Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance

Paolo Di Vecchia

Niels Bohr Institute, Copenhagen and Nordita, Stockholm

Schloss Ringberg, 01.08.2014

Foreword

- This talk is based on the work done together with Zvi Bern, Scott Davies and Josh Nohle

  "Low-Energy Behavior of Gluons and Gravitons: from Gauge Invariance", arXiv:1406:6987 [hep-th].

  See also the related work by J. Broedel, M. de Leeuw, J. Plefka and M. Rosso "Constraining subleading soft gluon and graviton theorems", arXiv:1406.6574 [hep-th]

Plan of the talk

1. Introduction
2. Scattering of a photon and n scalar particles
3. Scattering of a graviton and n scalar particles
4. Soft limit of n-gluon amplitude
5. Soft limit of n-graviton amplitude
6. Comments on loop corrections: gauge theory
7. Comments on loop corrections: gravity
8. What about soft theorems in string theory?
9. Conclusions
10. Outlook

Introduction

- Three kinds of symmetries with different physical consequences.
- Global unbroken symmetries as isotopic spin or $SU(3)_V$ in three-flavor QCD.
- Unique vacuum annihilated by the symmetry gener.: $Q_a|0\rangle = 0$
- Particles are classified according to multiplets of this symmetry and all particles of a multiplet have the same mass.
- If isotopic spin were an exact symmetry, the proton and the neutron would have the same mass.
- This would have happened in QCD if the lowest two quarks would have had the same mass.
- This is not the case because the mass matrix of the quarks breaks explicitly $SU(2)$ and even more $SU(3)$ flavor symmetry.

- Then, we have the global spontaneously broken symmetries as $SU(3)_L \times SU(3)_R$ (broken to $SU(3)_V$) symmetry in QCD for zero mass quarks.
- Degenerate vacua: $Q_a|0\rangle = |0'\rangle$.
- Not realized in the spectrum, but it implies the presence of massless particles, called Goldstone bosons.
- They are the pions in QCD with 2 flavors.
- This is one physical consequence of the spontaneous breaking.
- Another one is the existence of low-energy theorems.
- The $\pi\pi$ scattering amplitude is fixed at low energy.
- One gets the two scattering lengths:

$$a_0 = \frac{7m_\pi}{32\pi F_\pi^2} \quad ; \quad a_2 = -\frac{m_\pi}{16\pi F_\pi^2}$$

explicit breaking by a mass term.

- Scattering amplitude is zero for massless pions at low energy because Goldstone bosons interact with derivative coupling implying a shift symmetry.

- Finally, we have the local gauge symmetries for massless spin 1 and spin 2 particles.
- Local gauge invariance is necessary to reconcile the theory of relativity with quantum mechanics.
- It allows a fully relativistic description, but eliminating, at the same time, the presence of negative norm states in the spectrum of physical states.
- Although described by A_μ and $G_{\mu\nu}$, both photons and gravitons have only two physical degrees of freedom in d=4.
- and respectively

$$d-2 \quad and \quad \frac{(d-2)(d-1)}{2} - 1$$

  in d space-time dimensions.
- Another consequence of gauge invariance is charge conservation that, however, follows from the global part of the gauge group.
- Yet another physical consequence of local gauge invariance is the existence of low-energy theorems for photons and gravitons: [F. Low, 1958; S. Weinberg, 1964]

- Let us consider Compton scattering on spinless particles.
- The scattering amplitude $M_{\mu\nu}$ is gauge invariant:

$$k_1^{\mu} M_{\mu\nu} = k_2^{\nu} M_{\mu\nu} = 0$$

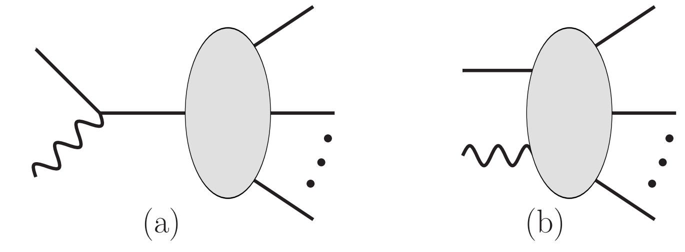
- The previous conditions determine the scattering amplitude for zero frequency photons and one gets the Thompson cross-section:

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2 = \frac{8\pi}{3} r_{cl} \qquad (1)$$

where r_{cl} is the classical radius of a point particle of mass m and charge e.

- The interest on the soft theorems was recently revived by [Cachazo and Strominger, arXiv:1404.1491[hep-th]].
- They study the behavior of the n-graviton amplitude when the momentum q of one graviton becomes soft ($q \sim 0$).
- They suggest a universal formula for the subleading term $O(q^0)$.
- The leading term $O(q^{-1})$ was shown to be universal by Weinberg in the sixties.
- In a previous paper Strominger et al derived the Weinberg universal behavior from the Ward identities of the BMS transformations.
- They speculate that also the next to the leading term follows from the BMS transformations.
- In the following we show that the first three leading terms are a direct consequence of gauge invariance.

One photon and n scalar particles



- The scattering amplitude $M_\mu(q; k_1 \dots k_n)$, involving one photon and n scalar particles, consists of two pieces:

$$A_n^\mu(q; k_1, \dots, k_n) = \sum_{i=1}^{n} e_i \frac{k_i^\mu}{k_i \cdot q} T_n(k_1, \dots, k_i + q, \dots, k_n) + N_n^\mu(q; k_1, \dots, k_n) \,.$$

- and must be gauge invariant for any value of q:

$$q_\mu A_n^\mu = \sum^{n} e_i T_n(k_1, \dots, k_i + q, \dots, k_n) + q_\mu N_n^\mu(q; k_1, \dots, k_n) = 0$$

- Expanding around $q = 0$, we have

$$0 = \sum_{i=1}^{n} e_i \left[T_n(k_1, \ldots, k_i, \ldots, k_n) + q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_i, \ldots, k_n) \right]$$

$$+ q_\mu N_n^\mu(q = 0; k_1, \ldots, k_n) + \mathcal{O}(q^2) .$$

- At leading order, this equation is

$$\sum_{i=1}^{n} e_i = 0 ,$$

which is simply a statement of charge conservation [Weinberg, 1964]

- At the next order, we have

$$q_\mu N_n^\mu(0; k_1, \ldots, k_n) = - \sum_{i=1}^{n} e_i q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n) .$$

- This equation tells us that $N_n^\mu(0; k_1, \ldots, k_n)$ is entirely determined in terms of T_n up to potential pieces that are separately gauge invariant.
- However, it is easy to see that the only expressions local in q that vanish under the gauge-invariance condition $q_\mu E^\mu = 0$ are of the form,

$$E^\mu = (B_1 \cdot q) B_2^\mu - (B_2 \cdot q) B_1^\mu \, ,$$

  where B_1^μ and B_2^μ are arbitrary vectors (local in q) constructed with the momenta of the scalar particles.
- The explicit factor of the soft momentum q in each term means that they are suppressed in the soft limit and do not contribute to $N_n^\mu(0; k_1, \ldots, k_n)$.
- We can therefore remove the q_μ leaving

$$N_n^\mu(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n) \, ,$$

  thereby determining $N_n^\mu(0; k_1, \ldots, k_n)$ as a function of the amplitude without the photon.

- Inserting this into the original expression yields

$$A_n^\mu(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu}\right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q)\,,$$

  where

$$J_i^{\mu\nu} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right)\,,$$

  is the orbital angular-momentum operator and $T_n(k_1, \ldots, k_n)$ is the scattering amplitude involving n scalar particles (and no photon).
- The amplitude with a soft photon with momentum q is entirely determined in terms of the amplitude without the photon up to $\mathcal{O}(q^0)$.
- This goes under the name of F. Low's low-energy theorem.

- Low's theorem is unchanged at loop level for the simple reason that even at loop level, all diagrams containing a pole in the soft momentum are of the form shown, with loops appearing only in the blob and not correcting the external vertex.
- Can we get any further information at higher orders in the soft expansion?
- One order further in the expansion, we find the extra condition,

$$\frac{1}{2}\sum_{i=1}^{n} e_i q_\mu q_\nu \frac{\partial^2}{\partial k_{i\mu}\partial k_{i\nu}} T_n(k_1,\ldots,k_n) + q_\mu q_\nu \frac{\partial N_n^\mu}{\partial q_\nu}(0;k_1,\ldots,k_n) = 0\,.$$

- This implies

$$\sum_{i=1}^{n} e_i \frac{\partial^2}{\partial k_{i\mu}\partial k_{i\nu}} T_n(k_1,\ldots,k_n) + \left[\frac{\partial N_n^\mu}{\partial q_\nu} + \frac{\partial N_n^\nu}{\partial q_\mu}\right](0;k_1,\ldots,k_n) = 0\,,$$

- Gauge invariance determines only the symmetric part of the quantity $\frac{\partial N_n^\nu}{\partial q_\mu}(0; k_1, \ldots, k_n)$.
- The antisymmetric part is not fixed by gauge invariance.
- Indeed, this corresponds exactly to the gauge invariant terms considered above.
- Then, up to this order, we have

$$
\begin{aligned}
& A_n^\mu(q; k_1, \ldots, k_n) \\
& = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu} \left(1 + \frac{1}{2} q_\rho \frac{\partial}{\partial k_{i\rho}} \right) \right] T_n(k_1, \ldots, k_n) \\
& + \frac{1}{2} q_\nu \left[\frac{\partial N_n^\mu}{\partial q_\nu} - \frac{\partial N_n^\nu}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) + O(q^2) \, .
\end{aligned}
$$

- It is straightforward to see that one gets zero by saturating the previous expression with q_μ.

- In order to write our universal expression in terms of the amplitude, we contract $A_n^\mu(q; k_1, \ldots, k_n)$ with the photon polarization $\varepsilon_{q\mu}$.
- Finally, we have the soft-photon limit of the single-photon, *n*-scalar amplitude:

$$A_n(q; k_1, \ldots, k_n) \to \left[S^{(0)} + S^{(1)} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q) \,,$$

where

$$S^{(0)} \equiv \sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q} \,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} e_i \frac{\varepsilon_{q\mu} q_\nu J_i^{\mu\nu}}{k_i \cdot q} \,,$$

where $J_i^{\mu\nu}$ is the angular momentum.

One graviton and n scalar particles

- In the case of a graviton scattering on *n* scalar particles, one can write

$$M_n^{\mu\nu}(q;k_1,\ldots,k_n) = \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{k_i \cdot q} T_n(k_1,\ldots,k_i+q,\ldots,k_n) + N_n^{\mu\nu}(q;k_1,\ldots,k_n)\,,$$

- $N_n^{\mu\nu}(q;k_1,\ldots,k_n)$ is symmetric under the exchange of μ and ν.
- For simplicity, we have set the gravitational coupling constant to unity.
- On-shell gauge invariance implies

$$0 = q_\mu M_n^{\mu\nu}(q;k_1,\ldots,k_n) = \sum_{i=1}^{n} k_i^\nu T_n(k_1,\ldots,k_i+q,\ldots,k_n) + q_\mu N_n^{\mu\nu}(q;k_1,\ldots,k_n)\,.$$

- At leading order in q, we then have

$$\sum_{i=1}^{n} k_i^{\mu} = 0\,,$$

- It is satisfied due to momentum conservation.
- If there had been different couplings to the different particles, it would have prevented this from vanishing in general.
- This shows that gravitons have universal coupling [Weinberg, 1964]).
- At first order in q, one gets

$$\sum_{i=1}^{n} k_i^{\nu} \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n) + N_n^{\mu\nu}(0; k_1, \ldots, k_n) = 0\,,$$

- while at second order in q, it gives

$$\sum_{i=1}^{n} k_i^{\nu} \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\rho}} T_n(k_1, \ldots, k_n) + \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} + \frac{\partial N_n^{\rho\nu}}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) = 0\,.$$

- As for the photon, this is true up to gauge-invariant contributions to $N_n^{\mu\nu}$.
- However, the requirement of locality prevents us from writing any expression that is local in q and not sufficiently suppressed in q.
- Using the previous equations, we write the expression for a soft graviton as

$$
\begin{aligned}
&M_n^{\mu\nu}(q; k_1 \dots k_n) \\
&= \sum_{i=1}^{n} \frac{k_i^\nu}{k_i \cdot q} \left[k_i^\mu - i q_\rho J_i^{\mu\rho} \left(1 + \frac{1}{2} q_\sigma \frac{\partial}{\partial k_{i\sigma}} \right) \right] T_n(k_1, \dots, k_n) \\
&+ \frac{1}{2} q_\rho \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} - \frac{\partial N_n^{\rho\nu}}{\partial q_\mu} \right] (0; k_1, \dots, k_n) + \mathcal{O}(q^2)\,.
\end{aligned}
$$

- This is essentially the same as for the photon except that there is a second Lorentz index in the graviton case.
- Unlike the case of the photon, the antisymmetric quantity in the second line of the previous equation can also be determined from the amplitude $T_n(k_1, \dots, k_n)$ without the graviton.

- Saturating the previous expression with q^μ we get of course zero.
- If we instead saturate it with q^ν, we get

$$q_\nu M_n^{\mu\nu}(q; k_1, \ldots, k_n)$$
$$= \frac{1}{2} q_\rho q_\sigma \Big\{ \sum_{i=1}^{n} \left(k_i^\mu \frac{\partial}{\partial k_{i\rho}} - k_i^\rho \frac{\partial}{\partial k_{i\mu}} \right) \frac{\partial}{\partial k_{i\sigma}} T_n(k_1, \ldots, k_n)$$
$$+ \left[\frac{\partial N_n^{\mu\sigma}}{\partial q_\rho} - \frac{\partial N_n^{\rho\sigma}}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) \Big\} = 0 \, ,$$

- The vanishing follows from the equation above (implied by gauge invariance), remembering that $N_n^{\mu\nu}$ is a symmetric matrix.
- Therefore the amplitude is gauge invariant.

- The same equation allows us to write the relation ,

$$-i \sum_{i=1}^{n} J_i^{\mu\rho} \frac{\partial}{\partial k_{i\sigma}} T_n(k_1, \ldots, k_n) = \left[\frac{\partial N_n^{\rho\sigma}}{\partial q_\mu} - \frac{\partial N_n^{\mu\sigma}}{\partial q_\rho} \right] (0; k_1, \ldots, k_n) \, ,$$

which fixes the antisymmetric part of the derivative of $N_n^{\mu\nu}$ in terms of the amplitude $T_n(k_1, \ldots, k_n)$ without the graviton.

- Using the previous equation, we can then rewrite the terms of $\mathcal{O}(q)$ as follows:

$$
\begin{aligned}
& M_n^{\mu\nu}(q; k_1, \ldots, k_n)\big|_{\mathcal{O}(q)} \\
& = -\frac{i}{2}\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{k_i \cdot q}\left[k_i^\nu J_i^{\mu\rho}\frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma J_i^{\mu\rho}\frac{\partial}{\partial k_{i\nu}}\right] T_n(k_1, \ldots, k_n) \\
& = -\frac{i}{2}\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{k_i \cdot q}\left[J_i^{\mu\rho}k_i^\nu\frac{\partial}{\partial k_{i\sigma}} - \left(J_i^{\mu\rho}k_{i\nu}\right)\frac{\partial}{\partial k_{i\sigma}}\right. \\
& \left. -J_i^{\mu\rho}k_i^\sigma\frac{\partial}{\partial k_{i\nu}} + \left(J_i^{\mu\rho}k_i^\sigma\right)\frac{\partial}{\partial k_{i\nu}}\right] T_n(k_1, \ldots, k_n) \\
& = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{k_i \cdot q}\left[\Big((k_i \cdot q)(\eta^{\mu\nu}q^\sigma - q^\mu\eta^{\nu\sigma}) - k_i^\mu q^\nu q^\sigma\Big)\frac{\partial}{\partial k_i^\sigma}\right. \\
& \left. - q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}\right] T_n(k_1, \ldots, k_n) \,.
\end{aligned}
$$

- Finally, we wish to write our soft-limit expression in terms of the amplitude, so we contract with the physical polarization tensor of the soft graviton, $\varepsilon_{q\mu\nu}$.
- We see that the physical-state conditions set to zero the terms that are proportional to $\eta^{\mu\nu}$, q^μ and q^ν.
- We are then left with the following expression for the graviton soft limit of a single-graviton, n-scalar amplitude:

$$M_n(q; k_1, \ldots, k_n) \to \left[S^{(0)} + S^{(1)} + S^{(2)} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q^2)\,,$$

- where

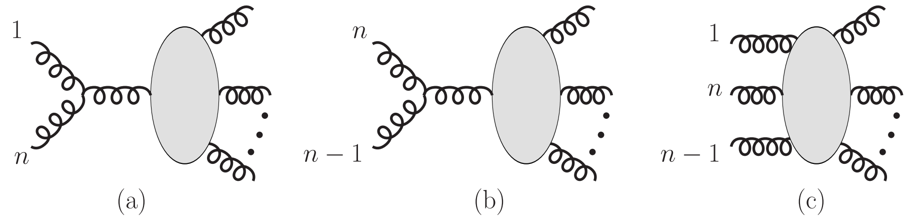
$$S^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q}\,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q}\,,$$

$$S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q}\,.$$

- These soft factors follow from gauge invariance and agree with those computed by Cachazo and Strominger.
- We have also looked at higher-order terms and found that gauge invariance does not fully determine them in terms of derivatives acting on $T_n(k_1, \ldots, k_n)$.

Soft limit of n-gluon amplitude



- We consider a tree-level color-ordered amplitude where gluon n becomes soft with $q \equiv k_n$.
- Being the amplitude color-ordered, we have to consider only two poles.

- We get

$$A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1})$$
$$= \frac{\delta_\rho^{\mu_1} k_1^\mu + \eta^{\mu\mu_1} q_\rho - \delta_\rho^\mu q^{\mu_1}}{\sqrt{2}(k_1\cdot q)} A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\ldots,k_{n-1})$$
$$- \frac{\delta_\rho^{\mu_{n-1}} k_{n-1}^\mu + \eta^{\mu_{n-1}\mu} q_\rho - \delta_\rho^\mu q^{\mu_{n-1}}}{\sqrt{2}(k_{n-1}\cdot q)} A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-2},k_{n-1}+q)$$
$$+ N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1}).$$

- We have dropped terms from the three-gluon vertex that vanish when saturated with the external-gluon polarization vectors in addition to using the current-conservation conditions,

$$(k_1+q)_\rho A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\ldots,k_{n-1}) = 0\,,$$
$$(k_{n-1}+q)_\rho A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-2},k_{n-1}+q) = 0\,,$$

which are valid once we contract with the polarization vectors carrying the μ_j indices.

- By introducing the spin-one angular-momentum operator,

$$(\Sigma_i^{\mu\sigma})^{\mu_i\rho} \equiv i\left(\eta^{\mu\mu_i}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\mu_i\sigma}\right) ,$$

  we can write the total amplitude as

$$\begin{aligned} A_n^{\mu;\mu_1\cdots\mu_{n-1}}&(q;k_1,\ldots,k_{n-1}) \\ &= \frac{\delta_\rho^{\mu_1}k_1^\mu - iq_\sigma(\Sigma_1^{\mu\sigma})^{\mu_1}{}_\rho}{\sqrt{2}(k_1\cdot q)} A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\ldots,k_{n-1}) \\ &- \frac{\delta_\rho^{\mu_{n-1}}k_{n-1}^\mu - iq_\sigma(\Sigma_{n-1}^{\mu\sigma})^{\mu_{n-1}}{}_\rho}{\sqrt{2}(k_{n-1}\cdot q)} A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-2},k_{n-1}+q) \\ &\quad + N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1}) . \end{aligned}$$

- Notice that the spin-one terms independently vanish when contracted with q_μ.

- On-shell gauge invariance requires

$$0 = q_\mu A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1})$$
$$= \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\ldots,k_{n-1})$$
$$-\frac{1}{\sqrt{2}} A_{n-1}^{\mu_1\cdots\mu_{n-2}\mu_{n-1}}(k_1,\ldots,k_{n-2},k_{n-1}+q)$$
$$+ q_\mu N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1}) \,.$$

- For $q = 0$, this is automatically satisfied.
- At the next order in q, we obtain

$$-\frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial k_{1\mu}} - \frac{\partial}{\partial k_{n-1\mu}}\right] A_{n-1}^{\mu_1\cdots\mu_{n-1}}(k_1,k_2\ldots k_{n-1})$$
$$= N_n^{\mu;\mu_1\cdots\mu_{n-1}}(0;k_1,\ldots,k_{n-1}) \,.$$

- Similar to the photon case, we ignore local gauge-invariant terms in $N_n^{\mu;\mu_1\cdots\mu_{n-1}}$ because they are necessarily of a higher order in q.
- Thus, $N_n^{\mu;\mu_1\cdots\mu_{n-1}}(0;k_1,\ldots,k_{n-1})$ is determined in terms of the amplitude without the soft gluon.

- With this, the total expression becomes

$$
\begin{aligned}
&A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1\ldots k_{n-1}) \\
&= \left(\frac{k_1^\mu}{\sqrt{2}(k_1\cdot q)} - \frac{k_{n-1}^\mu}{\sqrt{2}(k_{n-1}\cdot q)}\right) A_{n-1}^{\mu_1\cdots\mu_{n-1}}(k_1,\ldots,k_{n-1}) \\
&\quad - i\frac{q_\sigma (J_1^{\mu\sigma})^{\mu_1}{}_\rho}{\sqrt{2}(k_1\cdot q)} A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1,\ldots,k_{n-1}) \\
&\quad + i\frac{q_\sigma (J_{n-1}^{\mu\sigma})^{\mu_{n-1}}{}_\rho}{\sqrt{2}(k_{n-1}\cdot q)} A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-1}) + \mathcal{O}(q)\,,
\end{aligned}
$$

where

$$
(J_i^{\mu\sigma})^{\mu_i\rho} \equiv L_i^{\mu\sigma}\eta^{\mu_i\rho} + (\Sigma_i^{\mu\sigma})^{\mu_i\rho},
$$

with

$$
L_i^{\mu\sigma} \equiv i\left(k_i^\mu\frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma\frac{\partial}{\partial k_{i\mu}}\right) \quad ; \quad (\Sigma_i^{\mu\sigma})^{\mu_i\rho} \equiv i\,(\eta^{\mu\mu_i}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\mu_i\sigma})
$$

- In order to write the final result in terms of full amplitudes, we contract with external polarization vectors.
- We must pass polarization vectors $\varepsilon_{1\mu_1}$ and $\varepsilon_{n-1\mu_{n-1}}$ through the spin-one angular-momentum operator such that they will contract with the ρ index of, respectively, $A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1,\ldots,k_{n-1})$ and $A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-1})$.
- It is convenient write the spin angular-momentum operator as

$$\varepsilon_{i\mu_i}(\Sigma_i^{\mu\sigma})^{\mu_i}{}_{\rho}A^{\rho} = i\left(\varepsilon_i^{\mu}\frac{\partial}{\partial\varepsilon_{i\sigma}} - \varepsilon_i^{\sigma}\frac{\partial}{\partial\varepsilon_{i\mu}}\right)\varepsilon_{i\rho}A^{\rho}\,.$$

- We may therefore write

$$A_n(q; k_1, \ldots, k_{n-1}) \to \left[S_n^{(0)} + S_n^{(1)}\right] A_{n-1}(k_1, \ldots, k_{n-1}) + \mathcal{O}(q)\,,$$

where

$$S_n^{(0)} \equiv \frac{k_1 \cdot \varepsilon_n}{\sqrt{2}\,(k_1 \cdot q)} - \frac{k_{n-1} \cdot \varepsilon_n}{\sqrt{2}\,(k_{n-1} \cdot q)}\,,$$

$$S_n^{(1)} \equiv -i\varepsilon_{n\mu} q_\sigma \left(\frac{J_1^{\mu\sigma}}{\sqrt{2}\,(k_1 \cdot q)} - \frac{J_{n-1}^{\mu\sigma}}{\sqrt{2}\,(k_{n-1} \cdot q)} \right).$$

- Here

$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma}\,,$$

where

$$L_i^{\mu\nu} \equiv i\left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}}\right) \quad , \quad \Sigma_i^{\mu\sigma} \equiv i\left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}}\right).$$

Soft limit of n-graviton amplitude

- As before the amplitude is the sum of two pieces:

$$
\begin{aligned}
& M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1}) \\
&= \sum_{i=1}^{n-1} \frac{1}{k_i\cdot q}\left[k_i^{\mu}\eta^{\mu_i\alpha} - iq_\rho(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\left[k_i^{\nu}\eta^{\nu_i\beta} - iq_\sigma(\Sigma_i^{\mu\sigma})^{\nu_i\beta}\right] \\
& \times M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i+q,\ldots,k_{n-1}) \\
& \qquad\qquad + N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1}),
\end{aligned}
$$

where

$$
(\Sigma_i^{\mu\rho})^{\mu_i\alpha} \equiv i\left(\eta^{\mu\mu_i}\eta^{\alpha\rho} - \eta^{\mu\alpha}\eta^{\mu_i\rho}\right) .
$$

- On-shell gauge invariance implies

$$0 = q_\mu M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\dots,k_{n-1})$$

$$= \sum_{i=1}^{n-1}\left[k_i^\nu \eta^{\nu_i\beta} - iq_\rho(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right] M_{n-1}^{\mu_1\nu_1\cdots\mu_i}{}_{\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\dots,k_i+q,\dots,k_{n-1})$$

$$+ q_\mu N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\dots,k_{n-1}) .$$

- Proceeding as before we end up getting

$$M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\dots,k_{n-1})$$

$$= \sum_{i=1}^{n-1}\frac{1}{k_i\cdot q}\Big\{k_i^\mu k_i^\nu \eta^{\mu_i\alpha}\eta^{\nu_i\beta}$$

$$-\frac{i}{2}q_\rho\Big[k_i^\mu\eta^{\mu_i\alpha}\left[L_i^{\nu\rho}\eta^{\nu_i\beta}+2(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right]+k_i^\nu\eta^{\nu_i\beta}\left[L_i^{\mu\rho}\eta^{\mu_i\alpha}+2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\Big]$$

$$-\frac{1}{2}q_\rho q_\sigma\Big[\left[L_i^{\mu\rho}\eta^{\mu_i\alpha}+2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\left[L_i^{\nu\sigma}\eta^{\nu_i\beta}+2(\Sigma_i^{\nu\sigma})^{\nu_i\beta}\right]-2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}(\Sigma_i^{\nu\sigma})^{\nu_i\beta}\Big]\Big\}$$

$$\times M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\dots,k_i,\dots,k_{n-1}) + \mathcal{O}(q^2) .$$

- In order to write our expression in terms of amplitudes, we saturate with graviton polarization tensors using $\varepsilon_{\mu\nu} \to \varepsilon_\mu \varepsilon_\nu$ where ε_μ are spin-one polarization vectors.
- As we did for the case with gluons, we must pass the polarization vectors through the spin-one operators.
- $$M_n(q; k_1, \ldots, k_{n-1}) = \left[S_n^{(0)} + S_n^{(1)} + S_n^{(2)} \right] M_{n-1}(k_1, \ldots, k_{n-1}) + \mathcal{O}(q^2)$$

  where

$$S_n^{(0)} \equiv \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q} \, ,$$

$$S_n^{(1)} \equiv -i \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q} \, ,$$

$$S_n^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q} \, .$$

- Here

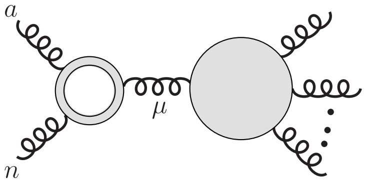
$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma} ,$$

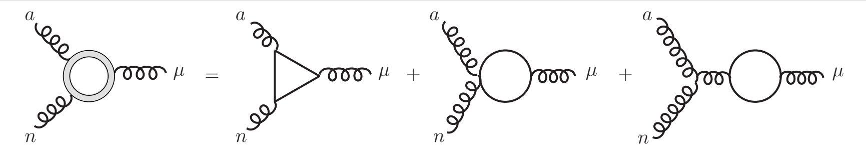
with

$$L_i^{\mu\sigma} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}} \right) , \qquad \Sigma_i^{\mu\sigma} \equiv i \left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}} \right) .$$

Comments on loop corrections: gauge theory

- At one-loop the amplitude will have in general IR and UV divergences.
- We are not giving here a complete study of them.
- The one-loop contributions have been classified into the factorizing ones and the non-factorizing ones.
- We will concentrate here to the factorizing ones.
- They modify the vertex present in the pole term.
- For the gauge theory they are of the type shown in the figure.





- They have been computed in QCD and are given by:

$$D^{\mu,\text{fact}} = \frac{i}{\sqrt{2}}\frac{1}{3}\frac{1}{(4\pi)^2}\left(1-\frac{n_f}{N_c}+\frac{n_s}{N_c}\right)(q-k_a)^\mu\left[(\varepsilon_n\cdot\varepsilon_a)-\frac{(q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n)}{(k_a\cdot q)}\right]$$

[Z. Bern, V. Del Duca, C.R. Schmidt, 1998]
[Z. Bern, V. Del Duca, W.B. Kilgore, C.R. Schmidt, 1999]

- It is both IR and UV finite and the limit $\epsilon \to 0$ has been taken.
- It is non-local because of the pole in (qk_a).
- It is gauge invariant under the substitution $\epsilon_q \to q$.
- It does not contribute to the leading soft behavior.

- Attaching to it the rest of the amplitude

$$D_\mu^{\text{fact}} \frac{-i}{2q \cdot k_a} \mathcal{J}^\mu \,,$$

- $\mathcal{J}^\mu$ is a conserved current:

$$(q + k_a)_\mu \mathcal{J}^\mu = 0 \,,$$

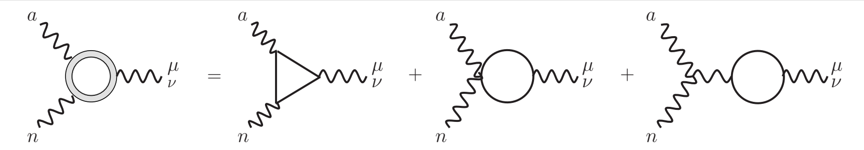
assuming that all the remaining legs are contracted with on-shell polarizations.

- We can trade k_a with q and we get immediately:

$$D_\mu^{\text{fact}} \frac{-i}{2q \cdot k_a} \mathcal{J}^\mu = \mathcal{O}(q^0) \,,$$

- No leading $\mathcal{O}(\frac{1}{q})$ correction from the factorizing contribution to the one-loop soft functions.

Comments on loop corrections: gravity



- A similar calculation can be done for the gravity case.
- We consider only the case in which scalar fields circulate in the loop.
- The result of this calculation is:

$$\mathcal{D}^{\mu\nu,\text{fact,s}} = \frac{i}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^3 \frac{1}{30}\left[(\varepsilon_n\cdot\varepsilon_a) - \frac{(q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n)}{(q\cdot k_a)}\right]$$
$$\times\Big((q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n) - (\varepsilon_n\cdot\varepsilon_a)(q\cdot k_a)\Big)k_a^\mu k_a^\nu + \mathcal{O}(q^2)\,,$$

- As in the gauge-theory case, the diagrams $\mathcal{D}^{\mu\nu,\text{fact,s}}$ contract into a conserved current:

$$(k_a+q)^\mu \mathcal{J}_{\mu\nu} = f(k_i,\epsilon_i)(k_a+q)_\nu \, , \;\; (k_a+q)^\nu \mathcal{J}_{\mu\nu} = f(k_i,\epsilon_i)(k_a+q)_\mu \, .$$

- This means

$$k_a^\mu k_a^\nu \mathcal{J}_{\mu\nu} = (k_a+q)^\mu (k_a+q)^\nu \mathcal{J}_{\mu\nu} + \mathcal{O}(q)$$
$$= f(k_i,\epsilon_i)(k_a+q)^2 + \mathcal{O}(q) = 2f(k_i,\epsilon_i) q\cdot k_a + \mathcal{O}(q) = \mathcal{O}(q)$$

- We therefore have

$$\mathcal{D}^{\mu\nu,\text{fact,s}} \frac{i}{2q\cdot k_a} \mathcal{J}_{\mu\nu} = \mathcal{O}(q) \, .$$

- No modification of the two first leading terms.
- As in QCD, we expect that the contribution of other particles circulating in the loop will not modify this result.

What about soft theorems in string theory?

- In superstring the soft theorems have been investigated by B.U.W. Schwab, arXiv:1406.4172 and M. Bianchi, Song He, Yu-tin Huang and Congkao Wen, arXiv:1406.5155.
- Here we give just few examples in the bosonic string.
- One gluon and three tachyons:

$$A_\mu(p_1, p_2, q, p_3) \sim \sqrt{2\alpha'}\frac{\Gamma(1+2\alpha' p_3 q)\Gamma(1+2\alpha' p_2 q)}{\Gamma(1+2\alpha'(p_2+p_3)q)} \times \left(\frac{p_{2\mu}}{2\alpha' p_3 q} - \frac{p_{3\mu}}{2\alpha' p_3 q}\right)$$

- One graviton(dilaton) and three tachyons ($p_1 + p_2 + p_3 = -q$):

$$A_{\mu\nu}(p_1, p_2, p_3, q) \sim \left(\frac{p_{1\mu}p_{1\nu}}{p_1 q} + \frac{p_{2\mu}p_{2\nu}}{p_2 q} + \frac{p_{3\mu}p_{3\nu}}{p_3 q}\right) \times \frac{\Gamma(1+\frac{\alpha'}{2}p_1 q)\Gamma(1+\frac{\alpha'}{2}p_2 q)\Gamma(1+\frac{\alpha'}{2}p_3 q)}{\Gamma(1-\frac{\alpha'}{2}p_1 q)\Gamma(1-\frac{\alpha'}{2}p_2 q)\Gamma(1-\frac{\alpha'}{2}p_3 q)}$$

- One gluon and 4 tachyons
  With [R. Marotta]

$$A_\mu(p_1, p_2, p_3, q, p_4) \sim \int_0^1 dz_3 (1-z_3)^{2\alpha' p_2 p_3} z_3^{2\alpha' p_3 p_4}$$
$$\times \int_0^{z_3} dz_4 (1-z_4)^{2\alpha' p_2 q} (z_3 - z_4)^{2\alpha' p_3 q} z_4^{2\alpha' p_4 q}$$
$$\times \left[\frac{p_{2\mu}}{1-z_4} + \frac{p_{3\mu}}{z_3 - z_4} - \frac{p_{4\mu}}{z_4} \right]$$

- It is gauge invariant: $q^\mu A_\mu = 0$.
- The last two lines are equal to $(z_4 = z_3 t)$

$$z_3^{2\alpha'(p_3+p_4)q} \int_0^1 dt (1-t)^{2\alpha' p_3 q} t^{2\alpha' p_4 q} (1 - z_3 t)^{2\alpha' p_2 q}$$
$$\times \left[\frac{z_3 p_{2\mu}}{1 - z_3 t} + \frac{p_{3\mu}}{1-t} - \frac{p_{4\mu}}{t} \right]$$

- They are equal to

$$
z_3^{2\alpha'(p_3+p_4)q} \left[\frac{\Gamma(1+2\alpha' p_4 q)\Gamma(2\alpha' p_3 q)}{\Gamma(2+2\alpha'(p_3+p_4)q)} z_3 \right.
$$

$$
\times {}_2F_1(1-2\alpha' p_2 q, 1+2\alpha' p_4 q; 2+2\alpha'(p_3+p_4)q; z_3)
$$

$$
+ \frac{\Gamma(2\alpha' p_4 q+1)\Gamma(1+2\alpha' p_3 q)}{\Gamma(1+2\alpha'(p_3+p_4)q)} \left(- \frac{p_{4\mu}}{2\alpha' p_4 q} \right.
$$

$$
\times {}_2F_1(-2\alpha' p_2 q, 2\alpha' p_4 q; 1+2\alpha'(p_3+p_4)q; z_3)
$$

$$
+ \frac{p_{3\mu}}{2\alpha' p_3 q} (1-z_3)^{2\alpha' p_2 q}
$$

$$
\left. \left. \times {}_2F_1(-2\alpha' p_2 q, 2\alpha' p_3 q; 2\alpha'(p_3+p_4)q+1; -\frac{z_3}{1-z_3}) \right) \right]
$$

- In the soft limit up to the order q^0 we can forget the ratio of Γ-functions, we can approximate the last two ${}_2F_1$ with 1 and the first one with: ${}_2F_1(1,1;2;z_3)z_3 = -\log(1-z_3)$.

- In this way we get:

$$\int_0^1 dz_3 (1-z_3)^{2\alpha' p_2 p_3} z_3^{2\alpha' p_3 p_4} \Big[-\log(1-z_3) p_{2\mu} + z_3^{2\alpha'(p_3+p_4)q} \left(\frac{p_{3\mu}}{2\alpha' p_3 q} (1-z_3)^{2\alpha' p_2 q} - \frac{p_{4\mu}}{2\alpha' p_4 q} \right) \Big]$$

- It can be written as follows:

$$\frac{1}{2\alpha'} \left[\frac{p_{3\mu}}{p_3 q} - \frac{p_{4\mu}}{p_4 q} + \frac{q^\rho J^{(3)}_{\mu\rho}}{p_3 q} - \frac{q^\rho J^{(4)}_{\mu\rho}}{p_4 q} \right] \times \int_0^1 dz_3 (1-z_3)^{\alpha'(p_2+p_3)^2 - 2} z_3^{\alpha'(p_3+p_4)^2 - 2}$$

- The last integral is the amplitude for four tachyons and

$$J^{(3,4)}_{\mu\rho} = p_{(3,4)\mu} \frac{\partial}{\partial p_{(3,4)\rho}} - p_{(3,4)\rho} \frac{\partial}{\partial p_{(3,4)\mu}}$$

Conclusions

- We have extended Low's proof of the universality of sub-leading behavior of photons to non-abelian gauge theory and to gravity.
- On-shell gauge invariance can be used to fully determine the first sub-leading soft-gluon behavior at tree level.
- In gravity the first two subleading terms in the soft expansion can also be fully determined from on-shell gauge invariance.
- We have considered the factorizing contribution to both gauge theories and gravity.
- In non-abelian gauge theories the leading term is not affected by it, but the next to the leading is affected.
- Similarly in gravity the first two leading terms are not affected by the factorizing contribution, but the next term is affected.

Outlook

- It would be nice to have under control, together with the factorizing contribution, also the ones involving both the IR and the UV divergences at one loop.
- In gauge theory they are well established, but in gravity some more work has to be done.
- It would be very nice to extract everything from string theory in the limit of $\alpha' \to 0$.