WiSe 2006 01/25/07

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T VI: Soft Matter and Biological Physics
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Problem set 9

Problem 9.1 *Doob's theorem*
Consider a stochastic process $\{Y_t : t \geq 0\}$, characterized by the three properties

(i) *(stationary)* For times $t_1 < t_2 < \ldots < t_n$ and $\tau > 0$ the random n vectors $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ and $(Y_{t_1+\tau}, Y_{t_2+\tau}, \ldots, Y_{t_n+\tau})$ are identically distributed, i.e. time shifts leave joint probabilities invariant.

(ii) *(gaussian)* For times $t_1 < t_2 < \ldots < t_n$ the vector $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ is multivariate normally distributed.

(iii) *(markovian)* For $t_1 < t_2 < \ldots < t_n$ the conditional probabilities depend only on the most recent event $p(Y_{t_n}|Y_{t_1}, \ldots, Y_{t_{n-1}}) = p(Y_{t_n}|Y_{t_{n-1}})$.

1. Show that the most general bivariate gaussian probability distribution is given by

$$p(\Delta x, \Delta y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(\Delta x)^2}{\sigma_x^2} - \frac{2\rho\Delta x\Delta y}{\sigma_x\sigma_y} + \frac{(\Delta y)^2}{\sigma_y^2}\right)\right], \quad \Delta x = x - \langle x\rangle\,,\ \Delta y = y - \langle y\rangle\,,$$

   with variances $\sigma_x^2 = \langle(\Delta x)^2\rangle$, $\sigma_y^2 = \langle(\Delta y)^2\rangle$, and correlation coefficient $\rho = \langle\Delta x\Delta y\rangle/\sigma_x\sigma_y$.

2. Demonstrate that for the stochastic process under consideration, the conditional probability is represented by

$$p(Y_t|Y_0) = \frac{1}{\sigma\sqrt{2\pi(1-\kappa(t)^2)}} \exp\left[-\frac{(Y_t - \kappa(t)Y_0)^2}{2\sigma^2(1-\kappa(t)^2)}\right],$$

   with stationary mean $\mu = \langle Y_t\rangle$ and variance $\sigma = \langle(Y_t-\mu)^2\rangle$ and the time-dependent correlation coefficient $\kappa(t) = \langle(Y_t-\mu)(Y_0-\mu)\rangle/\sigma^2$.

3. Use the Chapman-Kolmogorov equation

$$p(Y_{t+\tau}|Y_0) = \int \mathrm{d}Y_\tau\, p(Y_{t+\tau}|Y_\tau)p(Y_\tau|Y_0)\,,$$

   to show that the correlation coefficient satisfies the functional equation

$$\kappa(t+\tau) = \kappa(\tau)\kappa(t)\,,$$

   and conclude that $\kappa(t) = \exp(-\gamma t)$ with a suitable decay constant $\gamma \geq 0$.

In summary you have proven Doob's theorem, i.e. every stationary markovian gaussian process is characterized by an exponentially decaying correlation function $C(t) \equiv \langle(Y_t-\mu)(Y_0-\mu)\rangle = \sigma^2\exp(-\gamma t)$.

Problem 9.2 *Cambpell's theorem*

We are interested in characterizing the stochastic process

$$Y(t) = \sum_n \nu(t - t_n)$$

where $\nu(t)$ is a given rapidly decaying function for $t \to \pm\infty$ called a spike. Then $Y(t)$ is a spike train and the events where the spikes occur follow from the distribution of the times t_n. We assume that the corresponding stochastic properties are given by a Poisson distribution: The probability distribution for N events at times $\{t_i : i = 1, \dots, N\}$, $0 \le t_i \le T$ is prescribed by

$$\mathrm{d}Q(t_1, .., t_N) = Q(t_1\ \dots, t_N)\mathrm{d}t_1 \dots \mathrm{d}t_N = \frac{1}{N!\,\tau^N} e^{-T/\tau} \mathrm{d}t_1 \dots \mathrm{d}t_N\,.$$

Here the events may occur in any order.

1. Check the normalization

$$\sum_{N=0}^{\infty} \int_0^T \mathrm{d}t_1 \dots \int_0^T \mathrm{d}t_N\, Q(t_1, \dots, t_N) = 1\,.$$

2. Verify that the average of the spike train reads

$$\langle Y(t) \rangle = \frac{1}{\tau} \int_0^T \mathrm{d}\bar{t}\, \nu(t - \bar{t})\,.$$

and interpret the result.

3. Determine the variance

$$\langle \delta Y(t) \delta Y(t') \rangle = \frac{1}{\tau} \int_0^T \mathrm{d}\bar{t}\, \nu(t - \bar{t}) \nu(t' - \bar{t})\,, \qquad \delta Y(t) = Y(t) - \langle Y(t) \rangle\,.$$

The formulae for the mean and variance are known as *Campbell's theorem.*

4. Show that the moment generating functional can be evaluated explicitly to

$$M[\xi(t)] \equiv \left\langle \exp\left(\int_0^T \mathrm{d}t\ \xi(t) Y(t) \right) \right\rangle = \exp\left\{ \frac{1}{\tau} \int_0^T \mathrm{d}\bar{t}\ \left[\exp\left(\int_0^T \mathrm{d}t\, \xi(t) \nu(t - \bar{t}) \right) - 1 \right] \right\}$$

and evaluate the mean and variance of $Y(t)$ again by performing functional derivatives of the cumulant generating functional $\kappa[\xi(t)] = \ln M[\xi(t)]$.