Double Field Theory, Algebroids and Membrane Sigma-Models

Richard Szabo

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

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Quantum Structure of Spacetime

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Geometry and Strings

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Outline

- Introduction/Motivation
- Courant algebroids & sigma-models
- DFT algebroids & sigma-models
- Unification of geometric & non-geometric fluxes (T-duality)
- Unification of topological A- & B-models (S-duality)

with A. Chatzistavrakidis, L. Jonke & F. S. Khoo [arXiv:1802.07003]

and Z. Kökényesi & A. Sinkovics [arXiv:1805.11485]

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- Double field theory (DFT) doubles the spacetime:

$$M \longrightarrow \mathcal{M} = M \times \widetilde{M}$$

  Solving strong constraint (polarisation) reduces DFT structure to standard Courant algebroid

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- Is there a membrane sigma-model involving generalized complex structures unifying the topological A- and B-models with manifest S-duality invariance?

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- $n=3$: Symplectic Lie 2-algebroids $\Longleftrightarrow$ Courant algebroids

  (Roytenberg '02)

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3. Compatibility:

$$\rho(C)\langle A,B\rangle = \langle [C,A] + \mathcal{D}\langle C,A\rangle, B\rangle + \langle [C,B] + \mathcal{D}\langle C,B\rangle, A\rangle$$

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Additional properties:

4. Homomorphism: $\rho([A,B]) = [\rho(A), \rho(B)]$
5. "Strong constraint": $\langle \mathcal{D}f, \mathcal{D}g \rangle = 0$ $(\rho \circ \mathcal{D} = 0)$

for $A,B,C \in \Gamma(E)$ and $f,g \in C^\infty(M)$

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$$S = \int_{\Sigma_3} \Big(F_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\eta_{IJ}\,A^I \wedge \mathrm{d}A^J - {\rho_I}^i(X)\,F_i \wedge A^I + \frac{1}{3!}\,T_{IJK}(X)\,A^I \wedge A^J \wedge A^K \Big)$$

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- When $E = \mathbb{T}M = TM \oplus T^*M$ with natural frame $(e_I) = (\partial_i, \mathrm{d}x^i)$ and $O(d,d)$-invariant metric $\langle \partial_i, \mathrm{d}x^j\rangle = \delta_i{}^j$, axioms give fluxes and Bianchi identities of supergravity

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  For $\pi = 0$ reduces with exact BV gauge-fixing to R-twisted open membrane σ-model (Mylonas, Schupp & Sz '12):

$$\oint_{\partial\Sigma_3} A_i \wedge \mathrm{d}X^i + \frac{1}{3!} \int_{\Sigma_3} R^{ijk}\, A_i \wedge A_j \wedge A_k$$

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- $O(d,d)$-structure / generalized metric $\mathcal{H} \in O(d,d)/O(d) \times O(d)$:

$$\eta = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \qquad , \qquad \mathcal{H} = \begin{pmatrix} g - B\,g^{-1}\,B & B\,g^{-1} \\ -g^{-1}\,B & g^{-1} \end{pmatrix}$$

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- C-bracket: Closure $[\mathsf{L}_{\epsilon_1}, \mathsf{L}_{\epsilon_2}] = \mathsf{L}_{[\![\epsilon_1, \epsilon_2]\!]}$ after strong constraint:

$$[\![\epsilon_1, \epsilon_2]\!]^J \ = \ \epsilon_1^K\, \partial_K \epsilon_2^J - \tfrac{1}{2}\, \epsilon_1^K\, \partial^J \epsilon_{2\,K} - (\epsilon_1 \leftrightarrow \epsilon_2)$$

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- Frame: $e_I^{\pm} = \partial_I \pm \eta_{IJ}\,\mathrm{d}\mathbb{X}^J$ defines splitting $\mathbb{E} = L_+ \oplus L_-$

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- Frame: $e_I^{\pm} = \partial_I \pm \eta_{IJ}\, \mathrm{d}\mathbb{X}^J$ defines splitting $\mathbb{E} = L_+ \oplus L_-$
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- Other approaches:
  - Supergeometry (Deser & Stasheff '14; Deser & Sämann '16; Heller, Ikeda & Watamura '16)
  - Para-Hermitian geometry (Vaisman '12; Freidel, Rudolph & Svoboda '17; Svoboda '18)

DFT Membrane Sigma-Models

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- Project doubled Courant σ-model to get $O(d,d)$-invariant σ-model:

(Chatzistavrakidis, Jonke & Lechtenfeld '15)

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- Gives unified description of geometric and non-geometric fluxes realising T-duality chain (Shelton, Taylor & Wecht '05)

$$H_{ijk} \xleftrightarrow{\ \mathsf{T}_k\ } f_{ij}{}^k \xleftrightarrow{\ \mathsf{T}_j\ } Q_i{}^{jk} \xleftrightarrow{\ \mathsf{T}_i\ } R^{ijk}$$

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Twisted Poisson bivector $\Theta = \frac{1}{2}\,\Theta^{IJ}(\mathbb{X})\,\partial_I \wedge \partial_J$ on $\mathcal{M} = T^*M$ for noncomm/nonassoc phase space (Blumenhagen & Plauschinn '10; Lüst '10):

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  Twisted Poisson bivector $\Theta = \frac{1}{2}\,\Theta^{IJ}(\mathbb{X})\,\partial_I \wedge \partial_J$ on $\mathcal{M} = T^*M$ for noncomm/nonassoc phase space (Blumenhagen & Plauschinn '10; Lüst '10):

$$\Theta = \begin{pmatrix} R^{ijk}\,\tilde{X}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \qquad \begin{aligned} \{\mathbb{X}^I, \mathbb{X}^J\}_\Theta &= \Theta^{IJ}(\mathbb{X}) \\ \{X^i, X^j, X^k\}_\Theta &= R^{ijk} \end{aligned}$$

  $(\rho_+)_I{}^J = (0, 0, \delta^i{}_j, 0)$ gives T-dual of H-flux Wess–Zumino coupling under $X^i \longleftrightarrow \tilde{X}_i$: $\frac{1}{3!}\int_{\Sigma_3} R^{ijk}\,\mathrm{d}\tilde{X}_i \wedge \mathrm{d}\tilde{X}_j \wedge \mathrm{d}\tilde{X}_k$

- H-, f-, Q-, R-flux backgrounds correspond to standard Courant algebroid over different submanifolds of doubled space; doesn't include noncomm/nonassoc models which violate strong constraint

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(Witten '88; Bershadsky, Cecotti, Ooguri & Vafa '93)

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$$S_{\mathrm{B}} = \int_{\Sigma_2} \mathrm{d}^2 z \, \big(g_{ij} \, \partial X^i \, \bar{\partial} X^j + \mathrm{i} \, \eta_z^{\bar{a}} \, (D\rho_z^a + \bar{D}\rho_{\bar{z}}^a) \, g_{a\bar{a}} + \mathrm{i} \, \theta_a \, (\bar{D}\rho_z^a - D\rho_{\bar{z}}^a) - R_{a\bar{a}b\bar{b}} \, \rho_z^a \, \rho_{\bar{z}}^b \, \eta_z^{\bar{a}} \, \theta_c \, g^{c\bar{b}} \big)$$

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- DFT projection of doubled Poisson Courant sigma-model with generalized complex structure $\Pi^{IJ} = (\pi^{ij}, J^i{}_j, -J^j{}_i, 0)$ and 3-vector $\mathcal{R}^{IJK}$ gives unified description in generalized geometry, and allows for coupling A/B-models to geometric and non-geometric fluxes

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- Exchanges Poisson and complex structure Courant algebroids within Courant algebroid for generalized complex geometry