\left[\left\| [\beta_t^\Lambda(X), Y] \right\|\right] \leq c_1 |t|\, \|X\|\, \|Y\|\, e^{-\eta \operatorname{dist}(\mathrm{I}, \Omega)}$.

Let $X \in \mathfrak{S}_{\mathrm{I}}$, $Y \in \mathfrak{S}_{\Omega}$, and define the function

$$f(t) = [\beta_t^\Lambda(X), Y]$$

with derivative

$$\begin{aligned} f'(t) &= \mathrm{i}\left[\beta_t^\Lambda([\widetilde{H}_{\Xi}^{\Gamma}, X]), Y\right] = \mathrm{i}\left[\beta_t^\Lambda([\widetilde{H}_{\Xi}^{\Gamma}(\mathrm{I}), X]), Y\right] = \mathrm{i}\left[[\beta_t^\Lambda(\widetilde{H}_{\Xi}^{\Gamma}(\mathrm{I})), \beta_t^\Lambda(X)], Y\right] \\ &= -\mathrm{i}\left[[\beta_t^\Lambda(X), Y], \beta_t^\Lambda(\widetilde{H}_{\Xi}^{\Gamma}(\mathrm{I}))\right] - \mathrm{i}\left[[Y, \beta_t^\Lambda(\widetilde{H}_{\Xi}^{\Gamma}(\mathrm{I}))], \beta_t^\Lambda(X)\right] \end{aligned}$$

Here, $\widetilde{H}_{\Xi}^{\Gamma}(\mathrm{I})$ only contains the terms in $\widetilde{H}_{\Xi}^{\Gamma}$ that do not commute with X. The last line is an application of the Jacobi identity for commutators. Thus, f satisfies the differential equation

$$f'(t) = -\mathrm{i}\left[f(t), \beta_t^\Lambda(\widetilde{H}_{\Xi}^{\Gamma}(\mathrm{I}))\right] + \mathrm{i}\left[[\beta_t^\Lambda(\widetilde{H}_{\Xi}^{\Gamma}(\mathrm{I})), Y], \beta_t^\Lambda(X)\right]$$

Lemma 3.13. Let $A(t)$, $t \in \mathbb{R}$, be a family of linear, norm preserving operators[6] in some Banach space $\mathfrak{B}$. For any function $B : \mathbb{R} \to \mathfrak{B}$, the solution of

$$\partial_t Y(t) = A(t) Y(t) + B(t)$$

with boundary condition $Y(0) = 0$, satisfies the bound

$$\|Y(t)\| \leq \int_0^t \|B(s)\|\, \mathrm{d}s.$$

Proof. A proof of this can be found in Appendix A of NACHTERGAELE *et al.* [NOS06] (Lemma A.1 therein). ∎

[6] i.e. the mapping $\gamma_t : \mathfrak{B} \to \mathfrak{B}$ which maps $x_0 \in \mathfrak{B}$ to the solution $X(t)$ of the differential equation $\partial X(t) = A(t)X(t)$ with initial datum $X(0) = x_0$ is an isometry, $\|\gamma_t(x_0)\| = \|x_0\|$ for all $t \in \mathbb{R}$.

By the above lemma, one obtains

$$\|f(t)\| = \left\| \left[\beta_t^\Lambda(X), Y\right] \right\| \leq \int_0^{|t|} \left\| \left[[Y, \beta_s^\Lambda(\widetilde{H}_\Xi^\Gamma(\mathrm{I}))], \beta_s^\Lambda(X) \right] \right\| \, \mathrm{d}s$$
$$\leq 2\|X\| \int_0^{|t|} \left\| \left[\beta_t^\Lambda(\widetilde{H}_\Xi^\Gamma(\mathrm{I})), Y\right] \right\| \, \mathrm{d}s$$

Since $\widetilde{H}_\Xi^\Gamma(\mathrm{I})$ is given by

$$\widetilde{H}_\Xi^\Gamma(\mathrm{I}) = \mu \sum_{x \in \mathrm{I}^-} \Big(\tilde{S}_\Xi^\Gamma(x_\xi, (x+e_1)_\eta) + \tilde{S}_\Xi^\Gamma(x_\xi, (x+e_2)_\eta)$$
$$- \tilde{S}_\Xi^\Gamma(x_\eta, (x+e_1)_\xi) - \tilde{S}_\Xi^\Gamma(x_\eta, (x+e_2)_\xi) \Big) + \sum_{x \in \mathrm{I}} \nu_x \tilde{S}_\Xi^\Gamma(x_\xi, x_\eta)$$

where $\mathrm{I}^- = [\mathrm{I} \cup (\mathrm{I} - e_1 - e_2))] \cap \Lambda$, one has

$$\mathbb{E} \left\| \left[\beta_t^\Lambda(X), Y\right] \right\| \leq 2\|X\| \int_0^{|t|} \mathbb{E} \left\| \left[\beta_t^\Lambda(\widetilde{H}_\Xi^\Gamma(\mathrm{I})), Y\right] \right\| \, \mathrm{d}s$$
$$\leq 8\mu \|X\| \sum_{x \in \mathrm{I}^-} C'' \, \|Y\| e^{-\eta \mathrm{dist}(x,\Omega)} |t|$$
$$+ 2 \max_{x \in \Lambda} |\nu_x| \|X\| \sum_{x \in \mathrm{I}} 2C' \, \|Y\| e^{-\eta \mathrm{dist}(x,\Omega)} |t|$$
$$\leq c_1 \, |t| \, \|X\| \, \|Y\| e^{-\eta \mathrm{dist}(\mathrm{I},\Omega)}$$

Explicitly, the constant is $c_1 = 8\mu C'' + 4C' \max_{x \in \Lambda} |\nu_x|$.

Part 2: $\mathbb{E}\left[\left\| [\beta_t^\Lambda(X), Y] \right\| \right] \leq c_2 \|X\| \|Y\| e^{-\eta \mathrm{dist}(\mathrm{I},\Omega)}$ for all $t \in \mathbb{R}$.

By lemma 3.11, the inequality holds for X being a link operator, $X = \tilde{S}_\Xi^\Gamma(x_\zeta, y_\varsigma)$, with $x, y \in \mathrm{I}$. If X is the sum of such link operators, the claim follows by triangle inequality. If X is the product of link operators corresponding to links with vertices in I, a straightforward application of Leibniz's rule yields the inequality.

It follows that for general $X \in \mathfrak{S}_\mathrm{I}$ there exists a constant $c_2 < \infty$ such that $\mathbb{E}\left[\left\| [\beta_t^\Lambda(X), Y] \right\| \right] \leq c_2 \|X\| \|Y\| e^{-\eta \mathrm{dist}(\mathrm{I},\Omega)}$ holds for all $t \in \mathbb{R}$.

Part 3: Choosing $c = \max(c_1, c_2)$ concludes the proof of inequality 3.29. ■

This is a first step in proving zero-velocity Lieb Robinson bounds for the constrained, extended Gamma matrix model. The proof still relies on the additional assumption of the relative positions of the sets I and Ω. It is also an open question, whether or not the local algebras generated by the link operators cover all physically interesting observables. Nevertheless, it has been demonstrated how the methods in [HSS12] may be applied in the context of an approach via the generalised Jordan Wigner transformation due to Wosiek and Szczerba.

Chapter 4

Conclusion and Outlook

In this thesis the Jordan Wigner transformation has been discussed as an important tool in studying properties of one-dimensional quantum spin systems. Various possible extensions of this transformation have been introduced. It turned out that the appearance of statistical gauge fields after the transformation is related to the structure of the underlying graph. In particular, one (surprising) result was that *only* simple path graphs allow for a special Jordan Wigner transformation in the sense of definition 2.2.

An alternative approach, going back to Nambu in the 1950's and reformulated in terms of link operators is based upon the observation that the Pauli spin matrices are generators of a Clifford algebra. Using higher dimensional Clifford algebras in expressing the link operators is the main idea behind the Wosiek-Szczerba approach.

One of the most basic applications of Szczerba's theorem 2.10 – the Gamma matrix model – was introduced, and its connection to the link operators established. After slight modifications, the constrained Gamma matrix model allows for a description by Jordan Wigner fermions. This relation allowed one to study the constrained GMM in a random external field and prove localisation bounds on the Heisenberg evolution – a zero-velocity Lieb Robinson type bound in disorder average (3.29). Even though the bound presented here relies on additional assumptions, it is nevertheless interesting in that it provides another example of a quantum spin system where some of the techniques developed by Hamza, Sims and Stolz can be applied.

As hinted at the end of the general discussion of Szczerba's theorem and the Gamma matrix model, these methods can also be applied to appropriate subspaces of higher dimensional quantum spin systems or quantum spin systems on general graphs (keeping in mind the restriction on the vertex degrees), using higher dimensional generalisations of the Gamma matrices.

Appendix A

Some missing calculations in Sczcerba's theorem

Some properties of the link and path operators that were left out in the presentation of the proof of theorem 2.10 and lemma 2.11 are presented here for completeness, since they mainly involve calculations and cannot be found in the original paper [Szc85].

Lemma A.1 (Path operators). Let $\{S(\ell) : \ell \in \tilde{E}\}$ be a family of link operators satisfying properties (i) and (ii) of definition 2.8, and $S(\ell^{-1}) = -S(\ell)$. Then the path operators $S(\gamma)$ (see definition 2.8, (iv)) have the following properties:

(i) $S(\gamma_1 \circ \gamma_2) = -\mathrm{i}S(\gamma_1)S(\gamma_2)$ if the vertex at the end of γ_1 is the first vertex of γ_2 and $\gamma_1 \circ \gamma_2$ denotes their concatenation,

(ii) $\{S(\gamma_1), S(\gamma_2)\} = 0$ if γ_1 and γ_2 are simple paths with exactly one common edge-point (beginning or end),

(iii) $[S(\gamma), S(\ell)] = 0$ for all closed paths γ and $\ell \in \tilde{E}$.

(iv) $S(\gamma)^* = S(\gamma)$ and $(S(\gamma)^2) = \mathbb{1}$ if γ is not a closed path,

(v) $S(\gamma)^* = -S(\gamma)$ and $(S(\gamma)^2) = -\mathbb{1}$ if γ is a closed path,

(vi) $S(\gamma^{-1}) = -S(\gamma)^{-1}$ for all paths γ.

Proof. Throughout the proof let $\gamma = \gamma_1 = \ell_1 \circ \cdots \circ \ell_m$, $\gamma_2 = \ell_{m+1} \circ \cdots \circ \ell_{m+n}$ for $m, n \in \mathbb{N}$, be paths in $\widetilde{\mathbb{L}}$ with the property that ℓ_m and ℓ_{m+1} are neighbouring edges.

(i) Then $\gamma_1 \circ \gamma_2 = \ell_1 \circ \cdots \ell_{m+n}$ and

$$\begin{aligned} S(\gamma_1 \circ \gamma_2) &= (-\mathrm{i})^{m+n-1} S(\ell_1) \cdots S(\ell_m) S(\ell_{m+1}) \cdots S(\ell_{m+n}) \\ &= (-\mathrm{i}) \left((-\mathrm{i})^{m-1} S(\ell_1) \cdots S(\ell_m) \right) \left((-\mathrm{i})^{n-1} S(\ell_{m+1}) \cdots S(\ell_{m+n}) \right) \\ &= -\mathrm{i} S(\gamma_1) S(\gamma_2). \end{aligned}$$

(ii) Assume additionally that γ_1 and γ_2 are simple paths. Then $\{S(\ell_m), S(\ell_{m+1})\} = 0$, implying

$$\begin{aligned} S(\gamma_1) S(\gamma_2) &= (-\mathrm{i})^{m+n-2} S(\ell_1) \cdots S(\ell_m) S(\ell_{m+1}) \cdots S(\ell_{m+n}) \\ &= -(-\mathrm{i})^{m+n-2} S(\ell_1) \cdots S(\ell_{m+1}) S(\ell_m) \cdots S(\ell_{m+n}) \\ &= -(-\mathrm{i})^{m+n-2} S(\ell_{m+1}) \cdots S(\ell_{m+n}) S(\ell_1) \cdots S(\ell_m) = -S(\gamma_1) S(\gamma_2), \end{aligned}$$

the latter equality being a consequence of the fact that all other permutations involved in this calculation involve only link operators to non-neighbouring edges, which commute by assumption on the link operators.

(iii) Suppose that γ is a closed path, i.e. ℓ_1 and ℓ_m are neighbouring edges, and let $\ell \in \tilde{E}$. If ℓ and γ have no common vertex, $[S(\gamma), S(\ell)] = 0$ is trivial. In case they share exactly one common vertex, there are two adjoining edges to ℓ in γ, say ℓ_k and ℓ_{k+1} for some $k \in \{1, \dots, m\}$ (identifying $m+1 \equiv 1$). Hence

$$\begin{aligned} S(\gamma) S(\ell) &= (-\mathrm{i})^{m-1} S(\ell_1) \cdots S(\ell_k) S(\ell_{k+1}) \cdots S(\ell_m) S(\ell) \\ &= (-1)^2 S(\ell) (-\mathrm{i})^{m-1} S(\ell_1) \cdots S(\ell_m) = S(\gamma) S(\ell) \end{aligned}$$

If γ and ℓ have two common vertices, i.e. $\ell \equiv \ell_k \in \gamma$, then

$$\begin{aligned} S(\gamma) S(\ell) &= (-\mathrm{i})^{m-1} S(\ell_1) \cdots S(\ell_{k-1}) S(\ell_k) S(\ell_{k+1}) \cdots S(\ell_m) S(\ell) \\ &= (-1)^2 S(\ell) (-\mathrm{i})^{m-1} S(\ell_1) \cdots S(\ell_m) = S(\gamma) S(\ell). \end{aligned}$$

(iv) Assume γ is *not* a closed path. Then

$$\begin{aligned} S(\gamma)^* &= \left((-\mathrm{i})^{m-1} S(\ell_1) \cdots S(\ell_m) \right)^* = \mathrm{i}^{m-1} S(\ell_m) \cdots S(\ell_1) \\ &= \mathrm{i}^{m-1} (-1) S(\ell_1) S(\ell_m) \cdots S(\ell_2) = \mathrm{i}^{m-1} (-1)^{m-1} S(\ell_1) S(\ell_2) \cdots S(\ell_m) \\ &= S(\gamma) \end{aligned}$$

and

$$\begin{aligned} S(\gamma)^2 &= (-1)^{m-1} S(\ell_1) \cdots S(\ell_m) S(\ell_1) \cdots S(\ell_m) \\ &= -(-1)^{m-1} S(\ell_1)^2 S(\ell_2) \cdots S(\ell_m) S(\ell_2) \cdots S(\ell_m) \\ &= (-1)^{m-1} (-1)^{m-1} S(\ell_1)^2 S(\ell_2)^2 \cdots S(\ell_m)^2 = \mathbb{1}. \end{aligned}$$

(v) If γ is a closed path, then the steps in (iv) stay essentially the same, apart from the fact that $\{S(\ell_1), S(\ell_m)\} = 0$ instead of $[S(\ell_1), S(\ell_m)] = 0$, leading to an additional minus sign in both calculations.

(vi) Now let γ be arbitrary.

$$\begin{aligned} S(\gamma)S(\gamma^{-1}) &= (-\mathrm{i})^{m-1} S(\ell_1) \cdots S(\ell_m) \cdot (-\mathrm{i})^{m-1} S(\ell_m^{-1}) \cdots S(\ell_1^{-1}) \\ &= (-1)^{m-1}(-1)^m S(\ell_1) \cdots S(\ell_m) S(\ell_m) \cdots S(\ell_1) = -\mathbb{1}. \end{aligned}$$

■

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Declaration of Authorship

I hereby declare that I have written this thesis without any help from others and without the use of documents and aids other than those stated above. I have mentioned all used sources and cited them correctly according to established academic citation rules.

Hiermit versichere ich, die vorliegende Arbeit selbstständig und lediglich unter Zuhilfenahme der genannten Quellen verfasst zu haben.

Munich, 31st August 2013

Tobias Ried