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T II: Elektrodynamik

(Prof. E. Frey)

Problem set 2

Tutorial 2.1 *Sheet, cylinder, and sphere*

Consider scalar fields $\rho(\vec{x})$ specified in cartesian coordinates $\vec{x} = (x, y, z)$ by $\rho(\vec{x}) = \rho_0$ for

a) a sheet: $|z| \leq d$, b) a cylinder: $\sqrt{x^2 + y^2} \leq d$, c) a sphere: $\sqrt{x^2 + y^2 + z^2} \leq d$,

and $\rho(x) = 0$ elsewhere. Construct vector fields $\vec{E}(\vec{x})$ such that $\operatorname{div} \vec{E}(\vec{x}) = 4\pi\rho(\vec{x})$ and $\operatorname{curl} \vec{E}(\vec{x}) = 0$ and that reflect the symmetries of the problem. Determine appropriate scalar potentials $\varphi(\vec{x})$ with $\vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x})$ and sketch their functional forms.

Note: The gradient and divergence operator in cylindrical coordinates (r, ϕ, z) and in spherical coordinates (r, ϑ, ϕ) read

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\vec{e}_\phi + \frac{\partial\psi}{\partial z}\vec{e}_z\,, \qquad \vec{\nabla}\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z}\,,$$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\vec{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\psi}{\partial\phi}\vec{e}_\phi\,, \qquad \vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, V_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial V_\phi}{\partial\phi}\,.$$

Tutorial 2.2 *Momentum conservation law*

Defining the symmetric tensor field (Maxwell stress tensor)

$$T_{ik}(\vec{x}, t) = \frac{1}{4\pi}\left[\frac{1}{2}\delta_{ik}(\vec{E}^2 + \vec{B}^2) - E_iE_k - B_iB_k\right] \qquad (i, k = 1, 2, 3),$$

show that Maxwell's equations imply a local balance law for the momentum density,

$$\frac{1}{c^2}\partial_t S_i + \nabla_k T_{ik} = -F_i\,,$$

where $\vec{S} = (c/4\pi)\vec{E}\times\vec{B}$ denotes the Poynting vector. Determine the mechanical force density $\vec{F}$.

Hint: The following vector identity may prove useful,

$$\left[\vec{V}\times(\vec{\nabla}\times\vec{V})\right]_i = -\nabla_k\left(V_iV_k - \frac{1}{2}\delta_{ik}\vec{V}^2\right) + V_i \operatorname{div}\vec{V}\,.$$

Problem 2.3 *Dipole field*

Consider the (static) electric field $\vec{E}(\vec{x})$ of an electric dipole $\vec{p}$

$$\vec{E}(\vec{x}) = \frac{3\vec{x}(\vec{x}\cdot\vec{p}) - r^2\vec{p}}{r^5}, \qquad r = |\vec{x}|\,.$$

a) Demonstrate explicitly that the field may be represented by a scalar potential, $\vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x})$.
b) Show that $\vec{E}(\vec{x})$ allows for a representation in terms of a vector potential, i.e. $\vec{E}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x})$.
c) Argue that the dipole field is a homogenous function of the coordinates, i.e. $\vec{E}(\lambda\vec{x}) = \lambda^\zeta\vec{E}(\vec{x})$ where ζ denotes the degree of the homogeneous function. Conclude that the field is *scale-free*, i.e., zooming in (change of length scale) may be compensated by a simultaneous change of units for the field. What does this imply for the field lines?
d) Find a suitable scalar potential $\varphi(\vec{x})$ and vector potential $\vec{A}(\vec{x})$ corresponding to $\vec{E}(\vec{x})$. Choose $\varphi, \vec{A}$ such that they are again scale-free of appropriate degree. Verify your results explicitely.

   Hint: Since the electric field is linear in $\vec{p}$, one may choose φ and $\vec{A}$ that have the same property. Rotational symmetry dictates there is a unique scalar/pseudo vector that can be built from $\vec{x}$ and $\vec{p}$ up to a prefactor.
e) Discuss the field lines of the electric field as well as the vector potential. Discuss the surfaces of constant scalar potential.

Problem 2.4 *Vector potential*

The vector potential $\vec{A}$ corresponding to a solenoidal field $\vec{B}$, div $\vec{B}=0$, $\vec{B} = \vec{\nabla}\times\vec{A}$, may be obtained by evaluating the line integral (Poincaré's lemma)

$$\vec{A}(\vec{x}) = -\int_0^1 u(\vec{x}-\vec{x}_0)\times\vec{B}(\vec{x}(u))\,\mathrm{d}u \qquad (*)$$

for straight lines $\vec{x}(u) = x_0 + u(\vec{x}-\vec{x}_0)$.

a) Recall Ampère's law of magnetostatics, $\vec{\nabla}\times\vec{B} = 4\pi\vec{j}/c$. Thus in the case of a current-free region, $\vec{j}=0$, a scalar magnetostatic potential φ_M may by introduced, $\vec{B} = -\vec{\nabla}\varphi_M$, where $\nabla^2\varphi_M = 0$. Empoly Poincaré's lemma to determine a vector potential $\vec{A}$ of a magnetic octupole field corresponding to the potential

$$\varphi_M(\vec{x}) = z^3 - \frac{3}{2}(x^2+y^2)z.$$

b) Evaluate the curl of the integral representation $(*)$ for $\vec{A}$ to prove that indeed $\vec{B} = \vec{\nabla}\times\vec{A}$ provided div $\vec{B}=0$.

Problem 2.5 *Minimal coupling*

Consider the non-relativistic motion of a particle characterized by the Lagrangian

$$L(\vec{x},\dot{\vec{x}},t) = \frac{m}{2}\dot{\vec{x}}^2 + \frac{q}{c}\dot{\vec{x}}\cdot\vec{A}(\vec{x},t) - q\varphi(\vec{x},t)\,,$$

where $\varphi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ are a time-dependent scalar and vector field, respectively.

a) Derive the corresponding Euler-Lagrange equations and interpret the force terms in terms of electric and magnetic fields, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$.

b) Recall that a change

$$L(\vec{x}, \dot{\vec{x}}, t) \mapsto L(\vec{x}, \dot{\vec{x}}, t) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{q}{c}\chi(\vec{x}, t) = L(\vec{x}, \dot{\vec{x}}) + \frac{q}{c}\dot{\vec{x}} \cdot \vec{\nabla}\chi(\vec{x}, t) + \frac{q}{c}\partial_t\chi(\vec{x}, t)\,,$$

does not affect the principle of least action. Show that the additional terms can be absorbed by defining new fields $\varphi', \vec{A}'$. What does this imply for the electric and magnetic fields?

c) Perform a Legendre transform, $\vec{p} = \partial\mathcal{L}/\partial\dot{\vec{x}}$, to derive the corresponding Hamilton function, $\mathcal{H} = \vec{p} \cdot \dot{\vec{x}} - \mathcal{L}$. Distinguish carefully between the *canonical* momentum $\vec{p}$ and the *kinetic* momentum $m\dot{\vec{x}}$. Derive the canonical equations of motion.

Due date: Tuesday, 5/8/07, at 9 a.m.