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SoSe 05

Set 2
4/25/05

T IV: Thermodynamik und Statistik
(Prof. E. Frey)

Problem set 2

Problem 2.1 *Surface of a sphere*
Calculate the surface K_d of a d-dimensional unit sphere by evaluating

$$\int d^d x \exp(-\mathbf{x}^2)\,, \qquad \mathbf{x}^2 = x_1^2 + .. + x_d^2$$

in cartesian and polar coordinates, respectively [Answer: $K_d = 2\pi^{d/2}/\Gamma(d/2)$].

Problem 2.2 *ideal gas*
Calculate for a gas of N structureless non-interacting particles the extensive part of the entropy in the microcanonical ensemble

$$s(e,v) = S(E,V,N)/N\,, \qquad N,E,V \to \infty\,,$$

for fixed energy density $e = E/V$ and fixed particle density $n = 1/v = N/V$. Show that your result does not depend on the resolution of the energy shell.

Problem 2.3
The fundamental relation of some system is found to be

$$s(e,v) = k_B \ln[e^{3/2} v] + s_0$$

where s, e, v are the entropy, energy and volume per particle, respectively, and s_0 some constant independent of e and v. Calculate the temperature T and the pressure p. Find the mechanical equation of state $p = p(v,T)$.

Problem 2.4 *Information theory*
The probability to find a system of N microstates in state i is $w_i, i = 1, .., N$. Define the functional

$$H = H(w_1, .., w_N) = -\sum_{i=1}^{N} w_i \ln w_i\,,$$

and find the probability distribution that maximizes H provided that
(a) only the normalization of the probabilities is enforced.
(b) the average of some random function A is known to be $a = \langle A \rangle = \sum_{i=1}^{N} A_i w_i$.
Hint: Use Lagrange multpliers for the constraints.

Problem 2.5 *Poisson's theorem*
For small values of p the Poisson distritbution provides a good approximation to the Bernoulli distribution. Let

$$P_N(k) = \binom{N}{k} p^k (1-p)^{N-k}\,, \qquad 0 \le p \le 1\,, \quad k = 0, 1, .., N$$

and consider p as a function $p(N)$ of N. Consider the limit $p(N) \to 0, N \to \infty$ in such a way that $Np \to \lambda$, where $\lambda > 0$. Show that for $k = 0, 1, 2, ..$

$$P_N(k) \to \pi_k = \frac{\lambda^k e^{-\lambda}}{k!}\,, \qquad N \to \infty\,.$$

Problem 2.6 *Macmillan's theorem*

Consider a system of N spins where each spin has a probability p and $q = 1 - p$ of being up or down, respectively. A microstate ω, e.g. $\omega = [\uparrow_1 \downarrow_2 \downarrow_3 .. \uparrow_N]$, is called *typical* if the number of up spin k fulfills $|k/N - p| \leq \epsilon$ and denote T the set of all typical states. Show that:

(a) the probability of a state to be typical approaches unity, $P(T) \to 1$ as $N \to \infty$.

(b) the probability $P(\omega)$ for a single typical microstate ω is bounded by

$$e^{-N(H+\tilde{\epsilon})} \leq P(\omega) \leq e^{-N(H-\tilde{\epsilon})}\,,$$

where $H = p\ln(1/p) + q\ln(1/q)$ and $\tilde{\epsilon} = \epsilon[\ln(1/p) + \ln(1/q)]$.

(c) the number of typical microstates $\#T$ can be estimated for large N as

$$(1-\epsilon)e^{N(H-\tilde{\epsilon})} \leq \#T \leq e^{N(H+\tilde{\epsilon})}\,.$$