# Sublinearly reinforced Pólya urns on graphs of bounded degree

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**Abstract**

Consider reinforced Pólya urns on the vertices of infinite graphs with bounded degree, where the edges of each vertex correspond to a colour in the respective urn and weights on the edges correspond to the number of balls of the respective colour. Increments happen based on atoms in independent homogeneous Poisson clocks with inhomogeneous intensities. For this highly interactive model, consider the case of sublinear reinforcement, i.e. with a Pólya-exponent $\alpha \in (0, 1)$. After exploring a sensible notion of equilibrium distributions for these dynamics, this thesis shows that for infinite graphs of bounded degree and a subset of $\alpha$ the edge weights converge to the unique equilibrium distribution.

**Acknowledgments**

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# Contents

# Chapter 1

# Introduction

In [1] van der Hofstadt et al. define a model of Pólya urns with graph-based competition on finite graphs and discuss the case of strong reinforcement. The interest in this model comes from its interpretation as a simplified model of the learning process in the human brain with the edge weights standing for the strength of a synapse. Hirsch et al. generalize the discussion to classes of infinite graphs and in particular all graphs of bounded degree in [2].
This thesis seeks to extend the discussion on infinite graphs to the weak reinforcement case. For this, this thesis extends the notion of equilibrium presented in [1] to infinite graphs and it shows existence of non-trivial equilibrium distributions on graphs of bounded degree. Convergence to a generalised homogeneous phase, described by the equilibrium distribution, is shown for all graphs of bounded degree for a Pólya coefficient of $\alpha \in (0, 1/2)$, for vertex transitive graphs for $\alpha \in (0, \alpha_d)$ with $\alpha_d > 1/2$ and for $\mathbb{Z}$ for $\alpha \in (0, 1)$. This is in contrast to the results in the strong reinforcement case in which Hirsch et al. show that, for graphs of bounded degree and even some random graphs, the weights localize on forests where each component has a bounded number of edges (whisker trees) for all $\alpha > 1$.
The existence of the equilibrium distributions on graphs of bounded degree result from the application of fixed point theory and in particular the Schauder fixed point theorem on topological vector spaces. The convergence to the homogeneous phase is achieved by bounding the possible accumulation points of the edge weights away from zero and then iteratively tightening the bound through a repeated worst-case estimate which ends with almost surely determined edge weights.
In Section 1.1 the model for the graph based Pólya urns is defined, Section 1.2 lists the main results of this thesis, Section 1.3 discusses the results and gives an outlook on open problems and Section 1.4 gives an overview of the structure of this thesis.

## 1.1 Model definition

Let $\mathcal{G} = (V, E, \sim, d)$ denote an infinite graph with countable vertex set $V$, bounded degree and distance metric $d : V \times V \to \mathbb{N}$ Denote edges by $e = \{v, w\}$ for $v, w \in V$. Then, $v \in e$ and $w \in e$. The explicit mention of $\sim$ and $d$ is usually omitted so that $G = (V, E)$. If not explicitly stated otherwise, the distance metric will be the standard graph metric $d(v, w)$ of number of vertices contained in the shortest paths from $v$ to $w$. For the sake of convenience, define the distance between two edges as the shortest distance between the vertices contained in the edges. Denote by $E_v = \{e \in E | v \in e\}$ the set of edges incident to $v \in V$. Investigate a system of random variables

$$N_t = \{N_t^e\}_{e \in E}$$

of interacting Pólya-type urns on the edge set $E$ at continuous time $t \in \mathbb{R}_+ := [0, \infty)$ on a probability space $(\Omega, \mathbb{P})$. The dynamics of $N_t$ are a continuous-time analog of the process considered in [1]. Loosely speaking, every vertex has a Poisson clock and whenever that clock rings the dynamics choose and increment the weights on one of the adjacent edges by one.
The choice of the edge happens using

$$\text{pol}_{v,e}(N_t) := \frac{(N_t^e)^\alpha}{\sum_{e' \in E_v} (N_t^{e'})^\alpha} \quad \alpha > 0 \; , \tag{1.1}$$

which is the power-weighted Pólya-scheme considered in [1]. More precisely, initially the starting weight distribution $\{N_0^e\}_{e \in E}$ is a family of independent and identically distributed (i.i.d.) bounded random variables with $N_0^e > N_0$ for every edge $e \in E$ and some $N_0 > 0$. Denote the Lebesgue measure by $\mathcal{L}$ and let $\{\lambda_v\}_{v \in V}$ be a family of bounded positive real values $\lambda_v \in [\underline{\Lambda}, \overline{\Lambda}]$, where $\overline{\Lambda} \geq 1 \geq \underline{\Lambda} > 0$. Then, the dynamics of $N_t$ are governed by an i.i.d family of Poisson point processes $\{P_v\}_{v \in V}$ on $[0, \infty) \times [0, 1]$ giving the probability

$$\mathbb{P}\left(P_v\left(T \times U\right) = n\right) = \text{Poi}(\lambda_v \cdot \mathcal{L}(T) \cdot \mathcal{L}(U); n)$$

of having $n \in \mathbb{N}$ events fire on the vertex $v \in V$ for $T \times U \subset [0, \infty) \times [0, 1]$ and $v \in V$. If the process $P_v$ contains an atom of the form $(t, u)$ for some $u \in [0, 1]$ and $t \in [0, \infty)$ increment the mass of an edge $e_i \in \{e_i\}_{i \in [1, |E_v|]}$ by 1 if $u \in U_{v,e_i}$, where $\{U_{v,e_i}\}_{i \in [1, |E_v|]}$ is a partition of $[0, 1]$ given by

$$U_{v,e_1} = [0, \text{pol}_{v,e_1}(N_t)] \tag{1.2}$$

$$U_{v,e_i} = (\text{pol}_{v,e_{i-1}}(N_t), \text{pol}_{v,e_{i-1}}(N_t) + \text{pol}_{v,e_i}(N_t)] \; . \tag{1.3}$$

## 1.2 Main results

Let

$$X_t = \frac{N_t}{t}$$

denote the normalized particle system. This thesis concerns itself with the existence and convergence of the limiting configuration

$$X_\infty = \lim_{t \to \infty} X_t$$

and in particular will analyse the smallest interval

$$\mathcal{C}_e := \left[\liminf_{t \to \infty} X_t^e, \limsup_{t \to \infty} X_t^e\right]$$

containing all accumulation points of $X_t^e$ for $\alpha \in (0, 1)$ and $e \in E$.

The limiting edge weight distribution as $t \to \infty$ is, in general, not constant but will follow an *equilibrium distribution.* State the definition of equilibrium here for completeness and refer to Section 2.4 for a detailed discussion.

**Definition 1.1.** Call a measure $\mu_G \in \mathbb{R}_+^E$ an *equilibrium distribution* for the graph $G$ with firing rates $\{\lambda_v\}_{v \in V}$ of the Poisson processes on the vertices if

$$\mu_G(e) = \sum_{v \in e} \lambda_v \frac{\mu_G(e)^\alpha}{\sum_{e' \in E_v} \mu_G(e')^\alpha}$$

for all $e \in E$. If there is no ambiguity, write $\mu$ instead of $\mu_G$.

This thesis proves that equilibrium distributions with $\mu(e) > 0$ for all $e \in E$ exist for all graphs of bounded degree for any distribution of firing rates. The proof employs fixed point theory results and is contained in Section 3.2.

**Theorem 1.2.** *Let $\alpha \in (0, 1/2)$ and $G$ be infinite, connected and of bounded degree and $N_t$ as in Section 1.1. Then, there exists a unique equilibrium distribution $\mu$ with $\mu(e) > 0$ for all $e \in E$ and $X_t$ converges almost surely to the deterministic process distributed according to the equilibrium distribution.*

This is the result with the most general graph class. By limiting the analysis to graphs of vertex transitive graphs it is possible to extend the range of valid Pólya coefficients $\alpha$. In particular the case $G = \mathbb{Z}$ allows for $\alpha \in (0, 1)$.

**Theorem 1.3.** *Let $G = \mathbb{Z}$, $\alpha \in (0, 1)$ and the firing rates $\{\lambda_v\}_{v \in V}$ be constant $\lambda_v \equiv \lambda$. Furthermore, let all starting values $N_0^e$ be equal to a bounded random variable $N_0^E$ with $N_0^E > N_0$. Then, $X_t$ converges almost surely to the deterministic process having weight $\lambda$ on every edge.*

This statement generalizes to vertex transitive graphs at the cost of limiting the valid $\alpha$ again.

**Theorem 1.4.** *Let $G$ be vertex transitive with degree $d$, let the firing rates $\{\lambda_v\}_{v \in V}$ be constant $\lambda_v \equiv \lambda$. Furthermore, let all starting values $N_0^e$ be equal to a bounded random variable $N_0^E$ with $N_0^E > N_0$. Then, $X_t$ converges* *almost surely to the deterministic process having weight $2\lambda/d$ on every edge for $\alpha \in (0, \alpha_d)$, where $\alpha_d$ is the choice for $\alpha$ such that*

$$\max_{x \in (0,2)} \left( \frac{\mathrm{d}}{\mathrm{d}x} \varphi \circ \varphi(x) \right) = 1$$

*where*

$$\varphi(x) := \left( \frac{2\lambda}{(d-1)x^\alpha + (2\lambda/d)^\alpha} \right)^{1/1-\alpha} .$$

## 1.3 Discussion of results and open problems

This thesis marks the beginning of the work on the weakly reinforced process on infinite graphs and hence does not provide solutions to all the problems it gives. This sections intends to discuss the new results contained in this thesis and the problems encountered when trying to generalise the proofs.

Section 3.2 contains the proof for the existence of an equilibrium distribution on graphs of bounded degree. Hirsch et al. prove the existence of the process $N_t$ in [2] on graphs of bounded degrees but also on random graphs, e.g. Galton Watson trees with finite mean offspring. This is achieved by introducing a more general condition on the graphs (no infinite *descending chains*) which implies the existence on those graph classes. The proof for the existence of equilibrium distributions relies on a fairly direct calculation and the Schauder fixed point theorem. In order to bring together the graph classes for which $N_t$ exists and for which an equilibrium distribution $\mu$ with $\mu(e) > 0$ for all $e \in E$ exists it might be worthwhile to try to generalise the fixed point theory approach to random graphs (especially the ones discussed in [2]) or find a new approach to this problem.

The fixed point theory approach is noteworthy as the Banach fixed point theorem cannot be applied for all $\alpha \in (0, 1/2)$ (see Lemma 3.10). Thus, the uniqueness statements in the theorems in this thesis can be rephrased as statements on the uniqueness of fixed points for the functional in Definition 1.1 solved with the help of stochastic processes.

The main caveat of the results in this thesis is that not all $\alpha \in (0, 1)$ are covered. If the discussion of the weak reinforcement regime were extended to all $\alpha \in (0, 1)$, the combination with the results on the $\alpha > 1$ case presented in [2] might give insights into the linear case $\alpha = 1$. The problem with generalising the results is that worst-case estimates do not allow for better results so it might be fruitful to devise a more intricate estimate to cover $\alpha \in (1/2, 1)$ and ideally get insights into the linear case $\alpha = 1$. The vertex transitive case allows for an iterative repetition of the worst case estimate ending in an almost sure limit but the proof explicitly uses that the vertices are indistinguishable such that even the generalisation to quasi vertex transitive graphs proves difficult. Since the case $G = \mathbb{Z}$ worked out with the same technique for all $\alpha \in (0, 1)$

there might be appropriate estimates for higher-dimensional graphs which are not direct generalisations of the $G = \mathbb{Z}$ case.

The fact that the range for vertex transitive graphs is broader than for bounded degree graphs and that the boundary $\alpha_d$ in Theorem 1.4 is a seemingly arbitrary consequence of the worst case estimate suggests the following conjecture.

**Conjecture 1.5.** *Let $\alpha \in (0,1)$ and $G$ be of bounded degree and $N_t$ as in Section 1.1. Then, there exists a unique $\mu$ with $\mu(e) > 0$ for all $e \in E$ such that $X_t$ converges almost surely to the deterministic process distributed according to the equilibrium distribution.*

## 1.4 Structure

Chapter 2 gives an overview of mostly standard material from fixed point theory, graph theory and the theory of Poisson point processes. This chapter can be skipped if some basic background knowledge is present with the exception of Section 2.4 as it motivates and explores the notion of equilibrium distributions for $X_t$.

Chapter 3 contains existence proofs for the process on graphs of bounded degree, adapted from [2], in Section 3.1 and for equilibrium distributions in Section 3.2.

Finally, Chapter 4 contains the proofs for Theorem 1.2, Theorem 1.3 and Theorem 1.4.

# Chapter 2

# Preliminaries

This chapter concerns itself with some preliminary results not directly related to the theorems which are stated, mostly without proof, for completeness. Section 2.1 and Section 2.2 discusses core concepts of fixed point respectively graph theory appearing in this thesis. Section 2.3 adapts standard results given in [3–5] for Poisson point processes which are relevant for the theorem proofs. Finally, Section 2.4 defines and motivates the notion of an equilibrium distribution. Apart from the definition of edge classes, Definition 2.10, Section 2.1, Section 2.2 and 2.3 add no new material to standard literature and can safely be skipped.

## 2.1 Fixed point theorems and related notions

This section reiterates some basic results in fixed point theory which the proof for existence of equilibrium distributions and the proof of Theorem 1.2 need. This section does not reiterate basic notions of topology, like topological spaces, continuity of functions and the definition of compact sets but it will go over the definition and results for topological vector spaces as they are a less common object in mathematics. First, state a standard theorem of fixed point theory for later reference.

**Theorem 2.1** (Banach's fixed point theorem)**.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ a contraction mapping with Lipschitz constant $K \in (0, 1)$, i.e.*

$$d(T(x), T(y)) \leq Kd(x, y)$$

*for any $x, y \in X$. Then $T$ has a unique fixed point in $X$.*

*Proof.* See Theorem 5.1 in [6]. □

The rest of this section deals with topological vector spaces leading up to the Schauder fixed point theorem Theorem 2.5. For a more detailed account of the results, see for example [7].

**Theorem 2.2** (Tychonov Theorem)**.** *The topological product of any family of compact spaces is compact.*

*Proof.* See [7] for the statement of the theorem and references for topology books. □

**Definition 2.3.** A *topological vector space* $X$ is a vector space over a topological field $\mathbb{K}$ (in this thesis $\mathbb{K} = \mathbb{R}$) with a topology such that vector addition $X \times X \to X$ and scalar multiplication $\mathbb{K} \times X \to X$ are continuous on their respective spaces.

*Remark* 2.4. Let $G = (V, E)$ be a graph. The space

$$\{\mu : E \to \mathbb{R}_+\}$$

of functions mapping edges to positive real numbers (weight distributions in the context of this thesis) is a topological vector space. This follows by identifying the function space with the product space $\mathbb{R}_+^E$ and endowing it with the product topology.

**Theorem 2.5** (Schauder's fixed point theorem)**.** *Let $X$ be a locally convex Hausdorff topological vector space, $C$ a non empty closed convex subset of $X$ and $T$ a continuous mapping of $C$ into a compact subset of $C$. Then $T$ has a fixed point in $K$.*

*Proof.* See the proof due to Singbal in the Appendix of [8]. □

## 2.2 Graph theory related concepts

Most of the notation and notions in this section stem from [9]. Following that notation, let $G = (V, E)$ denote a graph with its vertex set $V$ and edge set $E$ and $\emptyset = (\emptyset, \emptyset)$ the *empty graph*. Note that $\emptyset$ also denotes the empty set, but this ambiguity should not be further confusing based on the context in which $\emptyset$ appears. Let the *degree* $\deg(v)$ of a vertex $v \in V$ be the cardinality of the set $E_v := \{e \in E \mid v \in e\}$.

**Definition 2.6.** Let $G = (V, E)$, $G' = (V', E')$ and $G'' = (V'', E'')$ be graphs.

- The graph $G'$ is a *subgraph* of $G$ if $V' \subset V$ and $E \subset E'$.
- For $G', G''$ subgraphs of $G$, define the *intersection of two subgraphs* as $G' \cap G'' = (V' \cap V'', E' \cap E'')$.
- The subgraphs $G'$ and $G''$ are *disconnected*, or $G' \cup G''$ is *disjoint*, if $G' \cap G'' = \emptyset$, otherwise they are *connected*.
- Define the *union* of two graphs as $G' \cup G'' = (V' \cup V'', E' \cup E'')$.

*Remark* 2.7. Note that this notion of intersection is dependent on $G'$ and $G''$ being subgraphs of the same graph $G$ and $G$ having a countable vertex set because then $G'$ and $G''$ inherit a counting from $G$ which allows for the unique comparison of vertices and edges.

Contrasting this, the notion of union is directly generalizable: The union of two arbitrary graphs $G', G''$, which are not explicitly given as subgraphs, is the disjoint union $G = G' \cup G''$ such that $G' \cap G'' = \emptyset$.

In this thesis a map $f : G \to G'$ between two graphs $G = (V, E), G' = (V', E')$ denotes a map between their respective vertex sets, i.e. $f : V \to V'$ unless explicitly stated otherwise. Similarly $v, w \in G$ is equivalent to $v, w \in V$. This notation helps reduce the clutter by not having to introduce the vertex sets explicitly.

**Definition 2.8.** Consider two graphs $G = (V, E)$ and $G' = (V', E')$ and a map $f : V \to V'$.

- The map $f$ is an *isomorphism* from $G$ to $G'$ if it preserves neighbourhoods, i.e. if $\{v, w\} \in E$ if and only if $\{f(v), f(w)\} \in E'$.
- The graphs $G$ and $G'$ are *isomorphic* if there exists an isomorphism between $G$ and $G'$.
- An isomorphism from $G$ to itself is an *automorphism* of $G$.

This thesis does not differentiate between isomorphic graphs and identifies equivalence classes of isomorphic graphs with any of their representatives. Denote by $[n]$ for $n \in \mathbb{N}$ the set $\{1, \ldots, n\}$ of natural number smaller or equal to $n$.

**Definition 2.9.** A graph $G = (V, E)$ is *vertex transitive* if, for any two vertices $v, w \in V$, there is an automorphism of $G$ mapping $v$ to $w$.

The graph $G$ is *quasi vertex transitive* if there exists a partition $\{V_i\}_{i \in [m]}$ for some $m \in \mathbb{N}$ of pairwise disjoint subsets of the vertex set $V$ such that for any pair of vertices $v, w \in V_j$ for any $j \in [m]$ there exists a graph automorphism $f$ of $G$ such that $f(v) = w$.

An example for a vertex transitive graph is $\mathbb{Z}^d$ or the graph given in Figure 1 and for a quasi vertex transitive graph in Figure 2 in Section 3.3.

Quasi vertex transitivity gives a finite amount of vertices with differing neighbourhoods, which implies that there is an analogous notion of finite amounts of equivalent neighbourhoods of edges. Formalize this concept for edges as edge classes.

**Definition 2.10.** A graph $G = (V, E)$ has $m \in \mathbb{N}$ *edge classes* if there exists a partition $\{E_i\}_{i \in [m]}$ of pairwise disjoint subsets of the edge set $E$ such that for any pair $e = \{v, w\}, e' = \{v', w'\} \in E_j$ for any $j \in [m]$ there exist graph automorphisms $f, g$ of $G$ such that either

$$f(v) = v' \text{ and } g(w) = w'$$

or

$$f(v) = w' \text{ and } g(w) = v \ .$$

*Remark* 2.11. Using two graph homomorphisms $f, g$ instead of one $f = g$ ensures that graphs with mirror symmetries have lower numbers of edge classes. An example for this is the ladder graph missing every second rung as in Figure 2. If $f = g$ were a condition in Definition 2.10 the graph would have three edge classes event though intuitively, based on mirror symmetry, it should have two. Definition 2.10 fulfills this heuristic expectation.

*Remark* 2.12. Refer to the vertex subsets $V_j$ respectively edge subsets $E_j$ (or any representative thereof) as vertex classes respectively edge classes.

Edge classes provide an equivalent definition of quasi vertex transitive graphs.

**Lemma 2.13.** *A graph $G = (V, E)$ is quasi vertex transitive if and only if it has a finite number of edge classes.*

*Proof.* First, assume that $G$ is quasi vertex transitive and let $\{V_i\}_{i\in[m]}$ be the partition in vertex classes. Then the partition

$$\{\{(v, w) \mid v \in V_j \text{ and } w \in V_k\} \cap E\}_{j,k\in[m],j\geq k}$$

forms a finite partition of pairwise disjoint subsets of the edge set $E$. The graph homomorphism property from Definition 2.10 follows immediately from the graph homomorphism property of $\{V_i\}_{i\in[m]}$ and thus $G$ has at most $\frac{1}{2}m(m+1)$ edge classes.

Assume now that $G$ has $m$ edge classes $\{E_j\}_{j\in[m]}$. Then, per definition, there are two vertex sets $V_{j1}$ and $V_{j2}$ of vertices contained in the edges $e$ in $E_j$ for any $j \in [m]$ such that for any pair of vertices $v, w \in V_{j1}$ or $v, w \in V_{j2}$ there exists a graph homomorphism such that $v = f(w)$. Thus, $G$ has at most $2m$ vertex classes and is quasi vertex transitive. □

**Corollary 2.14.** *Let $G$ be vertex transitive, then $G$ has one edge class.*

*Proof.* Follows from the previous proof as $G$ has 1 vertex class and thus at most $1^2 = 1$ edge class. □

*Remark* 2.15. Note that having one edge class is not equivalent to the graph being edge symmetric. To see this consider the graph in Figure 1. As every vertex has the same neighbourhood it is vertex transitive and thus has a single

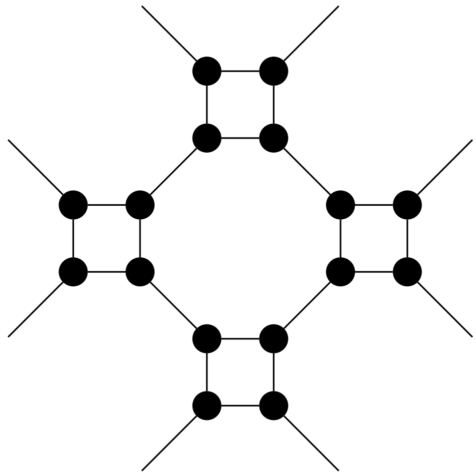

Figure 1: Example for a vertex transitive graph which is not edge transitive.

edge class. It is not edge transitive though, as there is no graph homomorphism mapping the edges connecting the four nodes in a square formation to the edges connecting the square formations.

The difference between edge transitivity and edge classes is that edge transitivity requires the edges to have identical neighbourhoods while edge classes only ask for identical neighbourhoods on the vertices contained in the respective edges.

An important consequence of Lemma 2.13 is that graphs of bounded degree, in general, do not have a finite number of edge classes. This complicates the handling of non-quasi vertex transitive graphs. For example, proving the existence of equilibrium distributions on quasi vertex transitive graphs with $m$ edge classes amounts to solving a nonlinear equation system on an $m$ dimensional space. Generalising this to infinite edge classes in Lemma 3.7 requires the concept of topological vector spaces.

One such class of non-quasi vertex transitive graphs arises from attaching a finite amount of vertices to quasi vertex transitive graphs, for example $\mathbb{Z} + \{v\}$, given in Figure 3. Formalise this concept of graphs that are quasi vertex transitive apart from on a finite set.

**Definition 2.16.** Let $G = (V, E)$ be a graph. $G$ is *almost (quasi) vertex transitive* if there exists a finite subgraph $G_s = (V_s, E_s)$ of $G$ and a *corresponding* (quasi) vertex transitive graph $\mathfrak{G}$ such that for any neighbourhood $G_v$ around a $v \in V \setminus V_s$ such that $G_s \cap G_v = \emptyset$ there exists a graph automorphism $f : G_v \to \mathfrak{G}_v$ to a neighbourhood $\mathfrak{G}_v \subset \mathfrak{G}$.

## 2.3 Standard results for Poisson Point Processes

Throughout this section, let $(\mathbb{R}^d, \mathcal{B})$ and $(\mathbb{X}, \mathcal{X})$ be metric spaces and let $(\mathbb{N}, \mathcal{N})$ be a measurable space where $\mathcal{B}$ denotes the set of bounded Borel sets and $(\Omega, \mathbb{P})$ a probability space.

**Definition 2.17.** A measure $\mu$ on a state space $\mathbb{X}$ is *locally finite* if any point $x \in \mathbb{X}$ has a neibhourhood $N_x$ such that $\mu(N_x) < \infty$.

**Definition 2.18.** Let $\mu$ be a locally finite measure on $\mathbb{X}$. A *Poisson point process* with intensity measure $\mu$ is a point process $\Pi$ on $\mathbb{X}$ with the following two properties:

1. For every $B \in \mathcal{X}$ the distribution of $\Pi(B)$ is Poisson with parameter $\mu(B)$, that is to say $\mathbb{P}(\Pi(B) = k) = \mathrm{Poi}(\mu(B); k)$ for all $k \in \mathbb{N}$.

2. For every $m \in \mathbb{N}$ and all pairwise disjoint sets $B_1, \ldots, B_m \in \mathcal{X}$ the random variables $\Pi(B_1), \ldots, \Pi(B_m)$ are independent.

The point process $\Pi$ is a *homogeneous Poisson point process* with intensity $\lambda \in \mathbb{R}_+$ if it is a Poisson point process with intensity measure $\mu(B) = \lambda\mathcal{L}(b)$.

**Theorem 2.19** (Existence theorem)**.** *Let $\mu$ be a locally finite measure on $\mathbb{X}$. Then there exists a Poisson process on $\mathbb{X}$ with intensity measure $\mu$.*

*Proof.* See Theorem 3.6 in [5]. □

The following results allows the identification of the process on the whole set of vertices and the set of processes $\{P_v\}_{v \in V}$.

**Theorem 2.20** (Restriction theorem)**.** *Let $\Pi$ be a Poisson process on $\mathbb{X}$ with locally finite intensity measure $\mu$ and let $C_1, C_2, \ldots \in \mathcal{X}$ be pairwise disjoint. Then $\Pi_{C_1}, \Pi_{C_2}, \ldots$ are independent Poisson processes with intensity measures $\mu_{C_1}, \mu_{C_2}, \ldots$, respectively.*

*Proof.* See Theorem 5.2 in [5]. □

The following is the analogue of the strong law of large numbers for Poisson point processes. The proof is adapted to the case at hand from the Law of Large numbers in [3].

**Lemma 2.21.** *Let $P$ be a homogeneous Poisson process with intensity $\lambda \in \mathbb{R}^+$ on $(0, \infty) \times [0, 1]$. Then*

$$\lim_{t \to \infty} \frac{P((0, t] \times U)}{t} = \mathcal{L}(U)\lambda \text{ almost surely,}$$

*for any Borel set $U \subset [0, 1]$.*

*Proof.* Let $U \subset [0,1]$ Borel. Recall that

$$\mathbb{E}\left[P((0,t] \times U)\right] = \mathcal{L}(U)\lambda t \quad \mathrm{var}\left[P((0,t] \times U)\right] = \mathcal{L}(U)\lambda t \ .$$

Insert this into the Chebyshev inequality (see [10] for example) to get

$$\mathbb{P}\left[\left|\frac{P((0,t] \times U)}{t} - \mathcal{L}(U)\lambda\right| \geq \varepsilon\right] \leq \frac{\mathcal{L}(U)\lambda}{\varepsilon^2 t} \tag{2.1}$$

for any $\varepsilon > 0$. Taking $t_k = k^2$ and inserting it into Eq. (2.1) implies

$$\sum_{k=1}^{\infty} \mathbb{P}\left[\left|\frac{P((0,k^2] \times U)}{k^2} - \mathcal{L}(U)\lambda\right| \geq \varepsilon\right] < \infty$$

with which the Borel-Cantelli Lemma (see [10])) gives that

$$\left|\frac{P((0,k^2] \times U)}{k^2} - \mathcal{L}(U)\lambda\right| \geq \varepsilon$$

almost surely only for a finite number of integer values $k$ and thus

$$\lim_{k\to\infty} \frac{P((0,k^2] \times U)}{k^2} = \mathcal{L}(U)\lambda$$

almost surely. To regain the statement for $t \in \mathbb{R}$ approximate $t$ by natural numbers, i.e. take $k$ to be the integer part of $\sqrt{t}$, so that, for $t > 1$,

$$P\left((0,k^2] \times U\right) \leq P\left((0,t] \times U\right) \leq P\left((0,(k+1)^2] \times U\right)$$

and

$$k^2 \leq t < (k+1)^2 \ .$$

Thus,

$$\frac{(k+1)^2}{k^2} \to 1$$

implies

$$\begin{aligned} \liminf_{t\to\infty} \frac{P((0,t] \times U)}{t} &> \liminf_{k\to\infty} \frac{P((0,k^2] \times U)}{(k+1)^2} \\ &= \lim_{k\to\infty} \frac{k^2}{(k+1)^2} \frac{P((0,k^2] \times U)}{k^2} \\ &= \mathcal{L}(U)\lambda \end{aligned}$$

And an analogous calculation for lim sup gives the claim. □

The following direct implication of Markov's inequality (see Theorem 5.11 in [10]) is necessary to bind $X_t^e$ away from zero (for a more general version see Lemma 1.2 in [4]).

**Lemma 2.22.** *Let $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$ such that $k \leq \lambda$ then*

$$\mathbb{P}\left[\mathrm{Poi}(\lambda) \leq k\right] \leq \exp\left(-\left(\lambda - k + k\log\left(\frac{k}{\lambda}\right)\right)\right) .$$

*Proof.* Let $X \sim \mathrm{Poi}(\lambda)$. By Markov's inequality applied to the monotonically decreasing function $f(x) = z^x$ for $z \leq 1$

$$\mathbb{P}\left[X \leq k\right] \leq z^{-k}\mathbb{E}\left[z^X\right] = z^{-k}e^{\lambda(z-1)} .$$

Putting $z = k/\lambda$ completes the proof. □

## 2.4 Equilibrium distributions

This section intends to extend the notion of equilibrium presented in [1] to the case of infinite graphs and analyse the case at hand.

Let $G = (V, E)$ be a finite graph and $\mu \in \Delta_E := \{\mu : E \to \mathbb{R}$ with $\mu \geq 0$ and $\int_E \mu(e)\mathrm{d}e = 1$ for all $e \in E\}$ and let

$$\begin{aligned} f : \Delta_E \times E &\longrightarrow \mathbb{R} \\ (\mu, e) &\longmapsto -\mu(e) + \sum_{v \in e} p_{E_v} \cdot \frac{\mu(e)^\alpha}{\sum_{e' \in E_v} \mu(e')^\alpha} , \end{aligned}$$

where $p_{E_v}$ is the probability that the *next* incremented edge is in $E_v$, i.e. that the *next* Poisson process that will have an atom is $P_v$.

The notion of next is well-defined since Poisson point processes are memoryless (see Theorem 7.4 in [5]). Thus, the probability of being the next firing vertex is independent of time.

For finite graphs, $p_{E_v} \equiv \lambda_v/|V|$. This is not directly extendible to infinite graphs as $V$ is an infinite set. Multiplying $f$ by $|V|$ gives a new map

$$\begin{aligned} F(\mu, e) : \Delta \times E &\longrightarrow \mathbb{R} \\ (\mu, e) &\longrightarrow -\mu(e) + \sum_{v \in e} \lambda_v \frac{\mu(e)^\alpha}{\sum_{e' \in E_v} \mu(e')^\alpha} \end{aligned} \tag{2.2}$$

on $\Delta := \{\mu : E \to \mathbb{R}\}$ which has no explicit mention of $p_{E_v}$ anymore. Thus results the definition given in Definition 1.1.

**Definition 2.23.** Call a measure $\mu_G \in \Delta$ an *equilibrium state* for the graph $G$ if $F(\mu, e) = 0$ for all $e \in E$. If there is no ambiguity, write $\mu$ instead of $\mu_G$.

The following remark justifies calling this a generalisation of the notion of equilibrium on finite graphs.

*Remark* 2.24. Let $G$ be a finite graph and $\lambda_v \equiv 1$. $\mu \in \Delta$ is an equilibrium distribution if and only if $\mu/|V| \in \Delta_E$ and $f(\mu/|V|, e) = 0$ for all $e \in E$. The heuristic presented in [1] for this definition of equilibrium is that $f(\mu/|V|, e) = 0$ when $\mu(e)/|V|$ is equal to the probability of incrementing the edge $e$ next. The equilibrium state is the weight distribution corresponding to the distribution $\mu/|V|$.

The general setting with $F(\mu, e) = 0$ does not query the probability for the next incrementation but the probability for multiple increments on the edge $e$. In the finite graph setting, for example, $F(\mu, e) = 0$ queries for how many increments out of $|V|$ on the whole graph fall onto $e$.

On infinite graphs the absolute value does not have a similar nice interpretation. The relation between two edges $\mu(e)/\mu(e')$ becomes more interesting as it looks at how many increments $e$ receives more than $e'$ which is a sensible measure independent of graph size. The equilibrium distribution reflects the ratios of incrementation of an edge in relation to every other edge in the graph. Finding equilibrium distributions is thus a highly non-local problem.

# Chapter 3

# Existence proofs

This section concerns itself mostly with existence proofs to show that the set of graphs to which Theorem 1.2 and Theorem 1.4 apply is not empty.

Section 3.1 contains arguments for the existence of the process $N_t$ on graphs of bounded degree and Section 3.2 shows the existence of equilibrium distributions for graphs of bounded degree and shows why the uniqueness statements in the theorems are noteworthy. Finally, Section 3.3 discusses some examples.

## 3.1 Existence of the process

This section seeks to reiterate the arguments given in [2] for the existence of the set $\{N_t^e\}_{e\in E}$ of edge weights at any time $t \in \mathbb{R}_+$. For this, the notion of the Poisson process on the vertex set is more convenient to construct the probability space on which those processes exist. Before coming to the process $N_t$ prove the existence of the Poisson process on the vertex set.

**Lemma 3.1.** *Let $G$ be a graph of bounded degree with a countable vertex set $V$ and $\{\lambda_v\}_{v\in V}$ be a set of bounded positive real numbers. Consider the measure*

$$M(V_s \times B) = \mathcal{L}(B) \sum_{v\in V_s} \lambda_v \tag{3.1}$$

*on the measurable space*

$$(V \times (0,\infty) \times [0,1], 2^V \times \mathcal{B}((0,\infty) \times [0,1]))$$

*with the power set $2^V$ of $V$. Then, the Poisson point process on $V \times (0,\infty) \times [0,1]$ with intensity measure $M$ exists.*

*Proof.* Let $(v, x, b)$ be a point in $V \times (0,\infty) \times [0,1]$. Consider the neighbourhood

$$N_{(v,x,b)} \left( \{v\} \times \left( \frac{x}{2}, \frac{3x}{2} \right) \times [0,1] \right)$$

with $\varepsilon \in (0, x)$. Note that $N_{(v,x,b)}$ is open in $V \times (0, \infty) \times [0, 1]$ and contains $(v, x, b)$. Then

$$M(N_{(v,x,b)}) = \lambda_v x < \infty \ .$$

Thus, $M$ is locally finite. The claim follows by the Existence theorem for Poisson point processes, Theorem 2.19. □

The problem with showing the existence of the process $N_t$ on infinite graphs is that the weight $N_t^e$ on an edge $e \in E$ at a time $t \in \mathbb{R}_+$ might not be determined by a finite subset of the graph but by infinitely many events in the past. Formalise this notion of dependence on past events as descending chains.

**Definition 3.2.** Let $M = \{(V_n, T_n, U_n)\}$ be a Poisson point process on $V \times (0, \infty) \times [0, 1]$ and $m \geq 2$ a natural number. A sequence $\{(V_{n_i}, T_{n_i}, U_{n_i})\}_{i \in [m]}$ is a *descending chain of length m* if

1. $V_{n_i}$ is adjacent to $V_{n_i+1}$, and
2. $T_{n_i} > T_{n_i+1}$,

for every $i \in [m-1]$.

**M** *admits infinite descending chains* if there exists an infinite sequence $\{(V_{n_i}, T_{n_i}, U_{n_i})\}_{i \geq 1}$ which fulfills the above conditions for all $i \geq 1$.

For each vertex $v \in V$ and its local weight distribution $\mathbf{n}_v = \{n(e)\}_{e \in E_v}$ define the *selection function*

$$\mathrm{sel}_v(\cdot, \mathbf{n}_v) : [0, 1] \to E_v$$

by setting $\mathrm{sel}_v(u, \mathbf{n}_v) = e$ if $u \in U_{v,e}$ for $e \in E_v$ and $U_{v,e}$ as in Equation (1.2) and Equation (1.3). Note that the sets $U_{v,e}$ require a counting $\{e_i\}_{e_i \in E_v}$ of $E_v$ which is not specified. An example would be the total ordering of the edges by their weights and having $\mathrm{sel}_v$ select edges with higher weights the higher $u$ is. Hirsch et al. use this total ordering of the edges in [2] to derive further results with applications in the strong reinforcement case. Leave the explicit counting ambiguous as the results are not relevant for the weak reinforcement case and thus any counting will do.

**Lemma 3.3.** *Let $G = (V, E)$ be such that it admits no infinite descending chains. Then the Poisson point process defined in Section 1.1 exists.*

*Proof.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and $\mathbf{P} = \{(V_n, T_n, U_n)\}_{n \geq 1}$ a Poisson point process on $V \times [0, \infty) \times [0, 1]$ with the intensity as in Equation (3.1).

Construct a family of approximations $(N_{t,i}^e)_{t \geq 0, i \in \mathbb{Z}_+}$ of the weights $N_t^e$ at time $t \geq 0$ and first set $N_{t,0}^e = N_0^e$ for all $e \in E$. Let $\mathcal{V}_{V_n}$ denote the set of vertices containing $V_n$ and all adjacent vertices. Define the initial layer of approximations

$$\mathbf{L}_1 = \{(V_m, T_m, U_m) \in \mathbf{P} \colon \mathbf{P} \cap (\mathcal{V}_{V_m} \times [0, T_m) \times [0,1]) = \emptyset\}$$

as the set containing all firing events such that no firing event occurs earlier on any $v \in \mathcal{V}_{V_m}$. This set is non-empty because there are no infinite descending chains. Thus, $\mathbf{L}_1$ contains the beginning points of all descending chains.

For every $(V_m, T_m, U_m) \in \mathbf{L}_1$ and $e \in E_{V_m}$ define the next weight approximation step as

$$N^e_{t,1} = N^e_{t,0} + \mathbb{1}_{\left\{t \geq T_m, \mathrm{sel}_{V_m}\left(U_m, \left\{N^{e'}_{T_m-,0}\right\}_{e' \in E_{V_m}}\right) = e\right\}}, \tag{3.2}$$

where $T_m-$ denotes a time infinitesimally smaller than $T_m$. Note that this is well defined because the probability for two atoms of $\mathbf{P}$ to occur at the same time is zero, which follows since the intensity measure of $\mathbf{P}$ has no atoms (for details see Section 2.1 of [3]). For the edges $e \in E \setminus E_{V_m}$ put $N^e_{t,1} = N^e_{t,0}$.

Proceed recursively for $i \geq 1$ by defining the $(i+1)$th layer

$$\mathbf{L}_{i+1} = \{(V_m, T_m, U_m) \in \mathbf{P} \colon \mathbf{P} \cap (\mathcal{V}_{V_m} \times [0, T_m) \times [0,1]) \subset \mathbf{L}_i\}$$

as the family of all firing events such that all earlier firing events at this or adjacent vertices are in layer $\mathbf{L}_i$, i.e. the next link in a descending chain. For every $(V_m, T_m, U_m) \in \mathbf{L}_{i+1} \setminus \mathbf{L}_i$ and $e \in E_{V_m}$ update the weights analogously to Eq. (3.2)

$$N^e_{t,i+1} = N^e_{t,i} + \mathbb{1}_{\left\{t \geq T_m, \mathrm{sel}_{V_m}\left(U_m, \left\{N^{e'}_{T_m-,i}\right\}_{e' \in E_{V_m}}\right) = e\right\}},$$

leaving the edges $e \in E \setminus E_{V_m}$ the same.

Since $N^e_{t,i}$ is increasing in $i$ and $N^e_{t,i} \leq |\{(X_m, T_m, U_m) : T_m \leq t, X_m \sim e\}|$, the limit

$$N^e_t := \lim_{i \to \infty} N^e_{t,i}$$

is well-defined and almost surely finite. Finally, as there are no infinite descending chains $\cup_{i \geq 0} \mathbf{L}_i = \mathbf{P}$ and thus these layers account for every firing.

The existence of the processes, where $\{P_v\}_{v \in V}$ governs the dynamics, follows by defining the processes

$$P_v = \mathbf{P} \cap \{v\} \times [0, \infty) \times [0,1]$$

as restrictions to the individual vertices using Theorem 2.20. Since the intensity measure of $\mathbf{P}$ contains the sum over the firing rates,

$$\sum_{v \in V_s} P_v \overset{d}{=} \mathbf{P}|_{V_s}$$

as both are Poisson point processes with the same intensity measure. Thus, the above arguments apply to $\{P_v\}_{v \in V}$ and the claim follows. □

The following is the key Lemma of this section stating that no graphs appearing in this thesis admit infinite descending chains.

**Lemma 3.4.** *Let $G$ be a graph of bounded degree, then $G$ does not admit infinite descending chains.*

*Proof.* Let $\mathbf{P} = \{(V_n, T_n, U_n)\}_{n\geq 1}$ be a Poisson point process on $V \times [0, \infty) \times [0, 1]$ with the intensity measure $M$ from Equation (3.1) with $\{\lambda_v\}_{v\in V}$ positive and bounded $\lambda_v \in [\underline{\Lambda}, \overline{\Lambda}]$ and let $d \in \mathbb{N}$ be the maximal degree of $G$. For a fixed finite set $V_0 \subset V$ the probability that there is an infinite descending chain $\{V_{n_i}, T_{n_i}, U_{n_i}\}_{i\geq 1}$ with $V_{n_i} \in V_0$ for all $i$ is 0.

To see this assume that there exists such an infinitely descending chain $\{V_{n_i}, T_{n_i}, U_{n_i}\}_{i\geq 1}$. Take a vertex $v_0 \in V_0$ such that $V_{n_i} = v_0$ for infinitely many $i$ and assume that there exists a vertex $v \in V$ with $V_{n_i} = v$ for at most finitely many $i$ and $v \sim v_0$. Then, on the event that the process forms a descending chain, the probability of choosing $v$ after $v_0$ equals

$$\mathbb{P}(V_{n_{i+1}} = v \mid V_{n_i} = v_0) = \frac{\lambda_v}{\sum_{v'\in\mathcal{V}_{v_0}} \lambda_{v'}}$$

as the Poisson Point Process chooses a vertex with a probability proportional to the firing rate $\lambda_v$. Denote by $\{\tilde{n}_i\}_{i\geq 0}$ the indices such that $V_{\tilde{n}_i} = v_0$ for all $i$ then

$$\sum_{i=0}^{\infty} \mathbb{P}(V_{\tilde{n}_{i+1}} = v) = \infty$$

and thus, by Borel-Cantelli, $V_{n_i} = v$ for infinitely many $i$ contrary to the assumption.

Since $G$ is countable, all finite subsets of $G$ have finite vertex sets. Thus, an infinite descending chain exists if and only if an infinite descending chain $\{V_{n_i}, T_{n_i}, U_{n_i}\}$ with $\{V_{n_i}\}_{i\geq 1}$ *all distinct* exists.

Let $t > 0$ and $v \in V$. Using the above argument, the claim follows by showing that the expected number of descending chains of *distinct vertices* and length $n \geq 1$ starting at $v$ before time $t$ tends to 0 as $n \to \infty$. Let $\langle v_1, \ldots, v_n\rangle$ be a fixed self-avoiding path of length $n$ in $G$ starting from $v_1 = v$. Then, let $L_{\langle v_1,\ldots,v_n\rangle}$ denote the number of descending chains $\{(V_{k_i}, T_{k_i}, U_{k_i})\}_{i\in[n]}$ such that $V_{n_i} = v_i$ and $t > T_{k_i} > T_{k_{i+1}}$ for every $i \in [n]$. By the multivariate Mecke formula

$$\begin{aligned}\mathbb{E}\left[L_{\langle v_1,\ldots,v_n\rangle}\right] &= \mathbb{E}\left[\sum_{\{(V_1,T_1,U_1),\ldots,(V_n,T_n,U_n)\}\subset\mathbf{P}} \mathbb{1}_{\{V_1=v_1,\ldots,V_n=v_n\}}\mathbb{1}_{\{T_1>\cdots>T_n\}}\right] \\ &= \sum_{v'_1,\ldots,v'_n\in V} \int_{[0,t]^n} \mathbb{1}_{\{v'_1=v_1,\ldots,v'_n=v_n\}}\mathbb{1}_{\{t_1>\cdots>t_n\}} \prod_{i=1}^{n} \mu(v'_i, \mathrm{d}t_i)\end{aligned}$$

$$= \frac{t^n}{n!} \prod_{i=1}^{n} \lambda_{v_i}$$
$$\leq \frac{t^n}{n!} \overline{\Lambda}^n \, .$$

Since the graph has maximum degree $d$ there are at most $d^n$ self-avoiding paths of length $n$ originating at the same vertex $v$ and hence by

$$\limsup_{n \to \infty} \frac{\left(dt\overline{\Lambda}\right)^n}{n!} = 0$$

there are almost surely no self-avoiding walks of infinite length originating from any $v \in V$, which completes the proof. □

The property that a graph does not admit infinite descending chains is less strict than the property of bounded degree. Even some random graphs admit no infinite descending chains (see [2]).

## 3.2 Existence of equilibrium

The existence of equilibrium distributions can be shown for the same class of graphs as the existence of processes, i.e. graphs of bounded degree. Furthermore, this section shows that the uniqueness statement for equilibrium distributions given by the Banach fixed point theorem on quasi vertex transitive graphs is weaker than Theorem 1.2.

The following condition specifies which existence is required.

**Condition 3.5.** *The graph $G$ is infinite, connected and of bounded degree such that it admits an equilibrium distribution $\mu$ with $\mu(e) > 0$ for all edges $e \in E$.*

Before treating the general case, the vertex transitive case is especially easy to handle and thus treated apart.

**Lemma 3.6.** *Let $G$ be vertex-transitive of degree $d$ and infinite with $\{\lambda_v\}_{v\in V}$ constant $\lambda_v \equiv \lambda > 0$. Then Condition 3.5 holds.*

*Proof.* For vertex transitive graphs the vertices are indistinguishable and thus there is an equilibrium distribution $\mu$ which is constant, i.e. where $\mu(e) \equiv \mu$ for $\mu > 0$ and $\alpha \in (0, 1)$. Inserting that into Eq. (2.2) gives

$$\mu = \sum_{v \in e} \lambda \frac{\mu^\alpha}{\sum_{e' \in E_v} \mu^\alpha}$$
$$= \sum_{v \in e} \lambda \frac{1}{\sum_{e' \in E_v} 1}$$
$$= \frac{2\lambda}{d} \, .$$

Thus, vertex transitive graphs admit a equilibrium distributions with non zero weights and hence Condition 3.5 holds. □

The case of bounded degree graphs does not admit such an easy analysis as the equilibrium distribution could theoretically take an infinite amount of different values.

**Lemma 3.7.** *Let $G$ be connected and of bounded degree. Then Condition 3.5 holds for all $\alpha \in (0,1)$.*

*Proof.* A map

$$\mu : E \to \mathbb{R}_+^d$$

is an equilibrium distribution if it fulfills

$$\mu(e) = \sum_{v \in e} \lambda_v \frac{\mu(e)^\alpha}{\sum_{e' \in E_v} \mu(e')^\alpha} .$$

The existence of an equilibrium distribution is thus a problem of finding a fixed point of the functional

$$\begin{aligned} g : \mathbb{R}_+^E &\longrightarrow \mathbb{R}_+^E \\ \mu(\cdot) &\longmapsto \sum_{v \in \cdot} \lambda_v \frac{\mu(\cdot)^\alpha}{\sum_{e' \in E_v} \mu(e')^\alpha} \end{aligned}$$

on the topological vector space $\mathbb{R}_+^E$ which is locally convex and Hausdorff as a product space of the locally convex and Hausdorff $\mathbb{R}_+$. Define the infinite product set

$$C = \left[ \left( \frac{\min\limits_{e \in E} \left\{ \sum\limits_{v \in e} \lambda_v \right\}}{\max\limits_{v \in V} \deg(v) \cdot \left( \max\limits_{e \in E} \left\{ \sum\limits_{v \in e} \lambda_v \right\} \right)^\alpha} \right)^{\frac{1}{1-\alpha}}, \max_{e \in E} \left\{ \sum_{v \in e} \lambda_v \right\} \right]^E$$

and note that $C$ is convex and closed as a product of convex and closed sets (using the product topology) and especially $\mu(e) > 0$ for $\mu \in C$. $C$ is also compact by the Tychonov theorem as a product of compact spaces. If

$$g \mid_C = C \tag{3.3}$$

holds then the claim follows by the Schauder fixed point theorem.

Note that, for $\mu \in \mathbb{R}_+^E$,

$$g(\mu(\cdot)) = \sum_{v \in \cdot} \lambda_v \frac{\mu(\cdot)^\alpha}{\sum_{e' \in E_v} \mu(e')^\alpha}$$

$$\leq \max_{e\in E}\left\{\sum_{v\in e}\lambda_v\right\} ,$$

which follows since $v \in e$ implies that $e \in E_v$ and thus

$$\frac{\mu(e)^\alpha}{\sum\limits_{e'\in E_v}\mu(e')^\alpha} \leq 1 .$$

Furthermore, for $\mu \in C$,

$$\begin{aligned}
g(\mu(\cdot)) &= \sum_{v\in\cdot}\lambda_v\frac{\mu(\cdot)^\alpha}{\sum_{e'\in E_v}\mu(e')^\alpha} \\
&\geq \sum_{v\in\cdot}\lambda_v\frac{\mu(\cdot)^\alpha}{\max\limits_{v'\in V}\deg(v')\cdot\left(\max\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}\right)^\alpha} \\
&\geq \left(\frac{\min\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}}{\max\limits_{v\in V}\deg(v)\cdot\left(\max\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}\right)^\alpha}\right)\mu(\cdot)^\alpha \\
&\geq \left(\frac{\min\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}}{\max\limits_{v\in V}\deg(v)\cdot\left(\max\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}\right)^\alpha}\right)^{\alpha+\frac{\alpha}{1-\alpha}} \\
&\geq \left(\frac{\min\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}}{\max\limits_{v\in V}\deg(v)\cdot\left(\max\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}\right)^\alpha}\right)^{\frac{1}{1-\alpha}} .
\end{aligned}$$

Hence, Equation (3.3) holds and the claim follows. □

The proof not only gives the existence but also the range for non-zero equilibrium distributions.

**Corollary 3.8.** *Any equilibrium distribution $\mu$ with $\mu(e) > 0$ for all $e \in E$ is bounded by*

$$\mu(e) \in \left[\left(\frac{\min\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}}{\max\limits_{v\in V}\deg(v)\cdot\left(\max\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}\right)^\alpha}\right)^{\frac{1}{1-\alpha}}, \max_{e\in E}\left\{\sum_{v\in e}\lambda_v\right\}\right]$$

*for all $e \in E$.*

The existence of equilibrium distributions on all graphs appearing in this thesis is the main result of this section. The rest of this section is a long remark on the importance of the uniqueness statement in Theorem 1.2, Theorem 1.3 and Theorem 1.4 since standard fixed point theory does not give an equivalent statement.

These results concern quasi vertex transitive graphs on which the related notion of quasi stationarity has to be defined first.

**Definition 3.9.** Let $\{\lambda_v\}_{v\in V}$ be a family of positive real values. Call $\{\lambda_v\}_{v\in V}$ *quasi stationary* if the family $\lambda_v = \lambda_{v'}$ for $v, v \in V$ if there exists a graph automorphism $f : G \to G$ such that $f(v) = v'$.

This allows the statement of the condition for uniqueness of equilibrium due to fixed point theory.

**Lemma 3.10.** *Let $G$ be a quasi vertex transitive graph with $m \in \mathbb{N}$ different edge classes. Then, for quasi stationary $\{\lambda_v\}_{v\in V}$*

1. *there exists an equilibrium distribution $\mu$ as in Condition 3.5 only taking at most $m$ different values for all $\alpha \in (0, 1)$.*
2. *there exists $\alpha_{\max} \in (0, 1)$ such that the equilibrium distribution $\mu$ as in Condition 3.5 is unique for all $\alpha \in (0, \alpha_{\max})$.*

*Proof.* Denote edges of the $m$ edge classes by $\{e_i\}_{i\in[m]} = \{(v_i, w_i)\}_{i\in[m]}$ and let $\mu$ be a weight distribution taking $m$ different values

$$\{\mu(e)\}_{e\in E} = \{\mu_j\}_{j\in[m]}$$

where $\mu_j$ denotes the values for the edge weights for the respective edge class $e_j$. Let $\{n_v^j\}_{j\in[m]}$ denote the amount of edges of class $j$ incident to the vertex $v \in V$. Then, the characterising function

$$\begin{aligned}\mu_j &= \sum_{v\in e_j} \lambda_v \frac{\mu_j^\alpha}{\sum_{e'\in E_v} \mu(e')^\alpha} \\ &= \lambda_{v_j} \frac{\mu_j^\alpha}{\sum_{i=1}^m n_{v_j}^i \mu_i^\alpha} + \lambda_{w_j} \frac{\mu_j^\alpha}{\sum_{i=1}^m n_{w_j}^i \mu_i^\alpha} \end{aligned} \tag{3.4}$$

for an equilibrium distribution $\mu \in \mathbb{R}_+^E$ is invariant under change of representatives of edge classes and thus the solutions of Equation (3.4) give an euqilibrium distribution as required in the first part of the claim.

The function $g$ from the proof of Lemma 3.7 becomes

$$\begin{aligned} g\colon \mathbb{R}_+^n &\longrightarrow \mathbb{R}_+^n \\ \mu &\longmapsto (g_1(\mu), \ldots, g_m(\mu)) \end{aligned}$$

where

$$g_j(\mu) := \lambda_{v_j} \frac{\mu_j^\alpha}{\sum_{i=1}^m n_{v_j}^i \mu_i^\alpha} + \lambda_{w_j} \frac{\mu_j^\alpha}{\sum_{i=1}^m n_{w_j}^i \mu_i^\alpha} \ .$$

The claim follows by showing that if there exists a convex subset $C \subset \mathbb{R}_+^d$ for which the Banach fixed point theorem is applicable, there exists a unique fixed point of $g$ on $C$.

The function $g$ is differentiable as a concatenation of differentiable functions and straight-up calculation gives the required result (proven later to keep the argument at hand concise).

**Lemma 3.11.** *There exists a nonempty, compact, convex subset $C$ of $\mathbb{R}_+^d$ which does not intersect any axes such that*

$$g|_C : C \to C$$

*for all $\alpha \in (0,1)$ and there exists an $\alpha_{\max} \in (0,1)$ and a $K \in (0,1)$ such that*

$$\|Dg(\mu)\|_{\mathrm{op}} := \sup_{\|x\|=1} \|Dg(\mu) \cdot x\| = \sup_{x \in \mathbb{R}^d} \frac{\|Dg(\mu) \cdot x\|}{\|x\|} \leq K$$

*for all $\mu \in C$, $\alpha \in (0, \alpha_{\max})$ where $Dg$ is the Jacobian matrix of $g$ and $Dg(\mu) \cdot x$ emphasizes that the product is a matrix-vector product.*

Using this Lemma the rest of the proof is standard for fixed-point theory. Let $C$, $\alpha_{\max}$ and $K$ as in Lemma 3.11. The first part of the claim follows by the Schauder fixed point theorem as a simplification of the proof for Lemma 3.7.

Let $\mu, \nu \in C$ and $\alpha \in (0, \alpha_{\max})$. As $C$ is convex, $C$ contains all points on the straight line $\mu + t(\nu - \mu)$ from $\mu$ to $\nu$ for $t \in [0,1]$. By the chain rule for Jacobian matrices applied to $G(t) := g(\mu + t(\nu - \mu))$

$$\frac{\mathrm{d}G}{\mathrm{d}t}(t) = G'(t) = Dg\left(\mu + t(\nu - \mu)\right) \cdot (\nu - \mu)$$

and thus by the fundamental theorem of calculus

$$\begin{aligned} g(\nu) - g(\mu) &= G(1) - G(0) \\ &= \int_0^1 G'(t) \mathrm{d}t \\ &= \int_0^1 Dg\left(\mu + t(\nu - \mu)\right) \cdot (\nu - \mu) \mathrm{d}t \ . \end{aligned}$$

Since $\|Ax\| \leq \|A\|_{\mathrm{op}} \|x\|$ for any linear operator $A$ on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, $g$ is a contraction on $C$ since $g \colon C \longrightarrow C$ and

$$\|g(\nu) - g(\mu)\| = \left\| \int_0^1 Dg\left(\mu + t(\nu - \mu)\right)(\nu - \mu) \mathrm{d}t \right\|$$

$$
\begin{aligned}
&\leq \int_0^1 \|Dg\left(\mu+t(\nu-\mu)\right)(\nu-\mu)\|\mathrm{d}t \\
&\leq \int_0^1 \|Dg\left(\mu+t(\nu-\mu)\right)\|_{\mathrm{op}}\|(\nu-\mu)\|\mathrm{d}t \\
&\leq K\|\nu-\mu\| \, .
\end{aligned}
$$

By the Banach fixed-point theorem there exists a unique fixed point $\mu$ and since $C$ does not intersect any axes $\mu(e) > 0$ for all $e \in E$ and thus the claim follows. ☐

*Proof of Lemma 3.11.* Take

$$
C := \left[\left(\frac{\min_{e\in E}\left\{\sum_{v\in e}\lambda_v\right\}}{\left(\max_{e\in E}\left\{\sum_{v\in e}\lambda_v\right\}\right)^{\alpha}\max_{v\in V}\deg(v)}\right)^{\frac{1}{1-\alpha}}, \max_{e\in E}\left\{\sum_{v\in e}\lambda_v\right\}\right]^d
$$

from Corollary 3.8 which is a convex subset of $\mathbb{R}_+^d$ that does not intersect any axes and, by analogous calculations to the proof of Lemma 3.7, $g\colon C \to C$.

The claim follows by showing that there is a $K \in (0,1)$ bounding the operator norm of the Jacobian $Dg$ of $g$ (formally the Jacobian of $g|_C$ but the leave out the explicit mention of $C$ for increased clarity). This is an explicit calculation. The entries of the Jacobian read as

$$
\begin{aligned}
\frac{\partial g_j}{\partial \mu_j}(\mu) &= \alpha\mu_j^{\alpha-1}\sum_{v\in e_j}\lambda_v\frac{\sum_{i=1;i\neq j}^m n_v^i\mu_i^\alpha}{\left(\sum_{i=1}^m n_v^i\mu_i^\alpha\right)^2} \\
\frac{\partial g_j}{\partial \mu_k}(\mu) &= -\alpha\mu_j^\alpha\mu_k^{\alpha-1}\sum_{v\in e_j}\frac{\lambda_v n_v^k}{\left(\sum_{i=1}^m n_v^i\mu_i^\alpha\right)^2}
\end{aligned}
$$

for $[m] \ni k \neq j$ and $\mu \in C$. Thus, for $x \in \mathbb{R}^d$

$$
\begin{aligned}
\|Dg(\mu)\cdot x\| &= \sqrt{\sum_{j=1}^m\left(\sum_{k=1}^m\frac{\partial g_j}{\partial \mu_k}(\mu)x_k\right)^2} \\
&= \sqrt{\sum_{j=1}^m\alpha^2\mu_j^{2\alpha}\left(\sum_{v\in e_j}\lambda_v\frac{\sum_{k=1;k\neq j}^m n_v^k\mu_k^\alpha\left(\frac{x_k}{\mu_k}-\frac{x_j}{\mu_j}\right)}{\left(\sum_{i=1}^m n_v^i\mu_i^\alpha\right)^2}\right)^2} \\
&= \sqrt{\sum_{j=1}^m\alpha^2\mu_j^{2\alpha}\left(\sum_{v\in e_j}\lambda_v\frac{\sum_{k=1}^m n_v^k\mu_k^\alpha\left(\frac{x_k}{\mu_k}-\frac{x_j}{\mu_j}\right)}{\left(\sum_{i=1}^m n_v^i\mu_i^\alpha\right)^2}\right)^2} \, .
\end{aligned}
$$

For $\mu \in C$

$$\mu_k^\alpha \left( \frac{x_k}{\mu_k} - \frac{x_j}{\mu_j} \right) < \left( \mu_k^{\alpha-1} + \frac{\mu_k^\alpha}{\mu_j} \right)$$
$$\leq 2 \left( \frac{\left( \max_{e \in E} \left\{ \sum_{e \in v} \lambda_v \right\} \right)^{\alpha(2-\alpha)} \max_{v \in V} \deg(v)}{\min_{e \in E} \left\{ \sum_{v \in e} \lambda_v \right\}} \right)^{\frac{1}{1-\alpha}}$$

for all $j, k \in [m]$ and any $x \in \mathbb{R}^d$ with $\|x\| = 1$. Furthermore, using that the number $n_v^i$ of edges of class $i \in [m]$ adjacent to a vertex $v$ fulfills

$$\sum_{i=1}^m n_v^i = \deg(v)$$

gives

$$\frac{\mu_j^\alpha}{\left( \sum_{i=1}^m n_v^i \mu_i^\alpha \right)^2} \leq \frac{\mu_j^\alpha}{\left( \sum_{i=1}^m n_v^i \right)^2} \left( \frac{\left( \max_{e \in E} \left\{ \sum_{v \in e} \lambda_v \right\} \right)^{\alpha} \max_{v \in V} \deg(v)}{\min_{e \in E} \left\{ \sum_{v \in e} \lambda_v \right\}} \right)^{\frac{2\alpha}{1-\alpha}}$$
$$\leq \frac{1}{\deg(v)^2} \left( \frac{\left( \max_{e \in E} \left\{ \sum_{v \in e} \lambda_v \right\} \right)^{\frac{1+\alpha}{2}} \max_{v \in V} \deg(v)}{\min_{e \in E} \left\{ \sum_{v \in e} \lambda_v \right\}} \right)^{\frac{2\alpha}{1-\alpha}}$$

and

$$\sqrt{\sum_{j=1}^m \left( \sum_{v \in e_j} \lambda_v \frac{\sum_{k=1}^m n_v^k}{\deg(v)^2} \right)^2} = \frac{1}{\min_{v \in V} \deg(v)} \sqrt{\sum_{j=1}^m \left( \sum_{v \in e_j} \lambda_v \right)^2}$$
$$\leq \frac{\sqrt{m} \max_{e \in E} \left\{ \sum_{v \in e} \lambda_v \right\}}{\min_{v \in V} \deg(v)} .$$

Then, the operator norm of the Jacobian has the upper limit

$$||Dg(\mu)||_{\text{op}} = \sup_{\|x\|=1} ||Dg(\mu) \cdot x||$$
$$< \alpha \sqrt{\sum_{j=1}^m \left( \sum_{v \in e_j} \frac{\lambda_v \mu_j^\alpha}{\left( \sum_{i=1}^m n_v^i \mu_i^\alpha \right)^2} \sum_{k=1}^m n_v^k \left( \mu_k^{\alpha-1} + \frac{\mu_k^\alpha}{\mu_j} \right) \right)^2}$$

$$\leq \alpha \frac{\sqrt{m}}{\min\limits_{v\in V}\deg(v)} \left( \frac{\max\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\} \max\limits_{v\in V}\deg(v)}{\min\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}} \right)^{\frac{1+2\alpha}{1-\alpha}} \tag{3.5}$$

for any $\mu \in C$. This gives

$$\|Dg(\mu)\|_{\mathrm{op}}\big|_{\alpha=0} = 0\ .$$

Since

$$\max_{v\in V}\deg(v), \frac{\max\limits_{e\in E}\left\{\sum\limits_{e\in v}\lambda_v\right\}}{\min\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}} \geq 1$$

and the map

$$\alpha \mapsto \frac{1+2\alpha}{1-\alpha}$$

is strictly increasing, the right hand side of Eq. (3.5) also increases with $\alpha$ increasing. Thus, there exists $\alpha_{\max} \in (0,1)$ such that for any $\alpha \in (0, \alpha_{\max})$ there exists $K \in (0,1)$ with

$$\|Dg(\mu)\|_{\mathrm{op}} \leq K$$

from which the claim follows. □

Lemma 3.11 gives a condition for $\alpha$ for the equilibrium distribution to be unique.

**Corollary 3.12.** *The upper limits of uniqueness $\alpha_{\max}$ in Lemma 3.10 is the minimum solution for $\alpha$ in*

$$1 = \alpha \frac{\sqrt{m}}{\min\limits_{v\in V}\deg(v)} \left( \frac{\max\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\} \max\limits_{v\in V}\deg(v)}{\min\limits_{e\in E}\left\{\sum\limits_{v\in e}\lambda_v\right\}} \right)^{\frac{1+2\alpha}{1-\alpha}}\ . \tag{3.6}$$

*Then, there exists a unique equilibrium distribution.*

*Remark* 3.13. This condition is restrictive as even in the vertex transitive case of Lemma 3.6 Equation (3.6) reads

$$\alpha = \frac{1}{d^{\frac{3\alpha}{1-\alpha}}}$$

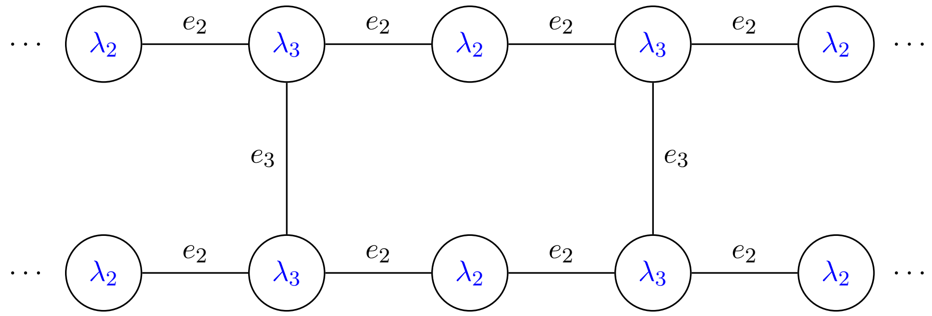


Figure 2: Ladder graph with every second rung missing and the notation used in Lemma 3.14.

which for the $\mathbb{Z}$ case with $d = 2$ numerically gives $\alpha_{\max} \approx 0.34$. Thus, even in the case where Theorem 1.3 proves convergence for all $\alpha \in (0, 1)$ to the unique equilibrium distribution, Lemma 3.10 does not give the uniqueness of the distribution on the same space for $\alpha$.

In general, Theorem 1.2 implies uniqueness of the equilibrium distribution for all $\alpha \in (0, 1/2)$ if Condition 3.5 holds for graphs of bounded degree. This shows that identifying this specific fixed point problem with weakly reinforcing dynamics on a graph gives results that are stronger than those given by Banach's fixed point theorem. While this is not immediately generalizable as it relies on the fixed point problem being a description for equilibrium, it is interesting as a case where probability theory helps solving a fixed point theory problem as opposed to the other way around.

## 3.3 Examples

This section will first discuss the quasi vertex transitive ladder graph missing every second rung to showcase an example allowing for the explicit calculation of an equilibrium distribution on non-vertex transitive graphs. After this, this section discusses limit values for $\mathbb{Z}$ with auxiliary vertices attached.

The ladder graph with missing rungs given in Figure 2 is a simple example for a quasi vertex transitive graph as it has two edge classes and two vertex classes. The existence of an equilibrium distribution as in Condition 3.5 can be explicitly calculated.

**Lemma 3.14.** *Let $G$ be the ladder graph given in Figure 2 with the corresponding firing rates $\{\lambda_2, \lambda_3\}$. Then there exists a unique equilibrium distribution taking up to two different values for $\alpha \in (0, 1)$.*

*Proof.* Let $\alpha \in (0, 1)$ first. The graph $G$ is quasi vertex transitive with vertex classes

$$V_i := \{v \in V | \lambda_v = \lambda_i\}$$

for $i \in \{2, 3\}$ as periodic translations connect those vertices. Then $G$ also has two edge classes $e_2$ and $e_3$. Let $\mu$ be an equilibrium distribution such that it assumes two values $c_2, c_3$ on the edge classes $e_2, e_3$ respectively. The claim follows by showing that the choice is unique. These values fulfill the relation

$$c_2 = \frac{\lambda_2}{2} + \lambda_3 \frac{c_2^\alpha}{2c_2^\alpha + c_3^\alpha} \tag{3.7}$$

$$c_3 = 2\lambda_3 \frac{c_3^\alpha}{2c_2^\alpha + c_3^\alpha} \ . \tag{3.8}$$

This implies, by an analogous calculation as in the proof of Lemma 3.7,

$$c_3 \in \left( 2\lambda_3 \left( 2^{1-2\alpha} \left( 1 + \frac{\lambda_2}{\lambda_3} \right)^\alpha + 1 \right)^{-\frac{1}{1-\alpha}}, 2\lambda_3 \right) \ .$$

Rearrange Equation (3.8) to get

$$\lambda_3 \frac{c_2^\alpha}{2c_2^\alpha + c_3^\alpha} = \frac{2\lambda_3 - c_3}{4} \ .$$

Inserting this into Equation (3.7) gives

$$c_2 = \frac{\lambda_2 + \lambda_3}{2} - \frac{c_3}{4}$$

which is a closed form expression for $c_2$ as a function of $c_3$ which, inserted into Equation (3.8), implies

$$c_3 = \frac{2\lambda_3 c_3^\alpha}{2\left(\frac{\lambda_2+\lambda_3}{2} - \frac{c_3}{4}\right)^\alpha + c_3^\alpha}$$

which rearranges to

$$0 = c_3^{1-\alpha} \left( 2 \left( \frac{\lambda_2 + \lambda_3}{2} - \frac{c_3}{4} \right)^\alpha + c_3^\alpha \right) - 2\lambda_3 \tag{3.9}$$

giving

$$0 = \frac{2}{4^\alpha} \left( \frac{2\lambda_2 + 2\lambda_3}{c_3} - 1 \right)^\alpha - \left( \frac{2\lambda_3}{c_3} - 1 \right) \ .$$

Substitute

$$x := \left( \frac{2\lambda_3}{c_3} - 1 \right)^{\frac{1}{\alpha}}$$

then

$$x \in \left(0, \left(\left(2^{1-2\alpha}\left(1+\frac{\lambda_2}{\lambda_3}\right)^{\alpha}+1\right)^{\frac{1}{1-\alpha}}-1\right)^{\frac{1}{\alpha}}\right)$$

and

$$\frac{4}{2^{\frac{1}{\alpha}}}x = \left(\frac{\lambda_2}{\lambda_3}+1\right)x^{\alpha}+\frac{\lambda_2}{\lambda_3}\,. \tag{3.10}$$

The right hand side of Equation (3.10) has constant slope while the left hand side has a slope that is inversely proportional to $x$. Furthermore, at $x = 0$, the right hand side is larger than the left hand side. Thus, there is at most one value for $x$ such that Equation (3.10) holds and thus at most one equilibrium distribution and, as Lemma 3.10 implies the existence of at least one equilibrium distribution, there is a unique equilibrium distribution for each $\alpha \in (0, 1)$. □

The previous proof even implies a closed form solution.

**Corollary 3.15.** *Let $G$ be the ladder graph given in Figure 2 with the corresponding firing rates $\{\lambda_2, \lambda_3\}$. Then*

$$\mu(e_3) = \frac{1}{2}\left(\lambda_2 + 3\lambda_3 - \sqrt{\lambda_2^2 + 6\lambda_2\lambda_3 + \lambda_3^2}\right)$$
$$\mu(e_2) = \frac{1}{8}\left(3\lambda_2 + \lambda_3 - \sqrt{\lambda_2^2 + 6\lambda_2\lambda_3 + \lambda_3^2}\right)$$

*is an equilibrium distribution for $\alpha = 1/2$.*

*Proof.* Let $\alpha = 1/2$. Then there is a closed form solution for $c_3$ as Equation (3.9) reads

$$\begin{aligned}
0 &= \sqrt{c_3}\left(2\sqrt{\frac{\lambda_2+\lambda_3}{2}-\frac{c_3}{4}}+\sqrt{c_3}\right)-2\lambda_3\\
0 &= \sqrt{2c_3(\lambda_2+\lambda_3)-c_3^2}+c_3-2\lambda_3\\
(2\lambda_3-c_3)^2 &= 2c_3(\lambda_2+\lambda_3)-c_3^2\\
0 &= c_3^2+c_3(3\lambda_3+\lambda_2)+2\lambda_3^2\,,
\end{aligned}$$

which gives two solutions for $c_3$

$$\begin{aligned}
c_{3,1/2} &= \frac{\lambda_2+3\lambda_3}{2}\pm\sqrt{\frac{(\lambda_2+3\lambda_3)^2}{4}-2\lambda_3^2}\\
&= \frac{1}{2}\left(\lambda_2+3\lambda_3\pm\sqrt{\lambda_2^2+6\lambda_2\lambda_3+\lambda_3^2}\right)
\end{aligned}$$

from which the claim follows. □

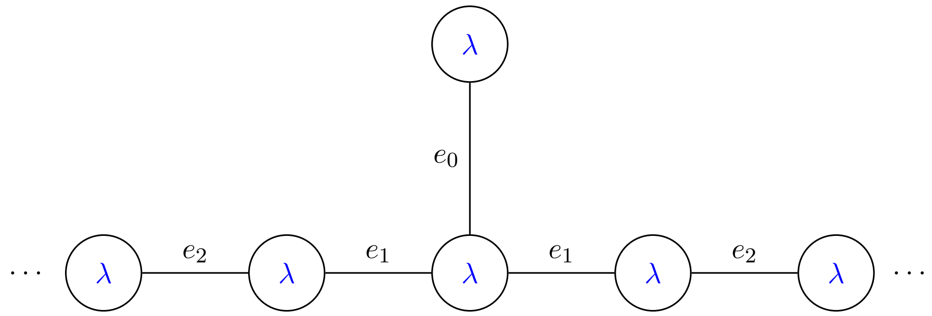


Figure 3: Graph of the integers with an auxiliary edge $e'$ attached to $\mathbb{Z}$ and the notation used in Lemma 3.17.

*Remark* 3.16. By Lemma 3.14 Condition 3.5 holds for the ladder graph with every second rung missing and $\alpha \in (0,1)$ if Equation (3.9) has a solution for that $\alpha$ as implied by Lemma 3.7.

The easiest example of an almost quasi vertex transitive graph results from attaching a new vertex to a vertex-transitive graph, e.g. $V = \mathbb{Z} + \{w\}$ with a new edge $e_0 = \{v, w\}$ attached to a new vertex $w$ and $v \in \mathbb{Z}$ which is displayed in Figure 3. The next few Lemmas deal with $\mathbb{Z}$ based almost vertex transitive graphs and show that the equilibrium distributions as in Condition 3.5, if they converge, converge to the distribution of $\mathbb{Z}$ the more the neighbourhood of an edge 'looks like' $\mathbb{Z}$. This gives insights into the behaviour of equilibrium distributions for non-quasi vertex transitive graphs.

**Lemma 3.17.** *Let* $G = (\mathbb{Z} + \{w\}, E + e_0)$ *and* $\{\lambda_v\}_{v \in \mathbb{Z}+\{w\}}$ *constant with* $\lambda_v \equiv \lambda$. *Then* $G$ *is almost vertex transitive and if the limit*

$$\lim_{d(e,f)\to\infty} \mu(e)$$

*exists for one pair* $e, f \in E$ *and* $\mu$ *as in Condition 3.5 then it exists for any pair and*

$$\lim_{d(e',f')\to\infty} \mu(e) = \lambda$$

*holds for any* $e', f' \in E$.

*Remark* 3.18. Note that the limit is the edge weight for the vertex transitive $\mathbb{Z}$ graph given in Lemma 3.6.

*Proof.* Removing the auxiliary vertex $w$ and its corresponding edge $e_0$ from the graph, which is a finite subgraph of $G$, gives back $\mathbb{Z}$ which is vertex transitive and thus $G$ is almost vertex transitive. In particular, $G$ is not quasi vertex transitive. This follows since any graph automorphism $f$ has to map $f(w) = w$

and since $\{f(v), f(v')\} \in E$ if and only if $\{v, v'\} \in E$ this means that $f$ has to be the identity meaning that the identity map is the only automorphism of $G$ which precludes quasi vertex transitivity.

Furthermore, the setup has a discrete spherical symmetry, i.e. the distance of an edge to $e_0$ determines its weight as the neighbourhoods of vertices of same distance to $e_0$ are identical. Let $c_n$ be the weight of the edge with distance $n$ from $e_0$ and let $c_0 = \mu(e_0) > 0$. Then

$$c_n^\alpha = c_{n-1}^\alpha \left( \frac{1}{\frac{1}{\lambda}\left(\sum_{i=1}^{n-1} c_i + \frac{1}{2}c_0\right) - (n-1)} - 1 \right) \qquad n \geq 2 \tag{3.11}$$

describes an equilibrium distribution. This follows by requiring that every edge fulfills the equilibrium equation leading to a recursion for $n \geq 1$

$$\begin{aligned} 0 &= -c_n + \lambda \left( \frac{c_n^\alpha}{c_n^\alpha + c_{n-1}^\alpha} + \frac{c_n^\alpha}{c_n^\alpha + c_{n+1}^\alpha} \right) \\ &= -c_n + \lambda - \lambda \frac{c_{n-1}^\alpha}{c_n^\alpha + c_{n-1}^\alpha} + \lambda \frac{c_n^\alpha}{c_n^\alpha + c_{n+1}^\alpha} \\ &= -c_n + \lambda - \sum_{i=1}^{n-1} c_i - \frac{c_0}{2} + \lambda(n-1) + \lambda \frac{c_n^\alpha}{c_n^\alpha + c_{n+1}^\alpha} \\ c_{n+1}^\alpha &= c_n^\alpha \left( \frac{1}{\frac{1}{\lambda}\left(\sum_{i=1}^{n} c_i + \frac{1}{2}c_0\right) - n} - 1 \right) . \end{aligned}$$

The weight $c_1$ is special, due to the concerned edges being incident to $e_0$. The equilibrium distribution equation for $e_0$ reads

$$0 = -c_0 + \lambda + \lambda \frac{c_0^\alpha}{c_0^\alpha + 2c_1^\alpha}$$

which implies

$$c_1^\alpha = \frac{c_0^\alpha}{2} \left( \frac{1}{\frac{c_0}{\lambda} - 1} - 1 \right) . \tag{3.12}$$

Thus, $c_0$ determines every edge weight in the equilibrium distribution and is the only free parameter left.

By Lemma 3.7 there exists a weight distribution $\mu \in \mathbb{R}_+^E$ such that Equation (3.11) and Equation (3.12) hold. Assume that there is a pair $e, f \in E$ as in the claim, then

$$\begin{aligned} \lim_{\mathrm{d}(e,f)\to\infty} \mu(e) &= \lim_{n\to\infty} c_n \\ &= \lim_{\mathrm{d}(e',f')\to\infty} \mu(e') \end{aligned}$$

for any other pair of edges $e', f' \in E$. So the claim follows by showing that

$$\lim_{n\to\infty} c_n = \lambda \,.$$

By Corollary 3.8 the sequence $\{c_n\}_{n\in\mathbb{N}}$ is a sequence of bounded, strictly positive real numbers so that the existence of the limit implies that

$$\lim_{n\to\infty} \left(\frac{c_{n+1}}{c_n}\right)^{\alpha} = L$$

for $L \in (0,1]$. If $L < 1$, then

$$\lim_{n\to\infty} c_n = 0$$

which is a contradiction to the strict positiveness and hence $L = 1$. Inserting this into Equation (3.11) gives

$$\begin{aligned} 1 &= \lim_{n\to\infty} \left(\frac{c_{n+1}}{c_n}\right)^{\alpha} \\ &= \frac{1}{\frac{1}{\lambda}\left(\sum_{i=1}^{\infty}(c_i - \lambda) + \frac{1}{2}c_0\right)} - 1 \end{aligned}$$

which rearranges to

$$\frac{1}{2}\left(\lambda - c_0\right) = \sum_{i=1}^{\infty}(c_i - \lambda) \,. \tag{3.13}$$

For the right hand side of Equation (3.13) to be finite, the summands have to converge to zero and thus

$$\lim_{n\to\infty} c_n = \lambda \,.$$

□

The previous proof gives a set of equations that uniquely characterize Condition 3.5 by eliminating all the free parameters if the equilibrium distribution converges. These expressions emphasize the non-linear nature of the equilibrium distribution. Even for this easy case of almost quasi vertex transitive graphs there is no clear way to say that $c_n$ has more *influence* on the value of $c_0$ than $c_m$ for $m > n$.

**Corollary 3.19.** *Consider $G$ and $\mu$ as in Lemma 3.17. If the equilibrium distribution $\mu$ converges then*

$$\frac{c_0}{2} = \frac{\lambda}{2} - \sum_{i=1}^{\infty}(c_i - \lambda)$$

$$c_1^\alpha = -\frac{\left(\lambda - \sum_{i=1}^{\infty}(2c_i - \lambda)\right)^\alpha}{2}\left(\frac{1}{\frac{2}{\lambda}\sum_{i=1}^{\infty}(c_i - \lambda)} + 1\right)$$
$$c_n^\alpha = c_{n-1}^\alpha\left(\frac{\frac{1}{2} + \frac{1}{\lambda}\sum_{i=n}^{\infty}(c_i - \lambda)}{\frac{1}{2} - \frac{1}{\lambda}\sum_{i=n}^{\infty}(c_i - \lambda)}\right)$$

*holds for $n > 1$.*

*Proof.* Rearrange Equation (3.13) to

$$\frac{c_0}{2} = \frac{\lambda}{2} - \sum_{i=1}^{\infty}(c_i - \lambda) \ .$$

Inserting this into Equation (3.12) gives

$$\begin{aligned} c_1^\alpha &= -\frac{\left(\lambda - \sum_{i=1}^{\infty}(2c_i - \lambda)\right)^\alpha}{2}\left(\frac{1}{\frac{2}{\lambda}\left(\frac{\lambda}{2} - \sum_{i=1}^{\infty}(c_i - \lambda)\right) - 1} - 1\right) \\ &= -\frac{\left(\lambda - \sum_{i=1}^{\infty}(2c_i - \lambda)\right)^\alpha}{2}\left(\frac{1}{\frac{2}{\lambda}\sum_{i=1}^{\infty}(c_i - \lambda)} + 1\right) \ . \end{aligned}$$

And, finally, inserting it into Equation (3.11) gives

$$\begin{aligned} c_n^\alpha &= c_{n-1}^\alpha\left(\frac{1}{\frac{1}{\lambda}\left(\sum_{i=1}^{n-1}c_i + \frac{\lambda}{2} - \sum_{i=1}^{\infty}(c_i - \lambda)\right) - (n-1)} - 1\right) \\ &= c_{n-1}^\alpha\left(\frac{\frac{1}{2} + \frac{1}{\lambda}\sum_{i=n}^{\infty}(c_i - \lambda)}{\frac{1}{2} - \frac{1}{\lambda}\sum_{i=n}^{\infty}(c_i - \lambda)}\right) \ . \end{aligned}$$

□

The proof of Lemma 3.17 generalizes to more general graphs.

**Corollary 3.20.** *Let $G$ be a connected, almost vertex transitive with corresponding vertex transitive graph $\mathbb{Z}$ and let $\mu$ be an equilibrium distribution as in Condition 3.5. If the limit*

$$\lim_{d(e,f)\to\infty} \mu(e)$$

*exists for one pair $e, f \in E$ then it exists for any pair and*

$$\lim_{d(e',f')\to\infty} \mu(e) = \lambda$$

*holds for any $e', f' \in E$.*

*Proof.* Assume that $\mu$ is an equilibrium distribution on $G$ with $\mu(e) > 0$ for all $e \in E$. Using that $G$ is connected, choose a partition $(\mathbb{Z}, G_s)$ such that the edge set $E_s$ does not intersect $\mathbb{Z}$ and that $G_s$ is as in Definition 2.16.

Denote by $\{c_i\}_{i\in\mathbb{Z}}$ the edge weights given by $\mu$ for the edges $\{v_i, v_{i+1}\}$ on $\mathbb{Z}$. Then, there exists $N \in \mathbb{N}$ such that

$$E_{v_n} \cap G_s = \emptyset$$

for all $n \geq N$. Then

$$c_n^\alpha = c_{n-1}^\alpha \left( \frac{1}{\frac{1}{\lambda}\sum_{i=N}^{n-1} c_i - (n-1)} - 1 \right) \qquad n \geq N\ .$$

and consequently

$$\lim_{n\to\infty} c_n = \lambda$$

by analogous arguments to the ones in the proof of Lemma 3.17. A similar calculation gives

$$\lim_{n\to-\infty} c_n = \lambda$$

and the claim follows. □

Thus, if the equilibrium distributions converge at infinity, it has to converge to the vertex transitive limit. These results fail to show that any equilibrium distribution has to converge.

These results further motivate the guess that equilibrium distributions on almost quasi vertex transitive graphs converge against the distribution on the corresponding quasi vertex transitive graph which would show that equilibrium distributions exhibit a sort of locality behaviour. This might be exploitable to extend the theorems beyond $\alpha \in (0, 1/2)$ as, for example, the dynamics $G = \mathbb{Z}$ converge to the equilibrium distribution for all $\alpha \in (0, 1)$.

# Chapter 4

# Proof of homogenization

This chapter contains the proofs and preparatory Lemmas for the theorems. Section 4.1 contains the proof for Theorem 1.2 and Section 4.2 contains the proofs for Theorem 1.3 and Theorem 1.4.

## 4.1 Proof of Theorem 1.2

Consider $G$ connected and such that it admits no infinite descending chains and the setup of $N_t$ as in Section 1.1. Especially, let the firing rates $\{\lambda_v\}_{v\in V}$ be an arbitrary set of positive real values. This preliminary section seeks to first bound $C_e$ away from zero and then allow for a tightening of the bounds to approach the equilibrium distribution successively.

As each edge is incident to two vertices firing at a rate bounded above by $\overline{\Lambda}$, the following lemma suggests itself intuitively.

**Lemma 4.1.** *Let $\alpha \in (0,1)$. Then, $\mathcal{C}_e \subset [0, 2\overline{\Lambda}]$ for all $e \in E$ almost surely.*

*Proof.* Recall the probability

$$\mathbb{P}(P_v([0,t] \times [0,1]) = n) = \mathrm{Poi}(\lambda_v t; n) = \frac{(\lambda_v t)^n}{n!} e^{-\lambda_v t},$$

of having had $n \in \mathbb{N}$ events fire on a vertex (independent of the incremented edge) for any time $t \in \mathbb{R}^+$. One extremal case is that each event that fires for $P_v$ increments the value for an edge $e = \{v, w\} \in E$ and that the rate of firing is $\overline{\Lambda}$, denote this process of maximal firing rate as $P_v^{\Lambda}$. The process

$$Y_t = \frac{P_w^{\Lambda}([0,t] \times [0,1]) + P_v^{\Lambda}([0,t] \times [0,1])}{t}$$

counts the normalised occurrences for this upper bound and has an expected value of

$$\mathbb{E}(Y_t) = \frac{\mathbb{E}(P_w^{\Lambda}([0,t] \times [0,1])) + \mathbb{E}(P_v^{\Lambda}([0,t] \times [0,1]))}{t} = \frac{2\overline{\Lambda}t}{t} = 2\overline{\Lambda}\,.$$

Then, the strong law of large numbers, Lemma 2.21, gives

$$\lim_{t\to\infty} Y_t = 2\overline{\Lambda} \ \mathbb{P}\text{-almost surely.}$$

Now,

$$X_t^e \leq Y_t + \frac{N_0^e}{t}$$

for all $t \in [0, \infty)$, implies that

$$\limsup_{t\to\infty} X_t^e \leq \lim_{t\to\infty} Y_t = 2\overline{\Lambda} \quad \mathbb{P}\text{-almost surely .}$$

□

The next goal is to bound $\mathcal{C}_e$ away from 0. This requires some preparatory lemmas. The following gives an idea about the growth speed of the edge weights under sufficiently big time steps.

**Lemma 4.2.** *Let $\alpha \in (0, 1)$, $k \geq 1$ a natural number, $v \in V$ and consider a sequence $\{a_{\ell,k,v}\}_{\ell\in\mathbb{N},k\geq 1,v\in V}$ of positive real numbers given by*

$$a_{\ell,k,v} = \underline{\Lambda}^{A_\ell^0}\overline{\Lambda}^{-A_\ell^1}\left(N_0 \wedge (\deg(v) - 1)^{\frac{1}{\alpha-1}}\right) 2^{k\ell - kA_\ell^1\alpha} ,$$

*where*

$$A_\ell^n = \sum_{j=n}^{\ell} \alpha^j .$$

*Then, there exists a constant $c > 0$ such that for any $\{v, w\} = e \in E$*

$$\begin{aligned} P_{\ell,k}^e :=&\mathbb{P}(N_{2^{k(\ell+1)}}^e \geq a_{\ell+1,k,v} \wedge a_{\ell+1,k,w} \mid N_{2^{k\ell}}^e \geq a_{\ell,k,v} \wedge a_{\ell,k,w}) \\ \geq& 1 - \exp\left(-c(a_{\ell+1,k,v} \wedge a_{\ell+1,k,w})\right) . \end{aligned}$$

*holds for $k$ large enough and all $\ell \geq 0$* .*

*Proof.* Consider an edge $e = \{v, w\}$ where $\deg(v) \geq \deg(w)$ without loss of generality and note that then $\deg(v) > 1$ since otherwise $v, w$ would form an isolated part in the graph in contradiction to the assumption that $G$ is

*Note the notation often used in probability theory:

$$\max(a, b) = a \vee b$$
$$\min(a, b) = a \wedge b$$

infinite and connected. Let $a_{\ell,k} := a_{\ell,k,v} \wedge a_{\ell,k,w} = a_{\ell,k,v}$. Then, for any $k \in \mathbb{N}$, Lemma 4.1 gives an $\varepsilon > 0$ such that

$$N_t^{e'} \leq 2\overline{\Lambda}(1+\varepsilon)t, \quad \text{for all } t \geq 2^k ,$$

for any $e' \in E_v \cup E_w$. Hence, under the event $N^e_{2^{k\ell}} \geq a_{\ell,k}$, Eq. (1.1) has a lower bound for times $t \in [2^{k\ell}, 2^{k(\ell+1)}] =: T_\ell$ given by

$$\begin{aligned}
\frac{1}{\mathrm{pol}_{v,e}(N_t)} &= \frac{\sum_{e' \in E_v} \left(N_t^{e'}\right)^\alpha}{(N_t^e)^\alpha} \\
&\leq \frac{{a_{\ell,k}}^\alpha + (\deg(v)-1)(1+\varepsilon)^\alpha \left(2\overline{\Lambda}\right)^\alpha 2^{k(\ell+1)\alpha}}{{a_{\ell,k}}^\alpha} \\
&= 1 + \Big((\deg(v)-1)(1+\varepsilon)^\alpha \underline{\Lambda}^{-A^1_{\ell+1}} \cdot \\
&\qquad \cdot \left(N_0 \wedge (\deg(v)-1)^{\frac{1}{\alpha-1}}\right)^{-\alpha} \overline{\Lambda}^{A^1_{\ell+1}} 2^{\alpha + kA^1_{\ell+1}}\Big) \\
&\leq \frac{(1+2^{-k\alpha})(\deg(v)-1)(1+\varepsilon)^\alpha 2^\alpha \overline{\Lambda}^{A^1_{\ell+1}} 2^{kA^1_{\ell+1}}}{\underline{\Lambda}^{A^1_{\ell+1}} \left(N_0 \wedge (\deg(v)-1)^{\frac{1}{\alpha-1}}\right)^\alpha} \\
&=: \frac{1}{G(k,\ell,\varepsilon)} .
\end{aligned}$$

By the assumption $\deg(v) \geq \deg(w)$ this implies

$$\mathrm{pol}_{v,e}(N_t) \wedge \mathrm{pol}_{w,e}(N_t) \geq G(k,\ell,\varepsilon) . \tag{4.1}$$

Now, calculate a lower bound for $P^e_{\ell,k}$. Let $t \in T_\ell$, the Poisson processes $P_v$ and $P_w$ having points in $[2^{k\ell}, t] \times U_e$ where $U_e \subset [0,1]$ and $\mathcal{L}(U_e) \geq G(k,\ell,\varepsilon)$ implies weight increases of the edge $e$ (by one or more) in the time frame $[2^{k\ell}, t]$. This follows since the bound in Eq. 4.1 gives a *lowest* value for $\mathrm{pol}_{v,e}$, which is equivalent to the probability of incrementation. Limit the considered increments to the ones in the time frame $T_\ell$ and use that the homogeneous Poisson Point Processes $\{P_e\}_{e\in E}$ are pairwise independent and thus their sums again homogeneous Poisson Point processes to get a lower bound for $P^e_{\ell,k}$. Explicitly

$$\begin{aligned}
P^e_{\ell,k} &= \mathbb{P}\left(N^e_{2^{k(\ell+1)}} \geq a_{\ell+1,k} \mid N^e_{2^{k\ell}} \geq a_{\ell,k}\right) \\
&\geq \mathbb{P}\left(N^e_{2^{k(\ell+1)}} - N^e_{2^{k\ell}} \geq a_{\ell+1,k} \mid N^e_{2^{k\ell}} \geq a_{\ell,k}\right) \\
&\geq \mathbb{P}\left(P_v(T_\ell \times U_e) + P_w(T_\ell \times U_e) \geq a_{\ell+1,k} \mid N^e_{2^{k\ell}} \geq a_{\ell,k}\right) \\
&= \mathbb{P}\left(\mathrm{Poi}\left((\lambda_v + \lambda_w)\,\mathcal{L}(T_\ell) G(k,\ell,\varepsilon)\right) \geq a_{\ell+1,k} \mid N^e_{2^{k\ell}} \geq a_{\ell,k}\right) \\
&\geq \mathbb{P}\left(\mathrm{Poi}(2\underline{\Lambda}\mathcal{L}(T_\ell) G(k,\ell,\varepsilon)) \geq a_{\ell+1,k} \mid N^e_{2^{k\ell}} \geq a_{\ell,k}\right) .
\end{aligned}$$

Note that this is the probability for a Poisson-distributed random variable and is independent of the weights on the graph, and thus independent from its condition so that

$$P^e_{\ell,k} \geq \mathbb{P}\left(\mathrm{Poi}(2\underline{\Lambda}\mathcal{L}(T_\ell)G(k,\ell,\varepsilon)) \geq a_{\ell+1,k}\right) . \tag{4.2}$$

Use Lemma 2.22 to handle this term. For the application, the order between the Poisson parameter in Eq. (4.2) and $a_{\ell+1,k}$ is required. Recall that

$$2\underline{\Lambda}\mathcal{L}(T_\ell)G(k,\ell,\varepsilon) = a_{\ell+1,k}\frac{\left(N_0 \wedge (\deg(v)-1)^{\frac{1}{\alpha-1}}\right)^{\alpha-1}}{\deg(v)-1}\frac{2^{1-\alpha}\left(1-2^{-k}\right)}{(1+2^{-k\alpha})(1+\varepsilon)^\alpha} ,$$

and that by Lemma 4.1, $\varepsilon$ is arbitrarily small for large enough $k \in \mathbb{N}$. Thus, let $k \to \infty$ and choose $\varepsilon$ such that the second fraction is arbitrarily close to 1. Furthermore, $\alpha \in (0,1)$ implies that $2^{1-\alpha} > 1$ and

$$\frac{\left(N_0 \wedge (\deg(v)-1)^{\frac{1}{\alpha-1}}\right)^{\alpha-1}}{\deg(v)-1} \geq 1 ,$$

which can be easily seen by noting that if $N_0 > (\deg(v)-1)^{1/(\alpha-1)}$ then the fraction evaluates to 1 and that the fraction is inversely proportional to $N_0$ as $\alpha - 1 < 0$. Hence, $K \in \mathbb{N}$ can be chosen such that the coefficient of $a_{\ell+1,k}$ is larger than or equal to 1 for all $k > K$. The resulting inequality

$$2\underline{\Lambda}\mathcal{L}(T_\ell)G(k,\ell,\varepsilon) \geq a_{\ell+1,k}$$

implies that, for fixed $k$ and $\varepsilon$, there exists $\tilde{c} > 1$ such that

$$2\underline{\Lambda}\mathcal{L}(T_\ell)G(k,\ell,\varepsilon) = \tilde{c} \cdot a_{\ell+1,k} .$$

Note that $c$ is not dependent on $\ell$. Thus, Lemma 2.22 gives

$$\begin{aligned} P^e_{\ell,k} &\geq 1 - \exp\left(a_{\ell+1,k} - 2\underline{\Lambda}\mathcal{L}(T_\ell)G(k,\ell,\varepsilon) + a_{\ell+1,k}\log\left(\frac{2\underline{\Lambda}\mathcal{L}(T_\ell)G(k,\ell,\varepsilon)}{a_{\ell+1,k}}\right)\right) \\ &= 1 - \exp\left((1-\tilde{c}+\log(\tilde{c}))a_{\ell+1,k}\right) . \end{aligned}$$

The function $1 - x + \log(x)$ has the maximum 0 at $x = 1$ and decreases monotonically for $x > 1$. Hence, the above coefficient of $a_{\ell+1,k}$ is negative and choosing

$$c = |1 - \tilde{c} + \log(\tilde{c})|$$

yields the claim for all $\ell$. □

*Remark* 4.3. For $\alpha \in (0, 1/2)$ and $\ell \geq 1$

$$\ell - A^1_\ell \alpha > 0$$

and thus

$$\lim_{k\to\infty} a_{\ell,k,v} = \infty$$

because the exponent of 2 is strictly positive.

Lemma 4.2 gives a propagation law for lower bounds on the edge weights through time. Taking the limit of $t \to \infty$ these lower bounds allow the exclusion of trivial edge weights in $X_\infty$ at the cost of already limiting $\alpha$ to $(0, 1/2)$.

**Lemma 4.4.** *Let $\alpha \in (0, 1/2)$. Then, there exists a $K \in \mathbb{N}$ such that*

$$\mathcal{C}_e \subset \left[ \left( \frac{\underline{\Lambda}}{\overline{\Lambda}^{\alpha}} \right)^{\frac{1}{1-\alpha}} \frac{N_0 \wedge ((\deg(v) \vee \deg(w)) - 1)^{\frac{1}{\alpha-1}}}{2^{K\alpha/(1-\alpha)}}, 2\overline{\Lambda} \right]$$

*almost surely for all $\{v, w\} = e \in E$.*

*Proof.* The upper bound follows from Lemma 4.1. Let $E \ni e = \{v, w\}$ where, without loss of generality, $\deg(v) \geq \deg(w)$ and set $a_{\ell,k} := a_{\ell,k,v} \wedge a_{\ell,k,w} = a_{\ell,k,v}$. The claim follows if there exists $K \in \mathbb{N}$ such that

$$\mathbb{P}\left( \liminf_{t \to \infty} X_t^e < \left( \frac{\underline{\Lambda}}{\overline{\Lambda}^{\alpha}} \right)^{\frac{1}{1-\alpha}} \frac{N_0 \wedge (\deg(v) - 1)^{\frac{1}{\alpha-1}}}{2^{K\alpha/(1-\alpha)}} \right) = 0 \ . \tag{4.3}$$

This follows by showing that there exists a $K \in \mathbb{N}$ such that

$$\sum_{\ell=1}^{\infty} \mathbb{P}\left( N^e_{2^{K\ell}} < a_{\ell,k} \right) < \infty \tag{4.4}$$

since then Borel-Cantelli implies that $\{N^e_{2^{K\ell}} < a_{\ell,k}\}$ is true for at most finitely many $\ell \in \mathbb{N}$ almost surely, implying that

$$\begin{aligned} 1 \leq \liminf_{\ell \to \infty} \frac{N^e_{2^{K\ell}}}{a_{\ell,k}} &= \liminf_{\ell \to \infty} \frac{X^e_{2^{K\ell}} \left( 2^K \overline{\Lambda} \right)^{\sum_{j=1}^{\ell} \alpha^j}}{\underline{\Lambda}^{\sum_{j=0}^{\ell} \alpha^j} \left( N_0 \wedge (\deg(v) - 1)^{\frac{1}{\alpha-1}} \right)} \\ &= \frac{\liminf_{\ell \to \infty} X^e_{2^{K\ell}} \left( 2^K \overline{\Lambda} \right)^{\alpha/(1-\alpha)}}{\underline{\Lambda}^{\frac{1}{1-\alpha}} \left( N_0 \wedge (\deg(v) - 1)^{\frac{1}{\alpha-1}} \right)} \end{aligned}$$

almost surely, which is the statement from Eq. (4.3).

It remains to show that Eq. (4.4) holds. To this end, use that for two events $A_1$ and $A_2$ in an arbitrary probability space $(\Omega, \mathbb{P})$

$$\begin{aligned} 1 - \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1^c \cup A_2^c) &= \mathbb{P}\left( A_1^c \cup (A_2^c \cap A_1) \right) \\ &\leq \mathbb{P}(A_1^c) + \mathbb{P}(A_2^c) \mathbb{P}(A_1 \mid A_2^c) \\ &\leq \mathbb{P}(A_1^c) + \mathbb{P}(A_1 \mid A_2^c) \ . \end{aligned}$$

Reordering this gives that

$$\mathbb{P}(A_1) \leq \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \mid A_2^c) \tag{4.5}$$

holds, whereby $A_i^c$ indicates the complement of $A_i$ in $\Omega$.

Using Lemma 4.2 with Equation (4.5) gives

$$\begin{aligned}\mathbb{P}\left(N^e_{2^{k\ell}} < a_{\ell,k}\right) &\overset{(4.5)}{\leq} \begin{pmatrix}\mathbb{P}\left(\left\{N^e_{2^{k\ell}} < a_{\ell,k}\right\} \cap \left\{N^e_{2^{k(\ell-1)}} < a_{\ell-1,k}\right\}\right)\\ + \mathbb{P}\left(N^e_{2^{k\ell}} < a_{\ell,k} \mid N^e_{2^{k(\ell-1)}} \geq a_{\ell-1,k}\right)\end{pmatrix}\\ &\leq \begin{pmatrix}\mathbb{P}\left(\left\{N^e_{2^{k\ell}} < a_{\ell,k}\right\} \cap \left\{N^e_{2^{k(\ell-1)}} < a_{\ell-1,k}\right\}\right)\\ + \exp(-ca_{\ell,k})\end{pmatrix}\\ &\leq \mathbb{P}\left(N^e_{2^{k(\ell-1)}} < a_{\ell-1,k}\right) + \exp(-ca_{\ell,k})\ .\end{aligned}$$

A recursive evaluation gives

$$\mathbb{P}\left(N^e_{2^{k\ell}} < a_{\ell,k}\right) \leq \sum_{j=1}^{\ell} \exp(-ca_{\ell,k}) + \mathbb{P}\left(N^e_1 < \underline{\Lambda}\left(N_0 \wedge (\deg(v)-1)^{\frac{1}{\alpha-1}}\right)\right)\ .$$

Since $\underline{\Lambda} \in (0,1)$ and $N^e_1 > N_0$

$$\mathbb{P}\left(N^e_1 < \underline{\Lambda}\left(N_0 \wedge (\deg(v)-1)^{\frac{1}{\alpha-1}}\right)\right) \leq \mathbb{P}(N^e_1 < N_0) = 0\ .$$

This is an increasing sequence of non-negative real numbers in $\ell$. Apply the monotone convergence theorem to get

$$\begin{aligned}\lim_{k\to\infty}\sum_{\ell=1}^{\infty}\mathbb{P}\left(N^e_{2^{k\ell}} < a_{\ell,k}\right) &\leq \lim_{k\to\infty}\sum_{\ell=1}^{\infty}\sum_{j=1}^{\ell}\exp(-ca_{\ell,k})\\ &\leq \sum_{\ell=1}^{\infty}\sum_{j=1}^{\ell}\underbrace{\lim_{k\to\infty}\exp(-ca_{\ell,k})}_{=0} = 0\end{aligned}$$

whereby the last equation follows by Remark 4.3. As the summands, and thus the sum, are continuous in $k$, there exists a $K$ such that the sum is finite giving the claim by Eq. (4.3). □

In preparation for the theorem proofs develop lemmas allowing the iterative approach to equilibrium if there exists a non-trivial lower bound on $\mathcal{C}_e$. For the equilibrium to exist assume $G$ with bounded degree from here on out.

**Lemma 4.5.** *Let $v, w \in V, v \sim w$, $\mu$ an equilibrium distribution on $G$ and $\alpha \in (0,1)$. Suppose that there is $\varrho \in \mathbb{R}^+$ such that $\varrho > 1$ and $\mathcal{C}_e \subset \left[\frac{1}{\varrho}\mu(e), \varrho\mu(e)\right]$ almost surely holds for all $e \in E_v \cup E_w$. Then,*

$$C_{\{v,w\}} \subset \left[\frac{1}{\varrho^{2\alpha}}\mu(\{v,w\}), \varrho^{2\alpha}\mu(\{v,w\})\right]\ .$$

*Proof.* Without loss of generality, let $\deg(v) \geq \deg(w)$. For $\varrho > 1$ define

$$G_\varrho = \bigcap_{e\in E_v\cup E_w}\left\{\mathcal{C}_e \subset \left[\frac{1}{\varrho}\mu(e), \varrho\mu(e)\right]\right\}$$

as the event that $\mathcal{C}_e \subset \left[\frac{1}{\varrho}\mu(e), \varrho\mu(e)\right]$ holds for all $e \in E_v \cup E_w$. Under this event, $X_t^{\{v,w\}}$ gains mass at a rate of at least

$$\begin{aligned} a_v &= \frac{\left(\frac{1}{\varrho}\mu(e)t\right)^\alpha}{\left(\frac{1}{\varrho}\mu(e)t\right)^\alpha + \sum\limits_{\substack{e' \in E_v \\ e' \neq e}} (\varrho\mu(e')t)^\alpha} \\ &= \frac{\left(\frac{1}{\varrho}\mu(e)\right)^\alpha}{\left(\frac{1}{\varrho}\mu(e)\right)^\alpha + \sum\limits_{\substack{e' \in E_v \\ e' \neq e}} (\varrho\mu(e'))^\alpha} \\ &= \frac{1}{1 + \sum\limits_{\substack{e' \in E_v \\ e' \neq e}} \varrho^{2\alpha} \left(\frac{\mu(e')}{\mu(e)}\right)^\alpha} \\ &\geq \frac{1}{\varrho^{2\alpha} + \sum\limits_{\substack{e' \in E_v \\ e' \neq e}} \varrho^{2\alpha} \left(\frac{\mu(e')}{\mu(e)}\right)^\alpha} \end{aligned}$$

per vertex for $t$ large enough and $e = \{v, w\}$. More precisely, under $G_\varrho$ find a function $\{\varepsilon_t\}_{t\geq 0}$ with $\varepsilon_t \searrow 0$ which bounds the mass $N_t^{\{v,w\}}$ by below by

$$N_t^{\{v,w\}} \geq P_v([0,t] \times U_{v,t}) + P_w([0,t] \times U_{w,t}) + N_0^{\{v,w\}}$$

for all $t > 0$ where $\mathcal{L}(U_{v,t}) = a_v - \varepsilon_t$ and $\mathcal{L}(U_{w,t}) = a_w - \varepsilon_t$. Analogous arguments to the one in the proof of Lemma 4.1 give

$$\begin{aligned} \liminf_{t\to\infty} X_t^{\{v,w\}} &\geq \lambda_v \frac{1}{\varrho^{2\alpha} + \sum\limits_{\substack{e' \in E_v \\ e' \neq e}} \varrho^{2\alpha} \left(\frac{\mu(e')}{\mu(e)}\right)^\alpha} + \lambda_w \frac{1}{\varrho^{2\alpha} + \sum\limits_{\substack{e' \in E_w \\ e' \neq e}} \varrho^{2\alpha} \left(\frac{\mu(e')}{\mu(e)}\right)^\alpha} \\ &= \frac{\mu(e)}{\varrho^{2\alpha}} . \end{aligned}$$

where the last equality follows by $\mu$ being an equilibrium distribution. A similar argument leads to

$$\limsup_{t\to\infty} X_t^{\{v,w\}} \leq \varrho^{2\alpha}\mu(e)$$

and the claim follows. □

The first part of this section binds $\mathcal{C}$ away from zero and the second part shows that if $\mathcal{C}$ is bound away from zero, its bounds can be iteratively tightened. Combine these results to get the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Lemma 3.7 implies that there exists an equilibrium distribution $\mu$ such that $\mu(e) > 0$ for all $e \in E$. Lemma 4.4 and 4.5 imply that there exists a $\varrho > 1$ such that $C_e \subset \left[\frac{1}{\varrho}\mu(e), \varrho\mu(e)\right]$ almost surely for all $e \in E$ which can be iteratively tightened, with the $n$-th tightening step given by the $\mathbb{P}$-almost sure event

$$A_n = \left\{ C_e \subset \left[ \frac{1}{\varrho^{(2\alpha)^n}} \mu(e), \varrho^{(2\alpha)^n} \mu(e) \right] \text{ for all } e \in E \right\} .$$

For $\alpha < 1/2$ this yields the claim by taking the limit as

$$1 = \lim_{n \to \infty} \mathbb{P}\left(A_n\right) = \mathbb{P}\left( \bigcap_{n \in \mathbb{N}} A_n \right) = \mathbb{P}(C_e = \mu(e) \text{ for all } e \in E) .$$

Thus, for an equilibrium distribution $\mu$, $X_t$ converges $\mathbb{P}$-almost surely against it. By the uniqueness of almost sure limits this implies that if there exists an equilibrium distribution it must be unique. □

## 4.2 Proof of Theorem 1.3 and Theorem 1.4

Let $G$ be vertex-transitive with degree $d$ and let the firing rates $\{\lambda_v\}_{v \in V}$ be constant

$$\lambda_v \equiv \lambda$$

for some $\lambda > 0$. Improve the previous results by proving a version of Lemma 4.5, explicitly using the fact that the vertices are indistinguishable, which allows for an iterated approach to the equilibrium which includes a broader range for $\alpha$ than $(0, 1/2)$.

**Lemma 4.6.** *Let $v, w \in V$ with $v \sim w$ and $\alpha \in (0,1)$. Suppose that there are $0 < a < 2\lambda/d < b < 2\overline{\Lambda}$ such that $\mathcal{C}_e \subset [a, b]$ almost surely holds for all $e \in E_v \cup E_w$. Then, there exist $a' \in [a, 2\lambda/d)$ and $b' \in (2\lambda/d, b]$ such that $b'/a' \leq (b/a)^{2\alpha}$ and $\mathcal{C}_{\{v,w\}} \subset [a', b']$ almost surely.*

*Proof.* Let

$$G_{a,b} = \bigcap_{e \in E_v \cup E_w} \{\mathcal{C}_e \subset [a, b]\}$$

denote the event that $\mathcal{C}_e \subset [a, b]$ holds for all $e \in E_v \cup E_w$. Under this event, $X_t^{\{v,w\}}$ gains mass at a rate of at least

$$a'' = 2\lambda \frac{(at)^\alpha}{(at)^\alpha + (d-1)(bt)^\alpha} = 2\lambda \frac{a^\alpha}{a^\alpha + (d-1)b^\alpha} ,$$

for $t$ large enough. One can find the desired $a'$ by setting $a' = a \vee a''$. More precisely, under $G_{a,b}$ one can find a sequence $\{\varepsilon_t\}_{t\geq 0}$ with $\varepsilon_t \searrow 0$ such that

$$N_t^{\{v,w\}} \geq P_v([0,t] \times U_{v,t}) + P_w([0,t] \times U_{w,t}) + N_0^{\{v,w\}}$$

bounds the mass $N_t^{\{v,w\}}$ by below for all $t > 0$ where $\mathcal{L}(U_{v,t}) = \mathcal{L}(U_{w,t}) = a'/2\lambda - \varepsilon_t$. Analogous arguments to the one in the proof of Lemma 4.1 give

$$\liminf_{t\to\infty} X_t^{\{v,w\}} \geq a' \quad \mathbb{P}\text{-a.s.}\ .$$

Similar arguments give the upper bound

$$b' = b \wedge b'' \text{ where } b'' = 2\lambda \frac{b^\alpha}{(d-1)a^\alpha + b^\alpha}\ .$$

Combining both expressions gives

$$\begin{aligned} \frac{b'}{a'} &\leq \frac{b^\alpha}{a^\alpha} \cdot \frac{(d-1)b^\alpha + a^\alpha}{b^\alpha + (d-1)a^\alpha} \qquad (4.6) \\ &\leq \frac{b^\alpha}{a^\alpha} \cdot \frac{d \cdot b^\alpha}{d \cdot a^\alpha} \\ &\leq \left(\frac{b}{a}\right)^{2\alpha} \end{aligned}$$

almost surely. □

*Remark* 4.7. Note that in the special case $d = 2$ Eq. (4.6) reduces to

$$\frac{b'}{a'} \leq \left(\frac{b}{a}\right)^\alpha\ .$$

*Remark* 4.8. Using that

$$a'' = 2\lambda \frac{a^\alpha}{a^\alpha + (d-1)b^\alpha} > \frac{2\lambda a^\alpha}{(d-1)\left(2\overline{\Lambda}\right)^\alpha + (2\lambda/d)^\alpha}$$

yields that

$$\frac{2\lambda a^\alpha}{(d-1)\left(2\overline{\Lambda}\right)^\alpha + (2\lambda/d)^\alpha} > a \quad \Longleftrightarrow \quad a < \varphi\left(2\overline{\Lambda}\right)\ ,$$

where

$$\varphi(x) := \left(\frac{2\lambda}{(d-1)x^\alpha + (2\lambda/d)^\alpha}\right)^{1/1-\alpha},$$

then $a' = a \vee a'' = a''$ and thus $a' \in (a, 2\lambda/d]$ since $a'' > a$.

Define the concept of a discrete sphere to expand the tightening of the bounds for one edge to tightening of the bounds for edges inside a sphere.

**Definition 4.9.** Define the sphere $\partial D_N(v) \subset E$ of radius N centered around $v \in V$ on $G$ as the maximal set of edges, such that for any $\{w, w'\} \in \partial D_N(v)$: $d(v, w) \wedge d(v, w') = N$.

Furthermore, define it's interior $D_N(v)$ as the set of all edges surrounded by $\partial D_N(v)$, i.e.

$$D_N(v) = \bigcup_{n=0}^{N-1} \partial D_n(v) \ .$$

and note that $\partial D_N(v) \cap D_N(v) = \emptyset$.

Discrete spheres have similar properties to their continuous analogues. The following statement shows that a sphere surrounds its interior.

**Lemma 4.10.** *Let $v, w, w' \in V$ with $w \sim w'$ and $n \in \mathbb{N}$ such that $\{w, w'\} \in D_N(v)$. Then $E_w \cup E_{w'} \subset \partial D_N(v) \cup D_N(v)$.*

*Proof.* Since $d(w, v) \vee d(w', v) \leq N - 1$, any edge $\{u, u'\} \in E_w \cup E_{w'}$ has $d(u, v) \vee d(u', v) \leq N$ which implies the claim per definition of the sphere. □

Spheres around $v$ of radius $N$ also exhaust the set of edges less or equal that distance to $v$.

**Lemma 4.11.** *Let $v, w, w' \in V$ with $w \sim w'$ and $N \in \mathbb{N}$ such that $d(v, w) \vee d(v, w') \leq N$, then $\{w, w'\} \subset \partial D_N(v) \cup D_N(v)$.*

*Proof.* Follows by the fact that the spheres are the maximal sets. □

Finally, spheres encircle a set of edges and paths to the outside need to intersect them.

**Lemma 4.12.** *For any $v, w \in V$ and $N \in \mathbb{N}$ such that $d(v, w) > N$ there is no path $P \subset E$ from $v$ to $w$ such that $P \cap \partial D_N(v) = \emptyset$.*

*Proof.* Assume a path $P$ with $P \cap \partial D_N(v) = \emptyset$ exists. Since $d(v, w) > N$, $P$ contains an edge $\{u, u'\}$ such that $d(u, v) \wedge d(u', v) = N$ but this contradicts $\partial D_N(v)$ being, per definition, the maximal set with such edges. □

With this find a tightening of $\mathcal{C}_e$ for edges in the interior of a sphere depending on the edges in the sphere having non-trivial weights.

**Lemma 4.13.** *Let $\alpha \in (0, 1)$. Let $N \in \mathbb{N}$, $v \in V$ and $a_0 > 0$ such that $\mathcal{C}_e \subset [a_0, 2]$ almost surely for $e \in \partial D_N(v)$ then $\mathcal{C}_e \subset [a_0', 2]$ almost surely for $e \in D_N(v)$ where*

$$a_0' = \begin{cases} 2\frac{{a_0}^\alpha}{{a_0}^\alpha + (d-1)(2\overline{\Lambda})^\alpha} & \text{if } a_0 < \varphi\left(2\overline{\Lambda}\right) \ , \\ \varphi\left(2\overline{\Lambda}\right) & \text{else.} \end{cases}$$

*Remark* 4.14. Note that in the $a_0 < \varphi\left(2\overline{\Lambda}\right)$ case $a_0' > a_0$ (per definition of $\varphi\left(2\overline{\Lambda}\right)$).

*Proof.* Lemma 4.4 gives almost sure lower bounds $a_e > 0$ for $\mathcal{C}_e$ with $e \in D_N(v)$. If

$$\min\left(\{a_e\}_{e \in D_N(v)}\right) \geq \varphi\left(2\overline{\Lambda}\right) \tag{4.7}$$

choose $a_0'$ such that the claim follows. Assume that Eq. (4.7) does not hold. Choose $\{w, w'\} \in D_N(v)$ such that

$$a_{\{w,w'\}} = \min(\{a_e\}_{e \in D_N(v)}) \, .$$

Then, $\mathcal{C}_e \subset [a_{\{w,w'\}}, 2]$ for all $e \in E_w \cup E_{w'}$ by Lemma 4.10. As $a_e < \varphi\left(2\overline{\Lambda}\right)$, Lemma 4.6 is applicable on the $\{w, w'\}$ edge and by Remark 4.8 there exists a new lower bound $a_e'$ where $a_e < a_e'$. The definition of $D_N(v)$ ensures that this procedure to increase the lowest lower bound is repeatable until the claim follows, whereby the $a_0'$ in the case $a_0 < \varphi\left(2\overline{\Lambda}\right)$ comes from $a'$ as in Lemma 4.6. □

Using Lemma 4.6 and 4.13, improve Lemma 4.4 by replacing the lower bound by a strictly positive value. First, a definition to ease the proof later.

**Definition 4.15.** Let $N \in \mathbb{N}$, $v \in V$ and $a, b \in \mathbb{R}_+^E$. Denote by

$$\mathrm{S}_N^v(a,b) = \bigcap_{n \geq N} \bigcup_{e \in \partial D_n(v)} \{C_e \not\subset [a(e), b(e)]\}$$

the event that there exists an edge in every sphere of radius $n \geq N$ around $v$ such that $\mathcal{C}_e$ is not in $[a, b]$. Say that the vertex $v$ has a *opening range* of $N$. The name comes from the fact that under $S_N^v(a,b)$ Lemma 4.13 is not applicable anymore and no statement can be made on the edges contained within $D_N(v)$.

The inverse statement $\neg S_N^v(a,b)$ states that there exists an $n \geq N$ such that $\mathcal{C}_e$ is in $[a, b]$ for all edges in $\partial D_n(v)$. Under this event, say that $v$ has a *closing range* or *infinite opening range* as $\neg S_N^v(a,b)$ implies that Lemma 4.13 can be applied to all edges contained within $D_n(v)$.

**Lemma 4.16.** *Let* $\alpha \in (0,1)$*. Then* $\mathcal{C}_e \subset \left[\varphi\left(2\overline{\Lambda}\right), 2\right]$ *almost surely holds for any* $e \in E$*.*

*Proof.* Let the opening and closing ranges in this proof refer to the range $[\varphi\left(2\overline{\Lambda}\right), 2]$. The claim is equivalent to showing that

$$\mathbb{P}(\text{All vertices have finite opening ranges}) = 0 \tag{4.8}$$

This follows since Eq. (4.8) rewrites to

$$\mathbb{P}(\text{There exists } v \in V \text{ with closing range } 0) = 1 \ .$$

From which Lemma 4.13 implies the claim since Lemma 4.11 implies that for any $e \in E$ there exists $N \in \mathbb{N}$ such that $e \in D_n(v)$ for all $n > N$. Prove Eq. (4.8) by contradiction, i.e. show that the assumption

$$\mathbb{P}(\text{All vertices have finite opening ranges}) > 0 \tag{4.9}$$

leads to a contradiction. Eq. (4.9) implies

$$\begin{aligned}
0 <& \mathbb{P}\left(\forall v \in V \exists N_0 \in \mathbb{N} \colon S^v_{N_0}\left(\varphi\left(2\overline{\Lambda}\right), 2\right)\right) \\
=& \mathbb{P}\left(\bigcap_{v \in V} \bigcup_{N_0=0}^{\infty} S^v_{N_0}\left(\varphi\left(2\overline{\Lambda}\right), 2\right)\right) \\
\leq& \sum_{N_0=0}^{\infty} \mathbb{P}\left(S^v_{N_0}\left(\varphi\left(2\overline{\Lambda}\right), 2\right)\right) \ ,
\end{aligned}$$

for any $v \in V$. Hence, there exists a summation index $N \in \mathbb{N}$ and $\delta \in (0, 1)$ such that

$$\mathbb{P}\left(S^v_N\left(\varphi\left(2\overline{\Lambda}\right), 2\right)\right) = \mathbb{P}\left(v \text{ has an opening range of } N\right) = \delta \ . \tag{4.10}$$

By vertex-transitivity each edge connects two indistinguishable vertices and since the initializations of the edge weights are equal $\{\mathcal{C}_e\}_{e \in E}$ is stationary and Eq. (4.10) holds for any $N \in \mathbb{N}$. Furthermore, the event is decreasing with $N$ and taking the limit gives

$$\begin{aligned}
\delta =& \lim_{N \to \infty} \mathbb{P}\left(v \text{ has an opening range of } N\right) \\
=& \mathbb{P}\left(\bigcap_{N=0}^{\infty} S^v_N\left(\varphi\left(2\overline{\Lambda}\right), 2\right)\right) \\
=& \mathbb{P}\left(\forall n \in \mathbb{N} \exists e \in \partial D_n(v) \colon \mathcal{C}_e \not\subset \left[\varphi\left(2\overline{\Lambda}\right), 2\right]\right) \\
=& \mathbb{P}\left(v \text{ has an opening range of } 0\right) \ .
\end{aligned}$$

This gives

$$\delta \leq \mathbb{P}\left(C_e \not\subset \left[\varphi\left(2\overline{\Lambda}\right), 2\right] \text{ for some } e \in \partial D_n(v)\right) \tag{4.11}$$

for any $n \in \mathbb{N}$. Now, take a decreasing sequence $\{b_m\}_{m \in \mathbb{N}}$ of positive real numbers smaller than $2\lambda/d$ such that $b_m \searrow 0$. Using Lemma 4.4 and the above $n$ gives

$$1 = \mathbb{P}\left(\mathcal{C}_e \subset (0, 2\overline{\Lambda}] \text{ for all } e \in \partial D_{n+M}(v)\right)$$

$$= \mathbb{P}\left(\bigcup_{m=0}^{\infty} \{\mathcal{C}_e \subset (b_m, 2] \text{ for all } e \in \partial D_{n+M}(v)\}\right) \tag{4.12}$$
$$= \lim_{m\to\infty} \mathbb{P}\left(\mathcal{C}_e \subset [b_m, 2] \text{ for all } e \in \partial D_{n+M}(v)\right) ,$$

where $M \in \mathbb{N}$ and stationarity of $\mathcal{C}_e$ has been used to get Equation (4.12). Thus, there exists $\tilde{m} \in \mathbb{N}$ such that

$$\mathbb{P}\left(\mathcal{C}_e \subset [b_{\tilde{m}}, 2] \text{ for all } e \in \partial D_{n+M}\right) > 1 - \delta . \tag{4.13}$$

Note that by the stationarity of $\mathcal{C}_e$, $\tilde{m}$ is independent of $M$. If $b_{\tilde{m}} > \varphi\left(2\overline{\Lambda}\right)$ the claim follows for $M = 0$ as $\mathcal{C}_e \subset [b_{\tilde{m}}, 2] \Rightarrow \mathcal{C}_e \subset [\varphi\left(2\overline{\Lambda}\right), 2]$ which, inserted into Eq. (4.13), gives the desired contradiction to Eq. (4.11). In the case $b_{\tilde{m}} < \varphi\left(2\overline{\Lambda}\right)$, Lemma 4.13 gives $b^1_m > b_m$ such that

$$\begin{aligned} 1 - \delta &< \mathbb{P}\left(\mathcal{C}_e \subset [b_{\tilde{m}}, 2] \text{ for all } e \in D_{n+M}\right) \\ &\leq \mathbb{P}\left(\mathcal{C}_e \subset [b^1_{\tilde{m}}, 2] \text{ for all } e \in \partial D_{n+M-1}\right) . \end{aligned}$$

Choose $M \in \mathbb{N}$ and iterate the process to get a sequence $\left\{b^{\tilde{M}}_{\tilde{m}}\right\}_{\tilde{M}\in[1,M]}$ such that $b^M_{\tilde{m}} > \varphi\left(2\overline{\Lambda}\right)$ and thus

$$1 - \delta < \mathbb{P}\left(\mathcal{C}_e \subset [\varphi\left(2\overline{\Lambda}\right), 2] \text{ for all } e \in \partial D_n\right)$$

which gives the desired contradiction to Eq. (4.11) again and thus the claim. □

Finally,

*Proof of Theorem 1.3.* Note that the case $G = \mathbb{Z}$ corresponds to $d = 2$. Then, Lemma 4.16, 4.6 and Remark 4.7 imply that there exist sequences $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 1}$ such that

1. $\{a_i\}_{i\geq 1}$ is increasing and bounded above by $\lambda$,
2. $\{b_i\}_{i\geq 1}$ is decreasing and bounded below by $\lambda$,
3. $b_{i+1}/a_{i+1} \leq (b_i/a_i)^\alpha < b_i/a_i$, and
4. $\mathcal{C}_e \subset [a_i, b_i]$ almost surely holds for all $i \geq 0, e \in E$.

Complete the proof by observing that the first three items imply that $a_i$ and $b_i$ converge to 1. Since $\{\mathcal{C}_e \subset [a_i, b_i]\}$ is a decreasing set the claim follows for any $e \in E$.

$$\begin{aligned} 1 &= \lim_{i\to\infty} \mathbb{P}\left(\mathcal{C}_e \subset [a_i, b_i]\right) \\ &= \mathbb{P}\left(\bigcap_{i\geq 0} \{\mathcal{C}_e \subset [a_i, b_i]\}\right) \\ &= \mathbb{P}\left(\mathcal{C}_e = 1\right) . \end{aligned}$$

□

*Proof of Theorem 1.4.* Intuitively, the $\varphi\left(2\overline{\Lambda}\right)$ from Lemma 4.6 is the worst-case estimate for the lower bound on $\mathcal{C}_e$ using the previous bounds $\mathcal{C}_e \subset (0, 2\overline{\Lambda}]$. With these improved bounds, $\mathcal{C}_e \subset [\varphi\left(2\overline{\Lambda}\right), 2]$, update the worst-case estimate on the upper bound by $\varphi \circ \varphi\left(2\overline{\Lambda}\right)$ and repeating analogous proofs to get $C_e \subset [\varphi\left(2\overline{\Lambda}\right), \varphi \circ \varphi\left(2\overline{\Lambda}\right)]$. Define the sequence of improved lower, resp. upper, bounds as

$$x_n = \varphi(\varphi(x_{n-1}))$$

and note that the only fixed point on $(0, 2\overline{\Lambda}]$ of that sequence is $2\lambda/d$, whereby uniqueness follows by monotonicity. Since the map $\varphi \circ \varphi$ describes one improvement step of either the lower or the upper bound on $\mathcal{C}_e$.

Complete the proof by iteratively improving $\mathcal{C}_e$ for all $\alpha < \alpha_d$ using Banach's fixed point theorem until it converges to the unique fixed point, choosing $\alpha_d > 1/2$ such that $\varphi \circ \varphi$ is contracting on $(0, 2\overline{\Lambda}]$. The rest of this proof concerns itself with showing the existence of $\alpha_d$.

$\varphi$ is strictly monotonically decreasing and thus $\varphi \circ \varphi$ is strictly monotonically increasing. Furthermore, for $\alpha = 1/2$

$$\frac{\varphi \circ \varphi(x)}{x} = \left( \frac{2\lambda}{\frac{2\lambda(d-1)}{d-1+\sqrt{\frac{2\lambda}{xd}}} + \sqrt{\frac{2\lambda x}{d}}} \right)^2 ,$$

which is again strictly monotonically decreasing. Thus the slope of $\varphi \circ \varphi$ is strictly lower than one, i.e.

$$\Phi_\alpha := \max_{x \in (0, 2\overline{\Lambda}]} (\varphi \circ \varphi)'(x) \in (0, 1) .$$

Using the fundamental theorem of calculus, this implies that $\varphi \circ \varphi(x)$ is a contraction on $(0, 2\overline{\Lambda}]$ with the standard euclidean metric $\mathrm{d}_{\mathbb{R}}$ since

$$\begin{aligned} \mathrm{d}_{\mathbb{R}}(\varphi(\varphi(x)), \varphi(\varphi(y))) &= |\varphi(\varphi(x)) - \varphi(\varphi(y))| \\ &= \left| \int_y^x (\varphi \circ \varphi)'(z) \mathrm{d}z \right| \\ &\leq \Phi_\alpha \left| \int_y^x \mathrm{d}z \right| \\ &= \Phi_\alpha \cdot \mathrm{d}_{\mathbb{R}}(x, y) . \end{aligned}$$

As the slope of $\varphi \circ \varphi$ increases with increasing $\alpha$ and $\varphi \circ \varphi$ is continuous in $\alpha$ there exists an $\alpha_d \in (1/2, 1)$ such that $\Phi_{\alpha_d} = 1$ and thus $\varphi \circ \varphi$ is contracting for all $\alpha \in (0, \alpha_d)$. □

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# Statement of Authorship

I hereby declare that I am the sole author of this master thesis and that I have not used any sources other than those listed in the bibliography and identified as references. I further declare that I have not submitted this thesis at any other institution in order to obtain a degree.

Munich, September 17, 2018

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