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Übung zur Vorlesung
Mathematische Statistische Physik
Sommersemester 2008
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Problem 30
Let A be a positive definite and symmetric $\mathbb{R}^N \times \mathbb{R}^N$ matrix. Show that

$$\int \prod_{i=1}^{N} dx_i \exp\left(-\sum_{i=1}^{N} j_i x_i - \frac{1}{2}\sum_{i,j=1}^{N} x_i A_{ij} x_j\right) = a \cdot \exp\left(\frac{1}{2} j^T A^{-1} j\right) ,$$

for some a that does not depend on the j. Hence show that the partition sum of the Ising-Hamiltonian $H(\{\sigma_i\}) = -J\sum_{<i,j>} \sigma_i \sigma_j$ (where $< i,j >$ denotes the sum over nearest neighbours) can be rewritten in the form

$$Z = \widetilde{a} \int \prod dm_i \exp\left(-\widetilde{H}(\{m_i\})\right) ,$$

with $\widetilde{a}$ independent of the σ_i. Develop $\widetilde{H}$ in a power series up to the order $O(m^4)$.

Problem 31
The fieldtheoretic description of critical phenomena leads to functionals of the form

$$F[\phi(y)] = \int dy f(\phi(y)) ,$$

where $F[\phi(y)]$ is a function depending on the field $\phi(x)$. Therefore the concepts differentiation and integration need to be extended. We define the functional derivative of the functional $F[\phi(x)]$ as the limit of the derivative on a discrete lattice:

$$\frac{\delta F}{\delta \phi(y)} = \lim_{a\to 0} \frac{1}{a}\frac{\partial F}{\partial \phi_i} ,$$

where ϕ_i is the field at a discrete point i on the lattice with lattice constant a. Likewise the functional derivative can be defined continuously as usually done by physicists:

$$\frac{\delta F}{\delta \phi(y)} = \lim_{\epsilon\to 0} \frac{F[\phi(x) + \epsilon\delta(x-y)] - F[\phi(x)]}{\epsilon} .$$

(a) Derive the Euler-Lagrange equation by finding the extremum of the action $\delta S = 0$, where the action is defined by: $S = \int dt L(q(t), \partial_t q(t))$.

(b) Calculate the functional derivative for the following functional $F[\phi(x)]$

$$F = \int dy f(y)\phi^p(y) , \quad F = \int dy V(\phi(y)) , \quad F = \int dy \left(\frac{d\phi}{dy}\right)^2 , \quad F = \phi(y) .$$

Problem 32

The generating functional $\Gamma[\vec{m}]$ of vertex functions is obtained by Legendre-transforming the generating functional $W[\vec{h}] = \ln Z[\vec{h}]$ of cumulants:

$$\Gamma[\vec{m}] = -W[\vec{h}] + \int d^d x \sum_{i=1}^{n} m_i(x) h_i(x) ,$$

where

$$m_i(x) = \langle \phi_i(x) \rangle_{\vec{h}} = \frac{\delta W[\vec{h}]}{\delta h_i(x)} .$$

and where the order parameter $\vec{\phi}$ and the field $\vec{h}$ have n components. The N-point vertex functions are defined as

$$\Gamma^{(N)}_{i_1,\ldots,i_N}(x_1,\ldots,x_N) = \left. \frac{\delta^N \Gamma[\vec{m}]}{\delta m_{i_1}(x_1)\cdots \delta m_{i_N}(x_N)} \right|_{\vec{h}\equiv 0} .$$

(a) Show that

$$\frac{\delta \Gamma}{\delta m_i(x)} = h_i(x) .$$

(b) Deduce the following relation between 2-point vertex functions and 2-point cumulants

$$\int d^d x \sum_k G^{(2)}_{c\,ik}(x_1,x) \Gamma^{(2)}_{kj}(x,x_2) = \delta_{ij} \delta(x_1 - x_2) .$$

(Recall that the 2-point cumulant is $G^{(2)}_{c\,ik}(x_1,x_2) = \left. \frac{\delta W[\vec{h}]}{\delta h_i(x_1) \delta h_k(x_2)} \right|_{\vec{h}\equiv 0}$.) What is the corresponding relation in momentum space?

(c) Find analogous expressions for $\Gamma^{(N)}$ and $G_c^{(N)}$ in the cases $N = 3$ and $N = 4$.

(d) From (b) and (c), deduce that the vertex functions are given by the one-particle irreducible diagrams.