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AN EXACT MODEL FOR NON-EQUILIBRIUM PHENOMENA

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SPHERICAL GRAVITATIONAL WAVES

Long time ago Robinson–Trautman considered special class of metrics in four space-time dimensions describing outgoing radiation emitted from bounded sources in the form of spherical gravitational waves. Using retarded time u they assumed metrics of the form

$$ds^2 = 2r^2 e^{\Phi(z,\bar{z};u)} dz d\bar{z} - 2du dr - F(r,u,z,\bar{z}) du^2 \ .$$

and found that Einstein equations with cosmological constant Λ, i.e., $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$, can be partially integrated to yield the front factor F,

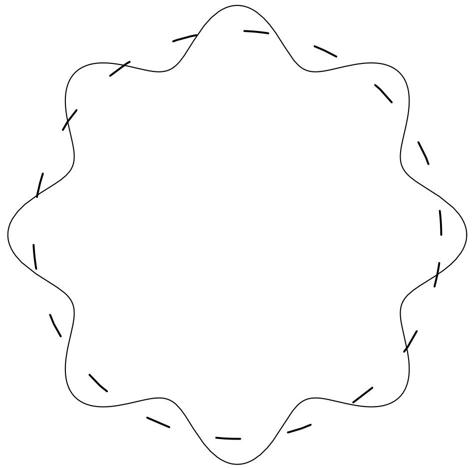
$$F = r\partial_u \Phi - \Delta\Phi - \frac{2m}{r} - \frac{\Lambda}{3} r^2 \ ,$$

where $\Delta = e^{-\Phi}\partial_z \partial_{\bar{z}}$ is the Laplace–Beltrami operator on S^2. $\Delta\Phi$ is the curvature of S^2 at fixed r, which, as it turns out, varies with u.

Φ satisfies a parabolic fourth-order non-linear differential equation, called Robinson–Trautman equation,

$$3m\partial_u\Phi + \Delta\Delta\Phi = 0 \ .$$

Given sufficiently smooth data, the metric on S^2 evolves by dissipating curvature perturbations trying to reach the constant curvature metric, as in heat flow equations



Proper study of the evolution with respect to u relies on the theory of geometric flow equation noting the connection with Calabi flow on S^2. Recall the general definition of Calabi flow on a Kähler manifold M, say compact without boundaries,

$$\partial_u g_{a\bar{b}} = \frac{\partial^2 R}{\partial z^a \partial \bar{z}^b} \ ,$$

deforming the metric $g_{a\bar{b}}$ by derivatives of the Ricci scalar curvature. On S^2 with metric

$$ds_2^2 = 2e^{\Phi(z,\bar{z};u)} dz d\bar{z}$$

Calabi flow is identical to Robinson–Trautman equation (with $3m = 2$) for $R = -2\Delta\Phi$.

Calabi flow on S^2 cannot be solved in closed form, but it exhibits some properties that are sufficient for our purposes:

- **For any given initial data at u_0, a solution exists for all $u \geq u_0$.**
- **All trajectories flow to a fixed point, as $u \to \infty$, associated to the constant curvature metric on S^2,**

$$e^{\Phi_0} = \frac{1}{(1 + z\bar{z}/2)^2} \ .$$

- **The area of S^2 and the average curvature $< R > \sim \chi(S^2)$ remain fixed throughout the evolution, but not higher moments of the curvature, like Calabi's functional $< R^2 > = \int_{S^2} \sqrt{g}\, R^2$ that acts as an entropy functional, decreasing monotonically along the flow lines.**

Close to the fixed point the equation linearizes and one can easily compute the damping rate of different harmonics. Thus, assuming axial symmetry (for simplicity) and parametrizing all perturbations of the round S^2 in terms of Legendre polynomials ($l \geq 2$), as

$$ds_2^2 = [1 + \epsilon_l(u) P_l(\cos\theta)] \left(d\theta^2 + \sin^2\theta d\phi^2 \right),$$

we find that the perturbations are damped exponentially as

$$\epsilon_l(u) = \epsilon_l(0)\exp\left(-\frac{u}{12m}(l-1)l(l+1)(l+2) \right) \equiv \epsilon_l(0)e^{-i\omega_{\rm S} u}$$

with fall-off rate given by the characteristic imaginary frequencies

$$\omega_{\rm S} = -i\frac{(l-1)l(l+1)(l+2)}{12m} .$$

Returning back to the Robinson-Trautman metric in four space-time dimensions, we can assign a definite meaning to all previous results:

- **Starting from sufficiently smooth data at u_0, the space-time metric exists for all $u \geq u_0$ and after infinitely long time it settles to a static solution that is nothing else but the exterior of a Schwarzschild black-hole with mass m and cosmological constant Λ:**

$$ds^2 = \frac{2r^2}{(1 + z\bar{z}/2)^2} dz d\bar{z} - 2dudr - \left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right) du^2$$

written in Eddington-Filkenstein frame. To pass to ordinary frame we set $u = t - r_\star$, using the tortoise coordinate $r_\star$ defined as $dr_\star = dr/f(r)$ with profile function $f(r) = 1 - 2m/r - \Lambda r^3/3$.

- **At late times, the four-dimensional solution is a small perturbation of the black-hole metric associated to the algebraically special modes which are purely out-going total transmission modes that vanish on the horizon at $r = r_{\rm h}$. Comparison with the theory of quasi-normal modes reveals that such modes are zero energy states of an effective Schrödinger problem for the polar (vs axial) perturbations of the black-hole,**

$$[\frac{d^2}{dr_\star^2} + W^2(r) + \frac{dW(r)}{dr_\star}]\Psi(r) = E\,\Psi(r)\,,$$

where $E = \omega^2 - \omega_{\rm s}^2$ and

$$W(r) = \frac{6mf(r)}{r[(l-1)(l+2)r + 6m]} + i\omega_{\rm S}\,.$$

The algebraically special modes are purely dissipative corresponding to "ringing" frequencies

$$\omega = \omega_{\rm S} = -i\,\frac{(l-1)l(l+1)(l+2)}{12m} = -\frac{2i}{m}, \quad -\frac{10i}{m}, \quad -\frac{30i}{m}, \cdots .$$

As in supersymmetric quantum mechanics, they satisfy a first order equation

$$Q\Psi(r_\star) = \left(-\frac{d}{dr_\star} + W(r_\star)\right)\Psi(r_\star) = 0\,.$$

Thus, the Robinson-Trautman metrics are formed by superposition of the algebraically special modes as compared to all other classes of radiative metrics that require making use of the entire set of quasi-normal modes.

Linear as well as not linear effects can be captured by the late time expansion of the solutions. Parametrizing deviations from equilibrium state as

$$e^{\Phi(z,\bar{z};u)} = \frac{1}{\sigma^2(z,\bar{z};u)\,(1+z\bar{z}/2)^2}$$

we may expand systematically as

$$\sigma(z,\bar{z};u) \;=\; 1 + \sigma_1(z,\bar{z})e^{-2u/m} + \sigma_2(z,\bar{z})e^{-4u/m} + \cdots$$

showing only the quadrupole and the first non-linear correction to it. For axially symmetric solutions we obtain recursively, setting $x = \cos\theta$,

$$\sigma_1(x) = a\left(x^2 - \frac{1}{3}\right), \quad \sigma_2(x) = -a^2\left(\frac{23}{78}x^4 - \frac{47}{39}x^2 + \frac{49}{234}\right), \quad \cdots .$$

In view of the holographic applications of AdS_4 Robinson-Trautman, we note the mixed boundary conditions satisfied by the algebraically special modes,

$$\frac{d}{dr_\star}\Psi_+^{(0)}(r_\star)\,|_{r_\star=0} \quad = \left(i\omega_{\rm S} - \frac{2m\Lambda}{(l-1)(l+2)}\right)\Psi_+^{(0)}(r_\star = 0)\,.$$

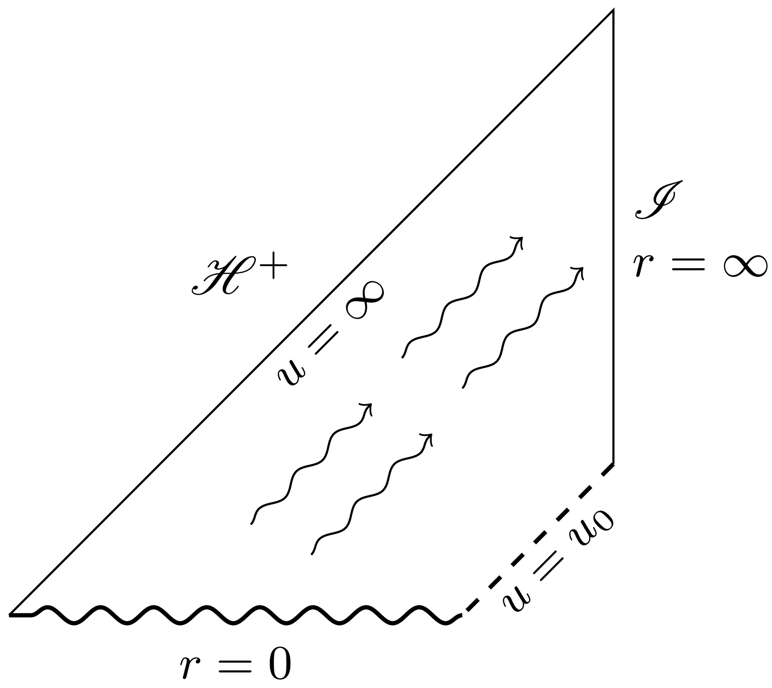
Here, $r_\star$ ranges from $-\infty$ to 0 as r ranges from $r_{\rm h}$ to $+\infty$.

The corresponding wave-functions are normalizable,

$$\int_{-\infty}^{0} dr_\star\,|\,\Psi_+^{(0)}(r_\star)\,|^2 < \infty \ .$$

They amount to a metric at conformal infinity $\mathcal{I} = \mathbb{R} \times S^2$ that is not conformally flat; its evolution is also governed by Calabi flow.

The Penrose diagram of AdS_4 Robinson–Trautman space-times is



but it turns out that Kruskal extension across the future horizon $\mathcal{H}^+$ breaks down completely for sufficiently large AdS_4 black-holes, which are thermodynamically favorable.

PENROSE (AND OTHER) INEQUALITIES

Robinson–Trautman space-times provide a nice example to illustrate the validity of Penrose (and other related) inequalities.

• **Bondi mass: Motivated by $\Lambda = 0$, we consider the mass formula**

$$\mathcal{M}_{\mathrm{Bondi}} = \frac{m}{4\pi} \int_{S^2} d\mu_0 \, \frac{1}{\sigma^3}$$

which can be shown to decrease monotonically with time u, reaching m as $u \to \infty$.

• **Past apparent horizon: It is a marginally trapped surface Σ defined by the embedding equation $r = U(z, \bar{z})$ for constant u with U satisfying a variant of Tod–Penrose equation in the presence of Λ**

$$2\Delta(\log U) + \Delta\Phi + \frac{2m}{U} + \frac{\Lambda}{3}U^2 = 0 \ .$$

The area of the apparent horizon is

$$\mathrm{Area}(\Sigma) = \int_{S^2} d\mu_0 \left(\frac{U}{\sigma}\right)^2 .$$

It can be shown, using Holder and Sobolev inequalities, that although $\mathcal{M}_{\mathrm{Bondi}}$ **and** $\mathrm{Area}(\Sigma)$ **both vary with** u**, the following version of Penrose inequality holds for all** $\Lambda \leq 0$**,**

$$16\pi\mathcal{M}^2_{\mathrm{Bondi}} \geq \mathrm{Area}(\Sigma)\left(1 - \frac{\Lambda}{3}\frac{\mathrm{Area}(\Sigma)}{4\pi}\right)^2 .$$

Another inequality is provided by Thorne's hoop conjecture stating (when generalized in the presence of cosmological constant)

$$4\pi\mathcal{M} \geq C\left(1 - \frac{\Lambda}{3}\left(\frac{C}{2\pi}\right)^2\right)$$

for appropriately defined mass $\mathcal{M}$ and circumference C lassoing the black-hole. The Robinson–Trautman space-times provide a realization of it letting $\mathcal{M}_{\rm Bondi}$ to be the mass and C given by the length of the shortest closed geodesic on the past apparent horizon Σ.

- **Inequalities turn into equalities at the equilibrium state, $u = \infty$.**

HOLOGRAPHIC RENORMALIZATION

The AdS_4 Robinson–Trautman metric is an example of asymptotically locally AdS space-time. The boundary metric (after rescaling) is

$$ds^2_{\mathcal{I}} = -dt^2 - \frac{6}{\Lambda} e^{\hat{\Phi}} dz d\bar{z}$$

and the corresponding energy-momentum tensor turns out to be

$$\kappa^2 T_{tt} = -\frac{2m\Lambda}{3} \,, \qquad \kappa^2 T_{tz} = -\frac{1}{2} \partial_z (\hat{\Delta} \hat{\Phi}) \,,$$

$$\kappa^2 T_{z\bar{z}} = m e^{\hat{\Phi}}, \qquad \kappa^2 T_{zz} = -\frac{3}{4\Lambda} \partial_t \left((\partial_z \hat{\Phi})^2 - 2 \partial_z^2 \hat{\Phi} \right) ,$$

whereas $T_{t\bar{z}} = \bar{T}_{tz} \,, \quad T_{\bar{z}\bar{z}} = \bar{T}_{zz}$ **.**

The energy-momentum tensor is traceless and conserved, as it should

$$T^a{}_a = 0\,, \qquad \nabla^a T_{ab} = 0$$

by the classical equations of motion provided by the boundary version of Calabi flow,

$$3m\partial_t \hat{\Phi} + \hat{\Delta}\hat{\Delta}\hat{\Phi} = 0\ ,$$

where $\hat{\Phi}$ **is the boundary value of** Φ

$$\hat{\Phi}(z, \bar{z}; t) = \lim_{r_\star \to 0} \Phi(z, \bar{z}; u)$$

and $\hat{\Delta} = e^{-\hat{\Phi}} \partial_z \partial_{\bar{z}}$ **is the corresponding Laplace–Beltrami operator on the spatial slices** S^2 **of** $\mathcal{I} = \mathbb{R} \times S^2$**.**

HYDRODYNAMIC CONSIDERATIONS

On the boundary we have a $2+1$ dimensional relativistic fluid that can be very far away from equilibrium. As $t \to \infty$, the system thermalizes.

To compare with first order hydrodynamics we have to linearize the energy-momentum tensor around the black-hole equilibrium state and determine the corresponding energy density ρ and the time-like unit vector u^a so that

$$T_{ab}\, u^b \,=\, -\rho\, u_a\,.$$

Then, T_{ab} takes the perfect fluid form plus a viscous term,

$$T^{ab} = \rho u^a u^b + p\Delta^{ab} + \Pi^{ab}\,,$$

where $\Delta^{ab} = u^a u^b + g^{ab}$**. The viscous term is**

$$\Pi^{ab} = -\eta\sigma^{ab} - \zeta\Delta^{ab}(\nabla_c u^c) \ ,$$

where

$$\sigma^{ab} = 2\nabla^{<a}u^{b>}$$

using the short-hand notation

$$A^{<ab>} = \frac{1}{2}\left(\Delta^{ac}\Delta^{bd}(A_{cd} + A_{dc}) - \Delta^{ab}\Delta^{cd}A_{cd}\right) .$$

The bulk viscosity coefficient $\zeta = 0$**, since the fluid is conformal. For that reason we also have** $\rho = 2p$**.**

The effective shear viscosity coefficient η can be determined for each algebraically spacial mode. Explicit computation shows that

$$\kappa^2 \eta = \frac{1}{4} l(l+1)$$

and so the ratio of shear viscosity to the entropy density for large AdS_4 black holes turns out to be

$$\frac{\eta}{s} = \frac{4}{r_{\mathrm{h}}^2}\left(-\frac{3}{\Lambda}\right)\eta = \frac{1}{4\pi} \cdot \frac{l(l+1)r_{\mathrm{h}}}{4m} .$$

As such, it violates the celebrated KSS bound for sufficiently low l. The modes that give rise to KSS bound have $\Omega_{\mathrm{s}} = -i(l-1)(l+2)/3r_{\mathrm{h}}$

Entropy production is established by considering the entropy current $s^a = su^a$, where s is the local entropy density that is related to the energy density ρ via the thermodynamic relations of $2+1$ dimensional conformal fluids,

$$s = \gamma \mathcal{T}^2, \qquad \rho = 2p = \frac{2\gamma}{3}\mathcal{T}^3$$

setting $\gamma = -4\pi^2/3\Lambda$ for large AdS_4 black-holes.

Expanding all fields to order $e^{-4t/m}$ (quadrupole plus first non-linear corrections) one finds that there is entropy production such that

$$\nabla_a s^a = \frac{\eta}{2\mathcal{T}_0}\sigma_{ab}\sigma^{ab}\,.$$

THANK YOU!