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T VI: Soft Matter and Biological Physics
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Problem set 8

Problem 8.1 *Wick's theorem*
The joint probability distribution for N gaussian variables $\{\varphi_i\}_{i=1,\dots,N}$ reads

$$P[\varphi] = \frac{1}{\sqrt{(2\pi)^N \det A}} \exp\left(-\frac{1}{2}\,\varphi^T \cdot A^{-1} \cdot \varphi\right),$$

with some real positive definite symmetric matrix A and the shorthand notation $\varphi^T \cdot A^{-1} \cdot \varphi = \sum_{i=1}^N \sum_{j=1}^N \varphi_i (A^{-1})_{ij} \varphi_j$ has been introduced.

1. Convince yourself that P is properly normalized.
2. Prove the following relation for the *generating functional*
$$Z[j] = \langle \exp\left(j^T \cdot \varphi\right)\rangle = \exp\left(\frac{1}{2}\,j^T \cdot A \cdot j\right).$$
3. Correlation functions are then readily obtain by taking appropriate derivatives
$$\langle \varphi_{i_1} \dots \varphi_{i_{2n}}\rangle = \left.\frac{\partial^{2n}}{\partial j_{i_1} \dots \partial j_{i_{2n}}}\right|_{j=0} Z[j]\,.$$
Evaluate explicitly the two- and four-point correlation functions $\langle \varphi_{i_1}\varphi_{i_2}\rangle, \langle \varphi_{i_1}\varphi_{i_2}\varphi_{i_3}\varphi_{i_4}\rangle$.
4. Prove that all only non-vanishing correlation functions contain an even number of random variables. Furthermore show that higher order correlation function can be related to two-point function via
$$\langle \varphi_{i_1} \dots \varphi_{i_{2n}}\rangle = \langle \varphi_{i_1}\varphi_{i_2}\rangle\langle \varphi_{i_3}\varphi_{i_4}\rangle \dots \langle \varphi_{i_{2n-1}}\varphi_{i_{2n}}\rangle + \text{ permutations}\,.$$

Problem 8.2 *cubic anisotropy*
Consider the modified Landau-Ginzburg Hamiltonian

$$\beta\mathcal{H} = \bar{\mathcal{H}} = \int \mathrm{d}^d x \left\{\sum_{i=1}^M \left[\frac{c}{2}\left(\vec{\nabla}\varphi_i\right)^2 + \frac{r}{2}\varphi_i^2\right] + u(\sum_{i=1}^M \varphi_i^2)^2 + v\sum_{i=1}^M \varphi_i^4\right\}$$

for an M-component vector $\varphi_i(\vec{x}), i = 1, \dots M$. The term $v\sum_{i=1}^M \varphi_i^4$ generates a *cubic anisotropy.*

1. Mean-field theory:

(a) The anisotropy breaks roatiational symmetry. Find the optimal direction for a fixed magnitude $\sum_{i=1}^{M} \varphi_i^2$ for $v > 0$ and for $v < 0$? What is the degeneracy of the easy magnetization axes in each direction?

(b) Provide conditions for the stability of the mean-field solution in the u–v plane.

(c) In general higher order terms, e.g. $w \left(\sum_{i=1}^{M} \varphi_i^2\right)^3$ with $w > 0$, ensure stability in the regions not allowed from part b). Sketch a phase diagram in the r–v plane for fixed $u > 0$ and indicate the ordered phases and nature of the phase transitions.

(d) Are there any Goldstone modes in the ordered phases?

2. ϵ-expansion: Perform a perturbation expansion up to second order, and inspect the resulting diagrams.

(a) Show that the first order correction yields recursion relations

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\ell} &= (d + 2\zeta) r + 4A[u(M+2) + 3v] + \mathcal{O}(u^2, uvv^2, \ldots)\,, \\
\frac{\mathrm{d}c}{\mathrm{d}\ell} &= (d - 2 + 2\zeta) c + \mathcal{O}(u^2, uv, v^2, \ldots)\,,
\end{aligned}
$$

where

$$
A = \int_p^{>} \frac{1}{r + cp^2} = \frac{1}{c} K_d \Lambda^{d-2} \mathrm{d}\ell + \mathcal{O}(r)
$$

Assume that the non-trivial fixed point are $\mathcal{O}(\epsilon)$ where $\epsilon = 4 - d$ to conclude that

$$
\zeta = \frac{2-d}{2} + \mathcal{O}(\epsilon^2)\,.
$$

The parameter c may then kept fixed at unity, $c = 1$.

(b) The second order perturbation yields the recursion relation for the couplings. Using the results derived so far show that

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}\ell} &= \epsilon u - 4u^2(M+8)K_4 - 24uvK_4 + \mathcal{O}(\epsilon^3)\,, \\
\frac{\mathrm{d}v}{\mathrm{d}\ell} &= \epsilon v - 36v^2 K_4 - 48uvK_4 + \mathcal{O}(\epsilon^3)\,.
\end{aligned}
$$

(c) Find all fixed points in the u–v plane, and draw the flow patterns for $M < 4$ and $M > 4$. Discuss the relevance of the cubic anisotropy term near the stable fixed point in each case. Calculate the exponent ν at the stable fixed point for the cases $M < 4$ and $M > 4$.

(d) Is the region of stability in the u–v plane calculated within mean-field approximation enhance or diminished by inclusion of fluctuations? Since in reality higher order terms will be present, what does this imply about say the nature of the phase transition for a small negative v and $M > 4$.

(e) Sketch schematic phase diagrams in the r–v plane for $M > 4, M < 4$ and $u > 0$, identifying the ordered phases. Are there Goldstone modes in any of these phases close to the phase transition?