Comments on Dilaton Actions

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Disclaimers/Apologies

- This is not a talk on string phenomenology
- This is not a talk on string theory
- The dilaton of this talk is not the string theory dilaton
- but it is inspired by holography in $D = d+1$ (here often $d=4$)

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based on work with A. Schwimmer

In this talk

dilaton:

Goldstone boson of spontaneously broken conformal symmetry

Our goal:
find a low-enery effective action for the dilaton along the lines of

spontaneously broken (anomalous) chiral symmetry

à la Wess, Zumino and Witten

Recall the following features of chiral anomalies ($d = 2n$):

Couple theory to external gauge fields (gauging the global symmetries) and integrate out the quarks.

Gauge non-invariance of the resulting (non-local) effective action signals the anomaly.

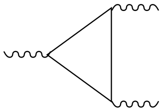
Because of the possibility of spontaneous symmetry breaking, we need to distiguish two phases:

the unbroken and the broken phase

- unbroken phase:

  anmalous part of the effective action is the local Chern-Simons action in $d+1$ dimensions with the property $(\mathcal{M}_d = \partial\mathcal{D}_{d+1})$

$$\delta_\lambda S_{\mathrm{CS}} = \delta_\lambda \int_{\mathcal{D}_{d+1}} \omega_{d+1} = \int_{\mathcal{M}_d} \mathrm{tr}(\lambda\, \mathcal{A})$$



$\mathcal{A}[A]$ is the anomaly and A are external gauge fields (sources) to which the (anomalous) currents are coupled.

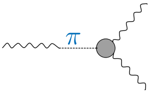
ω_{d+1} is the CS-form with the property $d\omega_{d+1} = F^{(d+1)/2}$

- spontaneously broken case:

$$G \xrightarrow{\langle q\bar{q}\rangle \neq 0} H \qquad \text{e.g. } SU(3) \times SU(3) \to SU(3)_{\rm diag}$$

with $\dim(G/H)$ Goldstone bosons (pions)

Effective action: (σ model action) $\underbrace{d\text{ - dim local term}}_{\text{gauge invariant}} + \underbrace{(d+1)\text{ - dim WZW term}}_{\text{reproduces anomaly}}$



$$\delta_\lambda S_{\rm WZW}[A,\pi] = \int_{\mathcal{M}_d} \underset{\substack{\delta_\lambda A=[A,\lambda]+d\lambda \\ \delta_\lambda \pi=\lambda}}{\uparrow}{\rm tr}(\lambda\, \mathcal{A}) \qquad \text{independent of } \pi$$

In this talk:

study analogous issues for trace (conformal, Weyl) anomalies

i.e. construct the effective actions, in particular for the broken phase.

Here the symmetry breaking pattern is

$$SO(d,2) \to SO(d-1,1) \times T_d$$

and there is one physical Goldstone boson, the dilaton (see below)

Aside on trace anomalies of CFTs:

- Couple CFT to an external metric g (source for T_{ij}): $S[\phi] \to S[\phi, g]$
- def. effective action $e^{W[g]} = \int \mathcal{D}\phi \, e^{-S[\phi,g]}$
- global symmetries of $S[\phi] \Rightarrow$ local symmetries of $W[g]$
  - Poincaré invariance $\Rightarrow$ diffeo invariance of $W[g]$
  - conformal invariance $\Rightarrow$ Weyl invariance $g_{ij} \to e^{2\sigma} g_{ij}$

modulo the trace anomaly

$$\delta_\sigma W[g] = \int \sqrt{g} \, \sigma \mathcal{A}$$

where (for $d = 4$)

$$\mathcal{A} = \langle T^i_i \rangle = a \, E_4 - c \, C^2 \qquad \int_{\mathcal{M}_4} \sqrt{g} E_4 \sim \chi$$

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Conformal invariance can be spontaneously broken by

$$\langle \mathcal{O}_\Delta \rangle \neq 0 \quad \text{if} \quad \Delta \neq d$$

$\Rightarrow$ two phases

Unbroken Phase

- the role of the CS action played by the dual gravitational bulk action $S[G]$ in $d+1$
- particular subgroup of $d+1$ dim diffeos plays the role of gauge transformations and produces the anomalies at the bundary

  This provides a different and more general view, along the lines explored here, of the holographic anomaly calculation of Henningson-Skenderis.

more specifically:

- use 'bulk' diffeos to bring G to Fefferman-Graham (FG) gauge

$$G_{\mu\nu} dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j$$

with $g(x,\rho) = g(x) + \rho\, g^{(1)}(x) + \ldots$ near boundary at $\rho = 0$
(FG expansion)

  - For $g_{ij} = \delta_{ij}$ .... AdS metric
  - metrics of this form are solutions of gravitational 'bulk' actions which admit AdS solutions
  - higher order terms $g^{(1)}, g^{(2)}, \ldots$ are covariant functionals of the boundary metric $g^{(0)} \equiv g$, largely fixed by symmetries (cf. below)

- gauge fixing not complete; residual diffeos ξ^μ (PBH [1]) such that

$$\mathcal{L}_\xi G_{\rho\rho} = \mathcal{L}_\xi G_{\rho i} = 0$$

$$\xi^\rho = \sigma(x)\rho\,, \qquad \xi^i \equiv -a^i[\sigma,\rho] = -\frac{1}{2}\int^\rho d\rho' g^{ij}(x,\rho')\partial_i\sigma(x)$$

$$\Rightarrow \quad \delta_\xi g_{ij}(x,\rho) = \ldots$$

- action on boundary metric g as Weyl transformations

$$g_{ij} \to e^{2\sigma} g_{ij}$$

- they largely fix $g^{(1)}$, $g^{(2)}$, etc.

- bulk action $S[G] = \int L(G)$ is invariant up to a bundary term

$$\delta_\xi S[G] = \int_{\mathcal{M}_5} \xi^\mu \partial_\mu L(G) = \int_{\mathcal{M}_4} \sqrt{\det g}\, \sigma\, \mathcal{A}(g)$$

In fact, for the a-anomaly the simplest action with $L = \sqrt{G}$ suffices (in any even dimension)

[1] Penrose-Brown-Henneaux

Broken Phase

Differences compared to the chiral symmetry case

- Conformal symmetry breaking

$$\text{conformal group} \simeq SO(4,2) \xrightarrow{\langle\mathcal{O}\rangle\neq 0} \text{Poincaré group}$$

5 broken generators but only one Goldstone boson — dilaton

- breaking of dilations and special conformal transfs. if $T_i^i \neq 0$
- gauging of $SO(4,2)$ currents is replaced by diffeo and Weyl invariance
- Goldstone's theorem $\#$ Goldstone bosons $=$ dim(G/H) does not hold for space-time symmetries
- more concretely, for space-time symmetries

$$\delta\langle\mathcal{O}\rangle = \sum_{\substack{\text{broken} \\ \text{generators}}} c_\alpha(x) Q_\alpha \langle\mathcal{O}\rangle = 0 \qquad c_\alpha(x): \quad \text{Goldstone fields}$$

may have nontrivial solutions.

- locally a special conformal transformation cannot be distinguished from a dilation ... both rescale the metric
- four of the five Goldstone bosons can be gauged away (inverse Higgs effect)

- The WZW action for the dilaton is local in $d = 2n$ and does not seem to have any higher dimensional origin. In fact by integrating the anomaly one finds

$$S_{\rm WZW} = \int d^4x \sqrt{g}\mathcal{L}(g,\tau)$$

which satisfies

$$\delta_\sigma S_{\rm WZW} \underset{\substack{\delta g_{ij}=2\sigma g_{ij} \\ \delta\tau=\sigma}}{\overset{\uparrow}{=}} \int d^4x \sqrt{g}\,\sigma\mathcal{A}(g)$$

Aim of the talk is to uncover its higher dimensional origin

But first ...

the invariant term

- Start with coset

$$SO(d+1,1)/[SO(d)\times\mathbb{R}_d]\simeq AdS_{d+1}$$

- coset of GBs can be parametrized by $d+1$ fields $X^\mu(x)$.
- want to construct analogue of a 'gauged sigma model' $\Rightarrow$ allow for more general metric $G_{\mu\nu}(X^\lambda)$ on which Weyl transformations act.

  We impose the following requirements:

  - $G_{\mu\nu}=F_{\mu\nu}[g]$ .... functional of d-dim metric
  - should admit action of a group isomorphic to Weyl transformations s.t.

$$\delta_\sigma G=F[e^{2\sigma}g_{ij}]-F[g_{ij}]$$

  - $g_{ij}=\delta_{ij}\quad\Rightarrow\quad G=\text{AdS metric}$

- the FG metrics from above satisfy these requirements.
- the transformations δ_σ are the residual diffeos (PBH)

Insisting on reparametrization invariance we propose the following (minimal) sigma-model action

$$S = \int d^d x \sqrt{\det h_{ij}} \qquad h_{ij}(x) = G_{\mu\nu}(X)\,\partial_i X^\mu(x)\,\partial_j X^\nu(x)$$

Reduction to dilaton action is achieved by the gauge choice

$$X^\mu(x) = \big(\Phi(x), X^i(x)\big) \quad \text{with} \quad X^i(x) = x^i$$

(four out of five GB's are gauged away (inverse Higgs effect)) for which the action becomes

$$S_{\text{g.f.}} = \int d^d x \frac{\sqrt{\det g_{ij}(x, \Phi(x))}}{\Phi^{d/2}(x)} \sqrt{1 + \frac{g^{ij}(x, \Phi(x))\partial_i \Phi(x)\, \partial_j \Phi(x)}{4\, \Phi(x)}}$$

For $g_{ij} = \delta_{ij}$ and $d = 4$ this is the action for the displacement of a D3 brane which breaks gauge and conformal symmetry on the N=4 SYM Coulomb branch.

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How does Φ transform under Weyl transformations and what is its relation to then dilaton?

Action S is invariant under

- PBH transformations: relate two different backgrund metrics $G_{\mu\nu}$; make explicit the variation under change of boundary metric g_{ij}:

$$\Phi \to \big(1+2\,\sigma(X)\big)\Phi\,, \quad X^i \to X^i - a^i(X^j,\Phi) \qquad \big(X^i = X^i(x),\, \Phi = \Phi(x)\big)$$

with
$$a^i(X,\Phi(x)) = \frac{1}{2}\int_0^{\Phi(x)} d\rho'\, g^{ij}(X,\rho')\,\partial_i\sigma(X)$$

- reparametrizations of the x^i

$$x^i \to x^i - \xi^i(x)$$

The invariance of the gauge fixed action $S_{\rm g.f}[g,\Phi]$ is that combination which leaves $X^i(x) = x^i$ invariant. It translates to

- $\delta g_{ij}(x) = 2\sigma(x)\, g_{ij}(x)$
- $\delta\Phi(x) = 2\sigma(x)\,\Phi(x) + a^i(x,\Phi(x))\,\partial_i\Phi(x) \qquad a^i$ as before

Define dilaton τ such that under Weyl transformations

$$\tau \to \tau + \sigma$$

Relation between Φ and τ can be found from the known FG expansions of $g(x, \rho)$ and $a^i(x, \rho)$:

$$\Phi = e^{2\tau} + \frac{1}{2} e^{4\tau} (\nabla\tau)^2 + e^{6\tau} \left(\ldots (\nabla\tau)^4 \ldots R(\nabla\tau)^2 \ldots \right) + \ldots$$

Inserting into $S_{\mathrm{g.f}}$ gives the invariant action (for d=4)

$$S_{\mathrm{inv}} = \int d^4x \sqrt{\hat{g}} \left(1 + \frac{1}{12} \hat{R} + \frac{1}{64} (\hat{E}_4 - \gamma\, \hat{C}) + \ldots \right)$$

where $\hat{g}$ is the Weyl invariant combination

$$\hat{g}_{ij} = e^{-2\tau} g_{ij}$$

From the symmetries of $S_{\mathrm{g.f}}$ is was a priori clear that the action had to be a functional of $\hat{g}$.

Aside:

We can easily specialize this discussion to a flat boundary metric (i.e. AdS bulk metric). In this case the above analysis shows that the D-brane probe action in AdS has the symmetry

$$
\begin{aligned}
x'^i &= x^i - \frac{1}{2}\epsilon^i\, x^2 + x^i\,(\epsilon \cdot x) - \frac{1}{2}\epsilon^i\,\Phi(x) \\
\Phi'(x') &= \Phi(x) + 2\,\epsilon \cdot x\,\Phi(x)
\end{aligned}
\tag{1}
$$

(cf. Maldacena 1997)

The WZW term

Requirements

- should reproduce the anomaly
- should be an action on a $d+1$ dim. manifold with boundary
- as the invariant term discussed above, should reduce to a functional of the dilaton and the boundary metric

Define

$$f_{\alpha\beta} = G_{\mu\nu}(X)\partial_\alpha X^\mu \, \partial_\beta X^\nu \qquad (\alpha,\beta,\mu,\nu=1,\ldots d+1)$$

where

- $G_{\mu\nu}$ bulk metric in FG gauge
- $X^\mu = X^\mu(x^i, \rho)$
- boundary of manifold at $\rho = 0$

Then

$$S_{\mathrm{WZW}} = \int d^d x \, d\rho \sqrt{\det f_{\alpha\beta}}$$

Comment

Since the embedding has the same dimension as the space, all relevant information is in the boundary conditions of the embedding fields X^μ

We split them into

$$X^\mu(x,\rho) = (X^i(x,\rho), \Phi(x,\rho))$$

Symmetries of $S_{\rm WZW}$:

- Field transformations relating backgrounds defined by different g_{ij}. They have the same form as before (PBH) as they are at fixed x.
- reparametrizations in $d+1$ dimensions

As before, choose special set of coordinates

$$X^i(x,\rho) = x^i$$

to obtain

$$S_{\rm WZW} = \frac{1}{2}\int d^dx\, d\rho\, \partial_\rho \Phi \frac{\sqrt{\det g_{ij}(x,\Phi(x,\rho))}}{\Phi(x,\rho)^{1+d/2}}$$

Note that this action

$$S_{\mathrm{WZW}} = \frac{1}{2} \int d^d x \, d\rho \, \partial_\rho \Phi \frac{\sqrt{\det g_{ij}(x, \Phi(x, \rho))}}{\Phi(x, \rho)^{1+d/2}}$$

through a change of variables $\rho \to \Phi(x, \rho)$ for fixed x, depends on $\Phi(x, \rho)$ only though its boundary value $\Phi(x, \rho = 0)$.

Symmetry of this gauge fixed action (cf. above)

$$\delta g_{ij}(x) = 2\,\sigma(x)\,g_{ij}(x)\,, \qquad \delta\Phi(x,\rho) = 2\sigma\,\Phi(x,\rho) + a^i(x,\Phi(x,\rho))\,\partial_i\Phi(x,\rho)$$

at boundary $\rho = 0$, $\Phi(x, \rho = 0)$ has same transformation property as $\Phi(x)$ of the discussion of $S_{\rm inv}$.

$\Rightarrow$ same field redefinition relating $\Phi(x,0)$ to dilaton $\tau(x)$ as before

From that transformation follows that in flat space $\tau = 0 \triangleq \Phi(x,0) = 1$.

Using the change of variables $\rho \to \Phi(x,\rho)$ we finally define the WZW action as

$$S_{\rm WZW} = \frac{1}{2}\int_1^{\Phi(x,0)} d\Phi\, d^d x \frac{\sqrt{\det g_{ij}(x,\Phi)}}{\Phi^{1+d/2}}$$

Final point: to show that S_{WZW} reproduces the trace anomalies and the anomalous dilaton action

- Under simultaneous variation of Φ and g_{ij} the action is invariant if we also transformed the lower limit $\delta(\Phi = 1) = 2\sigma(x)$. However we keep the lower limit fixed and thus

  $$\delta S_{\mathrm{WZW}} = -\frac{1}{2}\int d^d x\, 2\sigma(x)\sqrt{\det g_{ij}(x, \Phi = 1)}$$

  a functional purely of the boundary metric g_{ij}.

  The anomaly in $d = 2n$ is the term with $2n$ derivatives $\partial^{2n} g$.

- Using the relation between $\Phi(x,0)$ and τ in $S_{\rm WSW}$ one finds the dilaton action (in d=4)

$$S_{\rm WZW} = \int d^4x \Big\{ -\frac{1}{4}\left(\sqrt{\hat{g}} - \sqrt{g}\right) + \frac{1}{24}\left(\sqrt{\hat{g}}\,\hat{R} - \sqrt{g}\,R\right)$$

$$+\frac{1}{64}\Big(\tau\, E_4 + 4\big(R^{ij} - \frac{1}{2}R\, g^{ij}\big)\nabla_i\tau\,\nabla_j\tau - 4(\nabla\tau)^2\Box\tau + 2(\nabla\tau)^4\Big)$$

$$+\gamma\,\tau\, C^2\Big\}$$

Final comment:

- the normalization of $S_{\rm WZW}$ is fixed by the a-anomaly coefficient
- Both $S_{\rm inv}$ and $S_{\rm WZW}$ contain the dilaton potential $\int d^4x \sqrt{\det \hat{g}}$

These two contributions must cancel.

This fixes the relative coefficients and in particular fixes the normalization of $S_{\rm inv}$.

One finds in this way the same relative coefficient as for the D3-brane probe on AdS_5 where it is required by SUSY (no force condition) while SUSY played no role in our analysis.

Summary and comments

- In the construction of the sigma-model actions we used their diffeo invariance. This allowed to gauge away some of the 'would be Goldstone bosons', leaving only the dilaton.

  This might be a general feature whenever a space-time symmetry is spontaneously broken.

- We extensively used 'gauging'(= coupling to a general 'boundary' metric) of the sigma-model to study its symmetries. This gauging was a natural deformation of the AdS_{d+1} metric on the Goldstone boson space. This lead to a natural appearance of the holographic set-up.

- This gauging might not be the most natural one. In analogy to the chiral case a coupling of the sigma-model to $SO(d+1,1)$ gauge fields might be more natural. But this requires an understanding of the relation (if any) between the Weyl anomaly and the descent equations of the conformal group.