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Master's Thesis in Theoretical and Mathematical Physics

Tree tensor network approximations to conformal field theories

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Summary

Tree tensor networks (TTN) are often considered to be suitable a ansatz states for critical systems, however, this was not established rigorously. Here we construct an explicit TTN that yields multipoint functions of 2D CFTs and establish a general approximation scheme. We also give quantitative estimates for the approximation error for a rather large class of CFTs, namely WZW models, thus providing both suitable initial values and a rigorous justification of this class of variational methods.

Key words: tree tensor networks, CFT, multipoint function approximation.

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Chapter 1

Introduction

1.1 Tensor networks

Tensor networks (TN) were originally proposed as a family of numerical approaches to study strongly entangled quantum many-body systems, including quantum criticality and topological order. However, the range of applicability of the tensor network formalism has quickly extended well beyond the computational domain.

TN are currently also investigated as a natural framework to classify exotic phases of quantum matter, as the basis for new non-perturbative formulations of the renormalization group and interacting quantum field theories, and as a lattice realization of the AdS/CFT correspondence in quantum gravity (see e.g. [Orú14] or [ZCZW15] for a review).

We will use a term *tensor* to refer to a multidimensional array of complex numbers.

To understand why TN are natural and needed, let us consider a chain of n spin-$\frac{1}{2}$ fermions. If we are interested, for example, in the ground-state wave function, it is a superposition of computational basis vectors

$$|\psi\rangle_{GS} = \sum_{i_1,\ldots,i_n \in \{0,1\}} C_{i_1,\ldots i_N} |i_1 \ldots i_n\rangle. \tag{1.1}$$

In general it leaves us with $2^n - 1$ degrees of freedom - that is, the number of components of $C_{i_1,\ldots i_n}$ minus phase and normalization. This immediately yields a problem, as even relatively small systems of, say, a hundred particles do not fit into the memory of any existing classical computer. This means we need to work in some specific subspace of wave functions, and this subspace should contain wave functions which are arbitrarily close to ground-state ones and it should allow for entanglement.

To proceed, let us imagine that we can decompose $C_{i_1,\ldots i_n}$ into two smaller tensors

$$C_{i_1,\ldots i_n} = A_{i_1,\ldots i_k} B_{i_{k+1},\ldots i_n}. \tag{1.2}$$

This already yields a simplification, as now we have just $2^k + 2^{n-k} - 1$ degrees of freedom. This number can be much smaller than in the original case, however, it does not satisfy one of the conditions – the wave function is necessarily a product state, thus we have two **not** entangled parts

$$|\psi\rangle_{GS} = \left(\sum_{i_1,\dots,i_k\in\{0,1\}} A_{i_1,\dots i_n}|i_1\dots i_k\rangle\right) \cdot \left(\sum_{i_{k+1},\dots,i_n\in\{0,1\}} B_{i_{k+1},\dots i_n}|i_{k+1}\dots i_n\rangle\right). \tag{1.3}$$

However, we can introduce some additional complexity - let there be additional index α that run from 1 to χ

$$C_{i_1,\dots i_n} = \sum_{a\in\{0,\dots,\chi-1\}} A^a_{i_1,\dots i_k} B^a_{i_{k+1},\dots i_n}, \tag{1.4}$$

here a itself can be a multi-index. The quantity χ is called the *bond dimension.* Now there are $\chi(2^k + 2^{n-k}) - 1$ degrees of freedom. If χ is not too large, it can still be significantly smaller than $2^n - 1$ in the general case. To see that this ansatz can give entangled states, let us consider the GHZ state

$$|GHZ_n\rangle = \frac{|0\rangle^{\otimes n} + |1\rangle^{\otimes n}}{\sqrt{2}}, \tag{1.5}$$

which is known to be entangled (see e.g. [GHZ07]), and demonstrate that it can be obtained via the decomposition discussed above. Take $a \in \{0,1\}$, so that the bond dimension is 2. We can set

$$A^a_{i_1,\dots i_k} = \begin{cases} \frac{1}{\sqrt[4]{2}} & \text{if } a = i_1 = \dots = i_k \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

and use an analogous definition for $B^\alpha_{i_{k+1},\dots i_n}$ - it is non-zero and equals $\frac{1}{\sqrt[4]{2}}$ only if all of the indices are the same. It is trivial to check that with such definitions we get $|GHZ_n\rangle$.

As $\chi(2^k + 2^{n-k}) - 1$ can still be too large, we would like to reduce A and B further - that is, to introduce smaller tensors that are being contracted with each other. To simplify notation – and contractions of n tensors often start to look very complicated in standard notation, picture representation for tensors and their contraction is introduced. Also this notation is proved to be useful in introducing patterns of tensor contractions that correspond to physical intuition.

In *tensor network diagrams* tensors are represented by shapes and their indices by lines emerging from the shapes.

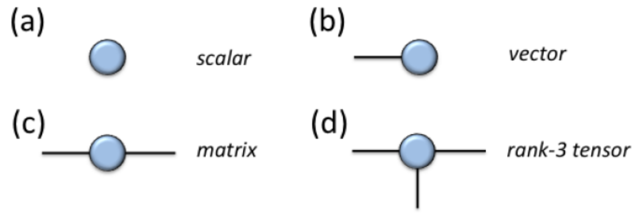


Figure 1.1: Tensor network diagrams: (a) a scalar, (b) a vector, (c) a matrix and (d) a tensor of rank 3. Image taken from [Orú14].

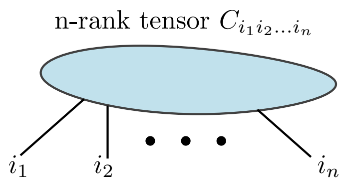


Figure 1.2: Diagram for some general tensor

When two shapes share a line, it corresponds to contraction of the corresponding index.

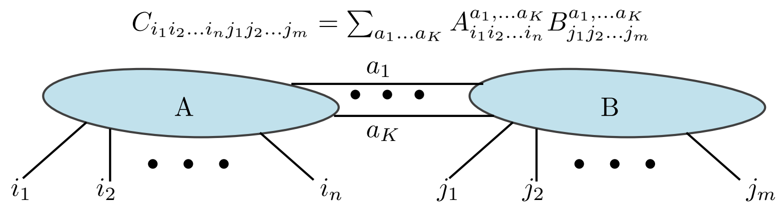


Figure 1.3: A shared line corresponds to a contracted index.

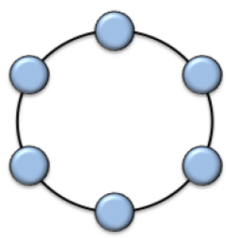


Figure 1.4: The trace of the product of 6 matrices. Image taken from [Orú14].

It is also possible to join two lines into 1 multi-index, if they are shared by the same two tensors, and it is possible to join shapes in such way that outer lines do not change.

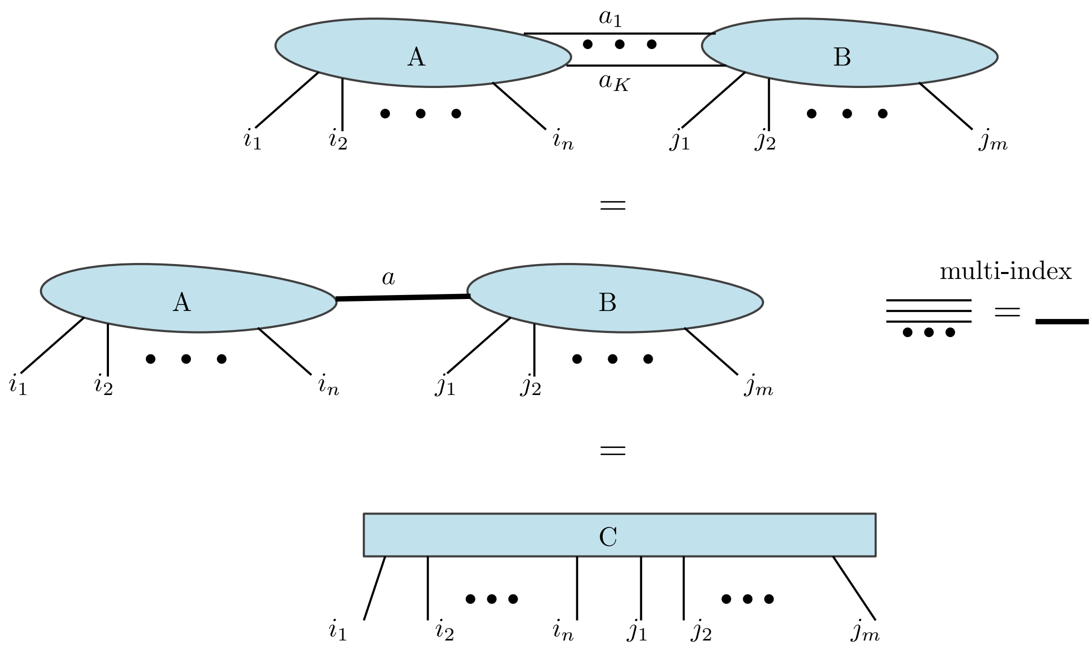


Figure 1.5: Joining shapes and introducing multi-indices

Consider a physical system that is defined on some arbitrary lattice. Let A be an operator which depends only on the lattice sites $k, \ldots, m$. Then we can also represent such an operator by some shape with $m-k$ ingoing and outgoing lines – $2(m-k)$ lines in total. If we have such a diagram for $|\psi\rangle$, then we will denote $\langle\psi|$ by the same diagram but which is flipped upside down. Then we will have the following figure 1.6 representing the expectation value $\langle\psi, A\psi\rangle$ of A with respect to ψ

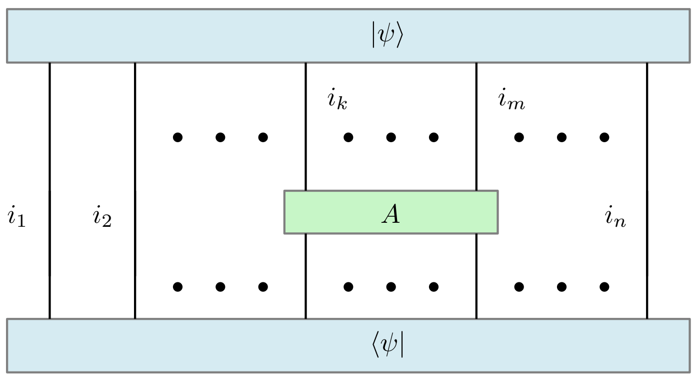


Figure 1.6: Diagrammatic expression for $\langle\psi, A\psi\rangle$

Suppose $A : V^n \to V^m$ is an operator that has n ingoing and m outgoing lines. Given positive definite Hermitian forms $\langle \cdot, \cdot \rangle_{V^n} : V^n \times V^n \to \mathbb{C}$ and $\langle \cdot, \cdot \rangle_{V^m} : V^m \times V^m \to \mathbb{C}$, one can define the adjoint of operator A, namely $A^\dagger : V^m \to V^n$, via

$$\langle A\psi, \phi \rangle_{V^m} = \langle \psi, A^\dagger \phi \rangle_{V^n} \quad \forall \psi \in V^n \text{ and } \forall \phi \in V^m \tag{1.7}$$

and such object is represented, just like $\langle \psi|$, by a flipped diagram (see figure 1.7).

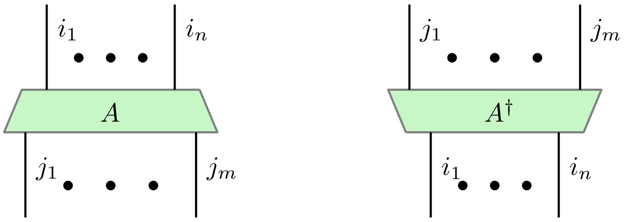


Figure 1.7: The adjoint of A is represented by flipped diagram.

We would like to represent objects like $C_{i_1,\ldots i_N}$ which have N outer lines. If we have a tensor with k outer lines, we can contract it with another tensor in such a way that the total number of outer lines increases. This requires at least tensors of rank 3 (see figure 1.8).

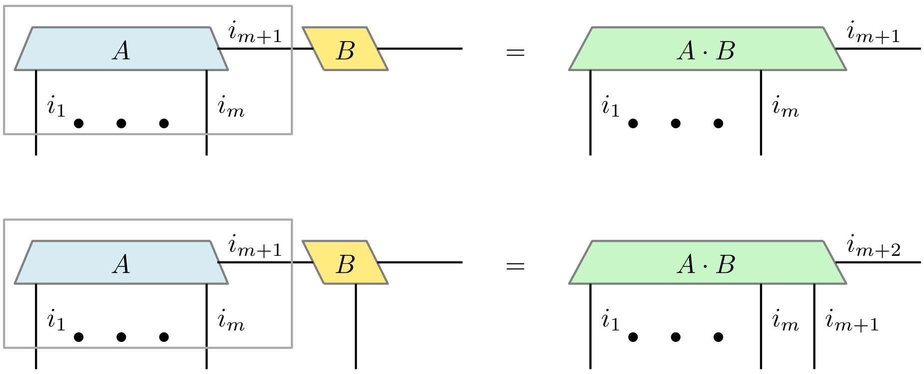


Figure 1.8: In order to construct a tensor with many legs from a low-rank connected tensors, one needs tensors of at least rank 3.

Two disconnected components correspond to a product state. We can obtain a rank n tensor with no disconnected parts by contraction of $O(n)$ rank-3 tensors. This gives a state which is potentially entangled and has only $O(n)$ parameters! This is a great simplification of the task, however, it is important to see that such states are indeed physically relevant.

1.1.1 Matrix Product States (MPS)

Matrix product states (MPS) are states of the form

$$|\psi\rangle_{GS} = \sum_{i_1,\dots,i_n\in\{0,1\}} \sum_{a_1,\dots,a_n} A_{i_1}^{a_1,a_2} \cdot A_{i_2}^{a_2,a_3} \cdot \ldots \cdot A_{i_n}^{a_n,a_1} |i_1 i_2 \ldots i_n\rangle \tag{1.8}$$

which corresponds to the picture:

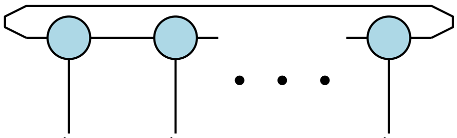


Figure 1.9: Diagrammatic representation of MPS.

If each $a_i \in \{1,\dots,\chi\}$ for a constant χ, then there are $2n\chi^2 - 1 = O(n)$ parameters in such an ansatz. The number χ is called the bond dimension of the MPS. We can cut the system in two parts by removing one of the connections between tensors that constitute the MPS. Then, we can use Schmidt decomposition

$$|\psi\rangle_{GS} = \sum_{\alpha=0}^{\infty} B_\alpha \psi_{L,GS} \otimes \psi_{R,GS} \tag{1.9}$$

with Schmidt coefficients B_α. As the bond dimension in the MPS approximation is χ, we can quantify error of the approximation as

$$\text{Error}_{MPS} = \sum_{\alpha=\chi}^{\infty} |B_i|^2. \tag{1.10}$$

It was proven by Hastings [Has07] that ground states of gapped Hamiltonians in one spatial dimensions can be arbitrarily well approximated by matrix product states in an efficient manner. That is, Error_{MPS} scales as χ^{-c}, where c is a constant that depends on the energy gap and the dimension of the Hilbert space on each site.

The MPS ansatz is used in various numerical algorithms including the *density matrix renormalization group* (DMRG) ([Whi92], for a more up-to-date review see e.g. [Sch05]). As of today it remains a method of choice for the analysis of a large number of one-dimensional systems.

An important generalization of the MPS to two-dimensional (or, in a similar way, higher-dimensional) systems are *projected entangled pair stares* (PEPS) [VC04]. This approach enables to not only describe the bulk of the material, but also the edge modes [YLP$^+$14].

There have been studies that generalize the MPS to a continuous number of variables [VC10]. These continuous MPS, or cMPS, are provide new way approach to quantum field theory and a fresh view on the real-space renormalization group methods (see e.g. [OEV10] and [JBH$^+$15]).

Recently it was proven by König and Scholz that CFTs can be described by the MPS [KS15]. To understand their result one should be familiar with the notion of transfer operator and truncation parameter that can be found in section 2.1. König and Scholz have proved that the error, namely, the operator norm of the difference between the transfer operator and truncated transfer operator, decreases with the *truncation parameter* N exponentially, while bond dimension increases as the number of vectors with weight less or equal to nN, where n is the number of (non-vacuum) fields (see section 1.2).

Our work is largely inspired by [KS15], however, it will deal with tree tensor networks.

1.1.2 Tree Tensor Networks (TTN) and MERA

Another approach which has at most rank-3 tensors is the tree tensor network (TTN). Unlike MPS it has a natural interpretation in terms of renormalization. TTN has following diagrammatic representation:

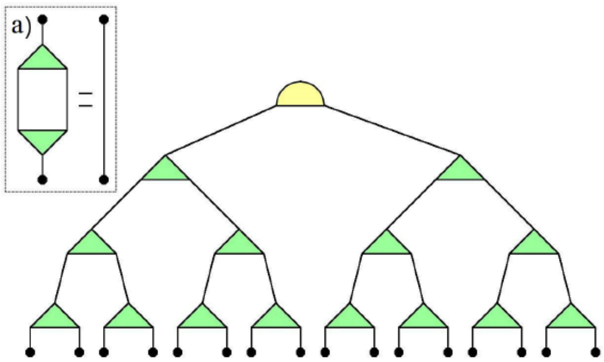


Figure 1.10: Tree tensor network for $16 = 2^m$ sites. Insert a) shows the isometric property (see figure 1.7). Image taken from [SGM$^+$10].

Indeed, this picture can be interpreted as two neighboring sites being mapped to one renormalized site per level of the tensor network. This approach has some similarity to the original spin-blocking procedure by Kadanoff (see e.g [EWKK14]). It was also shown that many-body states whose wave-function admits a representation in terms of a uniform binary tree tensor decomposition obey a power-law for the two-body correlations functions [SGM$^+$10]. Thus, TTN appear to be a reasonable ansatz for critical systems.

In order to have efficient contraction of a TTN - that is, to be able to compute expectation values of local operators fast, the isometry condition is introduced. Let us call the renormalization tensor ϵ. Suppose that every line on the lower edge of ϵ corresponds to a Hilbert space V with an inner product $\langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{C}$. We would like to have a positive definite Hermitian form

$\langle \cdot, \cdot \rangle_{V^2} : V^2 \times V^2 \to \mathbb{C}$ such that $\epsilon\epsilon^\dagger = 1$, where $\epsilon^\dagger$ denotes the adjoint (1.7). With isometry condition all tensors that do not take part in renormalization of the sites which local operator depend on can be contracted (see figure 1.11), thus the recourses needed for expectation value calculation can be significantly reduced.

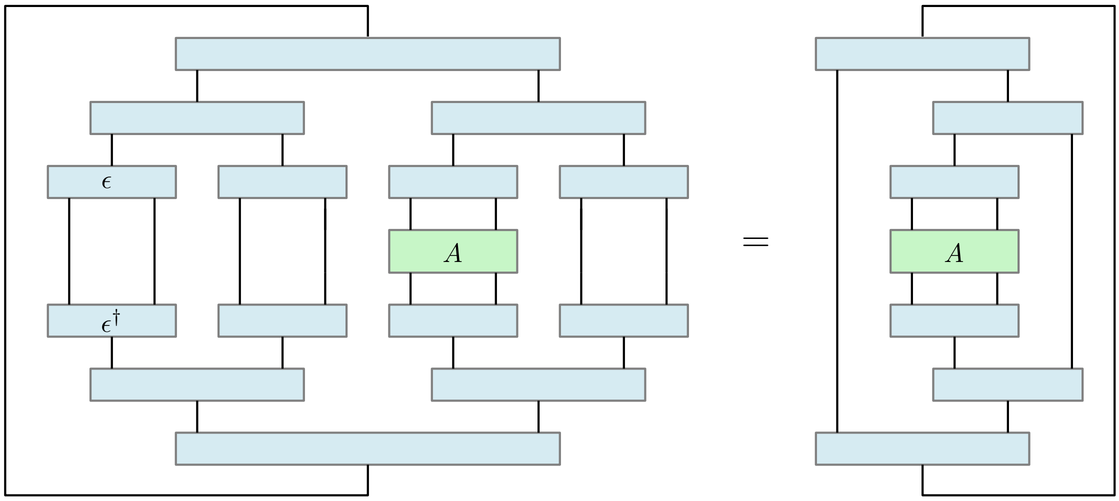


Figure 1.11: Usage of isometry condition $\epsilon\epsilon^\dagger = 1$ for TTN with 8 sites for local operator that depends only on 2 sites. As we can see, TTN simplifies significantly.

TTN can easily be generalized to non-binary trees, however for simplicity we will restrict ourselves to the binary case.

There are numerical studies which simulate small quantum chemical systems and support the idea that tree tensor networks can be much more efficient than current state-of-art MPS-based algorithms such as DMRG. This can be caused both by the polynomial behavior of the correlation function (see e.g [MVLN10]) and the fact that some systems, such as molecules, have tree-like structure (see e.g. [NC13]).

Another non-trivial generalization of TTN is the Multiscale Entanglement Renormalization Ansatz (MERA) [Vid08]. It also has an explicit interpretation as a renormalization group method, but, compared to TTN, MERA has an extra type of operation – disentanglers. Their purpose is to reduce entanglement of neighboring sites which are not being renormalized to one site in the next step. Recently MERA was applied to a large class of systems and it was shown that it can be exact for certain systems of interest (see for example [EV16] for an implementation of the lattice version of conformal maps and [KRV09] for exact MERA-like solution for string-net models). A continuous version of MERA called cMERA was developed [HOVV13]. Recently it was shown that cMERA can in principle correctly reproduce certain features of CFTs [HV17]. As MERA is more complicated than TTN, it makes sense to study simpler tensor networks first.

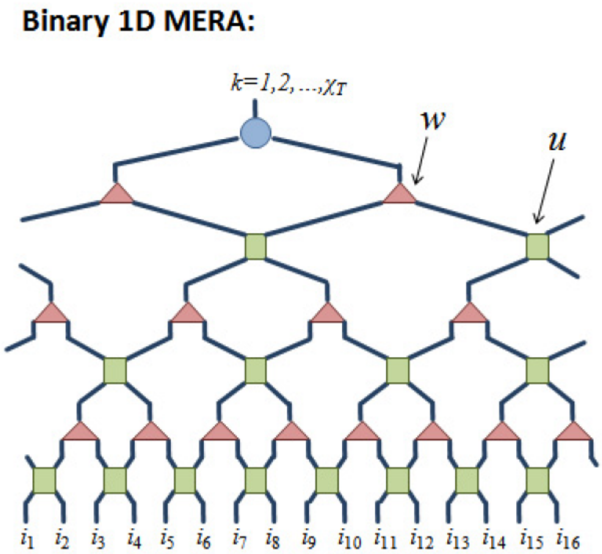


Figure 1.12: Example of a binary 1D MERA for a lattice with n = 16 sites. It contains two types of isometric tensors, organized in T = 4 layers. The *isometries* w are tensors of rank 3 and the *disentanglers* u are of type 4. We can prescribe renormalization meaning to MERA, that is we can say that isometries are used to replace each block of two sites with a single effective site and disentanglers are used to disentangle the blocks of sites before coarse-graining. We say that the tensor is of type (m, n) if it replaces n sites with m effective (i.e coarse-grained or with less entanglement) sites. The binary MERA is composed of tensors of type $(1, 2)$ and $(2, 2)$. It is easy to generalize the construction to m-ary MERA by using isometries of type $(1, m)$. Image taken from [EV09].

1.2 Conformal Field Theory (CFT)

Quantum field theory (QFT) is arguably one of the most versatile physical theories developed to date – with applications from condensed matter, subatomic and fundamental physics to cosmology and many other areas of physics. However, putting a general QFT on a rigorous footing remains an important research topic. An important exception in this regard is the class of conformal field theories (CFTs): the presence of conformal symmetry allows to provide a rigorous algebraic formulation (see e.g. [Gab00]). Moreover, CFTs are physically relevant (see e.g. [BP09]). One of their applications is the description of critical systems. Thus, not only do CFTs provide a great test bed for ideas related to general quantum field theories, but there also exists some intuition why tree tensor network approximations for CFTs can be a suitable description of an important class of physical systems [KS15]. Consequently, it is natural to first investigate TTNs in the context of CFTs. There are numerical results on the critical systems that support this claim (see e.g [TEV09] for simulation of quantum Ising model via TTN and general discussion).

Like any quantum field theory, a CFT is determined by its correlation functions which are physically interpreted as expectation values of products of basic observable quantities, or quantum fields. They depend continuously on certain parameters, specifying the degrees of freedom of the theory such as position or time. The correlation functions are postulated to transform in a simple manner under symmetry transformations of the theory. As the conformal group in two dimensional space is very rich – it is an infinite-dimensional Lie group, it is of special interest (see, for example, the pioneering work [BPZ84]).

Generally 2D CFTs are defined on a complex plane and the fields that constitute the theory depend on both a complex coordinate z and its conjugate $\bar{z}$. However, it can be proven for many CFTs of interest that the two variables decouple (see e.g [BP09], formalism that can be used to generalize this thesis to non-chiral case can be found in e.g. [KS15]). In this *chiral* case it is sufficient to study basic fields $Y(\psi, z)$.

As any quantum field theory, CFT is determined by its correlation functions. The theory is conformal when the correlation functions of certain fields are assumed to transform under conformal reparametrization in a simple manner. That is, let $w : \mathbb{C} \to \mathbb{C}$ be a holomorphic map. The conformal invariance is defined as the existence of real numbers h_i for $i = 1, \ldots, n$ such that

$$\langle Y(\psi_1, z_1) \ldots Y(\psi_n, z_n) \rangle = \prod_i \left(\frac{dw}{dz} \right)^{h_i} \Bigg|_{z_i} \langle Y(\psi_1, w(z_1)) \ldots Y(\psi_n, w(z_n)) \rangle, \tag{1.11}$$

for a set of fields called the *primary fields* (see e.g. [Gab00]). Computing these correlation functions is the main objective of this work. More generally we may be interested in expressions of the form

$$\mathrm{Tr}[Y(\psi_1, z_1) \ldots Y(\psi_n, z_n) \rho] \tag{1.12}$$

The object $Y(\psi, z)$ is an "insertion" of field ψ at the point z. In this thesis, we are mostly concerned with correlation functions evaluated on the real line, with insertion points $z_1, ..., z_{2^m}$ separated by minimal distance. By using appropriate conformal transformations, most configurations of insertion points can be brought into this standard form. To define the basic fields $Y(\psi, z)$ we need to introduce languages of vertex operator algebras, modules and intertwiners (see e.g. [KS15]).

1.2.1 Vertex Operator Algebras (VOA)

A vertex operator algebra $\mathcal{V}$ is a tuple $(\mathcal{V}, \mathcal{Y}, 1, \omega)$ consisting of an $\mathbb{N}_0$-graded vector space $\mathcal{V} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{V}_n$, a linear map $\mathcal{Y}(\cdot, z) : \mathcal{V} \to \mathsf{End}(\mathcal{V})[[z, z^{-1}]]$ into the space of formal Laurent series with coefficients in $\mathsf{End}(\mathcal{V})$, and two distinguished vectors: the *vacuum* $1 \in \mathcal{V}_0$ and the *conformal* or *Virasoro vector* $\omega \in \mathcal{V}_2$. Each vector space $\mathcal{V}_n$ is called a *weight space.* A vector $\nu \in \mathcal{V}_n$ is *homogeneous* of *weight* (or *level*) $\mathrm{wt}(\nu) = n$.

By definition, the *vertex operator* $\mathcal{Y}(\nu, z)$ associated with a vector $\nu \in \mathcal{V}$ can be written as

$$\mathcal{Y}(\nu, z) = \sum_{n \in \mathbb{Z}} \nu_n z^{-n-1}, \tag{1.13}$$

where $\nu_n \in \mathsf{End}(\mathcal{V})$ is referred to as a *mode operator* of ν. For every $\nu \in \mathcal{V}$, these satisfy

$$\nu_n = 0 \quad \text{for } n \text{ sufficiently large.} \tag{1.14}$$

The vacuum vector has the property

$$\mathcal{Y}(1, z) = \mathrm{id}_{\mathcal{V}}, \tag{1.15}$$

and the *creativity property*

$$\mathcal{Y}(\nu, z)1 = \nu + \sum_{n \in \mathbb{N}} \tilde{\nu}_n z^n \quad \text{for some } \tilde{\nu}_n \in \mathcal{V}. \tag{1.16}$$

For the conformal vector ω, which is homogeneous of weight $\mathrm{wt}(\nu) = 2$, the mode operators are denoted by $\omega_n = L_{n-1}$, i.e.,

$$\mathcal{Y}(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \tag{1.17}$$

The operators L_n satisfy Virasoro algebra relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \cdot \mathrm{id}_{\mathcal{V}} \quad \forall m, n \in \mathbb{N}_0, \tag{1.18}$$

where the constant c is the central charge. The object $\mathcal{Y}(\omega, z)$ is also called *the energy-momentum tensor.*

Every weight space $\mathcal{V}_n$ is finite-dimensional, the grading of $\mathcal{V}$ is given by the spectral decomposition of L_0 - that is, for every $n \in \mathbb{N}_0$, $\mathcal{V}_n$ is the eigenspace of L_0 with eigenvalue n. A homogeneous vector $\nu \in \mathcal{V}_n$ is *quasi-primary* if $L_1\nu = 0$ and *primary* if $L_n\nu = 0\ \forall n > 0$.

It is also possible to define a product of two vertex operators as a formal series. Then we will obtain $\mathcal{Y}(u, z_1)\mathcal{Y}(\nu, z_2) \in \mathsf{End}(\mathcal{V})[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$. A VOA satisfies *locality* or *weak commutativity* property, namely that there exists non-negative integer k such that:

$$(z_1 - z_2)^k [\mathcal{Y}(u, z_1), \mathcal{Y}(\nu, z_2)] = 0. \tag{1.19}$$

This implies the *Jacobi identity*

$$\begin{aligned} & \underset{z_1 - z_2}{Res}\,(\mathcal{Y}(\mathcal{Y}(u, z_1 - z_2)\nu, z_2)(z_1 - z_2)^m l_{z_2, z_1 - z_2}(z_2 + (z_1 - z_2))^n) \\ = \ & \underset{z_1}{Res}(\mathcal{Y}(u, z_1)\mathcal{Y}(\nu, z_2) l_{z_1, z_2}(z_1 - z_2)^m z_1^n) \\ & - \underset{z_1}{Res}(\mathcal{Y}(\nu, z_2)\mathcal{Y}(u, z_1) l_{z_2, z_1}(z_1 - z_2)^m z_1^n) \end{aligned} \tag{1.20}$$

holds for all $m, n \in \mathbb{Z},\ u, \nu \in \mathcal{V}$ and $Res_z(f(z))$ stands for residue – that is, a coefficient of z^{-1} in the Laurent expansion of $f(z)$. Here $l_{z_1,z_2} f(z_1, z_2)$ stands for the Laurent series expansion of the function $f(z_1, z_2)$ in the domain $|z_1| > |z_2|$.

In addition, a VOA has the *translation property*:

$$\frac{d}{dz}\mathcal{Y}(\nu, z) = \mathcal{Y}(L_{-1}\nu, z). \tag{1.21}$$

An important consequence of these axioms is the *duality theorem*

$$\mathcal{Y}(\psi, x)\mathcal{Y}(\phi, y) = \mathcal{Y}(\mathcal{Y}(\psi, x - y)\phi, y). \tag{1.22}$$

The operators $\{L_{-1}, L_0, L_1\}$ generate an action of $\mathrm{SL}(2, \mathbb{C})$ on the formal variable z by Möbius transformations

$$\gamma(z) = \frac{az + b}{cz + d} \quad \text{for} \quad a, b, c, d \in \mathbb{C},\ ad - bc = 1. \tag{1.23}$$

For $d \neq 0$ the action of $\mathrm{SL}(2, \mathbb{C})$ can be compactly described as

$$D_\gamma = \exp\left(\frac{b}{d}L_{-1}\right) d^{-2L_0} \exp\left(-\frac{c}{d}L_1\right). \tag{1.24}$$

The consequence of Virasoro algebra relations and translation property is

$$D_\gamma \mathcal{Y}(u, z) D_\gamma^{-1} = \mathcal{Y}\left(\gamma'(z)^{L_0} \exp\left(\frac{\gamma''(z)}{2\gamma'(z)}L_1\right)\nu, \gamma(z)\right), \tag{1.25}$$

In particular,

$$q^{L_0}\mathcal{Y}(u, z)q^{-L_0} = \mathcal{Y}(q^{L_0}u, qz) \tag{1.26}$$

corresponds to the dilation

$$\gamma(z) = qz. \tag{1.27}$$

1.2.2 Modules

We might be interested in structures that are similar to VOA with some more general operator $L_{A,n}$ instead of L_n. That is, a *module* of a VOA is a vector space carrying a structure satisfying almost all defining properties of a VOA and certain compatibility properties with the VOA. Namely, for a VOA $(\mathcal{V}, \mathcal{Y}, 1, \omega)$ a module $(A, \mathcal{Y}_A)$ is again a graded vector space $A = \bigoplus_{n \in \mathbb{N}_0} A_n$, together with a linear map

$$\mathcal{Y}(\cdot, z) : \mathcal{V} \to \mathsf{End}(A)[[z, z^{-1}]], \quad \mathcal{Y}_A(\nu, z) = \sum_{n \in \mathbb{Z}} \nu_n^A z^{-n-1}, \tag{1.28}$$

where $\nu_n^A \in \mathsf{End}(A)$ is called the *mode operator* of $\nu \in \mathcal{V}$. the subspace A_0 is called the *top level*, A_n - the n-th level of the module A.

Homogeneous vectors and mode operators are defined for modules analogously to VOAs. Weights are defined as eigenvalues of $L_{A,0}$. An important difference is that for modules weights need not be integers: they are of the form $\alpha + n$, where $\alpha \in I_A$ for some **finite** set $I_A \subset \mathbb{C}$ and $n \in \mathbb{N}_0$. Precisely this means that $\forall n \in \mathbb{N}_0$, we have $\forall a \in A_n$

$$L_{A,0}a = (\alpha + n)a \quad \text{for some } \alpha \in I_A. \tag{1.29}$$

A VOA $\mathcal{V}$ is called C_n-co-finite for $n \geq 2$ if $\mathcal{V}/C_n(\mathcal{V})$ is finite-dimensional, where

$$C_n(\mathcal{V}) = \text{span}\{v_{-n}w | v, w \in \mathcal{V}\}. \tag{1.30}$$

It was shown by Gaberdiel and Nietzke [GN03] as well as by Karel and Li [KL99] that the weight spaces A_n of an irreducible module of a VOA that satisfy the C_2-co-finitness condition are finite-dimensional. Moreover, the dimension of weight space is bounded by

$$\dim A_n \leq (\dim A_0) \cdot P(n, C_{\mathcal{V}}), \tag{1.31}$$

where $C_{\mathcal{V}} = \dim \mathcal{V}$ and $P(n, C_{\mathcal{V}})$ is the number of $C_{\mathcal{V}}$-component multi-partitions of the integer n. A multi-partition of n into r components is an r-tuple $(\lambda^{(1)}, \ldots, \lambda^{(r)})$ of partitions such that $|\lambda^{(1)}| + \ldots + |\lambda^{(r)}| = n$. For more information about partitions, multi-partitions and number of multi-partitions, see e.g [And08].

A VOA $\mathcal{V} = (\mathcal{V}, \mathcal{Y}, 1, \omega)$ is called *unitary* if there is an anti-linear involution $\phi : \mathcal{V} \to \mathcal{V}$ of $\mathcal{V}$ with

$$\phi(1) = 1, \quad \phi(\omega) = \omega, \quad \text{and} \quad \phi(\nu_n w) = \phi(\nu)_n \phi(w) \quad \forall \nu, w \in \mathcal{V}, \tag{1.32}$$

together with a positive definite Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ which is $\mathbb{C}$-linear in the second argument and anti-$\mathbb{C}$-linear in the first argument such that the invariance condition

$$\langle w_1, \mathcal{Y}(e^{zL_1}(-z^{-2})^{L_0}\nu, z^{-1})w_2 \rangle_{\mathcal{V}} = \langle \mathcal{Y}(\phi(\nu), z)w_1, w_2 \rangle_{\mathcal{V}} \tag{1.33}$$

holds $\forall \nu, w_1, w_2 \in \mathcal{V}$.

Combining (1.17) and (1.33), one can show that L_0 is a self-adjoint operator, namely

$$\forall \phi, \psi \in \mathcal{V} \quad \langle \phi, L_0 \psi \rangle_{\mathcal{V}} = \langle L_0 \phi, \psi \rangle_{\mathcal{V}}. \tag{1.34}$$

1.2.3 Intertwiners

Let A, B, C be modules of rational VOA $\mathcal{V}$. An *intertwining operator* Y *of type* $\binom{C}{A\ B}$ is a family of linear maps $Y(\cdot, z)$ from A to certain Laurent-like series with coefficients in $\mathsf{End}(B, C)$, i.e., it associates to every $a \in A$ a series

$$Y(a, z) = \sum_{\tau \in I, m \in \mathbb{Z}} a_{\tau,m} z^{-\tau - m}, \tag{1.35}$$

where $I = I_{AB}^C = \{\tau_1, \dots \tau_d\}$ is a finite collection of complex numbers (depending only on A, B and C) and $a_{\tau,m} \in \mathsf{End}(B, C)$ for $\tau \in I$ and $m \in \mathbb{Z}$. For all $b \in B$, the mode operators satisfy

$$a_{\tau,m} b = 0 \quad \text{for sufficiently large } m. \tag{1.36}$$

Like VOAs, intertwiners have the translation property

$$\frac{d}{dz} Y(a, z) = Y(L_{A,-1} a, z) \quad \forall a \in A, \tag{1.37}$$

and obey the Jacobi identity.

Intertwiners are generalization of vertex operators and correlation functions of intertwiners are exactly the objects that define the particular CFT. The axioms of VOA, and therefore, the definition of intertwiners have connection to Wightman's axioms for QFT (see e.g. [Sch09]).

In the following we will be constructing TTN to obtain correlation functions of intertwiners.

1.2.4 Wess-Zumino-Witten (WZW) models

One important subclass of CFTs are WZW models. They can be obtained by employing extra internal symmetries on the fields. The starting point is a compact simple Lie algebra $\mathfrak{g}$ with the corresponding structure constants f^{abc}. Then $\mathfrak{g}$ is turned into affine Lie algebra by via the affinization $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with the commutation rule (see e.g. [KS16])

$$[\mathfrak{a}(n), \mathfrak{b}(m)] = [\mathfrak{a}(n), \mathfrak{b}(m)](n+m) + kn\delta_{n+m,0} \mathrm{Tr}[\mathfrak{a}\mathfrak{b}] \tag{1.38}$$

where $\mathfrak{a}, \mathfrak{b} \in \mathfrak{g}$, $\mathfrak{a}(n) = \mathfrak{a} \otimes t^n$ and k is a positive integer defining the *level* of $\hat{\mathfrak{g}}$. For the case of *currents* - fields of conformal dimension 1, we get (see e.g. [BP09])

$$[j_n^a, j_m^b] = i \sum_c f^{abc} j_{n+m}^c + kn\delta^{ab}\delta_{n+m,0}. \tag{1.39}$$

The indices a, b, c run through the indices of $\mathfrak{g}$.

The VOA as a vector space is given be the full Fock space $\mathcal{V}$ generated by the negative modes of $\hat{\mathfrak{g}}$ acting on the vacuum vector 1 which is given by the identity element of $\mathfrak{g}$, e.g.

$$a_1(-n_1) \dots a_k(-n_k) 1. \tag{1.40}$$

Elements $\mathfrak{a}_n$ in $\hat{\mathfrak{g}}$ are identified with creation operators and the corresponding adjoints - with annihilation operators. Free fields are defined as

$$\mathfrak{a}(z) = \sum_{n \in \mathbb{Z}} \mathfrak{a}(n) z^{-n-1}. \tag{1.41}$$

The energy-momentum tensor and thus its Laurent modes L_m can be obtained via the *Sugawara construction*

$$L_m = \frac{1}{2(k + C_{\mathfrak{g}})} \sum_{a=1}^{\dim \mathfrak{g}} \left(\sum_{l \leq -1} j_l^a j_{m-l}^a + \sum_{l > 1} j_{m-l}^a j_l^a \right). \tag{1.42}$$

where $C_{\mathfrak{g}}$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$.

Given an irreducible representation of the Lie algebra $\mathfrak{g}$ with highest weight λ, we can repeat the above Fock construction with the vacuum replaced by the associated highest weight vector ϕ_λ resulting in the module $L_{k,\lambda}$ for the WZW VOA. It possesses a natural $\mathbb{N}_0$-grading

$$\mathrm{wt}\,[a_1(-n_1)\dots a_k(-n_k)\phi_\lambda] = \mathrm{wt}\lambda + \sum_i n_i \tag{1.43}$$

with wtλ a positive number depending on λ. The module is irreducible if $\langle\theta,\lambda\rangle \leq k$, with θ the maximal root of $\mathfrak{g}$. The set Λ_k of such highest weights λ is finite, implying that WZW theories are rational CFTs (see e.g. [DFMS97]).

Primary fields are defined as intertwiners between WZW modules: let V_{λ_i} $i = 1,2,3$ be three irreducible highest weight representations of $\mathfrak{g}$ with $\lambda_i \in \Lambda_k$. Turning the representations V_{λ_1}, V_{λ_2} into irreducible WZW VOA modules L_{k,ϕ_i}, $i = 1,2$, a primary field is a linear mappings of elements $\phi \in V_{\lambda_3}$ to linear z-dependent mappings $\phi(z) : L_{k,\phi_1} \to L_{k,\phi_i}$ such that there is a commutation rule with creation operators

$$[\mathfrak{a}(n), \phi(z)] = (\mathfrak{a}\phi)(z)z^n. \tag{1.44}$$

Yet another important property of WZW models are *linear energy bounds* [BSM90]. That is, suppose we have a mode expansion $Y_{L_{k,\lambda}}(a,z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$ then the operators $a_n, a \in \mathfrak{g}$ satisfy the bound

$$\|a(n)\chi\|_{L_{k,\lambda}} \leq c\cdot\|a\|_{L_{k,\lambda}}|n+1|\cdot\|(L_0+1)\chi\|_{L_{k,\lambda}} \tag{1.45}$$

for any $\chi \in L_{k,\lambda}$, where $c > 0$. The norm $\|a\|_{L_{k,\lambda}}$ is defined via

$$(\eta(a), a) = \langle a(-1)1, a(-1)1\rangle_{L_{k,\lambda}} = \|a\|^2_{L_{k,\lambda}}, \tag{1.46}$$

where $\eta(a)$ denotes the adjoint of a (see condition 1.33).

WZW models are not only well-understood, but also have many applications, for example in studies of string theory (see e.g. [BP09]), fractional quantum Hall effect (see e.g. [HST+15]), quantum spin chains (see e.g [NCS11]) and condensed matter theory (see e.g. [DFMS97]).

Chapter 2

Construction of the renormalization map

Suppose we have a lattice with *initial point* $z \in \mathbb{C}$ and *step* $q \in \mathbb{C}$, that is:

$$\{z, zq, zq^2, \dots\} \tag{2.1}$$

At these points there are *insertions* or *fields* $\psi_i \in V$. We would like to calculate an expectation value for products of intertwiners, as these are the relevant operators in CFTs. These are assumed to have insertions at lattice points. This leads to objects of the form

$$F^{(0)}_{v_{in},v_{out}}\left(\{(\psi_i, zq^{i-1})\}\right) \equiv \langle v_{in}, Y(\psi_1, z) \cdot Y(\psi_2, zq) \cdot \ldots \cdot Y(\psi_{2^m}, zq^{2^m-1})v_{out}\rangle, \tag{2.2}$$

where $v_{in}, v_{out}, \psi_i \in V, i \in \mathbb{N}$ (for the definition of $Y(\psi, z)$ see subsection 1.2.3). A generalization can be obtained by introducing a *density matrix* ρ. We have:

$$\langle A\rangle = \mathrm{Tr}_V[A\rho] \tag{2.3}$$

where the trace is calculated on each finite-dimensional level of V and hence is well-defined.

We consider **correlation functions**

$$Corr(\{\psi_1, \dots \psi_{2^m}\}, z, q, \rho) \equiv \mathrm{Tr}_V[Y(\psi_1, z) \cdot Y(\psi_2, zq) \cdot \ldots \cdot Y(\psi_{2^m}, zq^{2^m-1})\rho], \tag{2.4}$$

where $\psi_i \in V, i \in \mathbb{N}$, $\rho \in \mathsf{End}(V)$ and $Y(\psi_i, zq^{i-1})$ are intertwiners as defined in subsection 1.2.3, and we will seek to represent (2.4) as a tree tensor network.

Note that we do not demand a density matrix to be self-adjoint or obey $\mathrm{Tr}_V \rho = 1$.

Two particularly interesting examples are genus-0 correlation function, which corresponds to the density matrix $\rho = |v_{out}\rangle\langle v_{in}|$ and equals to the expression (2.2), and genus-1 correlation function which is defined as

$$F_p^{(1)}\left(\{(\psi_i, zq^{i-1})\}\right) \equiv \mathrm{Tr}_V\left[Y\left(z^{L_0}\psi_1, z\right)\cdot\ldots\cdot Y\left(\left(zq^{2^m-1}\right)^{L_0}\psi_{2^m}, zq^{2^m-1}\right)p^{L_0-c/24}\right]. \quad (2.5)$$

Both of this examples have important applications, for example, they correspond to tree-level and one-loop approximations in string theory (see e.g. [BP09] and [KS15]). As we will not specify density matrix, we will be able to get results that are applicable to both genus-0 and genus-1 case. It is also important to note that finiteness of genus-1 correlation function was established for a wide class of theories [Hua05].

This is a natural and rather general object to study, as CFTs, like any other QFTs, are specified by correlation functions. Moreover, a lot of systems can be well approximated by the system on equally spaced lattice $\{x, x+a, x+2a, \ldots\}$. Employing the usual exponential transform of coordinates (see e.g. [BP09])

$$x \to z = e^{x^0+ix^1}, \quad (2.6)$$

where x^0 and x^1 are time and space coordinates and $x = x^0 + ix^1$, we obtain exactly the lattice (2.1) with some z and q from the given equally spaced lattice. The number of insertions, namely, 2^m, is chosen for convenience for the tree tensor network construction. Of course, some of these fields can be vacuum insertions and $Y(1, z) = \mathrm{id}_V$, so this is not a restriction.

To compute an object of the form (2.4), we will use the *transfer operator* that was introduced in [KS15]. We will see that such an operator can be calculated in a tree-like fashion and that the corresponding bond dimension for the approximated computation of the transfer operator is asymptotically smaller - that is, the approximation error scales better when the number of variational parameters increases, than for the MPS-like computation that was done in [KS15].

2.1 Scaled intertwiners and transfer operator

Intertwiners are not in general bounded operators. In order to prove that the truncation error is small it is useful to have an operator that has finite norm. Moreover, it would be useful if composition of such n operators and an operator responsible for the density matrix would yield correlation function. The following operator has just the needed properties:

Definition 2.1.1. *A scaled intertwiner $W_q : V \to \mathsf{End}(V)[[z, z^{-1}]]$, $\psi \to W_q(\psi, z)$, $z \in \mathbb{C}$ and $q \in \mathbb{C}$ is*

$$W_q(\psi, z) = q^{L_0/2}Y(q^{L_0/2}\psi, z)q^{L_0/2}, \quad (2.7)$$

where Y's are ordinary intertwiners. We call it of type $\binom{\;C}{A\;B}$ if $W_q(\psi, z)$ is obtained from an intertwiner of type $\binom{\;C}{A\;B}$ (see section 1.2.3 for definition). We can associate the following picture to the object that appears in the map:

$$W_q(\psi, z) \quad \equiv$$

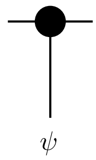


Figure 2.1: Diagram for scaled intertwiner as a building block for the tree tensor network.

If we multiply two such operators we get an operator which resembles a two point function of insertions $q^{-L_0/2}\psi_1$ at point z and $q^{-3L_0/2}\psi_2$ at point zq

$$W_q(\psi_1, z) \circ W_q(\psi_2, z) = q^{L_0/2} Y(q^{L_0/2}\psi_1, z) Y(q^{3L_0/2}\psi_1, qz) q^{3L_0/2} \tag{2.8}$$

where we have used property (1.26). Analogous expressions hold for n-point functions for all $n \in \mathbb{N}$ (see expression (2.10)). This leads to the following:

Definition 2.1.2. *A transfer operator* $T : V^n \to \mathsf{End}(V)[[z, z^{-1}]]$, $\{\psi_i\}_{i=1}^n \to T\left(\{\psi_i\}_{i=1}^n; z, q\right)$ *with insertions* $\{\psi_i\}_{i=1}^n$ *on the lattice with initial point* z *and step* q *is*

$$T\left(\{\psi_i\}_{i=1}^n; z, q\right) = W_q(\psi_1, z) \circ W_q(\psi_2, z) \circ \cdots \circ W_q(\psi_n, z). \tag{2.9}$$

To simplify notation we will sometimes denote the transfer operator as $T\left(\{\psi_i\}\right)$, $T\left(z, q\right)$ *or just* T *when the arguments are clear from the context.*

We can draw a diagram for the transfer operator using notation introduced in figure 2.1:

$$T(\{\psi_i\}_{i=1}^n; z, q) \quad =$$

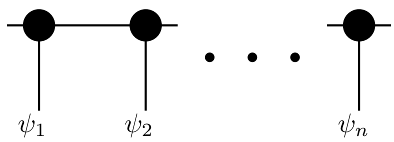


Figure 2.2: Diagram for transfer operator.

This diagram resembles MPS and it is not a coincidence – this is exactly the diagrammatic interpretation of MPS from [KS15]. One can check that

$$T\left(\{\psi_i\}_{i=1}^n; z, q\right) = q^{-L_0/2} Y_q(\tilde{\psi}_1, zq) \circ Y(\tilde{\psi}_2, zq) \circ \cdots \circ Y(\tilde{\psi}_n, zq^n) q^{(n+1/2)L_0}, \tag{2.10}$$

where

$$\tilde{\psi}_j = q^{(j+1/2)L_0}\psi_j. \tag{2.11}$$

Thus, we can write the correlation function (2.4) in terms of the transfer operator:

$$Corr(\{\tilde{\psi}\}_{i=1}^n, z, q, \tilde{\rho}) = Tr[T\left(\{\psi_i\}_{i=1}^n; z, q\right) \circ \rho]. \tag{2.12}$$

Here T has insertions $\{\psi_i\}_{i=1}^n$, $\tilde{\psi}$ defined via (2.11) and $\rho = q^{(n+1/2)L_0}\tilde{\rho}q^{-L_0/2}$. Moreover, one can do a global rescaling:

$$Corr(\{\tilde{\psi}_1, \ldots \tilde{\psi}_{2^m}\}, z, q, \tilde{\rho}) = Tr[q^{lL_0}Tq^{-lL_0} \circ q^{lL_0}\rho q^{-lL_0}] \tag{2.13}$$

for some $l \in \mathbb{C}$. In order to approximate correlation functions, one can approximate the rescaled transfer operator:

Notation 2.1.3. *We will denote by T_l (see figure 2.3) a product of the form*

$$T_l = q^{lL_0}Tq^{-lL_0}, \tag{2.14}$$

where T is the transfer operator, $l \in \mathbb{C}$ and $q \in \mathbb{C}$. We will be mostly interested in the case $l \in \mathbb{R}$, $l \geq 0$.

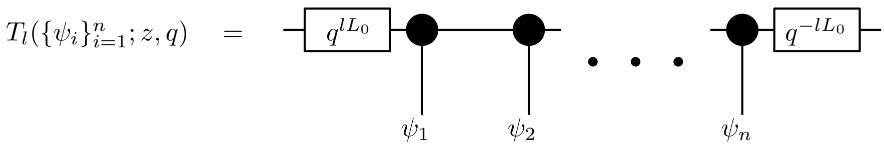


Figure 2.3: Diagram representation of T_l. Computing this object is equivalent to computing the transfer operator T. Due to technical reasons this will be the object that we will be obtaining and not T.

To make formulas and diagrams shorter and clearer, let us introduce:

Definition 2.1.4. *We say that $A, B \in \mathsf{End}V$ are equivalent and we will use the symbol $\simeq$ to denote it if there exist some number $l \in \mathbb{C}$ and $q \in \mathbb{C}$ such that:*

$$A \simeq B \Leftrightarrow A = q^{lL_0}Bq^{-lL_0}. \tag{2.15}$$

It is straightforward to check that $\simeq$ satisfy the axioms of equivalence relation.

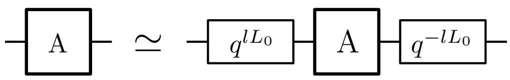


Figure 2.4: The notation $\simeq$ is used in order to suppress dependence on some physically unimportant transformation.

As we will see in sections 2.3 and 2.4, such a rescaled transfer operator T_l can be written in terms of the map

$$\epsilon_{z,q}(\psi_1 \otimes \psi_2) \equiv q^{\alpha L_0}W_q(\psi_1, z) \circ W_q(\psi_2, z)q^{-\alpha L_0}. \tag{2.16}$$

for some suitable $\alpha \in \mathbb{R}$. One can also prove bounds for norms for scaled intertwiners (proofs can be found in [KS15]) that we will use to prove that the transfer operator can be approximated via the TTN with finite dimensional tensors (see chapter 3). Let V be rational and C_2-co-finite VOA (see 1.30), A, B, C unitary modules of V, $W_q(a, z)$ be a scaled intertwiner of type $\binom{C}{A\ B}$ and $z \in \mathbb{C} \setminus \{0\}$, $0 < q < \min\{|z|^2, 1/|z|^2\}$. Fix $S \subset A$, $\dim(S) < \infty$. Then $\exists \vartheta_S : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ such that

$$\|W_q(a, z)\| \leq \vartheta_S(q, z)\|a\| \quad \forall a \in S. \tag{2.17}$$

As we can see, the bound requires that the field belongs to some finite-dimensional subspace S. Typically we will work with fields that have finite weight.

Definition 2.1.5. *Suppose $A = \bigoplus_{n\in\mathbb{N}_0} A_n$, $a \in A$. Then we will denote*

$$w_a = \min\{N \in \mathbb{N} | a \in \bigoplus_{n\leq N} A_n\} \tag{2.18}$$

if such minimum exists and $w_a = \infty$ otherwise.

For a family of fields $\{\psi_i\}$ we write $w_{\psi_i} \equiv w_i$.

As the grading of modules is given by the weights, usually we will consider subspaces

$$S = \bigoplus_{n\leq w_a} A_n \tag{2.19}$$

Notation 2.1.6. *For subspaces of the form (2.19) we will use notation*

$$\vartheta_S(q, z) \equiv \vartheta_{w_a}(q, z), \tag{2.20}$$

Using an orthonormal basis $\{a_j\}_{j=1}^{\dim S}$ of S one can write the bound:

$$\vartheta_S(q, z) \leq \sqrt{\dim S} \cdot \max_{1\leq j\leq \dim S} \|H(a_j)\|, \quad H(a_j) \equiv W_q(a_j, z). \tag{2.21}$$

It is known that $H(a_j)$ is Hilbert-Schmidt (see [KS15])

$$Tr_B(H(a)^\dagger H(a)) = p^{c/24} F_p^{(1)}\left((z_1^{-L_0} a_1, z_1), (z_2^{-L_0} a_2, z_2)\right). \tag{2.22}$$

where $F_p^{(1)}$ is genus-1 correlation function (2.5) and

$$z_1 := q^{1/2} z^{-1}, \quad z_2 := q^{3/2z}, \quad p = q^2. \tag{2.23}$$

By the assumption that the VOA V is rational and C_2-co-finite, $F_p^{(1)}\left((z_1^{-L_0} a_1, z_1), (z_2^{-L_0} a_2, z_2)\right)$ and, consequently, $\vartheta_S(q, z)$ is finite since $1 > |z_1| > |z_2| > p$ for any $0 < q < \min\{|z|^2, 1/|z|^2\}$ as was proven in [Hua05].

2.1.1 Truncated scaled intertwiners

In order to prove that the tensor network can be approximated by finite matrices, we will need a concept of truncated intertwiner that was introduced in [KS15]. Let N be a positive integer. We will refer to it as *truncation parameter*. A truncated intertwiner $Y^{[N]}(a,z)$ does not change a level by more than N, in the sense that if $C = \bigoplus_{n\in\mathbb{N}_0} C_n, \quad C_{n<0} = \emptyset$ and $B = \bigoplus_{n\in\mathbb{N}_0} B_n$ are the grading of the spaces C and B given by the spectral decomposition of L_0, then

$$Y^{[N]}(a,z)B_n \subset \bigoplus_{m\in\mathbb{Z},|m|\leq N} C_{n-m}[[z,z^{-1}]] \quad \forall n \in \mathbb{N}_0. \tag{2.24}$$

Likewise one can define the truncated scaled intertwiner:

$$W_q^{[N]}(a,z) = q^{L_0/2} Y^{[N]}(q^{L_0/2}a,z) q^{L_0/2} \tag{2.25}$$

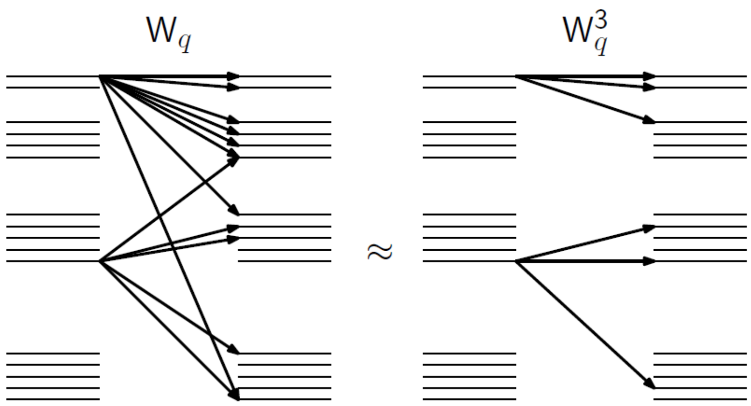


Figure 2.5: Scaled operators W_q create arbitrary superpositions of L_0 eigenstates, hence an infinite amount of entanglement is needed to implement their action. Truncated operators $W_q^{[N]}$ (right: $N = 3$) only create superpositions of at most N different energy eigenstates if applied to one element of the basis. Yet it can be shown that they approximate W_q exponentially well. Image taken from [KS16].

In [KS15] one can also find bounds for truncated scaled intertwiners and the difference of truncated and ordinary scaled intertwiners:

$$\|W_q^{[N]}(a,z)\| \leq \|a\|_A \cdot \frac{\sqrt{|I_B|} \cdot \vartheta_S(\sqrt{q},z)}{\sqrt{1-\sqrt{q}}}, \quad \forall a \in S, \tag{2.26}$$

for the scaled intertwiner of type $\binom{C}{A\ B}$ (see subsection 1.2.2 for the definition of I_B.). Note that this bound is independent of the truncation parameter N, and in fact holds for W_q itself. Also:

$$\|W_q(a,z) - W_q^{[N]}(a,z)\| \leq \|a\|_A \cdot k \cdot \vartheta_S(\sqrt{q},z) q^{N/4} \frac{1}{1-\sqrt{q}}, \tag{2.27}$$

where $k \in \mathbb{R}$, $k \geq 0$ is some constant that depends only on the modules A, B, C.

If $S = \bigoplus_{n \leq w_{O[N]_\psi}} A_n$ for some wield ψ such that w_ψ is finite, where O is some operator and N is a truncation parameter, then $\vartheta_S(q,z) \equiv \vartheta_{w_\psi + N}(q,z)$ (for w_i see definition 2.1.5).

2.2 Tree tensor network and renormalization

As been discussed in subsection 1.1.2, tree tensor network is a suitable ansatz for critical systems. Thus, one may want to compute the transfer operator in a tree-like manner. This means that one has to come up with a map that renormalizes fields – namely, outputs one field on renormalized scale from two fields on a smaller one. That is, we seek a map of the form

$$\epsilon_{z,q} : V \otimes V \to V, \tag{2.28}$$

where V is a space of fields. A priori it is not obvious which objects this renormalization map should act on. We have chosen it to act on fields (possibilities include intertwiners or some functions of intertwiners, we will show in appendix A that these possibilities are not useful).

We would like to "renormalize" fields. During this procedure parameters of lattice and density matrix may change at each step. We will call renormalized parameters

$$\rho^{(k)}, q^{(k)}, z^{(k)}. \tag{2.29}$$

Let us define initial wave functions as 0-th step of renormalization procedure, and every $k+1$-th step for $k \geq 0$ is obtained by applying $\epsilon_{z^{(k)},q^{(k)}}$ to a pair of neighboring fields

$$\begin{aligned} \psi_j^{(0)} &= \psi_i, \\ \psi_j^{(k+1)} &= \epsilon_{z^{(k)},q^{(k)}} \left(\psi_{2j-1}^{(k)} \otimes \psi_{2j}^{(k)} \right), \quad k = 0, 1, \dots \end{aligned} \tag{2.30}$$

This can be represented by the following picture:

$$\epsilon_{z^{(k)},q^{(k)}} \left(\psi_{2j-1}^{(k)} \otimes \psi_{2j}^{(k)} \right) \equiv$$

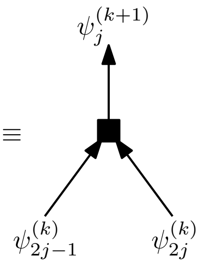


Figure 2.6: Renormalization map.

Transformations should have the following property:

$$Corr\left(\{\psi_1^{(k)}, \dots \psi_{2^{m-k}}^{(k)}\}, z^{(k)}, q^{(k)}, \rho^{(k)}\right) = Corr\left(\{\psi_1, \dots \psi_{2^m}\}, z, q, \rho\right) \qquad \forall k : 0 \leq k \leq m. \tag{2.31}$$

This condition can be represented pictorially. Indeed, computing $Corr$ is the same as computing matrix elements of transfer operators. Property (2.31) means that composing a transfer operator of renormalized fields should give the same operator as a transfer operator of initial fields. Of course, this should be true for any step of renormalization k. Pictorially we get:

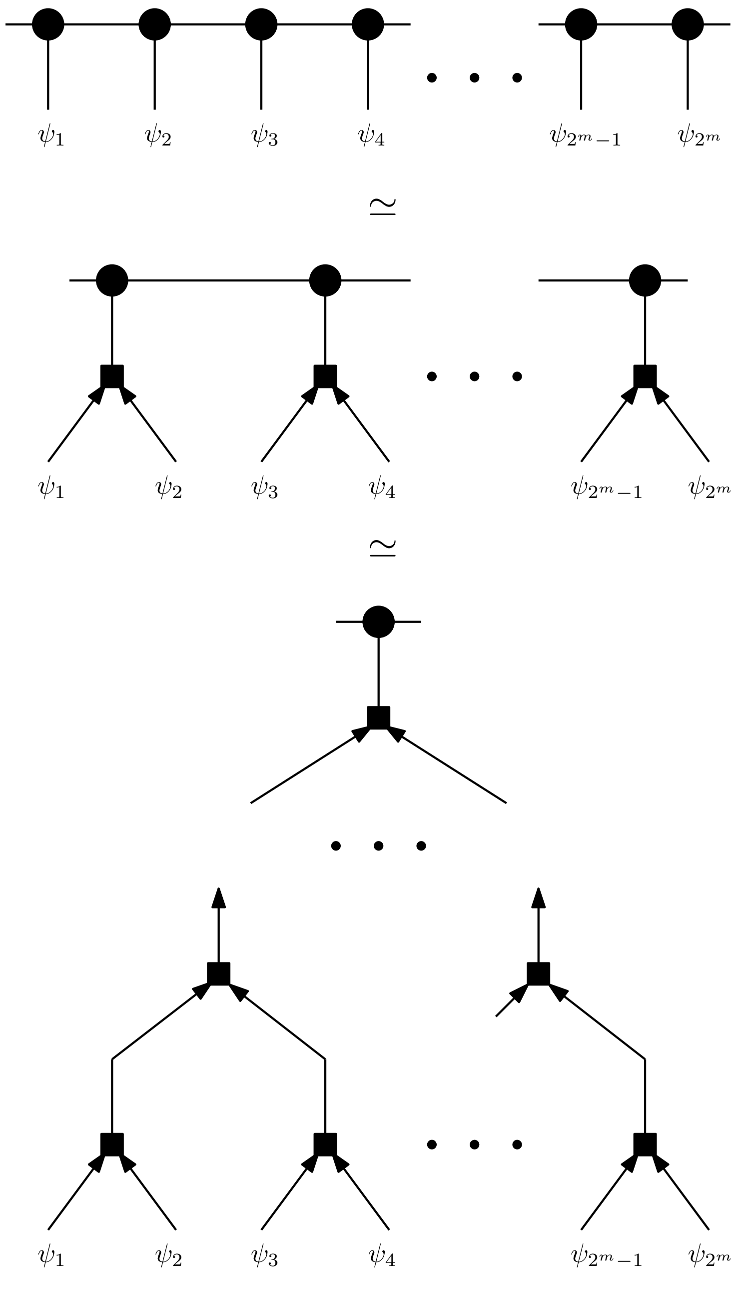


Figure 2.7: Tree tensor network scheme.

Now the task is to determine what $\psi_i^{(k)}, z^{(k)}, q^{(k)}, \rho^{(k)}$ are.

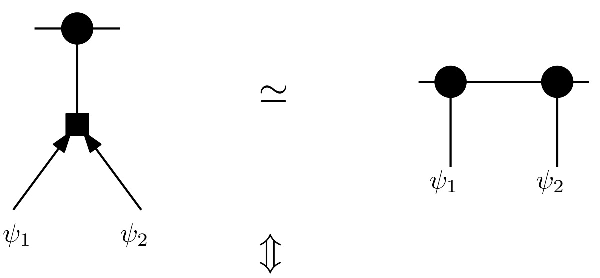
Figure 2.7 corresponds to property (2.31) when $\rho^{(k)} \simeq \rho$. In the section 2.3 we will see that the transformation (2.28) can be indeed chosen so that $\rho^{(k)}$ is equivalent to ρ for every $0 \leq k \leq m$.

The property (2.31) means that we would like to obtain such map $\epsilon_{z,q}$ (2.28), that exact correlation functions are obtained by applying $\epsilon_{z,q}$. Then we would like to truncate it – that is, to represent it by some finite-dimensional matrix. We would like this finite dimensional matrix to have the property that the difference between exact and truncated correlation functions is small. This is the motivation to introduce scaled intertwiners.

2.3 Renormalization transformation for fields

We would like to have a tree tensor network that provides T_l for some l when the initial fields $\{\psi_i\}$ are given. One could ask what properties the building block of TTN – the map should $\epsilon_{z,q}$ satisfy. As this map should provide a framework to compute transfer operators, one should be able to efficiently approximate it with finite matrices and it should obey an isometry condition for fast contraction. We may compose a list with the description of these three conditions.

1. In order for $\epsilon_{q,z}(\psi_1 \otimes \psi_2)$ to yield transfer operator it should have the following property in terms of diagrams for some $\alpha \in \mathbb{C}$ (see lemma 2.3.4 for more details):



$$W_{q^{(1)}}(\epsilon_{z,q}(\psi_1 \otimes \psi_2), z^{(1)}) \quad = \quad q^{\alpha L_0} W_q(\psi_1, z) \circ W_q(\psi_2, z) q^{-\alpha L_0}$$

Figure 2.8: Renormalization property.

2. The map $\epsilon_{q,z}(\psi_1 \otimes \psi_2)$ should be bounded if ψ_1 and ψ_2 have only finite weights in the weight decomposition. Indeed, if the map is unbounded, it is unclear how to estimate the error due to the truncation of the map. It is sufficient to demand that weights in decomposition of ψ's are finite as one typically has such fields as inputs for the algorithm that calculates correlation functions and the truncation procedure is such that if the weights are finite at the start, they are also finite at any step of the renormalization.

3. For fast contraction the map $\epsilon_{q,z}(\psi_1 \otimes \psi_2)$ should be an isometry. This requires the introduction of a suitable inner product in $V \otimes V$.

Now we will prove a theorem showing that a particular map satisfies the first two conditions. In section 2.5 we will see that the map also satisfies the third condition. Thus, with this map we will be able to construct an exact tree tensor network, and truncate it to obtain a TTN that requires finite-dimensional matrices.

For convenience, let us show a couple of useful facts:

Lemma 2.3.1. *There is a relation that is analogous to the duality theorem* $Y(\psi, x)Y(\phi, y) = Y(Y(\psi, x-y)\phi, y)$ *but for scaled intertwiners, namely*

$$W_q(\psi_1, z) \circ W_q(\psi_2, z) = W_{q^2}\left(Y\left(q^{-L_0}\psi_1, q^{-3/2}(1-q)z\right)\psi_2, q^{1/2}z\right). \tag{2.32}$$

Proof. For arbitrary $\beta \in \mathbb{C}$, we have

$$\begin{aligned} W_q(\psi_1, z) \circ W_q(\psi_2, z) &= q^{L_0/2}Y(q^{L_0/2}\psi_1, z)q^{L_0}Y(q^{L_0/2}\psi_2, z)q^{L_0/2} \\ &= \left(q^{L_0/2}q^{\beta L_0}\right)\left(q^{-\beta L_0}Y(q^{L_0/2}\psi_1, z)q^{\beta L_0}\right) \\ &\quad \cdot \left(q^{(1-\beta)L_0}Y(q^{L_0/2}\psi_1, z)q^{-(1-\beta)L_0}\right)\left(q^{(1-\beta)L_0}q^{L_0/2}\right). \end{aligned} \tag{2.33}$$

As we would like this expression to be a scaled intertwiner, let us fix $\beta = 1/2$, so that we have the same operator, namely q^{L_0}, at both sides of the expression. This leads to

$$\begin{aligned} W_q(\psi_1, z) \circ W_q(\psi_2, z) &= q^{L_0}Y(\psi_1, q^{-1/2}z)Y(q^{L_0}\psi_2, q^{1/2}z)q^{L_0} \\ &= W_{q^2}\left(q^{-L_0}Y(\psi_1, q^{-1/2}z - q^{1/2}z)q^{L_0}\psi_2, q^{1/2}z\right) \\ &= W_{q^2}\left(Y\left(q^{-L_0}\psi_1, q^{-3/2}(1-q)z\right)\psi_2, q^{1/2}z\right), \end{aligned} \tag{2.34}$$

which is the statement of the lemma. □

Theorem 2.3.2. *Consider the map* $\epsilon_{z,q} : V \otimes V \to V$ *for*

$$0 < q < \min\left\{|(1-q)z|^2, \frac{1}{|(1-q)z|^2}\right\} \tag{2.35}$$

defined on tensor products and linearly extended to the whole space

$$\epsilon_{z,q} : \psi \otimes \phi \quad \to \quad q^{2L_0}Y\left(q^{-L_0}\psi, q^{-3/2}(1-q)z\right)\phi \tag{2.36}$$

and the map

$$\begin{aligned} z &\to q^{5/2}z, \\ q &\to q^2. \end{aligned} \tag{2.37}$$

for parameters z and q. We will denote:

$$\begin{aligned} z^{(1)} &\equiv q^{5/2}z, \\ q^{(1)} &\equiv q^2, \\ \psi^{(1)} &\equiv q^{2L_0}Y\left(q^{-L_0}\psi, q^{-3/2}(1-q)z\right)\phi = W_q\left(\psi,(1-q)z\right)q^{L_0}\phi. \end{aligned} \tag{2.38}$$

Then following properties hold:

1. *Renormalization property (see figure 2.8, in this particular case $\alpha = 2$):*

$$W_{q^{(1)}}(\psi^{(1)}, z^{(1)}) = q^{2L_0}W_q(\psi, z)\circ W_q(\phi, z)q^{-2L_0} \tag{2.39}$$

2. *Boundedness:*

$$\textit{Let us fix } w. \textit{ Then } \forall\psi : w_\psi \le w \quad \exists C\in\mathbb{R} : \quad \|\epsilon_{z,q}(\psi\otimes\phi)\| \le C\|\psi\|\|\phi\| \tag{2.40}$$

(for w_ψ see definition 2.1.5).

Proof. For the renormalization property we get

$$q^{2L_0}W_q(\psi, z)\circ W_q(\phi, z)q^{-2L_0} = q^{2L_0}W_{q^2}\left(Y\left(q^{-L_0}\psi_1, q^{-3/2}(1-q)z\right)\psi_2, q^{1/2}z\right)q^{-2L_0} \tag{2.41}$$

Here we can use how intertwiners transform under the action of L_0, that is $q^{L_0}Y(u,z)q^{-L_0} = Y(q^{L_0}u, qz)$. Obviously, the same applies for the scaled intertwiners. We thus get

$$\begin{aligned} q^{2L_0}Y\left(q^{-L_0}\psi, q^{-3/2}(1-q)z\right)\phi &= \psi^{(1)}, \\ q^{2L_0}W_{q^2}\left(Y\left(q^{-L_0}\psi_1, q^{-3/2}(1-q)z\right)\psi_2, q^{1/2}z\right)q^{-2L_0} &= W_{q^2}\left(\psi^{(1)}, q^{5/2}z\right). \end{aligned} \tag{2.42}$$

Combining definitions (2.38) and formulas 2.42, we obtain

$$\begin{aligned} W_{q^{(1)}}(\psi^{(1)}, z^{(1)}) &= W_{q^2}\left(q^{2L_0}Y\left(q^{-L_0}\psi, q^{-3/2}(1-q)z\right)\phi, q^{5/2}z\right) \\ &= q^{2L_0}W_q(\psi, z)\circ W_q(\phi, z)q^{-2L_0} \end{aligned} \tag{2.43}$$

which is exactly the renormalization property.

To prove the boundedness, we will use formula (2.17)

$$\begin{aligned} \|\epsilon_{z,q}(\psi\otimes\phi)\| &= \|W_q\left(\psi,(1-q)z\right)q^{L_0}\phi\| \\ &\le \vartheta_w(q,(1-q)z)\left\|q^{L_0}\right\|\|\psi\|\|\phi\|, \end{aligned} \tag{2.44}$$

where ϑ_w corresponds to the notation 2.1.6. Due to the assumptions of the theorem (2.35), $0 < q < 1$, thus $\left\|q^{L_0}\right\| < 1$. We can conclude that $\epsilon_{z,q}$ is indeed bounded, as $\vartheta_w(q,(1-q)z)$ is finite for fixed w and z, q in range (2.35). This concludes the proof. □

As q^{L_0} does not change the weight of a vector, the is a natural way to truncate the map $\epsilon_{z,q}$

Definition 2.3.3. *Let $N \in \mathbb{N}$ be a truncation parameter (see subsection 2.1.1), $\psi, \phi \in V$ and $q, z \in \mathbb{C}$. Then a* ***truncated renormalization map*** *$\epsilon_{z,q}^{[N]} : V \times V \to V$ (see figure 2.9) is defined on tensor products and linearly extended to the whole space as follows*

$$\epsilon_{z,q}^{[N]} : \psi \otimes \phi \quad \to \quad W_q^{[N]} \left(\psi, (1-q)z\right) q^{L_0} \phi, \tag{2.45}$$

where $\epsilon_{z,q}$ is defined in (2.36).

We will denote it by following diagram:

$$\epsilon_{z^{(k)},q^{(k)}}^{[N]} \left(\psi_{2j-1}^{(k)} \otimes \psi_{2j}^{(k)}\right) \quad \equiv$$

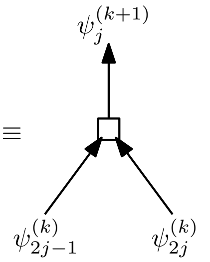


Figure 2.9: Truncated renormalization map.

Now that we have introduced a map $\epsilon_{z,q}$ (2.36) and have seen that it can be bounded and there is a natural way to truncate it, let us be more specific how renormalization property implies that the TTN yields transfer operator.

Lemma 2.3.4. *Let $\psi, \phi, \{\psi_i\}_{i=1}^{2n} \in V$ and $z, q, z^{(0)}, q^{(0)} \in \mathbb{C}$ be such that*

$$W_q(\psi, z) = T_l\left(\{\psi_i\}_{i=1}^{n}; z^{(0)}, q^{(0)}\right) \quad \text{and} \quad W_q(\phi, z) = T_l\left(\{\psi\}_{i=n+1}^{2n}; z^{(0)}, q^{(0)}\right) \tag{2.46}$$

for some $l \in \mathbb{C}$. Then

$$W_{q^2}\left(\epsilon_{z,q}(\psi \otimes \phi), q^{5/2} z\right) = T_{l+2\log_{q^{(0)}}(q)}\left(\{\psi_i\}_{i=1}^{2n}; z^{(0)}, q^{(0)}\right) \tag{2.47}$$

Proof. By the renormalization property (2.39) form the theorem 2.3.2 we obtain

$$\begin{aligned} W_{q^2}\left(\epsilon_{z,q}(\psi \otimes \phi), q^{5/2} z\right) &= q^{2l_0} W_q(\psi, z) \circ W_q(\phi, z) q^{-2L_0} \\ &= q^{2L_0} T_l\left(\{\psi_i\}_{i=1}^{n}; z^{(0)}, q^{(0)}\right) \circ T_l\left(\{\psi_i\}_{i=n+1}^{2n}; z^{(0)}, q^{(0)}\right) q^{-2L_0} \\ &= q^{2L_0}\left(q^{(0)}\right)^{lL_0} W_q(\psi_1, z) \circ \cdots \circ W_q(\psi_{2n}, z)\left(q^{(0)}\right)^{-lL_0} q^{-2L_0} \\ &= T_{l+2\log_{q^{(0)}}(q)}\left(\{\psi_i\}_{i=1}^{2n}; z^{(0)}, q^{(0)}\right) \end{aligned} \tag{2.48}$$

which is the statement of the lemma. It is straightforward to generalize the argument for the maps that obey renormalization property with arbitrary α (see figure 2.8). □

Let us introduce

$$z^{(m+1)} = q^{5/2} z^{(m)}, \quad q^{(m+1)} = \left(q^{(m)}\right)^2. \tag{2.49}$$

As $W_q(\psi_i, z) = T_0\left((\psi, q)\right)$, it is straightforwardly follows from the lemma 2.3.4 that if there are 2^m fields $\{\psi_i\}_{i=1}^{2^m}$ on the lattice $\{z, qz, \ldots, q^{2^m-1}z\}$, then the TTN composed of tensors that correspond to $\epsilon_{z^{(k)},q^{(k)}}$ on the k-th level of the tree yield some field $\psi^{(m)}$ such that

$$W_{q^{(m)}}(\psi^{(m)}, z^{(m)}) = T_l\left(\{\psi_i\}_{i=1}^{2n}; z^{(0)}, q^{(0)}\right) \tag{2.50}$$

for some l (for more discussion, including the exact expressions for $z^{(k)}$, $q^{(k)}$ and $l^{(k)}$, see subsection 3.2.1). In other words, such TTN reproduces the transfer operator and thus – the correlation functions for a given CFT.

Even though we have obtained the map that we will need in the following and that yields the tree tensor network, it is not yet clear what the intuition behind this map is. Moreover, one could have asked if there is a connection between this map and the MPS and if there are other maps that satisfy the needed three conditions, namely renormalization, boundedness and isometry. The next section is devoted to answering these questions.

2.4 Connection to the MPS and generalization

In order to compute correlation functions via the MPS the way it was done in [KS15], one has to compute scaled intertwiners of fields ψ_i and then multiply them. One can think about some map, let us call it ε_{MPS}, that does the job. Of course, it does not matter in which order one multiplies scaled intertwiners. Thus we will just specify a map that multiplies two scaled intertwiners to obtain a transfer operator of two fields. In the next step one can multiply these transfer operators of two fields and obtain a transfer operator of four fields. Moreover one does not need a new map, as a transfer operator of two fields is a scaled intertwiner itself, but with another scale – if the initial scale was q, the new scale is q^2. This way of computing the MPS transfer operator has a resemblance to the binary tree tensor network, thus we will use it to get the connection between the MPS and the TTN and to obtain a family of maps that are suitable for the TTN.

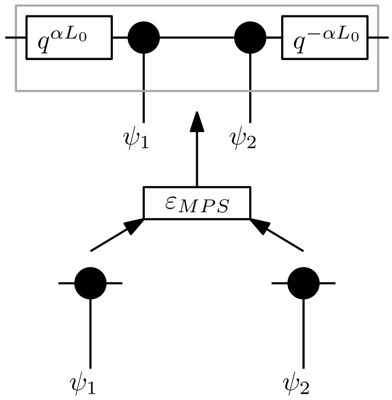


Figure 2.10: The map ε_{MPS} allows us to "glue together" two scaled intertwiners to obtain a transfer operator of two fields. This is the first step to get a transfer operator via the MPS using the scaled intertwiners as constituting tensors.

The obvious map that does the trick is simply a multiplication map. However, this map has problems with boundedness. To fix this, we will introduce an unphysical dilation – $q^{\alpha L_0}$ and $q^{-\alpha L_0}$.

Lemma 2.4.1. *Consider the map (see figure 2.10):*

$$\varepsilon_{MPS}(W_q(\psi_1, z), W_q(\psi_2, z)) \equiv q^{\alpha L_0} W_q(\psi_1, z) \circ W_q(\psi_2, z) q^{-\alpha L_0}. \tag{2.51}$$

Then,

$$\varepsilon_{MPS} : \mathcal{W}_q \times \mathcal{W}_q \rightarrow \mathcal{W}_{q^2} \tag{2.52}$$

$$\varepsilon_{MPS}\left(T_l(\{\psi\}_{i=1}^n), T_l(\{\psi\}_{i=n+1}^{2n})\right) = T_{l+\alpha}(\{\psi\}_{i=1}^{2n}). \tag{2.53}$$

where $\mathcal{W}_q$ is a space of scaled intertwiners with scale parameter q.

Proof. In order for such map to have the range stated in (2.52), it is necessary to have one scaled intertwiner at right-hand side. This is a direct consequence of Lemma 2.3.1 and the way scaled intertwiners transform under the action of L_0: $q^{L_0} W_{\tilde{q}}(u, z) q^{-L_0} = W_{\tilde{q}}(q^{L_0} u, qz)$.

We see that W_q's are mapped into W_{q^2}. This should be expected, as after the map is applied to all of the lattice with spacing q, a lattice with spacing q^2 is obtained.

Any transfer operator is in the domain of ε_{MPS} as a direct consequence of Lemma 2.3.1.

Applying ϵ to transfer operators is trivial:

$$\begin{aligned}\varepsilon_{MPS}\left(T_l(\{\psi\}_{i=1}^n), T_l(\{\psi\}_{i=n+1}^{2n})\right) &= q^{\alpha L_0} q^{lL_0} W_q(\psi_1, z) \circ \cdots \circ W_q(\psi_n, z) q^{-lL_0} \\ &\quad \circ q^{lL_0} W_q(\psi_{n+1}, z) \circ \cdots \circ W_q(\psi_{2n}, z) q^{-lL_0} q^{-\alpha L_0} \\ &= q^{(\alpha+l)L_0} W_q(\psi_1, z) \circ \cdots \circ W_q(\psi_{2n}, z) q^{-(\alpha+l)L_0} \\ &= T_{\alpha+l}(\{\psi\}_{i=1}^{2n}), \end{aligned} \tag{2.54}$$

which is the statement of the Lemma. □

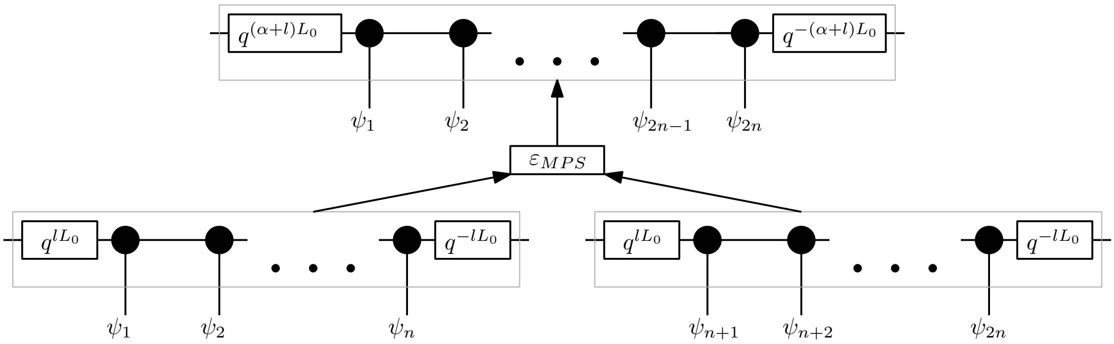


Figure 2.11: The map ε_{MPS} is actually the map that "glues together" two transfer operators of equal size, as Lemma 2.4.1 states. Even though ε_{MPS} actually corresponds to MPS, it resembles the structure of binary tree.

To draw the connection to TTN, we would like to obtain a map that acts on fields instead of scaled intertwiners. This can easily be obtained from equation (2.34). We can see, however, that now we also have an ordinary intertwiner at the r.h.s of (2.34). We would like to work with **only** scaled intertwiners as otherwise problems with non-boundedness arise – this is exactly the problem that extra factors $q^{\alpha L_0}$ in the definition of ε_{MPS} (2.51) solve. In other words, the r.h.s of (2.51) is:

$$\begin{aligned} &q^{\alpha L_0} W_{q^2}\left(Y\left(q^{-L_0}\psi_1, q^{-3/2}(1-q)z\right)\psi_2, q^{1/2}z\right) q^{-\alpha L_0} \\ &= W_{q^2}\left(q^{\alpha L_0} Y\left(q^{-L_0}\psi_1, q^{-3/2}(1-q)z\right)\psi_2, q^{1/2+\alpha}z\right) \end{aligned} \tag{2.55}$$

which leads to the map:

$$\epsilon_{z,q}^{\alpha} : \psi_1 \otimes \psi_2 \to q^{\alpha L_0} Y\left(q^{-L_0}\psi_1, q^{-3/2}(1-q)z\right)\psi_2, \tag{2.56}$$

which is a generalization of the map (2.36), so we use the same symbol to denote it. Even though this map obeys renormalization property – indeed, it was the way we derived it, we still have some work to do to understand which α is needed to also get boundedness.

2.4.1 Choice of α

We would like to work only with scaled intertwiners and bounded operators such as q^{L_0}, as this allows us to obtain error bounds for truncated operators. We compute

$$\begin{aligned}
\epsilon^{\alpha}_{z,q}(\psi\otimes\phi) &= q^{\alpha L_0} Y\left(q^{-L_0}\psi, q^{-3/2}(1-q)z\right)\phi \\
&= q^{(\alpha-\gamma)L_0} q^{\gamma L_0} Y\left(q^{-L_0}\psi, q^{-3/2}(1-q)z\right) q^{-\gamma L_0} q^{\gamma L_0}\phi \\
&= q^{(\alpha-\gamma)L_0} Y\left(q^{(\gamma-1)L_0}\psi, q^{-3/2+\gamma}(1-q)z\right) q^{\gamma L_0}\phi \\
&= q^{(\alpha-2\gamma+1)L_0} W_{q^{2(\gamma-1)}}\left(\psi, q^{-3/2+\gamma}(1-q)z\right) q^{L_0}\phi. \qquad (2.57)
\end{aligned}$$

Without the loss of generality, we can choose

$$\alpha = 2\gamma - 1, \qquad (2.58)$$

so we get the map

$$\begin{aligned}
\epsilon^{\alpha}_{z,q}(\psi\otimes\phi) &= W_{q^{2(\gamma-1)}}\left(\psi, q^{-3/2+\gamma}(1-q)z\right) q^{L_0}\phi; \\
z &\to q^{2\gamma-1/2} z; \\
q &\to q^2. \qquad (2.59)
\end{aligned}$$

The parameter γ is still free, but in order for all operators on the r.h.s. of (2.59) to be well-defined and bounded for $0 < q < 1$ we have following constraint:

$$\gamma > 1. \qquad (2.60)$$

As we will see in section 2.5, the following condition arises from the isometry condition:

$$|q^{3\gamma-5/2}(1-q)z| < 1. \qquad (2.61)$$

For each $\gamma > 1$ condition (2.61) can be fulfilled with a suitable choice of z and q. For simplicity we will choose

$$\gamma = 3/2 \quad \Leftrightarrow \quad \alpha = 2, \qquad (2.62)$$

which gives us the map (2.36): $\epsilon^{2}_{z,q} \equiv \epsilon_{z,q}$.

This means that the map $\epsilon_{z,q}$ that yields the TTN is intimately connected to the map ε_{MPS} that produces the MPS. Actually, both maps do more a less the same thing, but the TTN map acts on the space of fields while the MPS map acts on the space of scaled intertwiners. Another feature is that there is a whole one-parameter family of maps - $\epsilon^{\alpha}_{z,q}$, $\alpha > 1$ that satisfy the renormalization property (see figure 2.8) and boundedness from the list of conditions at the beginning of the section 2.3. We will also see that this family of maps satisfy isometry condition in an appropriate basis. In fact, any representative could be chosen in the family $\epsilon^{\alpha}_{z,q},\ \alpha > 1$ and it will be suitable for the construction of TTN. All the statements we will prove can be proven for any $\epsilon^{\alpha}_{z,q},\ \alpha > 1$ with minor modifications. We will work with $\epsilon_{z,q}$ only because it increases readability of formulas while modification of statements for general α do not require any new ideas.

2.5 Adjoint and isometry

For efficient contraction of the tree tensor network one needs an isometry condition. In the language of diagrams it reads:

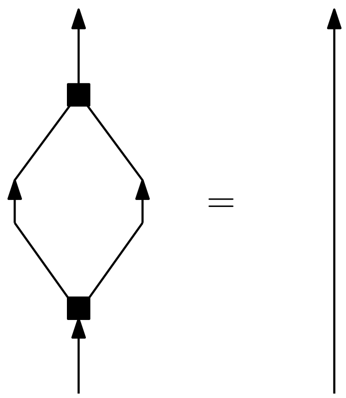


Figure 2.12: Isomerty condition.

However, it depends on the inner product of $A \otimes B$. One needs to define an inner product such that the isometry condition holds. Let $u, v, w \in \mathbb{R}$ be non-zero, $z \in \mathbb{C}$ and define

$$\iota^*(a, b) = u^{L_0} Y(v^{L_0} a, z) w^{L_0} b \tag{2.63}$$

Lemma 2.5.1. *Define*

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{u,v,w} := \langle \iota^*(a_1, b_1), \iota^*(a_2, b_2) \rangle \ . \tag{2.64}$$

Assume that $|u|^2 < 1/|z|^2$. *Then* $\langle \cdot, \cdot \rangle_{u,v,w}$ *can be extended to a sesquilinear, densely defined and positive semi-definite form on* $A \otimes B$.

Proof. We first show that this is well-defined: for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have

$$\begin{aligned}
\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{u,v,w} &= \langle u^{L_0} Y(v^{L_0} a_1, z) w^{L_0} b_1, u^{L_0} Y(v^{L_0} a_2, z) w^{L_0} b_2 \rangle \\
&= \langle Y(v^{L_0} a_1, z) w^{L_0} b_1, u^{2L_0} Y(v^{L_0} a_2, z) w^{L_0} b_2 \rangle \\
&= \langle w^{L_0} b_1, Y(e^{zL_1}(-z^{-2})^{L_0} \eta(v^{L_0} a_1), z^{-1}) u^{2L_0} Y(v^{L_0} a_2, z) w^{L_0} b_2 \rangle \\
&= \langle w^{L_0} b_1, Y(\tilde{a}_1, z^{-1}) u^{2L_0} Y(v^{L_0} a_2, z) w^{L_0} b_2 \rangle \\
&= \langle w^{L_0} b_1, Y(\tilde{a}_1, z^{-1}) Y(u^{2L_0} v^{L_0} a_2, u^2 z) u^{2L_0} w^{L_0} b_2 \rangle \\
&= \langle \tilde{b}_1, Y(\tilde{a}_1, z^{-1}) Y(\tilde{a}_2, u^2 z) \tilde{b}_2 \rangle \\
&= F^{(0)}_{\tilde{b}_1, \tilde{b}_2}((\tilde{a}_1, \underbrace{z^{-1}}_{=:z_1}), (\tilde{a}_2, \underbrace{u^2 z}_{=:z_2}))
\end{aligned} \tag{2.65}$$

where we have used the properties of the Hermitian form (1.33), the fact that L_0 is self-adjoint (1.34) and the way intertwiners transform under the action of L_0 (1.26) , $F^{(0)}_{\tilde{b}_1, \tilde{b}_2}$ was defined in (2.2) and we have introduced

$$\tilde{a}_1 := e^{zL_1}(-z^{-2})^{L_0} \eta(v^{L_0} a_1), \tag{2.66}$$

$$\tilde{a}_2 := u^{2L_0} v^{L_0} a_2, \tag{2.67}$$

$$\tilde{b}_1 := w^{L_0} b_1, \tag{2.68}$$

$$\tilde{b}_2 := u^{2L_0} w^{L_0} b_2. \tag{2.69}$$

In particular, $|z_1| > |z_2|$ if and only if $|u| < 1/|z|^2$, which is our assumption. The remainder of the proof is identical to the proof of Lemma 4.1 in [KS15]. Namely, extending the definition linearly to a finite sums of the from $\sum_i a_i \otimes b_i \in A \otimes B$, and using the facts that the latter are dense in $A \otimes B$ and that $Y(\cdot, z)$ is linear, it follows that $\langle \cdot, \cdot \rangle_{u,v,w}$ is indeed sesquilinear and densely defined form on $A \otimes B$.

Let $\{a_\alpha\}_\alpha \subset A$ and $\{b_\beta\}_\beta \subset B$ be finite families of elements in A and B, respectively. To show that $\langle \cdot, \cdot \rangle_{u,v,w}$ is positive semi-definite, it suffices to check that for any such families, the matrix

$$\langle \iota^*(a_{\alpha_1}, b_{\beta_1}), \iota^*(a_{\alpha_2}, b_{\beta_2}) \rangle_{(\alpha_1,\beta_1),(\alpha_2,\beta_2)} \tag{2.70}$$

is positive semi-definite. But this is the Gram matrix associated with the family of vectors $\{\iota^*(a_\alpha, b_\beta)\}_{(\alpha,\beta)}$ with entries given by the inner products, hence the claim follows. □

Using that $Y(1, z) = \mathsf{id}$ we get

$$\iota^*(1, b) = u^{L_0} w^{L_0} b \tag{2.71}$$

or

$$\iota^*(1, u^{-L_0} w^{-L_0} d) = d \ . \tag{2.72}$$

This means that

$$\langle d, \iota^*(a_2, b_2)\rangle = \langle \iota^*(1, u^{-L_0} w^{-L_0} d), \iota^*(a_2, b_2)\rangle \tag{2.73}$$

or

$$\langle 1 \otimes u^{-L_0} w^{-L_0} d, a_2 \otimes b_2\rangle_{u,v,w} = \langle d, \iota^*(a_2, b_2)\rangle \tag{2.74}$$

As a consequence, if

$$\iota(d) := 1 \otimes u^{-L_0} w^{-L_0} d \ , \tag{2.75}$$

then ι^* is the adjoint of ι. To check the isometry condition, observe that

$$\iota^* \circ \iota(d) = \iota^*(1 \otimes u^{-L_0} w^{-L_0} d) \tag{2.76}$$

$$= u^{L_0} Y(v^{L_0} 1, z) w^{L_0} u^{-L_0} w^{-L_0} d \tag{2.77}$$

$$= u^{L_0} w^{L_0} u^{-L_0} w^{-L_0} d \tag{2.78}$$

$$= d \tag{2.79}$$

for any $d \in C$. This shows that ι is an isometry.

2.5.1 Application to the actual map

Consider the map

$$X(a, b; z) = q^{\alpha L_0} Y(q^{-L_0} a, q^{-3/2}(1-q)\tilde{z}) b \ . \tag{2.80}$$

This is of the form $X = \iota^*$ with $u = q^\alpha$, $v = q^{-1}$ and $z = q^{-3/2}(1-q)\tilde{z}$, $w = 1$. This gives us an inner product for which the maps (2.56), including the map (2.36) are isometries. Thus, the network defined via the map (2.36) can be contracted efficiently.

Chapter 3

Approximation and error bounds

We now have the necessary components for the tree tensor network. However, the maps are infinite-dimensional. If one wants to simulate the model on a computer, a truncation to a finite-dimensional subspace is needed. In this section we will show that for a given 2^m fields on the lattice (2.1) the error, namely the norm of the difference between the transfer operator and the truncated transfer operator, decays exponentially in the truncation parameter N (see subsection 2.1.1) for a certain domain of z and q (see subsection 3.2.1).

Notation 3.0.1. *The symbol $\approx$ will be used as follows:*

$$A_N \approx A \Leftrightarrow \|A_N - A\| \leq q^{\Omega(N)}. \tag{3.1}$$

for a given parameters $0 < q < 1$ and N.

Then, given the truncation parameter, we will deduce necessary bond dimensions and thus – the number of parameters that are necessary to obtain tree tensor network.

3.1 Strategy of approximation

We call a tensor network **exact** if it is composed of the maps $\epsilon_{z,q}$ (2.36) and the intertwiners (see figure 2.7). We call the tensor network **truncated** if some of the maps $\epsilon_{z,q}$ are substituted by the truncated maps $\epsilon_{z,q}^{[N]}$ 2.45, and the network is **fully truncated** if all the maps $\epsilon_{z,q}$ are substituted by the truncated ones $\epsilon_{z,q}^{[N]}$.

It is not obvious what the norm of the difference of an exact tensor network and the fully truncated one is. However, one can use the inequality

$$\|AB\| \leq \|A\| \cdot \|B\|. \tag{3.2}$$

As TTN are formed by multiplication of the renormalization maps, we will need to obtain norm bounds for these elementary "building blocks" of the network (see figure 3.1). For the approximation scheme we will need norm bounds for the scaled intertwiner of the field renormalized by both exact and truncated map. As we will approximate the difference between exact and truncated TTN, we will also need a norm estimate for the difference between these two building blocks.

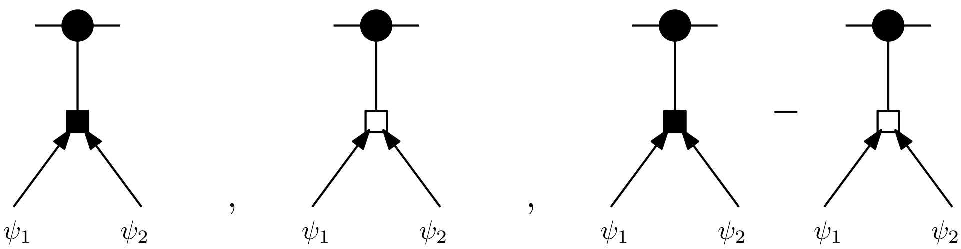


Figure 3.1: Elementary objects in the renormalization scheme. The "filled" box corresponds to the exact map $\epsilon_{z,q}$ that was introduced in (2.36), and the "empty" box corresponds to the truncated map $\epsilon_{z,q}^{[N]}$ that was introduced in (2.45). The remaining diagram corresponds to $\epsilon_{z,q} - \epsilon_{z,q}^{[N]}$, the norm of such object shows how good $\epsilon_{z,q}$ is approximated with $\epsilon_{z,q}^{[N]}$. The diagrammatic notation for $\epsilon_{z,q}$ was first introduced in figure 2.6, and for $\epsilon_{z,q}^{[N]}$ - in figure 2.9.

We can also use the intuition that norm of the difference of some network that consists of exact and truncated maps and the network that has *exactly one truncated map instead of the exact one* should be small. For this we will prove that norm of the difference of the exact and truncated building block should be exponentially small in the truncation parameter (see figure 3.2).

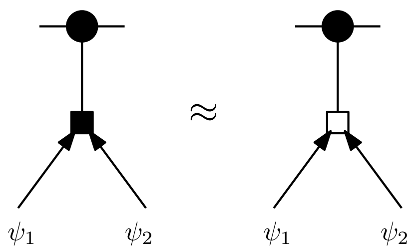


Figure 3.2: Truncation introduces error $\leq q^{\Omega(N)}$.

These truncation building blocks will be dealt with in section 3.2.

The next step in employing this intuition will be to use *the telescoping sum inequalities*

$$\|A_0 - A_n\| = \left\| \sum_{i=1}^{n} A_{i-1} - A_i \right\| \leq \sum_{i=1}^{n} \|A_{i-1} - A_i\| \tag{3.3}$$

There are plenty of ways to arrange terms in a telescoping sum expansion in such a way that every term is a difference of two TTNs with exactly one different map and A_0 being the exact TTN and A_n – the fully truncated one. We will use an arrangement where we will have truncated maps on the upper levels of the tree only if the lower level is fully truncated.

For this we will first show error bounds for the transfer operator where fields are renormalized only once. That is, we set

$$A_k = T_l^{[N,k]} \equiv \prod_{i=1}^{k} q^{(l-2)L_0} W_{q^2}\left(\epsilon_{z,q}^{[N]}(\psi_{2i-1}\otimes\psi_{2i}), q^{5/2}z\right) \circ \prod_{j=k+1}^{n} W_{q^2}(\epsilon_{z,q}\left(\psi_{2j-1}\otimes\psi_{2j-2}), q^{5/2}z\right) q^{-(l-2)L_0}. \tag{3.4}$$

in inequality (3.3) (see figure 3.3).

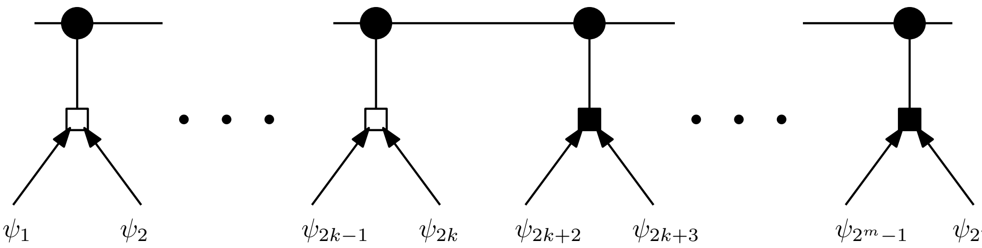


Figure 3.3: Diagrammatic representation of $T^{[N,k]}$.

These bounds will be obtained in section 3.3.

It is useful to obtain such bounds, as we will use it in analyzing the full tree using the telescoping sum expansion. Capitalizing on the discussion in section 2.2, we can define truncated renormalized fields via

$$\psi_i^{(0)[N]} \equiv \psi_i, \tag{3.5}$$

$$\psi_i^{(r)[N]} = \epsilon_{z^{(r-1)},q^{(r-1)}}^{[N]}\left(\psi_{2i-1}^{(r-1)[N]}\otimes\psi_{2i}^{(r-1)[N]}\right), \quad 1\le r\le m \tag{3.6}$$

(for the definition of $\epsilon_{z,q}^{[N]}$ see (2.45)) in analogy with the renormalized fields obtained via the exact map

$$\psi_i^{(0)} \equiv \psi_i, \tag{3.7}$$

$$\psi_i^{(r)} = \epsilon_{z^{(r-1)},q^{(r-1)}}\left(\psi_{2i-1}^{(r-1)}\otimes\psi_{2i}^{(r-1)}\right), \quad 1\le r\le m \tag{3.8}$$

(for the definition of $\epsilon_{z,q}$ see (2.36), this renormalized fields $\psi_i^{(r)}$ were first introduced in (2.30)). This construction allow to connect different levels of the tree via once truncated transfer operators (see definition 3.3.1).

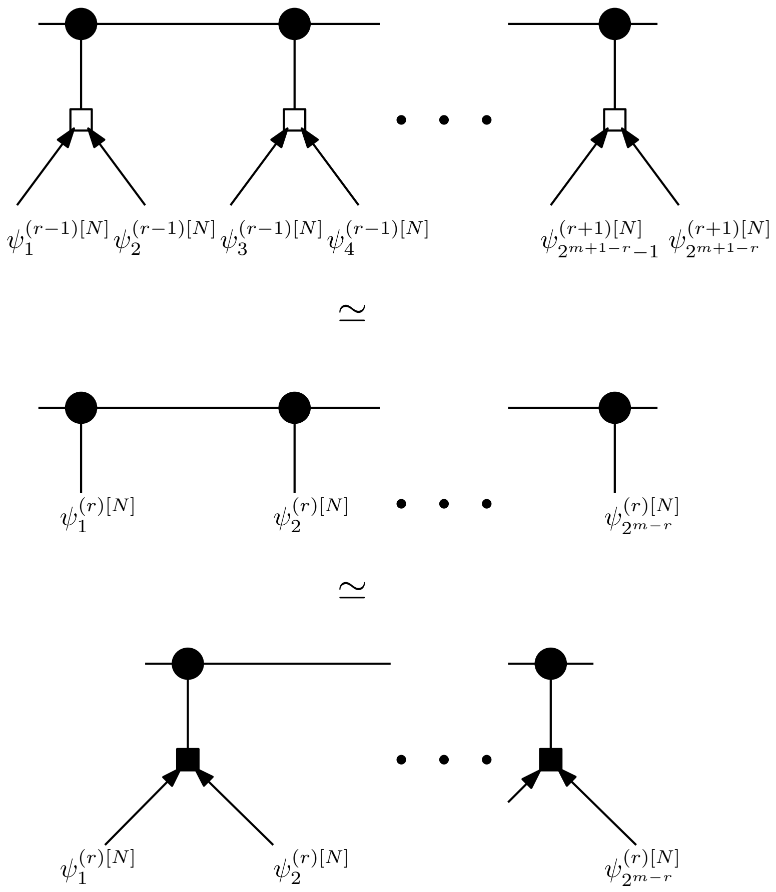


Figure 3.4: We can think of the transfer operator with renormalized fields both as one step of some truncated tensor network, or one step of non-truncated tensor network.

Finally, we will obtain the bound for the full tree. Again, we will use telescope expansion (3.3). For this, we will set

$$\begin{aligned} A_r = T_{l^{(r)}}^{(r)[N]} \quad &\equiv \quad \left(q^{(r)}\right)^{l^{(r)}L_0} W_{q^{(r)}}\left(\psi_1^{(r)[N]}, z^{(r)}\right) \circ \dots \\ &\quad \dots \circ W_{q^{(r)}}\left(\psi_{2^{m-r}}^{(r)[N]}, z^{(r)}\right)\left(q^{(r)}\right)^{-l^{(r)}L_0} \end{aligned} \tag{3.9}$$

(see figure 3.5).

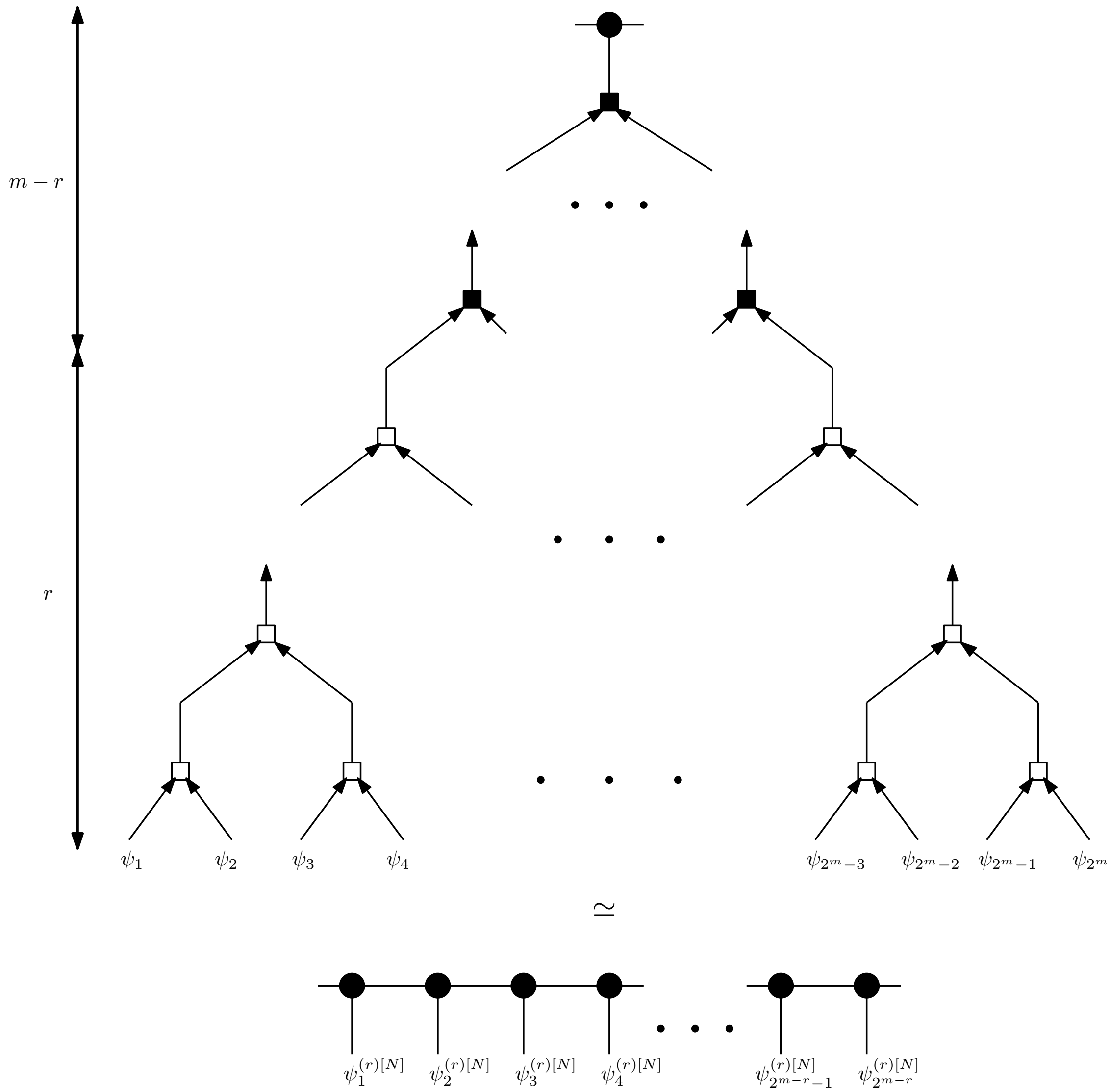


Figure 3.5: Diagrammatic representation of $T_{l^{(r)}}^{(r)[N]}$ and its relation to the transfer operator of truncated renormalized fields.

The relation between figures 3.5 and 3.4 allows us to use results of the section 3.3 for the proof of the error bound for the full tree. This error bound for truncated TTN will be obtained in section 3.4.

3.1.1 Linear energy bounds for WZW

As was discussed in section 2.1, there is a bound on the scaled intertwiner – for a given set of numbers $\{w_\psi, z, q\}$ such that $0 \le w_\psi < \infty$ and $0 < q < \min\{|z|^2, 1/|z|^2\}$ the inequality

$$\|W_q(\psi, z)\| \le \vartheta_{w_\psi}(q, z)\|\psi\| \tag{3.10}$$

holds for every $\psi \in \bigotimes_{n \le w_\psi} A_n$ in the module A and $\vartheta_{w_\psi}(q, z)$ is finite. However, if one tries to use such a bound for a renormalized field – that is, the field on the next level of the TTN, one would have:

$$\left\|W_{q^{(1)}}\left(\psi^{(1)}, z^{(1)}\right)\right\| \le \vartheta_{w_{\psi^{(1)}}}\left(q^{(1)}, z^{(1)}\right)\left\|\psi^{(1)}\right\|. \tag{3.11}$$

It is important to notice that $w_{\psi^{(1)}}$ actually grows. If one acts on fields at the first level of the TTN with maps that have truncation parameter N (see subsection 2.1.1), then typically

$$\max_i w_{\psi_i^{(1)}} = \max_i w_{\psi_i} + N. \tag{3.12}$$

While ϑ also depends on q and z, these do not depend on N. For the approximation scheme to be successful – that is, for the approximation error to fall sufficiently fast with the growth of N, one has to study how fast ϑ_w grows while w increases. Contrary to the MPS case [KS15], only the finiteness of ϑ_w is not enough, as each level of the TTN can increase w by N and all bounds start to depend on N, while for the MPS one needs these bounds only for the initial fields, so the resulting bound is N-independent.

Using linear energy bounds, König and Scholz have obtained exact expression for $\vartheta_S(q, z)$ in the case of WZW models [KS15]. That is, if a has a weight h, then

$$\begin{aligned}\|W_q(a, z)\|^2 \le c^2\left(\frac{q^h}{|z|}\right)^2\|a\|^2\Bigg[(1+2h)\Bigg(5!\frac{|z^2|}{q}\left[\log\left(\frac{|z|^2}{q}\right)\right]^{-6} \\ +\frac{1}{2|z|^2q^3}\left[\log\left(\frac{1}{q}\right)\log\left(\frac{1}{|z|^2q}\right)\right]^{-3}\Bigg) \\ +h^2\left(\frac{1}{1-|z|^{-2}q}+\frac{1}{2|z|^2q^3(1-q^2)(-\log(|z|^2q))^3}\right)\Bigg].\end{aligned} \tag{3.13}$$

As we can see, the multiplier q^h falls exponentially while h increases and the term in the square brackets grows polynomially. Thus the right hand side can be bounded by some constant dependent on z and q but not h.

Definition 3.1.1. *$d_A(M)$ is the dimension of the subspace obtained by keeping all the levels up to and including M in the module A*

$$d_A(M) = \dim\left(\bigoplus_{h \le M} A_h\right). \tag{3.14}$$

Then for the growth of $\vartheta_{w_\psi} = \vartheta_w,\ \psi \in A$ we have:

$$\vartheta_w(q,z) = \sum_{h \le w} q^h \text{Polynomial}(h) \cdot \dim(A_h) \le const \cdot d_A(w), \tag{3.15}$$

where *const* may depend on z and q but does not depend on w, $d_A(w)$ grows as the number of partitions of w (see inequality (1.31)), thus ϑ_w grows sub-exponentially with w.

To bound the truncation error one would need similar bounds for any CFT of interest. Unfortunately, we are unaware of bounds like linear energy bounds for more general classes of CFTs. However, we can conjecture weaker results than for WZW that will nevertheless be sufficient for the proof to work.

Conjecture 3.1.2. *The function $w \to \vartheta_w(q,z)$ grows sub-exponentially with the growth of w for q, z fixed.*

Let us note that even the weaker assumption that $\vartheta_w(q,z) \le q^{cw}$ for appropriate c will be sufficient, yet we will proceed with the sub-exponential conjecture as it makes the proofs technically simpler while the idea remains the same. Let us also note that even if the growth of the bound of type (3.13) were to be much faster – that is, sub-exponential, the argument (3.15) would still hold and thus, the conjecture 3.1.2 would hold.

3.2 Truncation building blocks

We would like to calculate bounds for norms of elementary objects that appear in the renormalized transfer operator. In these objects $l \ge 0$ is just some arbitrary parameter to be specified later. Let us treat them one at a time. In the proofs of the lemmas we will use the bounds (2.17) and (2.26).

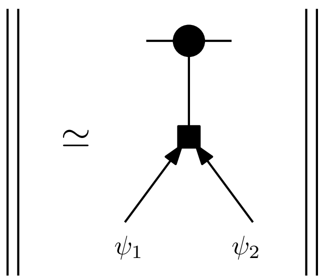


Figure 3.6: Diagram that correspond to $\mathcal{N}(z,q)$ in Lemma 3.2.1.

Lemma 3.2.1. *Suppose that w_{ψ_1} and w_{ψ_1} are finite (see definition 2.1.5). Let*

$$\mathcal{N}(z,q) \equiv \left\| q^{lL_0} W_{q^2}(\epsilon_{z,q}(\psi_1 \otimes \psi_2), q^{5/2} z) q^{-lL_0} \right\| \tag{3.16}$$

(see figure 3.6), $0 < q < \min\left\{|q^{2+l}z|^2, \frac{1}{|q^{2+l}z|^2}\right\}$ *and* $l \geq 0$*. Then*

$$\mathcal{N}(z,q) \leq \vartheta_{w_1}\left(q, q^{2+l}z\right) \cdot \vartheta_{w_2}\left(q, q^{2+l}z\right) \cdot \|\psi_1\| \cdot \|\psi_2\|. \tag{3.17}$$

(See section 2.1 for definition of $\vartheta_w(q,z)$*.)*

Proof.

$$\begin{aligned}\mathcal{N}(z,q) &= \left\|q^{(2+l)L_0} W_q(\psi_1, z) \circ W_q(\psi_2, z) q^{-(2+l)L_0}\right\| \\ &\leq \left\|W_q\left(q^{(2+l)L_0}\psi_1, q^{2+l}z\right)\right\| \cdot \left\|W_q\left(q^{(2+l)L_0}\psi_2, q^{2+l}z\right)\right\| \\ &\leq \vartheta_{w_1}\left(q, q^{2+l}z\right) \cdot \vartheta_{w_2}\left(q, q^{2+l}z\right) \cdot \left\|q^{(2+l)L_0}\psi_1\right\| \cdot \left\|q^{(2+l)L_0}\psi_2\right\|,\end{aligned} \tag{3.18}$$

where we have used the bound (2.17). As $\|q^{lL_0}\|$ is bounded by 1, we can write

$$\left\|q^{kL_0}\psi\right\| \leq \|\psi\| \quad \forall \psi; k > 0. \tag{3.19}$$

In particular we can use this for $k = 2 + l$ which implies the claim. □

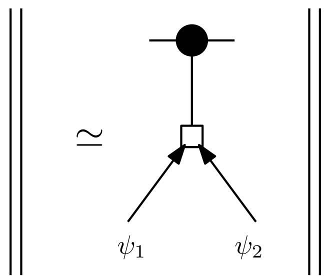


Figure 3.7: Diagram that correspond to $\mathcal{N}_{truncated}(z,q)$ in Lemma 3.2.2.

Lemma 3.2.2. *Suppose that* w_{ψ_1} *and* w_{ψ_1} *are finite (see definition 2.1.5). Let*

$$\mathcal{N}_{truncated}(z,q) \equiv \left\|q^{lL_0} W_{q^2}\left(\epsilon^{[N]}_{z,q}\left(\psi_1 \otimes \psi_2\right), q^{5/2}z\right) q^{-lL_0}\right\| \tag{3.20}$$

(see figure 3.7), $0 < q < \min\left\{q^{l+5/2}|z|, \frac{1}{q^{l+5/2}|z|}, ((1-q)|z|)^2, \frac{1}{((1-q)|z|)^2}\right\}$ *and* $l \geq 0$*. Then*

$$\mathcal{N}_{truncated}(z,q) \leq \frac{\sqrt{|I_V|}\vartheta_{w_1}(\sqrt{q}, (1-q)z)}{\sqrt{1-\sqrt{q}}} \vartheta_{w_2+N}\left(q^2, q^{5/2+l}z\right) \|\psi_1\|\|\psi_2\|. \tag{3.21}$$

(See section 2.1 for definition of $\vartheta_w(q,z)$ *and subsection 1.2.2 for the definition of* I_V*.)*

Proof. As $\epsilon^{[N]}_{z,q}\left(\psi_1 \otimes \psi_2\right)$ has finite weight and q^{lL_0} does not change the weight we have

$$\begin{aligned}\mathcal{N}_{truncated}(z,q) &\leq \vartheta_{w_2+N}\left(q^2, q^{5/2+l}z\right) \left\|W^{[N]}_q\left(\psi_1, (1-q)z\right) q^{L_0}\psi_2\right\| \\ &\leq \vartheta_{w_2+N}\left(q^2, q^{5/2+l}z\right) \left\|W^{[N]}_q\left(\psi_1, (1-q)z\right)\right\| \|q^{L_0}\|\|\psi_2\| \\ &\leq \frac{\sqrt{|I_V|}\vartheta_{w_1}(\sqrt{q}, (1-q)z)}{\sqrt{1-\sqrt{q}}} \vartheta_{w_2+N}\left(q^2, q^{5/2+l}z\right) \|\psi_1\|\|\psi_2\|\end{aligned} \tag{3.22}$$

which is the statement of the lemma. Here we have used the bounds (2.17) and (2.26). □

We have one more building block.

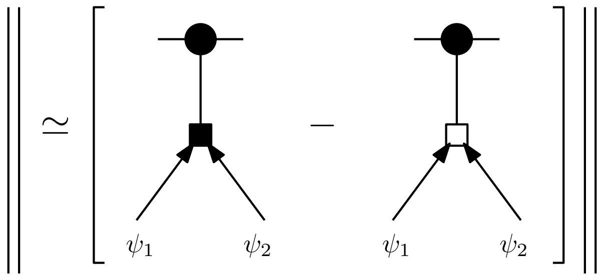


Figure 3.8: Diagram that correspond to $\mathcal{N}_{error}(z,q)$ in LemmaNerrBB.

To obtain a bound for it it is useful to work in a similar fashion to [KS15].

Lemma 3.2.3. *Suppose that w_{ψ_1} is finite (see definition 2.1.5). Let*

$$\mathcal{N}_{error}(z,q) \equiv \left\| q^{lL_0} \left(W_{q^2}\left(\epsilon_{z,q}(\psi_1 \otimes \psi_2), q^{5/2}z\right) - W_{q^2}\left(\epsilon^{[N]}_{z,q}(\psi_1 \otimes \psi_2), q^{5/2}z\right)\right) q^{-lL_0} \right\| \quad (3.23)$$

(see figure 3.8), $0 < q < \min\left\{ q^{l+5/2}|z|, \frac{1}{q^{l+5/2}|z|}, ((1-q)|z|)^4, \frac{1}{((1-q)|z|)^4} \right\}$, $l \geq 0$ *and assume that conjecture 3.1.2 holds. Then*

$$\mathcal{N}_{error}(z,q) \leq \vartheta_{w_1}\left(q^{1/2}, (1-q)z\right) \cdot \|\psi_1\| \cdot \|\psi_2\| \cdot \kappa(z,q) \cdot q^{\Omega(N)}. \quad (3.24)$$

For some $\kappa(z,q) > 0$. (See section 2.1 for definition of $\vartheta_w(q,z)$.)

Remark 3.2.4. *The assumption 3.1.2 is true for WZW models, as explained in section 3.1.1.*

Proof. For the graded module $B = \bigoplus B_n$ and $\phi \in B$ let us define:

$$\phi = \sum_{n \in \mathbb{N}_0} (\phi)_n, \quad \text{where } (\phi)_n \in B_n. \quad (3.25)$$

Let us define a projector

$$P_{[k,m]}\phi = \sum_{n=k}^{m} (\phi)_n \ \text{ for any } \phi \in B. \quad (3.26)$$

We have

$$\begin{aligned}
\mathcal{N}_{error}(z,q) &= \left\| \sum_{n \in \mathbb{N}_0} q^{lL_0} W_{q^2}\left((1 - P_{[n-N,n+N]})\epsilon_{z,q}(\psi_1 \otimes (\psi_2)_n), q^{5/2}z\right) q^{-lL_0} \right\| \\
&\leq \sum_{n \in \mathbb{N}_0} \left\| W_{q^2}\left((1 - P_{[n-N,n+N]}) q^{lL_0} \epsilon_{z,q}(\psi_1 \otimes (\psi_2)_n), q^{5/2+l}z\right) \right\| \\
&= \sum_{n \in \mathbb{N}_0} \Big\| W_{q^2}\big((1 - P_{[n-N,n+N]}) q^{lL_0} \\
&\quad \cdot \ W_q\left(\psi_1, (1-q)z\right) q^{L_0}(\psi_2)_n, q^{5/2+l}z\big) \Big\| \quad (3.27)
\end{aligned}$$

We can use

$$W_q\left(\psi_1,(1-q)z\right) = q^{L_0/4} W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right) q^{L_0/4} \tag{3.28}$$

as well as

$$q^{aL_0}(\phi)_n = q^{a(h_B+n)}(\phi)_n \quad \forall \phi \in B,\ a \in \mathbb{C}. \tag{3.29}$$

After expanding with vectors of different weights we get

$$\begin{aligned}\mathcal{N}_{error}(z,q) \;\; &\leq \;\; \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} \Big\| W_{q^2}\Big((1-P_{[n-N,n+N]})q^{L_0(l+1/4)} \\ &\cdot \left(W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right) q^{5L_0/4}(\psi_2)_n\right)_m, q^{5/2+l}z\Big)\Big\|. \end{aligned} \tag{3.30}$$

This leads to

$$\begin{aligned}\mathcal{N}_{error}(z,q) \;\; &\leq \;\; \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} q^{(m(4l+1)+5n)/4} \big\| W_{q^2}\big((1-P_{[n-N,n+N]}) \\ &\cdot \left(W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right) (\psi_2)_n\right)_m, q^{5/2+l}z\Big)\Big\|. \end{aligned} \tag{3.31}$$

By the definition of projector $P_{[n-N,n+N]}$ (3.26), we have

$$\begin{aligned}\mathcal{N}_{error}(z,q) \;\; &\leq \;\; \sum_{n\in\mathbb{N}_0}\sum_{0\leq m<n-N} q^{(m(4l+1)+5n)/4} \\ &\cdot \Big\| W_{q^2}\Big(\left(W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right) (\psi_2)_n\right)_m, q^{5/2+l}z\Big)\Big\| \\ &+ \sum_{n\in\mathbb{N}_0}\sum_{m>n+N} q^{(m(4l+1)+5n)/4} \\ &\cdot \Big\| W_{q^2}\Big(\left(W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right) (\psi_2)_n\right)_m, q^{5/2+l}z\Big)\Big\|. \end{aligned} \tag{3.32}$$

We can bound the norm of a scaled intertwiner now, as the insertion has finite weight. Using the bound (2.26), we obtain

$$\begin{aligned} &\Big\| W_{q^2}\Big(\left(W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right) (\psi_2)_n\right)_m, q^{5/2+l}z\Big)\Big\| \\ &\leq \vartheta_m\left(q^2, q^{5/2+l}z\right) \Big\|\left(W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right) (\psi_2)_n\right)_m\Big\| \\ &\leq \vartheta_m\left(q^2, q^{5/2+l}z\right) \Big\| W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right) (\psi_2)_n\Big\| \\ &\leq \vartheta_m\left(q^2, q^{5/2+l}z\right) \Big\| W_{\sqrt{q}}\left(q^{L_0/4}\psi_1,(1-q)z\right)\Big\| \cdot \|(\psi_2)_n\| \\ &\leq \vartheta_m\left(q^2, q^{5/2+l}z\right) \cdot \vartheta_{w_1}\left(q^{1/2},(1-q)z\right) \cdot \|\psi_1\| \cdot \|\psi_2\|, \end{aligned} \tag{3.33}$$

Putting everything together, we get

$$\begin{aligned}\mathcal{N}_{error}(z,q) \quad &\leq \quad \vartheta_{w_1}\left(q^{1/2},(1-q)z\right)\cdot\|\psi_1\|\cdot\|\psi_2\|\cdot \\ &\cdot\left(\sum_{n\in\mathbb{N}_0}\sum_{0\leq m<n-N} q^{(m(4l+1)+5n)/4}\vartheta_m\left(q^2,q^{5/2}z\right)\right. \\ &\left.+\sum_{n\in\mathbb{N}_0}\sum_{m>n-N} q^{(m(4l+1)+5n)/4}\vartheta_m\left(q^2,q^{5/2}z\right)\right) \qquad (3.34)\end{aligned}$$

Now in order to bound the sum we need to know how $q^{m(4l+1)/4}\vartheta_m\left(q^2,q^{5/2+l}z\right)$ behaves. Here we will use conjecture 3.1.2. (This conjecture holds for WZW models.) Then we can write

$$q^{m(4l+1)/4}\vartheta_m\left(q^2,q^{5/2+l}z\right)\leq C\cdot q^{cm} \qquad (3.35)$$

where C,c are some positive constants that may depend on z and q, but do not depend on m. The only remaining task is to evaluate the sums. Just like in [KS15] we get

$$\begin{aligned}\sum_{n\in\mathbb{N}_0}\sum_{0\leq m<n-N} q^{cm+5n/4}+\sum_{n\in\mathbb{N}_0}\sum_{m>n-N} q^{cm+5n/4} \quad &= \\ \sum_{n\in\mathbb{N}_0} q^{5n/4}\left(\sum_{0\leq m<n-N} q^{cm}+\sum_{m>n-N} q^{cm}\right) \quad &= \quad const\cdot q^{\Omega(N)}. \qquad (3.36)\end{aligned}$$

This yields the final result

$$\mathcal{N}_{error}(z,q)\leq\vartheta_{w_1}\left(q^{1/2},(1-q)z\right)\cdot\|\psi_1\|\cdot\|\psi_2\|\cdot\kappa(z,q)\cdot q^{\Omega(N)}. \qquad (3.37)$$

for some $\kappa(z,q)$. This is exactly the statement we aimed for. □

3.2.1 Choosing z, q and l

The assumption of Lemmas 3.2.1, 3.2.2 and 3.2.3 as well as the isometry condition (2.61) for $\gamma = 3/2$ all give constraints for z, q and l. In this subsection we will see that they all can be satisfied for all levels of the TTN. First let us note that these conditions imply that $q<1$ thus any extra constraint of type $q<$ r.h.s is only non-trivial if r.h.s <1. This means that the conditions in Lemma 3.2.3 are stronger than those in Lemma 3.2.2. Also we have

$$0<q<\min\left\{q^{l+5/2}|z|,\frac{1}{(q^{l+5/2}|z|)}\right\}\;\Leftrightarrow\;0<q<\min\left\{|q^{l+2}z|^2,\frac{1}{|q^{l+3}z|^2}\right\} \qquad (3.38)$$

and

$$0<q<\frac{1}{((1-q)|z|)^4}\;\Rightarrow\;|q^2(1-q)z|<1. \qquad (3.39)$$

Thus the conditions for lemmas 3.2.1, 3.2.2, 3.2.3 to be simultaneously applicable and isometry condition (2.61) for $\gamma = 3/2$ to be fulfilled simplify to:

$$\begin{cases} 0 < q < \min\left\{|q^{2+l}z|^2, \frac{1}{|q^{2+l}z|^2}\right\}, \\ \qquad q < \min\left\{((1-q)|z|)^4, \frac{1}{((1-q)|z|)^4}\right\}, \\ 0 \le l \ . \end{cases} \tag{3.40}$$

For convenience we will relabel parameter l:

$$l \to l - 2, \tag{3.41}$$

which gives:

$$\begin{cases} 0 < q < \min\left\{|q^{l}z|^2, \frac{1}{|q^{l}z|^2}\right\}, \\ \qquad q < \min\left\{((1-q)|z|)^4, \frac{1}{((1-q)|z|)^4}\right\}, \\ 2 \le l \ . \end{cases} \tag{3.42}$$

In order to show an error bound for the TTN we have to show that these conditions can be satisfied simultaneously for all levels of the tree. For the transfer operator, the renormalization maps (2.36) and (2.37) work as follows

$$\begin{aligned} T_l &= q^{lL_0} W_q(\psi_1, z) \circ \ldots \circ W_q(\psi_{2^m}, z) q^{-lL_0} \\ &= q^{2\frac{l-2}{2}L_0} W_{q^2}(\epsilon_{z,q}(\psi_1 \otimes \psi_2), q^{5/2}z) \circ \ldots \\ &\quad \ldots \circ W_{q^2}(\psi_{2^{m-1}-1} \otimes \psi_{2^{m-1}}, q^{5/2}z) q^{-2\frac{l-2}{2}L_0}. \end{aligned} \tag{3.43}$$

Thus we have

$$z^{(r+1)} = \left(q^{(r)}\right)^{5/2} z^{(r)}, \quad q^{(r+1)} = \left(q^{(r)}\right)^2, \quad l^{(r+1)} = \frac{l^{(r)} - 2}{2}. \tag{3.44}$$

Lemma 3.2.5. *There exist $z \in \mathbb{C}$, $q \in \mathbb{R}$, $l \in \mathbb{R}$, such that for every tree level $r \in \mathbb{N}$, $0 \le r < m$, where $m \in \mathbb{N}, m \ge 1$ is the total number of levels of the tree, and $q^{(r)}$, $z^{(r)}$, $l^{(r)}$ that are obtained via the TTN renormalization rules (3.44) from $z = z^{(0)}$, $z = z^{(0)}$, $l = l^{(0)}$, the following system of inequalities holds:*

$$\begin{cases} 0 < q^{(r)} < \min\left\{\left|(q^{(r)})^{l^{(r)}} z^{(r)}\right|^2, \frac{1}{\left|(q^{(r)})^{l^{(r)}} z^{(r)}\right|^2}, \right\}, \\ \qquad q^{(r)} < \min\left\{\left((1-q^{(r)})|z^{(r)}|\right)^4, \frac{1}{\left((1-q^{(r)})|z^{(r)}|\right)^4}\right\}, \\ 2 \le \ l^{(r)} \ . \end{cases} \tag{3.45}$$

Proof. The strategy of the proof can be divided in two steps:

1. Finding a small system of inequalities (in our case there will be only three inequalities) such that any solution gives rise to a solution of the original system (3.45).

2. Showing that the small system has a solution.

Employing this strategy, we will first radically decrease the number of inequalities. An obvious approach is to use the renormalization rules (3.44), which give

$$z^{(r)} = q^{\frac{5}{2}\cdot(2^r-1)}z, \quad q^{(r)} = q^{2^r}. \tag{3.46}$$

As $l^{(r)}$ is positive for every $r \geq 0$, we get $l^{(r+1)} < l^{(r)}$. This implies

$$2 \leq l^{(m-1)} \;\Rightarrow\; 2 \leq l^{(r)} \;\text{ for }\; 0 \leq r < m. \tag{3.47}$$

From the recursion relation (3.42) we get

$$2 \leq l^{(m-1)} \;\Leftrightarrow\; 2^{m+1} - 2 \leq l. \tag{3.48}$$

Let us note that any of the first tho rows of (3.45) implies that

$$0 < q^{(r)} < 1 \;\text{ for any r such that } 0 \leq r < m. \tag{3.49}$$

Further employing the renormalization rules (3.44) to reduce the number of inequalities, we note that the condition

$$q^{(r)} \quad < \quad \left|(q^{(r)})^{l^{(r)}} z^{(r)}\right|^2, \tag{3.50}$$

is independent of r. Indeed, using the equalities

$$\begin{aligned} q^{(r+1)} &= \left(q^{(r)}\right)^2, \\ \left|(q^{(r+1)})^{l^{(r+1)}} z^{(r+1)}\right|^2 &= \left|(q^{(r)})^{l^{(r)}+\frac{1}{2}} z^{(r)}\right|^2 \end{aligned} \tag{3.51}$$

we get

$$q^{(r)} < \left|(q^{(r)})^{l^{(r)}} z^{(r)}\right|^2 \quad\Leftrightarrow\quad q^{(r+1)} < \left|(q^{(r+1)})^{l^{(r+1)}} z^{(r+1)}\right|^2. \tag{3.52}$$

Likewise, the condition

$$q^{(r)} \quad < \quad \frac{1}{\left|(q^{(r)})^{l^{(r)}} z^{(r)}\right|^2} \tag{3.53}$$

is weaker for the level $r+1$ than for the level r, as $0 < q^{(r)} < 1$. Thus it is enough to solve to consider conditions (3.50) and (3.53) for the level $r = 0$. Let us fix $|z|$ (see remark 3.2.6):

$$|z| = \frac{1}{q^l}, \tag{3.54}$$

With a choice of $|z|$ as in (3.54), the system (3.45) becomes:

$$\begin{cases} 2^{m+1} - 2 \leq l, \\ 0 < q, \\ q^{l+\frac{5}{2}-\frac{9}{4}\cdot 2^r} - 1 + q^{2^r} < 0, \\ 1 - q^{2^r} - q^{l+\frac{5}{2}-\frac{11}{4}\cdot 2^r} < 0 \end{cases}$$

for every $r \in \mathbb{N},\ 0 \leq r < m$.

Even though the system (3.55) is considerably smaller than (3.45), it is still large for big m. To proceed further, let us use a following trick: consider a system of inequalities

$$f_r < 0 \quad \text{for every } r \in \mathbb{N},\ 0 \leq r < m. \tag{3.55}$$

If f_r is a differentiable function of r and

$$\partial_r f_r \leq 0 \quad \text{for every } r \in \mathbb{N},\ 0 \leq r < m, \tag{3.56}$$

then the system (3.55) is equivalent to

$$f_0 < 0. \tag{3.57}$$

Analogously, if

$$\partial_r f_r \geq 0 \quad \text{for every } r \in \mathbb{N}, 0 \leq r < m, \tag{3.58}$$

then the system (3.55) is equivalent to

$$f_{m-1} < 0. \tag{3.59}$$

We will use the facts (3.57) and (3.59) to reduce the number of inequalities. For technical simplicity, let us treat 2^r and not r as a parameter in the system (3.55) and differentiate with respect to it. Indeed, the argument does not change, as 2^r is monotonically increasing with r. Differentiating, we get:

$$\begin{aligned}
\partial_{2^r}\left(q^{l+\frac{5}{2}-\frac{9}{4}\cdot 2^r} - 1 + q^{2^r}\right) &= \ln(q)\left(q^{2^r} - \frac{9}{4}\cdot q^{l+\frac{5}{2}-\frac{9}{4}\cdot 2^r}\right),\\
\partial_{2^r}\left(1 - q^{2^r} - q^{l+\frac{5}{2}-\frac{11}{4}\cdot 2^r}\right) &= \ln(q)\left(\frac{11}{4}\cdot q^{l+\frac{5}{2}-\frac{11}{4}\cdot 2^r} - q^{2^r}\right).
\end{aligned} \tag{3.60}$$

As $0 < q < 1$ (see inequalities 3.49 for $r = 0$), we have $\ln(q) < 0$. Let us search for the solution of (3.55) in the set

$$l \geq \max\left\{-\frac{5}{2} + \frac{13}{4}\cdot 2^{m-1} - \log_q\left(\frac{9}{4}\right), -\frac{5}{2} + \frac{15}{4}\cdot 2^{m-1} - \log_q\left(\frac{11}{4}\right), 2^{m+1} - 2\right\}. \tag{3.61}$$

Then we have

$$\begin{aligned}
\partial_{2^r}\left(q^{l+\frac{5}{2}-\frac{9}{4}\cdot 2^r} - 1 + q^{2^r}\right) &\leq 0,\\
\partial_{2^r}\left(1 - q^{2^r} - q^{l+\frac{5}{2}-\frac{11}{4}\cdot 2^r}\right) &\geq 0.
\end{aligned} \tag{3.62}$$

Condition (3.61) holds if

$$l \geq 2^{m+1} - 2 - \log_q\left(\frac{11}{4}\right). \tag{3.63}$$

Let us denote

$$\begin{aligned}\tilde{l} &:= l - \left(2^{m+1} - 2 - \log_q\left(\frac{11}{4}\right)\right)\\ C_1 &:= 2^{m+1} - \frac{3}{4},\\ C_2 &:= \frac{5}{4}\cdot 2^{m-1} + \frac{1}{2}.\end{aligned} \tag{3.64}$$

Using the facts (3.57) and (3.59), we get that if the system of inequalities

$$\begin{cases} 0 \le \tilde{l}, \\ 0 < q, \\ 1 - q - \frac{4}{11}q^{\tilde{l}+C_1} < 0, \\ \frac{4}{11}q^{\tilde{l}+C_2} - 1 + q^{2^{m-1}} < 0. \end{cases} \tag{3.65}$$

has a solution, then system (3.45) also has a solution.

As $0 < q < 1$ (see inequalities 3.49 for $r = 0$),

$$\begin{cases} \frac{4}{11}q^{\tilde{l}+C_1} & > 1 - q, \\ \frac{4}{11}q^{\tilde{l}+C_2} & < 1 - q^{2^{m-1}}, \\ \tilde{l} & \ge 0. \end{cases} \Leftrightarrow \begin{cases} \tilde{l} & < \log_q\left(\frac{11}{4}(1-q)\right) - C_1, \\ \tilde{l} & > \log_q\left(\frac{11}{4}(1-q^{2^{m-1}})\right) - C_2, \\ \tilde{l} & \ge 0. \end{cases} \tag{3.66}$$

System (3.66) has a solution if and only if

$$\begin{cases} \log_q\left(\frac{11}{4}(1-q)\right) - C_1 & > \ 0, \\ \log_q\left(\frac{11}{4}(1-q)\right) - C_1 & > \ \log_q\left(\frac{11}{4}(1-q^{2^{m-1}})\right) - C_2 \end{cases} \tag{3.67}$$

This means that system (3.65), and, consequently, system (3.45) has a solution if and only if

$$\begin{cases} \frac{11}{4}(1-q) < q^{C_1}, \\ q^{C_2-C_1} < \frac{1-q^{2^{m-1}}}{1-q}, \\ 0 < q \end{cases} \tag{3.68}$$

has a solution. It is easy to see that such q exists: for every $m \ge 1$ it is easy to see that $C_2 < C_1$ and $C_1 > 0$, thus with growth of q from 0 to 1 the expression $\frac{11}{4}(1-q)$ monotonically decreases from $\frac{11}{4}$ to 0, while q^{C_1} monotonically increases from 0 to 1, $q^{C_2-C_1}$ monotonically decreases from infinity to 1 and $\frac{1-q^{2^{m-1}}}{1-q}$ monotonically increases from 1 to 2^{m-1}. This means that for q close enough to 1 all inequalities in (3.68) are satisfied.

As such q exists, there is also an l and z such that the system (3.45) holds for every r such that $0 \le r < m$. □

The fact that solution of (3.68) is such that q is close to 1 has an intuitive interpretation – the more levels in the tree we have, the more we have to truncate, thus the error increases. As we will see later, the error falls as $q^{\Omega(N)}$, where N is a truncation parameter (see subsection 2.1.1). For trees with more levels we have to use large q, as C_1 grows with m, thus error estimates fall faster for small trees.

Remark 3.2.6. *Every $|z|$ can be expressed via in terms of real parameter f as:*

$$|z| = \frac{q^{f/2}}{q^l}, \tag{3.69}$$

Then the conditions (3.45) are equivalent to

$$\begin{cases} -1 < \ f \ < 1, \\ 2^{m+1} - 2 \leq \ l \ , \\ 0 < q^{2^r} < \min\left\{ \left(1 - q^{2^r}\right)^4 q^{10(2^r-1)-4l+2f}, \frac{1}{(1-q^{2^r})^4 q^{10(2^r-1)-4l+2f}} \right\} \end{cases} \tag{3.70}$$

for every $r \in \mathbb{N}, 0 \leq r < m$.

Setting $f = 0$ in (3.69), we get (3.54). Conceptually similar proofs to Lemma 3.2.5 can be done for other values of f.

With suitable conformal mapping we can always arrange lattice $\{z, zq, zq^2, \ldots\}$ so that inequalities (3.45) hold and l can always be chosen as it, like a gauge, does not have a physical meaning. Thus, we can use bounds from section 3.2 for trees of any size.

3.3 One step truncation error bound

Let us look at truncation after one step of the tree tensor network – that is, when the number of fields is reduced by a factor of two.

$$\begin{aligned} T &\equiv W_q(\psi_1, z) \circ W_q(\psi_2, z) \circ \ldots \circ W_q(\psi_{2n-1}, z) \circ W_q(\psi_{2n}, z) \\ &= q^{2L_0} W_{q^{(1)}}\left(\psi_1^{(1)}, z^{(1)}\right) \circ \ldots \circ W_{q^{(1)}}\left(\psi_n^{(1)}, z^{(1)}\right) q^{-2L_0} \\ &= q^{2L_0} W_{q^2}\left(\epsilon_{z,q}(\psi_1 \otimes \psi_2), q^{5/2} z\right) \circ \ldots \circ W_{q^2}\left(\epsilon_{z,q}(\psi_{2n-1} \otimes \psi_{2n}), q^{5/2} z\right) q^{-2L_0}. \end{aligned} \tag{3.71}$$

It is natural to introduce the following truncation:

$T\left(\{\psi_i\}_{i=1}^{2n}; z, q\right) \simeq$

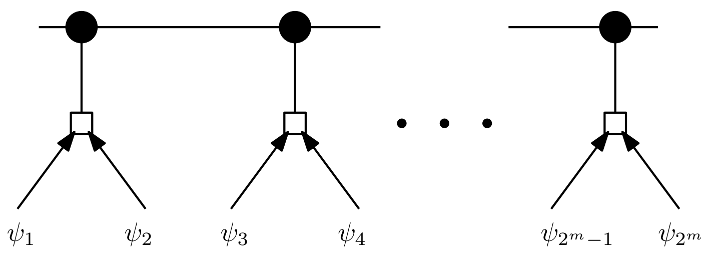


Figure 3.9: Truncated transfer operator diagram can be obtained from the exact transfer operator one by exchanging "filled" boxes to "empty" ones.

Let us remind that

$$\psi_i^{(1)} = \epsilon_{z,q}(\psi_{2i-1} \otimes \psi_{2i}) \tag{3.72}$$

(see (2.38)) and

$$\psi_i^{(1)[N]} = \epsilon_{z,q}^{[N]}(\psi_{2i-1} \otimes \psi_{2i}) \tag{3.73}$$

(see (3.5)).

Definition 3.3.1. *The once truncated transfer operator* $T^{[N]} : V^n \to \mathsf{End}(V)[[z, z^{-1}]]$, $\{\psi_i\}_{i=1}^{2n} \to T^{[N]}\left(\{\psi_i\}_{i=1}^{2n}; z, q\right)$ *is*

$$T^{[N]}\left(\{\psi_i\}_{i=1}^{2n}; z, q\right) = W_{q^{(1)}}\left(\psi_1^{(1)[N]}, z^{(1)}\right) \circ \cdots \circ W_{q^{(1)}}\left(\psi_n^{(1)[N]}, z^{(1)}\right) \tag{3.74}$$

(see (2.38) and figure 3.9).

Using the exact expression for $\epsilon_{z,q}$ as well as $z^{(1)}$ and $q^{(1)}$ (see equations (2.36)), we can write

$$\begin{aligned} T^{[N]}\left(\{\psi_i\}_{i=1}^{2n}; z, q\right) &= W_{q^2}\left(W_q^{[N]}\left(\psi_1, (1-q)z\right) q^{L_0}\psi_2, q^{5/2}z\right) \circ \ldots \\ &\cdots \circ W_{q^2}\left(W_q^{[N]}\left(\psi_{2n-1}, (1-q)z\right) q^{L_0}\psi_{2n}, q^{5/2}z\right). \end{aligned} \tag{3.75}$$

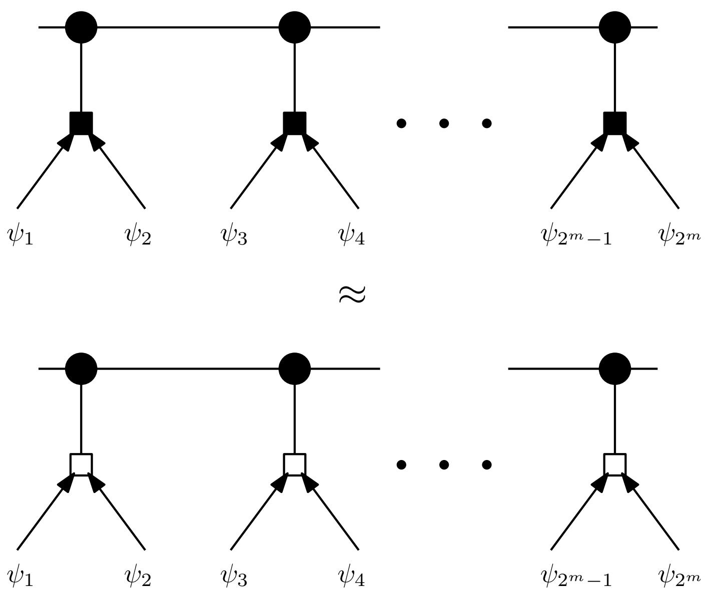


Figure 3.10: We would like to show that truncated transfer operator is almost as good for reproducing correlation functions as the exact one for large enough truncation parameter.

Lemma 3.3.2. *Consider a family of* $2n$ *fields that get renormalized to* n *fields. Let us fix a family of nonnegative finite numbers* $\{w_i\}_{i=0}^{2n}$. *Let* T *be the transfer operator (see definition 2.1.2) and* $T^{[N]}$ *be the once truncated transfer operator (see definition 3.3.1). If* z, q, l *satisfy*

$$\begin{cases} 0 < q < \min\left\{|q^l z|, \frac{1}{|q^l z|^2}\right\}, \\ \quad q < \min\left\{((1-q)|z|)^4, \frac{1}{((1-q)|z|)^4}\right\}, \\ 2 \leq l \ . \end{cases} \tag{3.76}$$

(see Lemma 3.2.5) and conjecture 3.1.2 holds, then for every family $\{\psi_i\}_{i=1}^{2n}$ *such that* w_i *corresponds to* ψ_i *(that is,* $\psi_i \in \bigoplus_{n \leq w_i} V_n$ *see definition 2.1.5) the following bound*

$$\left\| T_l\left(\{\psi_i\}_{i=1}^{2n}; z, q\right) - T_{l^{(1)}}^{[N]}\left(\{\psi_i\}_{i=1}^{2n}; z, q\right)\right\| \leq q^{\Omega(N)} \prod_{i=1}^{2n} \|\psi_i\| \tag{3.77}$$

holds (see figure 3.10).

Remark 3.3.3. *For the TTN we have* $n = 2^p$ *for* $p \in \mathbb{N}_0$.

Proof. As was discussed in section 3.1, we will use the telescoping sum bound (3.3). Let us recall that $\left(q^{(1)}\right)^{l^{(1)}} = q^{l-2}$ (see equation (3.44)). For this purpose, define

$$T_{l^{(1)}}^{[N,k]} \equiv \quad q^{(l-2)L_0}\left[\prod_{i=1}^{k} W_{q^{(1)}}\left(\psi_i^{(1)[N]}, z^{(1)}\right) \cdot \prod_{j=k+1}^{n} W_{q^2}\left(\psi_j^{(1)}, z^{(1)}\right)\right] q^{-(l-2)L_0} \tag{3.78}$$

(see (2.38)). It is easy to see that

$$T_{l^{(1)}}^{[N,0]} = T_l \quad \text{and} \quad T_{l^{(1)}}^{[N,n]} = T_{l^{(1)}}^{[N]}. \tag{3.79}$$

Then, as we have n renormalization fields $\psi_i^{(1)}$ or truncated renormalized fields $\psi_i^{(1)[N]}$

$$\left\| T_l - T_{l^{(1)}}^{[N]} \right\| \leq \sum_{k=0}^{n-1} \left\| T_{l^{(1)}}^{[N,k]} - T_{l^{(1)}}^{[N,k+1]} \right\|. \tag{3.80}$$

For every term we have

$$\begin{aligned} \left\| T_{l^{(1)}}^{[N,k]} - T_{l^{(1)}}^{[N,k+1]} \right\| \quad \leq \quad & \prod_{j=1}^{k} \left\| q^{(l-2)L_0} W_{q^2}(\epsilon_{z,q}(\psi_{2j-1} \otimes \psi_{2j}), q^{5/2} z) q^{-(l-2)L_0} \right\| \\ & \cdot \Big\| q^{(l-2)L_0} W_{q^2}\left(\epsilon_{z,q}(\psi_{2k+1} \otimes \psi_{2k+2})\right. \\ & \quad \left. - \epsilon_{z,q}^{[N]}(\psi_{2k+1} \otimes \psi_{2k+2}), q^{5/2} z\right) q^{-(l-2)L_0} \Big\| \\ & \cdot \prod_{j=k+2}^{n} \left\| q^{(l-2)L_0} W_{q^2}\left(\epsilon_{z,q}^{[N]}(\psi_{2j-1} \otimes \psi_{2j}), q^{5/2} z\right) q^{(l-2)L_0} \right\|. \end{aligned} \tag{3.81}$$

Basically, we would like to use that every term is sufficiently small. That is, if each term is of order $q^{\Omega(N)}$, then the sum should also be of this order. In every term there is only one map that is different – namely, truncated or not, in $T^{[N,k]}$ and $T^{[N,k+1]}$. This can be illustrated via the following diagram:

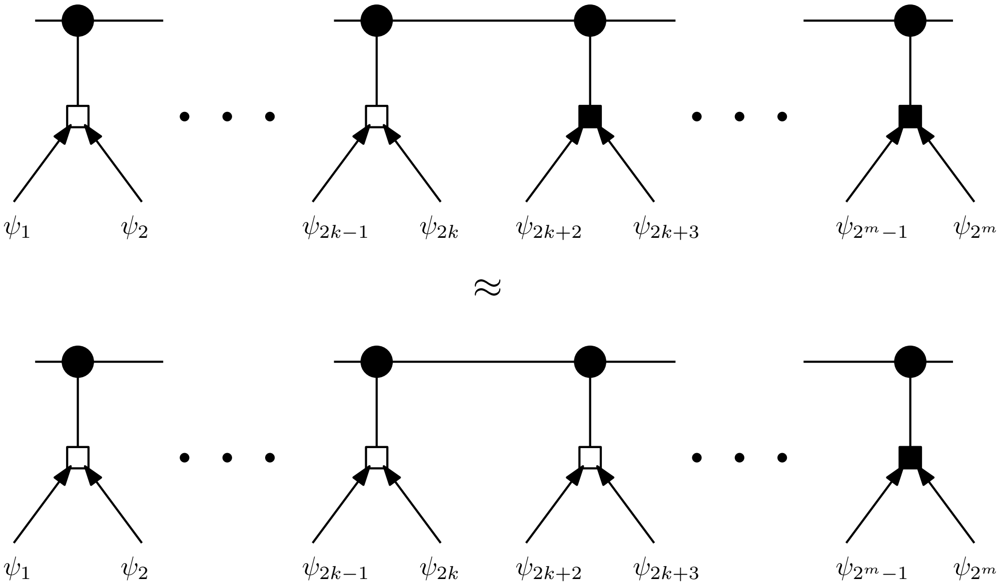


Figure 3.11: If we truncate one more map, the result does not change much.

Using lemmas 3.2.1, 3.2.2 and 3.2.3, we get

$$\begin{aligned}\|T^{[N,k]}_{l^{(1)}} - T^{[N,k+1]}_{l^{(1)}}\| \quad \leq \quad & \prod_{j=1}^{k} \vartheta_{w_{2j-1}}\left(q, q^l z\right) \cdot \vartheta_{w_{2j}}\left(q, q^l z\right) \cdot \|\psi_{2j-1}\| \cdot \|\psi_{2j}\| \\ & \cdot \vartheta_{w_{2k+1}}(\sqrt{q}, (1-q)z) \cdot \|\psi_{2k+1}\| \cdot \|\psi_{2k+2}\| \cdot \kappa(z,q) \cdot q^{\Omega(N)} \\ & \cdot \prod_{j=k+2}^{n} \vartheta_{w_{2j}+N}\left(q^2, q^{5/2+l} z\right) \\ & \cdot \frac{\sqrt{|I_V|}\vartheta_{w_{2j-1}}(\sqrt{q}, (1-q)z)}{\sqrt{1-\sqrt{q}}} \|\psi_{2j-1}\|\|\psi_{2j}\|. \end{aligned} \tag{3.82}$$

Definition 3.3.4. *Consider a family of $2n$ nonnegative finite numbers $\{w_i\}_{i=1}^{2n}$ (such that w_i corresponds to a field ψ_i, see the statement of the lemma 3.3.2 and definition (2.1.5)), a truncation parameter $N \in \mathbb{N}$ (see subsection 2.1.1), $q \in \mathbb{R}$ and $z \in \mathbb{C}$ are parameters of the lattice (2.1) such the conditions of the lemma 3.3.2 are satisfied. To simplify notation, let us introduce*

$$\begin{aligned}\Theta\left(\{w_i\}_{i=1}^{2n}, N, z, q\right) \equiv \max_{i \in [1,\ldots n]} \Big\{ & \vartheta_{w_{2i-1}}\left(q, q^l z\right) \cdot \vartheta_{w_{2i}}\left(q, q^l z\right), \vartheta_{w_{2i+1}}(\sqrt{q}, (1-q)z) \cdot \kappa(z,q), \\ & \vartheta_{w_{2j}+N}\left(q^2, q^{5/2+l} z\right) \cdot \frac{\sqrt{|I_V|}\vartheta_{w_{2j-1}}(\sqrt{q}, (1-q)z)}{\sqrt{1-\sqrt{q}}} \Big\} \end{aligned} \tag{3.83}$$

(See section 2.1 for definition of $\vartheta_w(q,z)$, subsection 1.2.2 for the definition of I_V and lemma 3.2.3 for $\kappa(z,q)$.)

Putting everything together we get an expression of the following type:

$$\|T_l - T_{l^{(1)}}^{[N]}\| \leq q^{\Omega(N)} \cdot \left[n\Theta\left(\{w_i\}_{i=1}^{2n}, N, z, q\right)\right] \cdot \prod_{i=1}^{2n} \|\psi_i\|. \tag{3.84}$$

A direct consequence of conjecture 3.1.2 is that $n\Theta\left(\{w_i\}_{i=1}^{2n}, N, z, q\right)$ increases at most sub-exponentially as N increases. Indeed, $q^{\Omega(N)} \cdot e^{O(\sqrt{N})} = q^{\Omega(N)}$. This leads to

$$\|T_l - T_{l^{(1)}}^{[N]}\| \leq q^{\Omega(N)} \cdot \prod_{i=1}^{2n} \|\psi_i\|. \tag{3.85}$$

□

3.4 Full truncation error bound

In this section we will show that a fully truncated TTN under certain assumptions give a good approximation to the exact TTN. To do so, we will use the telescope inequality (3.3). However, to use it, we will need to introduce a partially truncated TTN. If we want to compute n-point correlation function, we need a tree with $m = \log_2 n$ levels. We have the renormalization procedure (see sections 2.3 and 3.2.1):

$$\begin{aligned} \psi_i^{(0)} &\equiv \psi_i, \\ \{\psi_1^{(r)}, \ldots, \psi_{2^{m-r}}^{(r)}\} &\rightarrow \{\psi_1^{(r+1)}, \ldots \psi_{2^{m-r-1}}^{(r+1)}\}, \quad 0 \leq r < m, \end{aligned}$$

for each ψ_i^r we can write recursive relation, as well as for the lattice parameters z and q and additional scaling parameter l

$$\begin{aligned} \psi_i^{(r)} &= \epsilon_{z^{(r-1)}, q^{(r-1)}} \left(\psi_{(2r-1)}^{(r-1)} \otimes \psi_{(2r)}^{(r-1)}\right); \\ q^{(0)} &= q, \quad q^{(r)} = \left(q^{(r-1)}\right)^2, \\ z^{(0)} &= z, \quad z^{(r)} = \left(q^{(r-1)}\right)^{5/2} z_{r-1}. \\ l^{(0)} &= l, \quad l^{(r)} = \frac{l^{(r-1)} - 2}{2}. \end{aligned} \tag{3.86}$$

Using diagram language we get for $\psi_i^{(r)}$

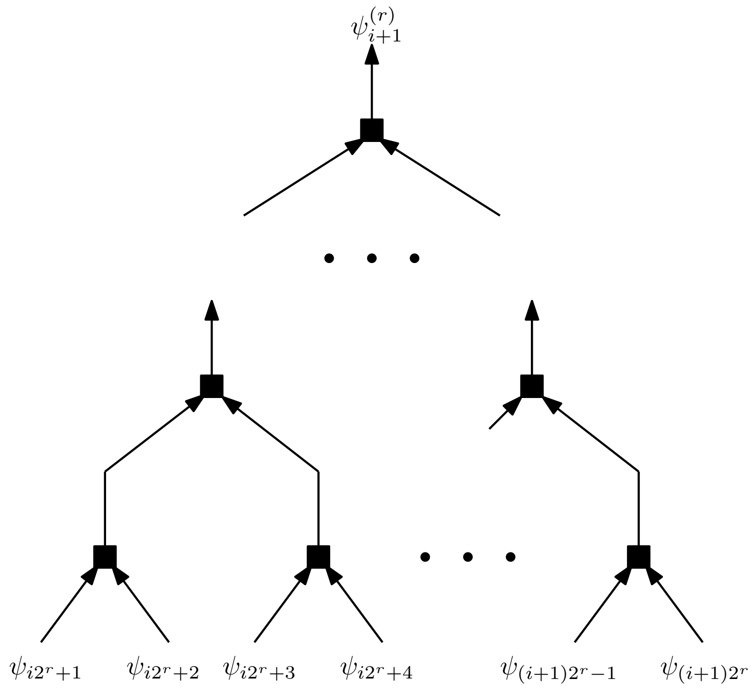


Figure 3.12: Renormalized field $\psi_i^{(r)}$

It is clear how to truncate this diagram – one just has to exchange "filled" boxes by the "empty" one. We can define r times renormalized fields via

$$\psi_i^{(0)[N]} \equiv \psi_i, \tag{3.87}$$

$$\psi_i^{(r)[N]} = \epsilon^{[N]}_{z^{(r-1)},q^{(r-1)}} \left(\psi_{2i-1}^{(r-1)[N]} \otimes \psi_{2i}^{(r-1)[N]}\right), \quad 1 \le r \le m. \tag{3.88}$$

Renormalization rules for $z^{(r)}$, $q^{(r)}$, $l^{(r)}$ remain the same. We have already used this construction for $r = 1$ in the section 3.3 and the construction (3.87) is a generalization of the truncation (3.73) for all $r \geq 1$.

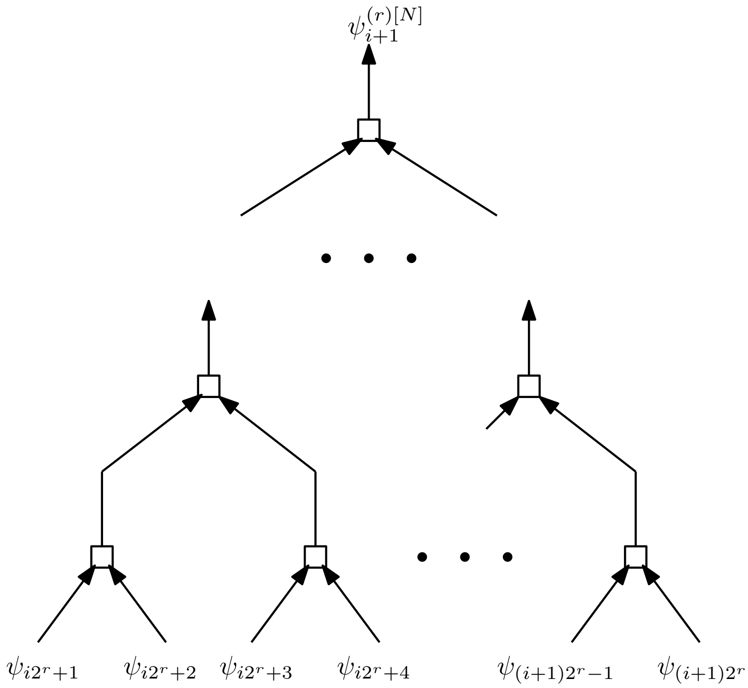


Figure 3.13: Truncated renormalized field $\psi_i^{(r)[N]}$

To use the telescoping sum expansion (3.3), let us define transfer operators that are truncated up to level r of the tree.

Definition 3.4.1. *A r-truncated transfer operator* $T^{(r)[N]} : V^n \to \mathsf{End}(V)[[z, z^{-1}]]$, $\{\psi_i\}_{i=1}^{2n} \to T^{[N]}\left(\{\psi_i\}_{i=1}^{2n}; z, q\right)$ *is*

$$T^{(r)[N]} \equiv W_{q^{(r)}}\left(\psi_1^{(r)[N]}, z^{(r)}\right) \circ \ldots \circ W_{q^{(r)}}\left(\psi_{2^{m-r}}^{(r)[N]}, z^{(r)}\right) \qquad (3.89)$$

(see figure 3.14).

It is easy to see that $T^{(0)[N]} = T$ and the fully truncated tree is $T^{(m)[N]}$.

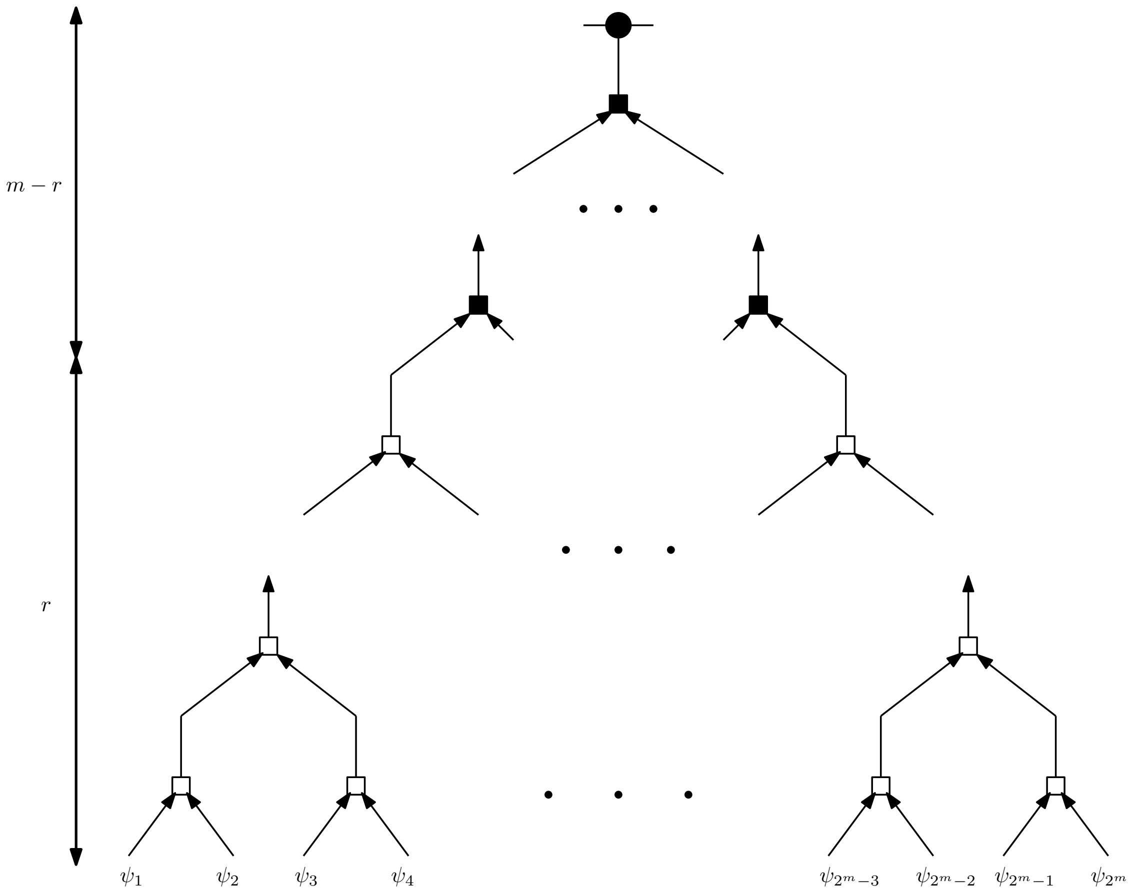


Figure 3.14: Diagram representation of a tree truncated up to the level r – that is $T^{(r)[N]}$.

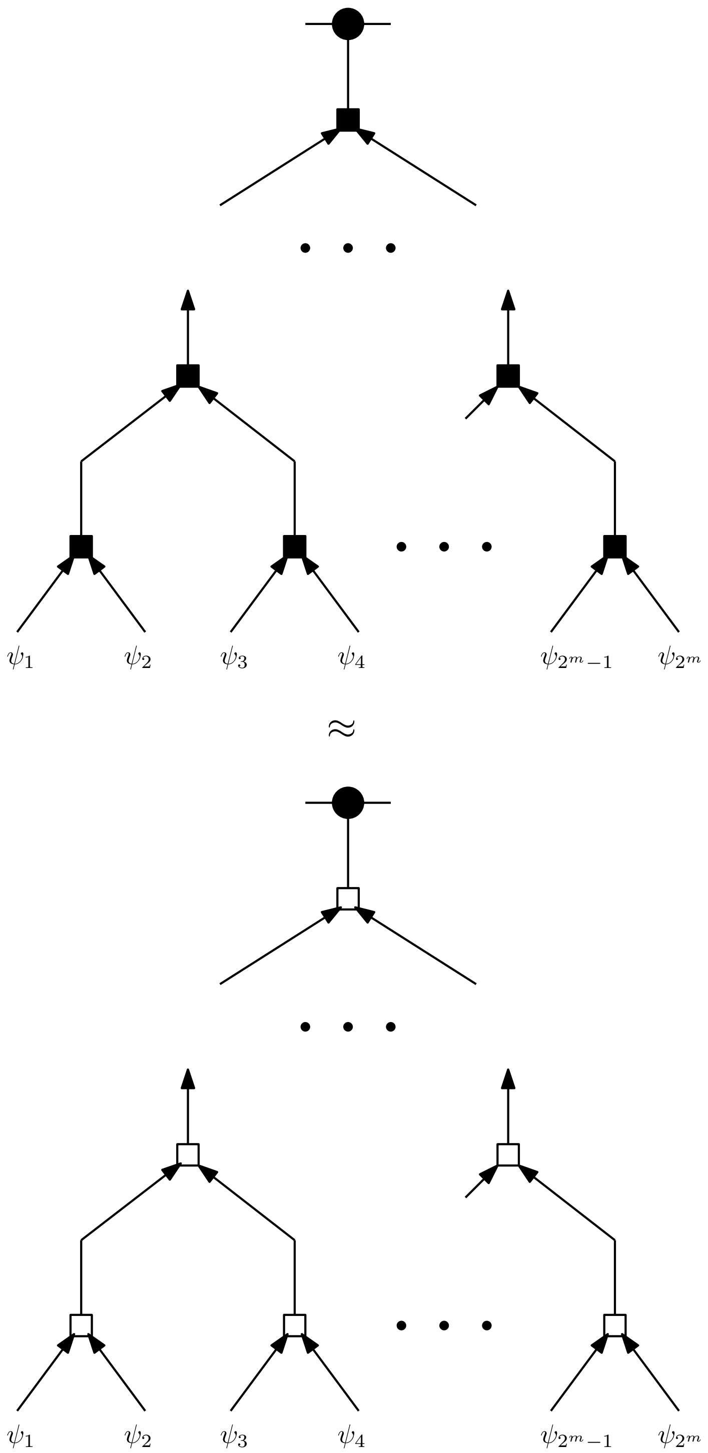


Figure 3.15: The fully truncated TTN $T_{l^{(m)}}^{(m)[N]}$ is close to the exact one T_l, that is $\left\| T_l - T_{l^{(m)}}^{(m)[N]} \right\| \leq q^{\Omega(N)} \cdot \prod_{i=1}^{2n} \|\psi_i\|$.

Lemma 3.4.2. *Let T be a transfer operator (see definition 2.1.2) and $T^{(m)[N]}$ be a m-truncated transfer operator (see definition 3.4.1). If z, q, l satisfy*

$$\begin{cases} l \geq 2^{m+1} - 2 - \log_q\left(\frac{11}{4}\right), \\ 0 < q, \\ 1 - q < \frac{4}{11} q^{l+\frac{5}{4}+\log_q\left(\frac{11}{4}\right)}, \\ 1 - q^{2^{m-1}} > \frac{4}{11} q^{l-\frac{11}{4}\cdot 2^{m-1}+\frac{5}{2}+\log_q\left(\frac{11}{4}\right)}, \\ |z| = q^{-l}, \end{cases} \tag{3.90}$$

conjecture 3.1.2 holds and for every ψ_i corresponding w_i is finite (see definition 2.1.5), then

$$\left\| T_l - T_{l^{(m)}}^{(m)[N]} \right\| \leq q^{\Omega(N)} \cdot \prod_{i=1}^{2n} \|\psi_i\| \tag{3.91}$$

(see figure 3.15).

Remark 3.4.3. *Such z, q, l that inequalities (3.90) are satisfied exist for every $m \geq 1$, see Lemma 3.2.5 for the proof and discussion.*

Proof. Let us use telescope expansion (3.3):

$$\left\| T_l - T_{l^{(m)}}^{(m)[N]} \right\| = \left\| T_{l^{(0)}}^{(0)[N]} - T_{l^{(m)}}^{(m)[N]} \right\| \leq \sum_{i=0}^{m-1} \left\| (T_{l^{(i)}}^{(i)[N]} - T_{l^{(i+1)}}^{(i+1)[N]}) \right\|. \tag{3.92}$$

However, we have already obtained a bound on every term in section 3.3:

$$\left\| T_{l^{(i)}}^{(i)[N]} - T_{l^{(i)}}^{(i+1)[N]} \right\| \quad \leq \quad q^{\Omega(N)} \cdot \prod_{j=1}^{2^{m-i}} \left\| \psi_j^{(i)[N]} \right\| \tag{3.93}$$

It is straightforward to obtain bounds of $\left\| \psi_j^{(i+1)[N]} \right\|$ using initial fields $\left\| \psi_j^{(i)[N]} \right\|$ and, consequently, $\|\psi_j\|$:

$$\begin{aligned} \left\| \psi_j^{(i+1)[N]} \right\| &= \left\| W_{q^{(i)}}^{[N]} \left(\psi_{2j-1}^{(i)[N]}, (1 - q^{(i)}) z^{(i)} \right) \left(q^{(i)} \right)^{L_0} \psi_{2j}^{(i)[N]} \right\| \\ &\leq \frac{\sqrt{|I_V|} \vartheta_{w_{2j-1}+iN} \left(\sqrt{q^{(i)}}, (1 - q^{(i)}) z^{(i)} \right)}{\sqrt{1 - \sqrt{q^{(i)}}}} \left\| \psi_{2j-1}^{(i)[N]} \right\| \left\| \psi_{2j}^{(i)[N]} \right\|. \end{aligned} \tag{3.94}$$

We can use conjecture 3.1.2 to observe that $\frac{\sqrt{|I_V|}\vartheta_{w_{2j-1}+iN}\left(\sqrt{q^{(i)}},(1-q^{(i)})z^{(i)}\right)}{\sqrt{1-\sqrt{q^{(i)}}}}$ grows only sub-exponentially with N. Thus we get:

$$\begin{aligned} \prod_{j=1}^{2^{m-i}} \left\| \psi_j^{(i)[N]} \right\| &\leq \prod_{j=1}^{2^{m-i}} \text{sub-exponential}(N) \prod_{j=1}^{2^{m-i+1}} \|\psi_j^{(i-1)[N]}\| \\ &\leq \prod_{l=1}^{i} \left(\prod_{j=1}^{2^{m-l}} \text{sub-exponential}(N) \right) \prod_{j=1}^{2^m} \|\psi_j\|. \end{aligned} \tag{3.95}$$

Now we have everything to get a bound on the truncated operator using $\|\psi_i\|$:

$$\left\|T^{(0)[N]}_{l^{(0)}} - T^{(m)[N]}_{l^{(m)}}\right\| \leq q^{\Omega(N)} \cdot \sum_{i=0}^{m-1} \left[\prod_{l=1}^{i} \left(\prod_{j=1}^{2^{m-l}} \text{sub-exponential}(N)\right)\right] \cdot \prod_{j=1}^{2^m} \|\psi_j\|. \tag{3.96}$$

A sum of products of sub-exponential functions is still a sub-exponential function. As a result we get the desired bound:

$$\left\|T_l - T^{(m)[N]}_{l^{(m)}}\right\| \leq q^{\Omega(N)} \cdot \prod_{j=1}^{2^m} \|\psi_j\|. \tag{3.97}$$

This precisely means that the truncated TTN is a good approximation of the exact TTN for large enough truncation parameter N. □

3.5 Bond dimension

For a correlation function of n fields, to any given fields the renormalization map with truncation parameter N (see subsection 2.1.1) is applied $m = \log_2 n$ times. After the first step of renormalization, the truncated fields will equal to (see equation (2.36)):

$$\psi_i^{(1)[N]} = W_q^{[N]} \left(\psi_{2i-1}, (1-q)z\right) q^{L_0} \psi_{2i} \tag{3.98}$$

Suppose the highest weight of ψ_i's is M. Then the highest weight of $\psi_i^{(1)[N]}$'s will be $M + N$. If one employs this reasoning for m steps, one gets that the weight of $\psi_1^{(m)[N]}$ is at most $M + Nm = M + N\log_2 n$. Consequently, a sufficient bond dimension is:

$$D_{TTN} = d_B(M + N\log_2 n), \tag{3.99}$$

that is, a dimension of the subspace obtained by keeping all the levels up to and including $M + N\log_2 n$ in the module B (for d_B see definition 3.1.1, and see bound (1.31) for the increase of D_{TTN} with the increase of N). For the MPS, on the other hand,

$$D_{MPS} = d_B(\tilde{M} + Nn), \tag{3.100}$$

for some constant $\tilde{M}$ (see [KS15] for details). With bond dimensions D_{TTN} and D_{MPS} these tensor networks yield error $\sim q^{\Omega(N)}$. This means that if one needs an approximation that will yield an error not more than δ and both the MPS and the TTN are applicable. For small enough δ known bounds yield a smaller number of parameters for the TTN than for the MPS. However, bounds for the MPS were proven for a bigger set of z and q. Thus, the answer to the question of whether bounds for the MPS or the TTN are tighter depends on the approximation algorithm.

One possible way to obtain even better bounds would be to use MERA. Disentanglers can be used to solve the issue with different conditions for z, q and l for each step of the renormalization more effectively. This is a subject for further research.

As the dimension $d_B(k)$ grows as a number of multi-partitions of k (see equation (1.31)), we see that

$$D_{TTN} = e^{\Theta(\sqrt{N})} \tag{3.101}$$

if m and M are kept fixed. As the error decreases with N as $q^{\Omega(N)}$, we can take a logarithm of both the dimension and the error and see, that while the bond dimension grows as $\sqrt{N}$, the error falls as $\sim \frac{1}{N}$. This means that TTN can represent correlation functions efficiently. This result is very similar to the result for the MPS.

Chapter 4

Conclusions

This thesis provides a step in drawing a connection between tensor networks and CFT. This is an important step in the program to unify two efficient approaches to many-body quatum systems – those are, tensor networks and QFT. While the thesis concentrates on TTN, it is complementary to previous studies of relations between MPS and CFT [KS15].

The work essentially consists of two parts. First, we have constructed a TTN isometry that has a meaningful truncation to a finite-dimensional space. This was done in chapter 2. As a result, a representation for correlation functions in two-dimensional CFTs in terms of a tree tensor network with isometric property was obtained and a general approximation scheme for TTN in terms of finite-dimensional matrices was developed.

In the second part of the work, we applied the truncation scheme to the full tree. We obtained error bounds for the representation of WZW models via the TTN. The main obstacle in generalizing this approach to other CFTs is the current limited knowledge about non-WZW theories. If one finds analogies to linear energy bounds for some other classes of models and they satisfy certain mild conditions (see subsection 3.1.1), then the construction can be straightforwardly generalized to these non-WZW cases. The construction and proof can be found in chapter 3.

Finally, we have observed that bounds for the bond dimension for the TTN can be asymptotically tighter than the current known bound for the bond dimension for the MPS. However, the proven area of validity of MPS in terms of initial parameters of the lattice is larger then that of TTN. (see section 3.5 for a discussion). It is conjectured that one can overcome this by considering more powerful tensor networks such as MERA.

Thus we have proven that one can indeed use TTN to describe one-dimensional critical systems that correspond to WZW models in an efficient manner and obtained analytic evidence that such results can be obtained for other classes of CFTs. This class of tensor networks is not just capable to reproduce power laws, but yields correct correlation functions while using a modest number of parameters.

One can obtain further interesting results by moving either to more general and interesting tensor networks – the primal example is MERA, or by relaxing the condition of conformal symmetry. While the formalism of vertex operators will not be applicable anymore, one can try to obtain conditions on C*-algebras for different tensor network methods to be applicable.

Appendix A

Other approaches to map construction

One may question, if we could choose vector spaces on which the map $\epsilon_{z,q}$ (2.36) acts differently. Indeed, it may seem that there are simpler choices for V in (2.28) – for example, one could consider a map that acts on space of intertwiners or scaled intertwiners. Unfortunately, several complications arise in these alternative approaches.

For example, if one considers space V to be the space of scaled intertwiners, than the maps of sort (2.51) require computing scaled intertwiners and multiplying them. However, this is exactly the way the MPS construction in [KS15] works! Thus, this approach does not yield any new or improved results unlike approach of (2.36), as we have seen in section 3.5.

One may consider V to be a space of intertwiners and choose different map to (2.51). Such maps indeed exist – it can be proved by induction that

$$\epsilon_{q^k}\left(Y(\psi_1,z)\otimes Y(\psi_2,z)\right) = Y(\psi_1,z)q^{kL_0}Y(q^{-kL_0}\psi_2,z)q^{-kL_0} \tag{A.1}$$

where k corresponds to a step of tensor network also yields $\simeq \prod_{i=0}^{2^m-1} Y(\psi_i, zq^i)$ via the TTN for a suitable choice of initial fields $\{\psi_i^{(0)}\}_{i=1}^{2^m}$. Even though it looks attractive at first sight – the map consists only of Möbius transformations, it has a big drawback – intertwiners are in general unbounded. Thus, proving that any finite-dimensional truncation of such map gives correct answer up to a small error is not straight-forward at all.

Finally, one could use the idea behind purely generated finally correlated states [FNW92], and try to represent the map (2.51) in the following form

$$\epsilon(A\otimes B) = U(A\otimes B)U^\dagger, \tag{A.2}$$

where A and B are either intertwiners or scaled intertwiners. Let us try to brute-force construct such U. In order for factors like q^{nL_0} to be diagonal, it is convenient to choose eigenbasis of L_0 for the calculation. In this basis we have

$$A = \sum A_{ab}|a\rangle\langle b|, \tag{A.3}$$
$$B = \sum B_{cd}|c\rangle\langle d|. \tag{A.4}$$

The general expression for U is

$$U = \sum U_{ijk}|i\rangle\langle j| \otimes \langle k|, \tag{A.5}$$
$$U^\dagger = \sum U^*_{ijk}|j\rangle\langle i| \otimes |k\rangle \tag{A.6}$$

This straightforwardly leads to

$$U(A \otimes B)U^\dagger = \sum(U_{iac}U^*_{jbd}A_{ab}B_{cd})|i\rangle\langle j|. \tag{A.7}$$

We can define $\epsilon(A \otimes B)$ with extra $q^{\alpha L_0}$ an one side and $q^{-\alpha L_0}$ on the other side and obtain equivalent map with the equivalence relation $\simeq$ (see definition 2.1.4), or ϵ can act on ordinary or scaled vertex operators – the only difference in all those cases are extra factors of type q^{nL_0}. Thus, we can write

$$\epsilon(A \otimes B) = C^1AC^2BC^3, \tag{A.8}$$

where

$$C^i = q^{n_iL_0} = \sum_j C^i_j|j\rangle\langle j|, \tag{A.9}$$

in basis of eigenvectors of L_0. So the map (2.51) written in the L_0 eigenbasis is

$$\epsilon(A \otimes B) = \sum(C^1_aC^2_bC^3_d)(A_{ab}B_{bd})|a\rangle\langle d|. \tag{A.10}$$

Using the condition (A.2), we get

$$\sum(U_{iac}U^*_{jbd}A_{ab}B_{cd})|i\rangle\langle j| = \sum(C^1_aC^2_bC^3_d)(A_{ab}B_{bd})|a\rangle\langle d|. \tag{A.11}$$

By observing the way the first index of A_{ab} and the ket-vector are being summed over, we conclude that

$$U = \sum U^{(1)}_{ij}|i\rangle\langle i| \otimes \langle j|, \tag{A.12}$$

and by observing the summation of the second index of B_{cd} and the bra-vector, we conclude

$$U^\dagger = \sum U^{(2)}_{ij}|j\rangle\langle i| \otimes |i\rangle. \tag{A.13}$$

The conditions (A.12) and (A.12) combined yield

$$U = \sum U_i |i\rangle\langle i| \otimes \langle i|. \tag{A.14}$$

However, if we input the result A.14 in (A.2), we get

$$\epsilon(A \otimes B) = U(A \otimes B)U^\dagger = \sum U_i U_j A_{ij} B_{ij} |i\rangle\langle j|. \tag{A.15}$$

This expression has more symmetry than the map (2.51) – it has one index less (and intertwiners are in general not diagonal in the basis of L_0), which means that our map can not be represented in the form (A.2) for general enough A and B. Of course, above considerations does not exclude other possible maps. However, these consideration mean that our choice of the map is rather simple and natural.

Bibliography

[And08] George E. Andrews. *A Survey of Multipartitions Congruences and Identities*, pages 1–19. Springer New York, New York, NY, 2008.

[BP09] R. Blumenhagen and E. Plauschinn. *Introduction to Conformal Field Theory.* Springer, 2009.

[BPZ84] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333 – 380, 1984.

[BSM90] Detlev Buchholz and Hanns Schulz-Mirbach. Haag duality in conformal quantum field theory. *Reviews in Mathematical Physics*, 02(01):105–125, 1990.

[DFMS97] P. Di Francesco, P. Mathieu, and D. Sénéchal. *Conformal Field Theory.* Graduate Texts in Contemporary Physics. Springer, 1997.

[EFK98] P. Etingof, I. Frenkel, and A. Kirillov. *Lectures on representation theory and Knizhnik-Zamolodchikov equations*, volume 58 of *Mathematical Surveys and Monographs.* AMS, 1998.

[EV09] G. Evenbly and G. Vidal. Algorithms for entanglement renormalization. *Phys. Rev. B*, 79(14):144108, April 2009.

[EV16] G. Evenbly and G. Vidal. Local Scale Transformations on the Lattice with Tensor Network Renormalization. *Physical Review Letters*, 116(4):040401, January 2016.

[EWKK14] E. Efrati, Z. Wang, A. Kolan, and L. P. Kadanoff. Real-space renormalization in statistical mechanics. *Reviews of Modern Physics*, 86:647–667, April 2014.

[FF96] Boris Feigin and Edward Frenkel. *Integrable Systems and Quantum Groups: Lectures given at the 1st Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, June 14–22, 1993*, chapter Integrals of motion and quantum groups, pages 349–418. Springer Berlin Heidelberg, Berlin, Heidelberg, 1996.

[FNW92] M. Fannes, B. Nachtergaele, and R. F. Werner. Finitely correlated states on quantum spin chains. *Comm. Math. Phys.*, 144(3):443–490, 1992.

[Gab00] M. R. Gaberdiel. An introduction to conformal field theory. *Reports on Progress in Physics*, 63:607–667, April 2000.

[GHZ07] D. M. Greenberger, M. A. Horne, and A. Zeilinger. Going Beyond Bell's Theorem. *ArXiv e-prints: quant-ph/0712.0921*, December 2007.

[GN03] R. Matthias Gaberdiel and Andrew Neitzke. Rationality, quasirationality and finite w-algebras. *Communications in Mathematical Physics*, 238(1):305–331, 2003.

[Has07] M B Hastings. An area law for one-dimensional quantum systems. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(08):P08024, 2007.

[HOVV13] Jutho Haegeman, Tobias J. Osborne, Henri Verschelde, and Frank Verstraete. Entanglement renormalization for quantum fields in real space. *Phys. Rev. Lett.*, 110:100402, Mar 2013.

[HR18] G. H. Hardy and S. Ramanujan. Asymptotic formulaæ in combinatory analysis. *Proceedings of the London Mathematical Society*, s2-17(1):75–115, 1918.

[HST+15] Benedikt Herwerth, Germán Sierra, Hong-Hao Tu, J. Ignacio Cirac, and Anne E. B. Nielsen. Edge states for the Kalmeyer-Laughlin wave function. *Phys. Rev. B*, 92:245111, Dec 2015.

[Hua05] Yi-Zhi Huang. Differential equations and intertwining operators. *Communications in Contemporary Mathematics*, 07(03):375–400, 2005.

[HV17] Qi Hu and Guifre Vidal. Spacetime symmetries and conformal data in the continuous multi-scale entanglement renormalization ansatz. *eprint arXiv: quant-ph/1703.04798*, 2017.

[JBH+15] David Jennings, Christoph Brockt, Jutho Haegeman, Tobias J Osborne, and Frank Verstraete. Continuum tensor network field states, path integral representations and spatial symmetries. *New Journal of Physics*, 17(6):063039, 2015.

[KL99] Martin Karel and Haisheng Li. Certain generating subspaces for vertex operator algebras. *Journal of Algebra*, 217(2):393 – 421, 1999.

[KRV09] Robert König, Ben W. Reichardt, and Guifré Vidal. Exact entanglement renormalization for string-net models. *Phys. Rev. B*, 79:195123, May 2009.

[KS15] R. Koenig and V. B. Scholz. Matrix product approximations to conformal field theories. *ArXiv e-prints: math-ph/1509.07414*, September 2015.

[KS16] Robert König and Volkher B. Scholz. Matrix product approximations to multipoint functions in two-dimensional conformal field theory. *Phys. Rev. Lett.*, 117:121601, Sep 2016.

[MVLN10] V. Murg, F. Verstraete, Ö. Legeza, and R. M. Noack. Simulating strongly correlated quantum systems with tree tensor networks. *Physical Review B*, 82(20):205105, November 2010.

[NC13] N. Nakatani and G. K.-L. Chan. Efficient tree tensor network states (TTNS) for quantum chemistry: Generalizations of the density matrix renormalization group algorithm. *J. Chem. Phys*, 138(13):134113–134113, April 2013.

[NCS11] Anne E. B. Nielsen, J. Ignacio Cirac, and German Sierra. Quantum spin Hamiltonians for the $SU(2)_k$ WZW model. *J. Stat. Mech.*, 1111:P11014, 2011.

[OEV10] Tobias J. Osborne, Jens Eisert, and Frank Verstraete. Holographic quantum states. *Phys. Rev. Lett.*, 105:260401, Dec 2010.

[Orú14] Román Orús. A practical introduction to tensor networks: Matrix product states and projected entangled pair states. *Annals of Physics*, 349:117–158, Oct 2014.

[Sch05] U. Schollwöck. The density-matrix renormalization group. *Rev. Mod. Phys.*, 77:259–315, Apr 2005.

[Sch09] Martin Schottenloher. *A Mathematical Introduction to Conformal Field Theory.* Springer-Verlag Berlin Heidelberg, 2009.

[SGM+10] P. Silvi, V. Giovannetti, S. Montangero, M. Rizzi, J. I. Cirac, and R. Fazio. Homogeneous binary trees as ground states of quantum critical Hamiltonians. *Phys. Rev. A*, 81:062335, Jun 2010.

[TEV09] L. Tagliacozzo, G. Evenbly, and G. Vidal. Simulation of two-dimensional quantum systems using a tree tensor network that exploits the entropic area law. *Phys.Rev.B*, 80(23):235127, December 2009.

[VC04] F. Verstraete and J. I. Cirac. Renormalization algorithms for Quantum-Many Body Systems in two and higher dimensions. *eprint arXiv: cond-mat/0407066*, July 2004.

[VC10] F. Verstraete and J. I. Cirac. Continuous matrix product states for quantum fields. *Phys. Rev. Lett.*, 104:190405, May 2010.

[Vid08] G. Vidal. Class of Quantum Many-Body States That Can Be Efficiently Simulated. *Physical Review Letters*, 101(11):110501, September 2008.

[Whi92] Steven R. White. Density matrix formulation for quantum renormalization groups. *Phys. Rev. Lett.*, 69:2863–2866, Nov 1992.

[YLP+14] S. Yang, L. Lehman, D. Poilblanc, K. Van Acoleyen, F. Verstraete, J. I. Cirac, and N. Schuch. Edge Theories in Projected Entangled Pair State Models. *Physical Review Letters*, 112(3):036402, January 2014.

[ZCZW15] B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen. Quantum Information Meets Quantum Matter – From Quantum Entanglement to Topological Phase in Many-Body Systems. *ArXiv e-prints: cond-mat.str-el/1508.02595*, August 2015.

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Statement of authorship

I certify that I am the author of this paper titled “Tree tensor network approximations to conformal field theories” and that any assistance I received in its preparation is fully acknowledged and disclosed in the paper. I have also cited any sources from which I used data or ideas, either quoted directly or paraphrased.

Date and place:____________________ Signature:____________________