Towards new non-geometric backgrounds

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Ringberg — 30.07.2014

this talk is based on ...

This talk is based on T-duality revisited [arXiv:1310.4194],
and on some work in progress [arXiv:1407.xxxx].

Non-geometric backgrounds

Non-geometric backgrounds :: why?

Non-geometric backgrounds :: why?

- Have non-commutative or non-associative features.
- Are part of the string-theory landscape.
- Provide uplifts for gauged supergravities.
- Can help with moduli stabilization & cosmology.
- …

What is a non-geometric background?

What is a non-geometric background? … apply T-duality to a three-torus:

$$H_{abc} \overset{T_c}{\longleftrightarrow} f_{ab}{}^{c} \overset{T_b}{\longleftrightarrow} Q_a{}^{bc} \overset{T_a}{\longleftrightarrow} R^{abc}$$

H_{abc}	$f_{ab}{}^{c}$	$Q_a{}^{bc}$	R^{abc}
↓	↓	↓	↓
flux background	"twisted torus"	T-fold	non-associative

Shelton, Taylor, Wecht - 2005

introduction :: main example

$$H_{abc} \xleftrightarrow{T_c} f_{ab}{}^{c} \xleftrightarrow{T_b} Q_a{}^{bc} \xleftrightarrow{T_a} R^{abc}$$

$$\boxed{H_{abc}} \xleftrightarrow{\;T_c\;} f_{ab}{}^{c} \xleftrightarrow{\;T_b\;} Q_a{}^{bc} \xleftrightarrow{\;T_a\;} R^{abc}$$

Consider string theory compactified on a three-torus with H-flux:

- The geometry is determined by

$$ds^2 = dx^2 + dy^2 + dz^2\,,$$

$$B_{yz} = N\,x\,,$$

$$x \sim x+1\,, \qquad y \sim y+1\,, \qquad z \sim z+1\,.$$

- The H-flux reads

$$H_{xyz} = N\,.$$

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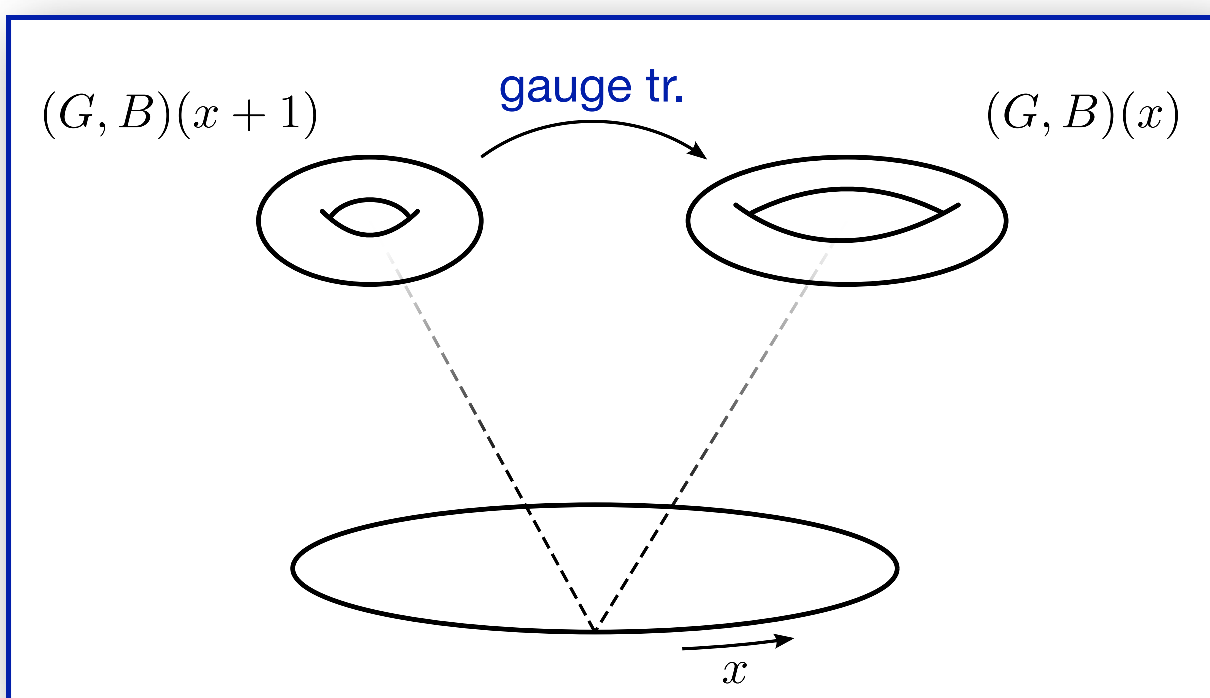

$= N\,.$

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After a T-duality in the *z*-direction, one arrives at a twisted torus:

- The geometry is determined by

$$ds^2 = dx^2 + dy^2 + (dz + Nxdy)^2\,,$$

$$B = 0\,,$$

$$(x, z) \sim (x + 1, z - N\,y)\,, \qquad y \sim y + 1\,, \qquad z \sim z + 1\,.$$

- The geometric flux reads

$$e^x = dx\,, \qquad e^y = dy\,, \qquad e^z = dz + N\,x\,dy\,,$$

$$\omega^z{}_{xy} = N/2\,,$$

$$[e_x, e_y] = -N\,e_z\,.$$

Scherk, Schwarz - 1979
Dasgupta, Rajesh, Sethi - 1999
Kachru, Schulz, Tripathy, Trivedi - 2002

introduction :: f-flux background

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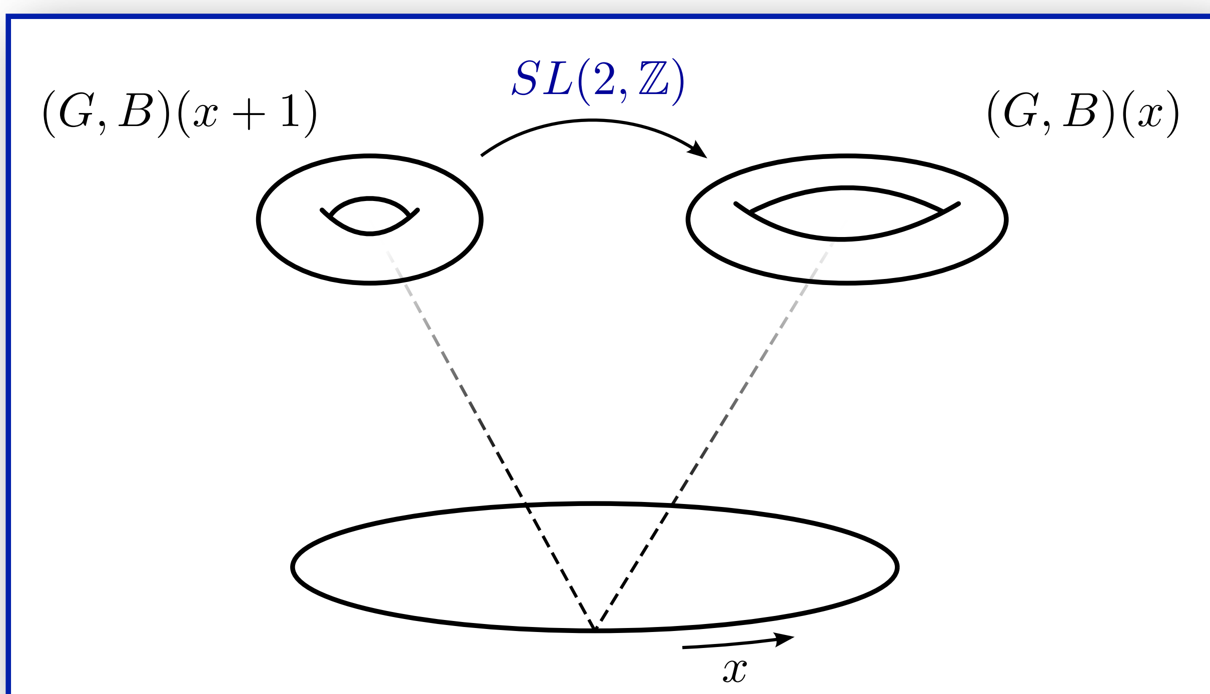

$dx \,, \qquad e^y = dy \,, \qquad e^z = dz + N\, x\, dy \,,$

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After a second T-duality in the *x*-direction, one arrives at a T-fold:

- The geometry is determined by

$$ds^2 = dy^2 + \frac{1}{1+(N\,y)^2}\left(dx^2 + dz^2\right),$$

$$B_{xz} = \frac{N\,y}{1+(N\,y)^2}\,,$$

$$x \sim x+1\,, \qquad z \sim z+1\,.$$

- The non-geometric flux reads

$$Q_y{}^{xz} = -N\,.$$

This space is locally geometry, but globally non-geometric.

Hellermann, McGreevy, Williams - 2002
Dabholkar, Hull - 2002
Hull - 2004

introduction :: q-flux background

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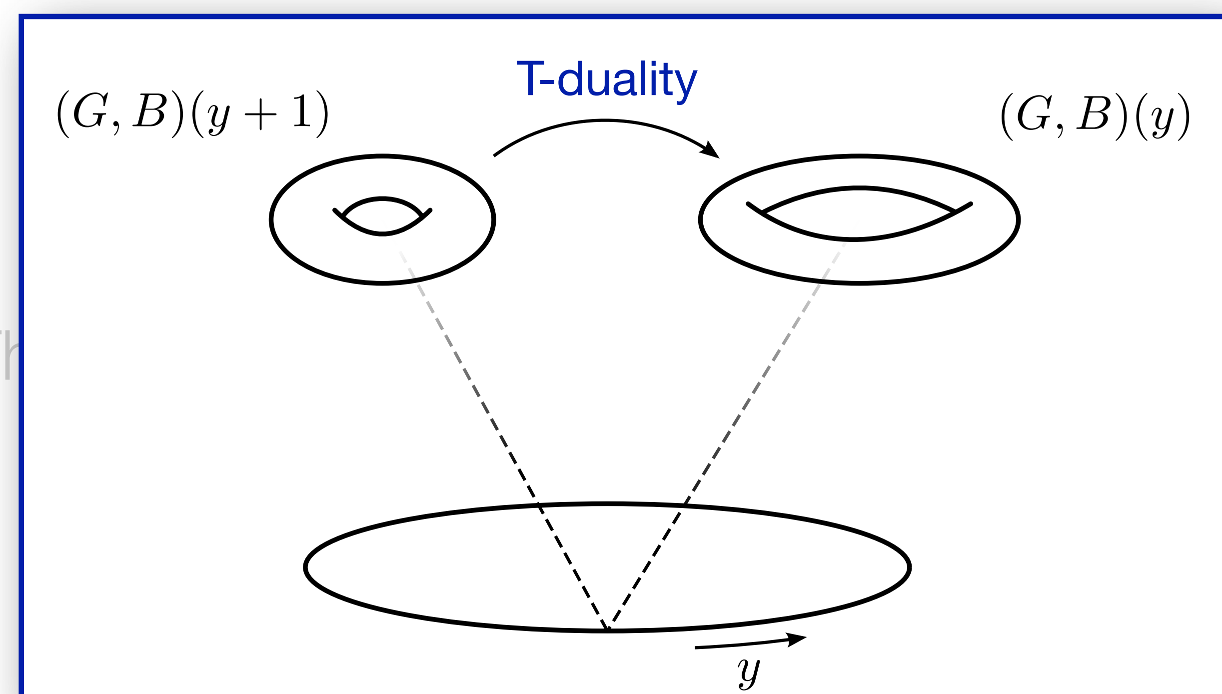

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After formally applying a third T-duality, one obtains an *R*-flux background:

- The geometry is not even locally defined.

- The non-geometric *R*-flux is obtained by raising the index of the *Q*-flux

$$Q_y{}^{xz} \longrightarrow R^{xyz} = N\,.$$

- This background gives rise to a non-associative structure.

Bouwknegt, Hannabuss, Mathai - 2004
Shelton, Taylor, Wecht - 2005
Ellwood, Hashimoto - 2006
Blumenhagen, EP & Lüst - 2010

But :: …

But :: what about other examples?

- The torus is the mainly (and only) studied background.
- Other — and better — examples are needed!

→ Consider the three-sphere.

Goal ::

- Construct new non-geometric backgrounds.

Plan ::

- Revisit (collective) T-duality.
- Review the three-torus.
- Consider the three-sphere.

outline

1. introduction
2. collective t-duality
3. three-torus
4. three-sphere
5. discussion

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To study T-duality for the three-sphere, a non-abelian version might be needed.

t-duality :: sigma-model action

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de la Ossa, Quevedo - 1992
Giveon, Rocek - 1993
Sfetsos - 1994
Alvarez, Alvarez-Gaume, [Barbon,] Lozano - 1993 & 1994

Consider the sigma-model action for the NS-NS sector of the closed string

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int_{\partial\Sigma}\Big[G_{ij}\,dX^i\wedge\star dX^j + \alpha' R\,\phi\star 1\Big] - \frac{i}{2\pi\alpha'}\int_{\Sigma}\tfrac{1}{3!}\,H_{ijk}\,dX^i\wedge dX^j\wedge dX^k\,.$$

This action is invariant under global transformations $\delta_\epsilon X^i = \epsilon^\alpha\, k^i_\alpha(X)$ if

$$\mathcal{L}_{k_\alpha}G = 0\,, \qquad \iota_{k_\alpha}H = dv_\alpha\,, \qquad \mathcal{L}_{k_\alpha}\phi = 0\,.$$

In general, the isometry algebra is non-abelian $[k_\alpha, k_\beta]_{\rm L} = f_{\alpha\beta}{}^{\gamma}\,k_\gamma\,.$

Hull, Spence - 1989 & 1991

t-duality :: gauged action

Following Buscher's procedure, the gauged sigma-model action is found as

$$\begin{aligned}\widehat{\mathcal{S}} = &-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} \tfrac{1}{2}\,G_{ij}(dX^i + k^i_\alpha A^\alpha)\wedge\star(dX^j + k^j_\beta A^\beta) \\ &-\frac{i}{2\pi\alpha'}\int_{\Sigma} \tfrac{1}{3!}\,H_{ijk}\,dX^i\wedge dX^j\wedge dX^k \\ &-\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\Big[\,(v_\alpha + d\chi_\alpha)\wedge A^\alpha + \tfrac{1}{2}\left(\iota_{k_{[\underline{\alpha}}}v_{\underline{\beta}]} + f_{\alpha\beta}{}^\gamma\chi_\gamma\right)A^\alpha\wedge A^\beta\,\Big]\,.\end{aligned}$$

Hull, Spence - 1989 & 1991
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$$-\frac{i}{2\pi\alpha'}\int_{\Sigma} \tfrac{1}{3!}H_{ijk}\,dX^i\wedge dX^j\wedge dX^k$$
$$-\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\Big[\,(v_\alpha + d\chi_\alpha)\wedge A^\alpha + \tfrac{1}{2}\big(\iota_{k_{[\underline{\alpha}}}v_{\underline{\beta}]} + f_{\alpha\beta}{}^\gamma\chi_\gamma\big)A^\alpha\wedge A^\beta\,\Big]\,.$$

Hull, Spence - 1989 & 1991
Alvarez, Alvarez-Gaume, Barbon, Lozano - 1994

The local symmetry transformations take the form

$$\hat\delta_\epsilon X^i = \epsilon^\alpha k^i_\alpha\,, \qquad \hat\delta_\epsilon A^\alpha = -d\epsilon^\alpha - \epsilon^\beta A^\gamma f_{\beta\gamma}{}^\alpha\,,$$
$$\hat\delta_\epsilon\chi_\alpha = -\iota_{k_{(\overline{\alpha}}}v_{\overline{\beta})}\epsilon^\beta - f_{\alpha\beta}{}^\gamma\epsilon^\beta\chi_\gamma\,.$$

EP (with F. Rennecke) - 2014

Following Buscher's procedure, the gauged sigma-model action is found as

$$\begin{aligned}\widehat{\mathcal{S}} = &- \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} \tfrac{1}{2}\, G_{ij}(dX^i + k^i_\alpha A^\alpha)\wedge\star(dX^j + k^j_\beta A^\beta)\\ &- \frac{i}{2\pi\alpha'}\int_{\Sigma} \tfrac{1}{3!}\, H_{ijk}\, dX^i\wedge dX^j\wedge dX^k\\ &- \frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\Big[\,(v_\alpha + d\chi_\alpha)\wedge A^\alpha + \tfrac{1}{2}\big(\iota_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} + f_{\alpha\beta}{}^{\gamma}\chi_\gamma\big)\, A^\alpha\wedge A^\beta\,\Big]\,.\end{aligned}$$

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Hull, Spence - 1989 & 1991
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This gauging is subject to the following constraints

$$\mathcal{L}_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} = f_{\alpha\beta}{}^\gamma v_\gamma\,, \qquad\qquad \iota_{k_{[\underline{\alpha}}} f_{\underline{\beta\gamma}]}{}^\delta v_\delta = \frac{1}{3}\,\iota_{k_\alpha}\iota_{k_\beta}\iota_{k_\gamma}H\,.$$

EP (with F. Rennecke) - 2014

t-duality :: recovering the original model

The original model is recovered via the equations of motion for χ_α

$$0 = dA^\alpha - \tfrac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma \,.$$

The gauge action can then be rewritten in terms of $DX^i = dX^i + k^i_\alpha A^\alpha$ as

$$\begin{aligned}\widehat{\mathcal{S}} = &- \frac{1}{4\pi\alpha'} \int_{\partial\Sigma} \Big[G_{ij}\, DX^i \wedge \star DX^j + \alpha' R\, \phi \star 1 \Big] \\ &- \frac{i}{2\pi\alpha'} \int_\Sigma \tfrac{1}{3!}\, H_{ijk}\, DX^i \wedge DX^j \wedge DX^k \,.\end{aligned}$$

Ignoring technical details, one replaces $DX^i \to dY^i$ and obtains the ungauged action.

t-duality :: obtaining the dual model I

The dual model is obtained via the equations of motion for A^α

$$A^\alpha = -\Big(\big[\mathcal{G} - \mathcal{D}\,\mathcal{G}^{-1}\mathcal{D}\big]^{-1}\Big)^{\alpha\beta}\Big(\mathbb{1} + i\star\mathcal{D}\,\mathcal{G}^{-1}\Big)_\beta{}^\gamma\big(k + i\star\xi\big)_\gamma\,,$$

where

$$\mathcal{G}_{\alpha\beta} = k^i_\alpha G_{ij} k^j_\beta\,, \qquad \xi_\alpha = d\chi_\alpha + v_\alpha\,,$$

$$\mathcal{D}_{\alpha\beta} = \iota_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} + f_{\alpha\beta}{}^\gamma\chi_\gamma\,, \qquad k_\alpha = k^i_\alpha G_{ij} dX^j\,.$$

The action of the dual sigma-model is found by integrating-out A^α and reads

$$\check{\mathcal{S}} = -\frac{1}{4\pi\alpha'}\int_{\partial\Sigma}\Big[\check{G} + \alpha' R\,\phi\star 1\Big] - \frac{i}{2\pi\alpha'}\int_\Sigma \check{H}\,,$$

where, with $\mathcal{M} = \mathcal{G} - \mathcal{D}\mathcal{G}^{-1}\mathcal{D}$ invertible,

$$\check{G} = G + \begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} -\mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} \\ +\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} & +\mathcal{M}^{-1} \end{pmatrix} \wedge \star \begin{pmatrix} k \\ \xi \end{pmatrix},$$

$$\check{H} = H + \tfrac{1}{2}\,d\left[\begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} +\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} & +\mathcal{M}^{-1} \\ -\mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} \end{pmatrix} \wedge \begin{pmatrix} k \\ \xi \end{pmatrix}\right].$$

t-duality :: obtaining the dual model II

Consider an enlarged target-space parametrized by coordinates X^i and χ_α .

The enlarged metric $\check{G}$ and field strength $\check{H}$ have null-eigenvectors (and isometries)

$$\iota_{\check{n}_\alpha}\check{G} = 0\,, \qquad \iota_{\check{n}_\alpha}\check{H} = 0\,, \qquad \check{n}_\alpha = k_\alpha + \mathcal{D}_{\alpha\beta}\,\partial_{\xi_\beta}\,.$$

The dual metric and field strength are obtained via a change of coordinates

$$\mathcal{T}^I{}_A = \begin{pmatrix} k & 0 \\ \mathcal{D} & \mathbb{1} \end{pmatrix}, \qquad \check{\mathsf{G}}_{AB} = (\mathcal{T}^T \check{G}\,\mathcal{T})_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & \mathsf{G}_{\alpha\beta} \end{pmatrix},$$

$$\check{\mathsf{H}}_{ABC} = \check{H}_{IJK}\,\mathcal{T}^I{}_A\,\mathcal{T}^J{}_B\,\mathcal{T}^K{}_C\,,$$

$$\check{\mathsf{H}}_{iBC} = 0\,.$$

The T-duality transformation rules are obtained via Buscher's procedure of

1. gauging isometries in the sigma-model action,
2. integrating-out the gauge field,
3. performing a change of coordinates.

The possible gaugings are restricted by (recall that $\iota_{k_\alpha} H = dv_\alpha$)

$$\mathcal{L}_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} = f_{\alpha\beta}{}^{\gamma} v_\gamma \,, \qquad\qquad \iota_{k_{[\underline{\alpha}}} f_{\underline{\beta\gamma}]}{}^{\delta} v_\delta = \frac{1}{3}\, \iota_{k_\alpha} \iota_{k_\beta} \iota_{k_\gamma} H \,.$$

The change of coordinates is performed using null-eigenvectors $\check{n}_\alpha$

$$\check{G}_{IJ}\, \check{n}_\alpha^J = 0 \,, \qquad\qquad \check{H}_{IJK}\, \check{n}_\alpha^K = 0 \,.$$

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Consider a three-torus with *H*-flux specified as follows

$$ds^2 = R_1^2\left(dX^1\right)^2 + R_2^2\left(dX^2\right)^2 + R_3^2\left(dX^3\right)^2\,, \qquad X^i \simeq X^i + \ell_{\mathrm{s}}\,,$$

$$H = h\, dX^1 \wedge dX^2 \wedge dX^3\,, \qquad h \in \ell_{\mathrm{s}}^{-1}\mathbb{Z}\,.$$

The Killing vectors (in the basis $\{\partial_1, \partial_2, \partial_3\}$) are abelian and can be chosen as

$$k_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad k_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The one-forms v_α (defined via $\iota_{k_\alpha} H = dv_\alpha$), up to exact terms take the form

$$\begin{aligned} v_1 &= h\,\alpha_1\, X^2 dX^3 - h\,\alpha_2\, X^3 dX^2\,, && \alpha_1 + \alpha_2 = 1\,, \\ v_2 &= h\,\beta_1\, X^3 dX^1 - h\,\beta_2\, X^1 dX^3\,, && \beta_1 + \beta_2 = 1\,, \\ v_3 &= h\,\gamma_1\, X^1 dX^2 - h\,\gamma_2\, X^2 dX^1\,, && \gamma_1 + \gamma_2 = 1\,. \end{aligned}$$

Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

$$v = h\,\alpha\,X^2\,dX^3 - h\,(1-\alpha)X^3\,dX^2\,, \qquad\qquad \alpha \in \mathbb{R}\,.$$

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

$$\begin{aligned} \mathcal{G} &= R_1^2\,, & \xi &= d\chi + v\,, \\ \mathcal{D} &= 0\,, & k &= R_1^2\,dX^1\,, \end{aligned} \qquad \longrightarrow \qquad \mathcal{M} = \mathcal{G} = R_1^2\,.$$

The metric and field strength are then computed as …

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The const

The geom

The metri

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$$\check{G} = G + \begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} -\mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} \\ +\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} & +\mathcal{M}^{-1} \end{pmatrix} \wedge \star \begin{pmatrix} k \\ \xi \end{pmatrix}$$

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$$= G + \begin{pmatrix} R_1^2\, dX^1 \\ \xi \end{pmatrix}^T \begin{pmatrix} -\frac{1}{R_1^2} & 0 \\ 0 & +\frac{1}{R_1^2} \end{pmatrix} \wedge \star \begin{pmatrix} R_1^2\, dX^1 \\ \xi \end{pmatrix}$$

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$$= G - R_1^2\, dX^1 \wedge \star dX^1 + \frac{1}{R_1^2}\, \xi \wedge \star \xi$$

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The geom

The metric

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$$= G - R_1^2\, dX^1 \wedge \star dX^1 + \frac{1}{R_1^2}\, \xi \wedge \star \xi$$

$$= \frac{1}{R_1^2}\, \xi \wedge \star \xi + R_2^2\, dX^2 \wedge \star dX^2 + R_3^2\, dX^3 \wedge \star dX^3$$

$\xi = d\chi + v$

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$$\check{H} = H + \frac{1}{2}\, d\left[\begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} +\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} & +\mathcal{M}^{-1} \\ -\mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} \end{pmatrix} \wedge \begin{pmatrix} k \\ \xi \end{pmatrix}\right]$$

The geom

The metri

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$$= H + \tfrac{1}{2}\, d\left[\begin{pmatrix} R_1^2\, dX^1 \\ \xi \end{pmatrix}^T \begin{pmatrix} 0 & +\frac{1}{R_1^2} \\ -\frac{1}{R_1^2} & 0 \end{pmatrix} \wedge \begin{pmatrix} R_1^2\, dX^1 \\ \xi \end{pmatrix} \right]$$

torus :: one t-duality I

Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

The const

The geom

The metric

$$\check{H} = H + \tfrac{1}{2}\, d\left[\begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} +\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} & +\mathcal{M}^{-1} \\ -\mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} \end{pmatrix} \wedge \begin{pmatrix} k \\ \xi \end{pmatrix}\right]$$

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$$= H + \tfrac{1}{2}\, d\left[\begin{pmatrix} R_1^2\, dX^1 \\ \xi \end{pmatrix}^T \begin{pmatrix} 0 & +\frac{1}{R_1^2} \\ -\frac{1}{R_1^2} & 0 \end{pmatrix} \wedge \begin{pmatrix} R_1^2\, dX^1 \\ \xi \end{pmatrix}\right]$$

$$= H + d\left[dX^1 \wedge \xi\right]$$

$$= 0$$

$$d\xi = d(d\chi + v) = h\, dX^2 \wedge dX^3$$

Consider one T-duality along $k_1 = \partial_1$. The corresponding one-form reads

$$v = h\alpha X^2 dX^3 - h(1-\alpha)X^3 dX^2 \,, \qquad\qquad \alpha \in \mathbb{R}\,.$$

The constraints for gauging the sigma-model are trivially satisfied.

The geometry of the dual background is determined from the quantities

$$\begin{aligned} \mathcal{G} &= R_1^2 \,, & \xi &= d\chi + v \,, \\ \mathcal{D} &= 0 \,, & k &= R_1^2 \, dX^1 \,, \end{aligned} \qquad \longrightarrow \qquad \mathcal{M} = \mathcal{G} = R_1^2 \,.$$

The metric and field strength are then computed as …

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The metric and field strength are then computed as

$$\check{G} = \frac{1}{R_1^2}\,\xi\wedge\star\xi + R_2^2\,dX^2\wedge\star dX^2 + R_3^2\,dX^3\wedge\star dX^3\,,$$

$$\check{H} = 0\,.$$

As expected, the dual background is a twisted torus (with $\alpha = 1$)

$$\check{ds}^2 = \frac{1}{R_1^2}\left(d\chi + h\, X^2\, dX^3\right)^2 + R_2^2\left(dX^2\right)^2 + R_3^2\left(dX^3\right)^2 ,$$

$$\check{H} = 0\,.$$

Consider two collective T-dualities along $k_1 = \partial_1$ and $k_2 = \partial_2$.

The constraints on gauging the sigma-model imply (for $\alpha \in \mathbb{R}$)

$$
\begin{aligned}
v_1 &= h\,\alpha\,X^2\,dX^3 - h\,(1-\alpha)\,X^3\,dX^2\,, \\
v_2 &= h\,(1+\alpha)\,X^3\,dX^1 + h\,\alpha\,X^1\,dX^3\,.
\end{aligned}
$$

The geometry of the dual background is determined from ($\alpha, \beta \in \{1,2\}$)

$$
\mathcal{G}_{\alpha\beta} = \begin{pmatrix} R_1^2 & 0 \\ 0 & R_2^2 \end{pmatrix}, \qquad \xi_\alpha = \begin{pmatrix} d\chi_1 + v_1 \\ d\chi_2 + v_2 \end{pmatrix},
$$

$$
\mathcal{D}_{\alpha\beta} = \begin{pmatrix} 0 & +h\,X^3 \\ -h\,X^3 & 0 \end{pmatrix}, \qquad k_\alpha = \begin{pmatrix} R_1^2\,dX^1 \\ R_2^2\,dX^2 \end{pmatrix}.
$$

torus :: two t-dualities II

The metric of the enlarged target space (in the basis $\{dX^1, dX^2, dX^3, \xi_1, \xi_2\}$) reads

$$\check{G}_{IJ} = \frac{1}{\rho}\left(\begin{array}{ccc|cc} \left[R_1 h X^3\right]^2 & 0 & 0 & 0 & -R_1^2 h X^3 \\ 0 & \left[R_2 h X^3\right]^2 & 0 & +R_2^2 h X^3 & 0 \\ 0 & 0 & \rho R_3^2 & 0 & 0 \\ \hline 0 & +R_2^2 h X^3 & 0 & R_2^2 & 0 \\ -R_1^2 h X^3 & 0 & 0 & 0 & R_1^2 \end{array}\right),$$

$$\rho = R_1^2 R_2^2 + \left[h X^3\right]^2 .$$

Performing then a change of basis one finds

$$\mathcal{T}^I{}_A = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & -h X^3 & 0 & 1 & 0 \\ +h X^3 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \check{\mathsf{G}}_{AB} = (\mathcal{T}^T \check{G}\, \mathcal{T})_{AB} = \frac{1}{\rho}\left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho R_3^2 & 0 & 0 \\ \hline 0 & 0 & 0 & R_2^2 & 0 \\ 0 & 0 & 0 & 0 & R_1^2 \end{array}\right).$$

torus :: two t-dualities III

Performing a similar analysis for the field strength and adjusting the notation, one finds

$$\check{\mathsf{ds}}^2 = \frac{1}{R_1^2 R_2^2 + \left[h\, X^3\right]^2} \left[R_1^2 \left(d\tilde{\chi}_1\right)^2 + R_2^2 \left(d\tilde{\chi}_2\right)^2\right] + R_3^2 \left(dX^3\right)^2 ,$$

$$\check{\mathsf{H}} = -h\, \frac{R_1^2 R_2^2 - \left[h\, X^3\right]^2}{\left[R_1^2 R_2^2 + \left[h\, X^3\right]^2\right]^2}\, d\tilde{\chi}_1 \wedge d\tilde{\chi}_2 \wedge dX^3 \,.$$

This is the familiar T-fold background.

Finally, consider three collective T-dualities along $k_1 = \partial_1$, $k_2 = \partial_2$ and $k_3 = \partial_3$.

The constraints on gauging the sigma-model require the *H*-flux to be vanishing

$$\iota_{k_\alpha}\iota_{k_\beta}\iota_{k_\gamma}H = 0 \qquad \longrightarrow \qquad H = 0\,.$$

The dual model is characterized by

$$\check{ds}^2 = \frac{1}{R_1^2}\left(d\chi_1\right)^2 + \frac{1}{R_2^2}\left(d\chi_2\right)^2 + \frac{1}{R_3^2}\left(d\chi_3\right)^2,$$

$$\check{H} = 0\,.$$

torus :: three t-dualities

Finally, consider three collective T-dualities along $k_1 = \partial_1$, $k_2 = \partial_2$ and $k_3 = \partial_3$.

The constraints on gauging the sigma-model require the *H*-flux to be vanishing

$$\iota_{k_\alpha}\iota_{k_\beta}\iota_{k_\gamma} H = 0 \qquad \longrightarrow \qquad H = 0\,.$$

$$\mathcal{L}_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} = f_{\alpha\beta}{}^{\gamma} v_\gamma$$

$$\iota_{k_{[\underline{\alpha}}} f_{\underline{\beta\gamma}]}{}^{\delta} v_\delta = \tfrac{1}{3}\, \iota_{k_\alpha}\iota_{k_\beta}\iota_{k_\gamma} H$$

The c

$$)^2 + \frac{1}{R_2^2}\,(d\chi_2)^2 + \frac{1}{R_3^2}\,(d\chi_3)^2\,,$$

$$\check{H} = 0\,.$$

Finally, consider three collective T-dualities along $k_1 = \partial_1$, $k_2 = \partial_2$ and $k_3 = \partial_3$.

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The dual model is characterized by

$$\check{ds}^2 = \frac{1}{R_1^2}\left(d\chi_1\right)^2 + \frac{1}{R_2^2}\left(d\chi_2\right)^2 + \frac{1}{R_3^2}\left(d\chi_3\right)^2\,,$$

$$\check{H} = 0\,.$$

torus :: summary

The formalism for T-duality introduced above works as expected.

three-torus with *H*-flux —(1 T-duality)→ twisted torus

—(2 T-dualities)→ T-fold

three-torus with *H=0* —(3 T-dualities)→ torus with $R \longrightarrow 1/R$

Bonus :: T-duality for the twisted torus with *H*-flux leads to twisted T-folds.

$$\check{\mathsf{G}} = \frac{1}{1+\left[\frac{R_1}{R_2}\, f\, X^3\right]^2}\left(R_1^2\, dX^1\wedge\star dX^1 + \frac{1}{R_2^2}\,\xi\wedge\star\xi\right) + R_3^2\, dX^3\wedge\star dX^3\,,$$

$$\check{\mathsf{H}} = -f\,\frac{R_1^2}{R_2^2}\,\frac{1-\left[\frac{R_1}{R_2}\, f\, X^3\right]^2}{\left(1+\left[\frac{R_1}{R_2}\, f\, X^3\right]^2\right)^2}\, dX^1\wedge\xi\wedge dX^3\,,$$

$$d\xi = -h\, dX^1\wedge dX^3\,.$$

This is a background with geometric and non-geometric flux.

outline

Consider a three-sphere with *H*-flux, specified by

$$ds^2 = R^2 \left[\sin^2\eta\,(d\zeta_1)^2 + \cos^2\eta\,(d\zeta_2)^2 + (d\eta)^2 \right], \qquad \zeta_{1,2} = 0 \ldots 2\pi\,,$$

$$H = \frac{h}{2\pi^2} \sin\eta \cos\eta\, d\zeta_1 \wedge d\zeta_2 \wedge d\eta\,, \qquad \eta = 0 \ldots \frac{\pi}{2}\,.$$

This model is conformal if $h = 4\pi^2 R^2$.

The isometry algebra is $\mathfrak{so}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2)$, and the Killing vectors satisfy (with $\alpha, \beta, \gamma \in \{1,2,3\}$)

$$[\mathsf{K}_\alpha, \mathsf{K}_\beta]_{\rm L} = \epsilon_{\alpha\beta}{}^{\gamma}\, \mathsf{K}_\gamma\,,$$

$$[\mathsf{K}_\alpha, \tilde{\mathsf{K}}_\beta]_{\rm L} = 0\,, \qquad |\mathsf{K}_\alpha|^2 = |\tilde{\mathsf{K}}_\alpha|^2 = \frac{R^2}{4}\,.$$

$$[\tilde{\mathsf{K}}_\alpha, \tilde{\mathsf{K}}_\beta]_{\rm L} = \epsilon_{\alpha\beta}{}^{\gamma}\, \tilde{\mathsf{K}}_\gamma\,,$$

sphere :: setting

Consider a three-sphere with H-flux, specified by

2π ,

$\frac{\pi}{2}$.

This mod

$$\mathsf{K}_1 = \frac{1}{2}\begin{pmatrix} +1 \\ -1 \\ 0 \end{pmatrix}, \qquad \tilde{\mathsf{K}}_1 = \frac{1}{2}\begin{pmatrix} +1 \\ +1 \\ 0 \end{pmatrix},$$

$$\mathsf{K}_2 = \frac{1}{2}\begin{pmatrix} -\sin(\zeta_1-\zeta_2)\cot\eta \\ -\sin(\zeta_1-\zeta_2)\tan\eta \\ \cos(\zeta_1-\zeta_2) \end{pmatrix}, \qquad \tilde{\mathsf{K}}_2 = \frac{1}{2}\begin{pmatrix} +\sin(\zeta_1+\zeta_2)\cot\eta \\ -\sin(\zeta_1+\zeta_2)\tan\eta \\ -\cos(\zeta_1+\zeta_2) \end{pmatrix},$$

$$\mathsf{K}_3 = \frac{1}{2}\begin{pmatrix} -\cos(\zeta_1-\zeta_2)\cot\eta \\ -\cos(\zeta_1-\zeta_2)\tan\eta \\ -\sin(\zeta_1-\zeta_2) \end{pmatrix}, \qquad \tilde{\mathsf{K}}_3 = \frac{1}{2}\begin{pmatrix} +\cos(\zeta_1+\zeta_2)\cot\eta \\ -\cos(\zeta_1+\zeta_2)\tan\eta \\ +\sin(\zeta_1+\zeta_2) \end{pmatrix}.$$

The isom

(with $\alpha,\beta,\gamma \in \{1,2,3\}$)

$$[\mathsf{K}_\alpha,\mathsf{K}_\beta]_{\mathrm{L}} = \epsilon_{\alpha\beta}{}^{\gamma}\,\mathsf{K}_\gamma\,,$$

$$[\mathsf{K}_\alpha,\tilde{\mathsf{K}}_\beta]_{\mathrm{L}} = 0\,,$$

$$[\tilde{\mathsf{K}}_\alpha,\tilde{\mathsf{K}}_\beta]_{\mathrm{L}} = \epsilon_{\alpha\beta}{}^{\gamma}\,\tilde{\mathsf{K}}_\gamma\,,$$

$$|\mathsf{K}_\alpha|^2 = |\tilde{\mathsf{K}}_\alpha|^2 = \frac{R^2}{4}\,.$$

Consider a three-sphere with *H*-flux, specified by

$$ds^2 = R^2 \left[\sin^2\eta\,(d\zeta_1)^2 + \cos^2\eta\,(d\zeta_2)^2 + (d\eta)^2 \right], \qquad \zeta_{1,2} = 0 \ldots 2\pi\,,$$

$$H = \frac{h}{2\pi^2} \sin\eta\cos\eta\, d\zeta_1 \wedge d\zeta_2 \wedge d\eta\,, \qquad \eta = 0 \ldots \frac{\pi}{2}\,.$$

This model is conformal if $h = 4\pi^2 R^2$.

The isometry algebra is $\mathfrak{so}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2)$, and the Killing vectors satisfy (with $\alpha, \beta, \gamma \in \{1,2,3\}$)

$$[\mathsf{K}_\alpha, \mathsf{K}_\beta]_{\rm L} = \epsilon_{\alpha\beta}{}^{\gamma}\,\mathsf{K}_\gamma\,,$$

$$[\mathsf{K}_\alpha, \tilde{\mathsf{K}}_\beta]_{\rm L} = 0\,, \qquad |\mathsf{K}_\alpha|^2 = |\tilde{\mathsf{K}}_\alpha|^2 = \frac{R^2}{4}\,.$$

$$[\tilde{\mathsf{K}}_\alpha, \tilde{\mathsf{K}}_\beta]_{\rm L} = \epsilon_{\alpha\beta}{}^{\gamma}\,\tilde{\mathsf{K}}_\gamma\,,$$

sphere :: one t-duality

Consider one T-duality along K_1. In this case, all constraints are satisfied:

- constraints from gauging the sigma-model ✓
- the matrix $\mathcal{G}_{\alpha\beta} = k^i_\alpha \, G_{ij} \, k^j_\beta$ is invertible ✓

The dual model is characterized by the metric and *H*-flux

$$\check{\mathsf{G}} = \frac{R^2}{4}\left[(d\tilde{\eta})^2 + \sin^2(\tilde{\eta})(d\tilde{\zeta})^2\right] + \frac{4}{R^2}\,\xi\wedge\star\xi\,,$$

$$\check{\mathsf{H}} = \sin\tilde{\eta}\; d\tilde{\zeta}\wedge d\tilde{\eta}\wedge\xi\,,$$

$$d\xi = -\frac{h}{16\pi^2}\,\sin\tilde{\eta}\,d\tilde{\eta}\wedge d\tilde{\zeta}\,.$$

This metric describes a circle fibered over a two-sphere.

Bouwknegt, Evslin, Mathai - 2003
incomplete in Alvarez, Alvarez-Gaume, Barbon, Lozano - 1993

For two collective T-dualities, consider the commuting Killing vectors K_1 and $\tilde{\mathsf{K}}_1$.

The constraints for this model are almost satisfied:

- constraints from gauging the sigma-model ✓
- the matrix $\mathcal{G}_{\alpha\beta} = k^i_\alpha\, G_{ij}\, k^j_\beta$ is invertible ✗ $\det \mathcal{G} = \frac{R^4}{16}\sin^2(2\eta)$

The dual model, via the above formalism, takes a form similar to the T-fold

$$\check{\mathsf{G}} = R^2\,(d\eta)^2 + \frac{1}{R^2}\,\frac{(d\tilde{\chi}_1)^2}{\sin^2\eta + \left[\frac{h}{4\pi^2 R^2}\right]^2\cos^2\eta} + \frac{1}{R^2}\,\frac{(d\tilde{\chi}_2)^2}{\cos^2\eta + \left(\frac{h}{4\pi^2 R^2}\right)^2\frac{\cos^4\eta}{\sin^2\eta}}\,,$$

$$\check{\mathsf{H}} = -8\,h\,\pi^2\big(h^2 - 16\pi^4 R^4\big)\,\frac{\sin\eta\cos\eta}{\left[16\pi^2 R^4\sin^2\eta + h^2\cos^2\eta\right]^2}\,d\eta\wedge d\tilde{\chi}_1\wedge d\tilde{\chi}_2\,.$$

sphere :: two t-dualities II

But, when starting from a conformal model with $h = 4\pi^2 R^2$, the background becomes

$$\overline{\mathsf{G}} = R^2\,(d\eta)^2 + \frac{1}{R^2}\left[(d\tilde{\chi}_1)^2 + \tan^2\eta\,(d\tilde{\chi}_2)^2\right],$$

$$\overline{\mathsf{H}} = 0\,.$$

With dual dilaton $\overline{\phi} = -\log\bigl(R^2\cos\eta\bigr) + \phi$, this is again a conformal model.

Consider finally a non-abelian T-duality along K_1, K_2 and K_3 .

- The constraints from gauging the sigma-model imply *H*=0,
- and the matrix $\mathcal{G}_{\alpha\beta} = k^i_\alpha G_{ij} k^j_\beta$ is invertible ✓

The dual model is obtained as (with $\rho \geq 0$ and $\phi_{1,2} = 0, \ldots, 2\pi$)

$$\check{\mathsf{G}} = \frac{4}{R^2}\, d\rho \wedge \star d\rho + \frac{R^2}{4}\, \frac{\rho^2}{\rho^2 + \frac{R^4}{16}} \left[d\phi_1 \wedge \star d\phi_1 + \sin^2(\phi_1)\, d\phi_2 \wedge \star d\phi_2 \right],$$

$$\check{\mathsf{H}} = \frac{\rho^2}{\left(\rho^2 + \frac{R^4}{16}\right)^2} \left[\rho^2 + 3\, \frac{R^4}{16} \right] \sin(\phi_1)\, d\rho \wedge d\phi_1 \wedge d\phi_2 \,.$$

partially in Alvarez, Alvarez-Gaume, Barbon, Lozano - 1993

sphere :: summary

In the formalism for T-duality introduced above, for a conformal model one finds:

three-sphere with H-flux	1 T-duality →	S^1 fibered over S^2
	2 T-dualities →	non-compact, geometric
three-sphere with $H=0$	3 T-dualities →	S^2 fibered over a ray

outline

A formalism for collective T-duality transformations was developed

- Restrictions on allowed transformations arise.

→ Reduction of the duality group?

For the three-torus with H-flux,

- known results have been reproduced, and
- a twisted T-fold has been obtained.

For the three-sphere with H-flux,

- new geometric backgrounds have been obtained,
- but their global structure is not clear.

discussion :: outlook

For two collective T-duality transformations it was found ::

three-torus with *H*-flux (not conformal)	2 T-dualities →	non-geometric compact
three-sphere with *H*-flux (conformal)	2 T-dualities →	geometric non-compact

Thus, the origin of non-geometry remains unclear …