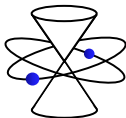



Double Field Theory, Algebroids and Membrane Sigma-Models

Richard Szabo



 **cost** Action MP 1405
Quantum Structure of Spacetime



Geometry and Strings
Ringberg Castle, Tegernsee

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Outline

- ▶ Introduction/Motivation
- ▶ Courant algebroids & sigma-models
- ▶ DFT algebroids & sigma-models
- ▶ Unification of geometric & non-geometric fluxes (T-duality)
- ▶ Unification of topological A- & B-models (S-duality)

with A. Chatzistavrakidis, L. Jonke & F. S. Khoo [arXiv:1802.07003]

and Z. Kökenyesi & A. Sinkovics [arXiv:1805.11485]

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- ▶ **Double field theory (DFT)** doubles the spacetime:

$$M \longrightarrow \mathcal{M} = M \times \tilde{M}$$

Solving strong constraint (polarisation) reduces DFT structure to standard Courant algebroid

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- ▶ Is there a membrane sigma-model involving generalized complex structures unifying the topological A- and B-models with manifest S-duality invariance?

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$$\rho(C)\langle A, B \rangle = \langle [C, A] + \mathcal{D}\langle C, A \rangle, B \rangle + \langle [C, B] + \mathcal{D}\langle C, B \rangle, A \rangle$$

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Additional properties:

4. Homomorphism: $\rho([A, B]) = [\rho(A), \rho(B)]$

5. "Strong constraint": $\langle \mathcal{D}f, \mathcal{D}g \rangle = 0$ ($\rho \circ \mathcal{D} = 0$)

for $A, B, C \in \Gamma(E)$ and $f, g \in C^\infty(M)$

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- ▶ Gauge invariance \implies axioms and properties of Courant algebroid
- ▶ When $E = \mathbb{T}M = TM \oplus T^*M$ with natural frame $(e_I) = (\partial_i, dx^i)$ and $O(d, d)$ -invariant metric $\langle \partial_i, dx^j \rangle = \delta_i^j$, axioms give fluxes and Bianchi identities of supergravity

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For $\pi = 0$ reduces with exact BV gauge-fixing to R -twisted open membrane σ -model (Mylonas, Schupp & Sz '12):

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- ▶ **C-bracket:** Closure $[\mathbb{L}_{\epsilon_1}, \mathbb{L}_{\epsilon_2}] = \mathbb{L}_{[\epsilon_1, \epsilon_2]}$ after strong constraint:

$$[[\epsilon_1, \epsilon_2]]^J = \epsilon_1^K \partial_K \epsilon_2^J - \frac{1}{2} \epsilon_1^K \partial^J \epsilon_{2K} - (\epsilon_1 \leftrightarrow \epsilon_2)$$

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- ▶ **Frame:** $e_I^\pm = \partial_I \pm \eta_{IJ} d\mathbb{X}^J$ defines splitting $\mathbb{E} = L_+ \oplus L_-$

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DFT Algebroids

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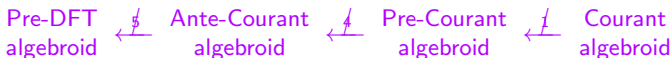
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- ▶ Other approaches:
 - Supergeometry (Deser & Stasheff '14; Deser & Sämann '16; Heller, Ikeda & Watamura '16)
 - Para-Hermitian geometry (Vaisman '12; Freidel, Rudolph & Svoboda '17; Svoboda '18)

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(Chatzistavrakidis, Jonke & Lechtenfeld '15)

$$S = \int_{\Sigma_3} \left(F_I \wedge d\mathbb{X}^I + \eta_{IJ} A^I \wedge dA^J - (\rho_+)_{I^J}(\mathbb{X}) F_J \wedge A^I \right. \\ \left. + \frac{1}{3!} T_{+IJK}(\mathbb{X}) A^I \wedge A^J \wedge A^K \right)$$

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- ▶ Gives unified description of geometric and non-geometric fluxes
 realising T-duality chain (Shelton, Taylor & Wecht '05)

$$H_{ijk} \xleftrightarrow{T_k} f_{ij}{}^k \xleftrightarrow{T_j} Q_i{}^{jk} \xleftrightarrow{T_i} R^{ijk}$$

DFT Membrane Sigma-Models — Examples

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- ▶ H-, f-, Q-, R-flux backgrounds correspond to standard Courant algebroid over different submanifolds of doubled space; doesn't include noncomm/nonassoc models which violate strong constraint

Topological String Theory

(Witten '88; Bershadsky, Cecotti, Ooguri & Vafa '93)

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$$S_A = \int_{\Sigma_2} d^2z \left(g_{a\bar{a}} \partial X^a \bar{\partial} X^{\bar{a}} - i \psi_{\bar{z}} D \chi_a - i \psi_z^{\bar{a}} \bar{D} \chi_{\bar{a}} + R_{a\bar{a}b\bar{b}} \psi_z^a \psi_z^{\bar{a}} \chi_b \chi_{\bar{b}} \right)$$
$$S_B = \int_{\Sigma_2} d^2z \left(g_{ij} \partial X^i \bar{\partial} X^j + i \eta_z^{\bar{a}} (D \rho_z^a + \bar{D} \rho_z^{\bar{a}}) g_{a\bar{a}} + i \theta_a (\bar{D} \rho_z^a - D \rho_z^{\bar{a}}) \right. \\ \left. - R_{a\bar{a}b\bar{b}} \rho_z^a \rho_z^{\bar{a}} \eta_z^{\bar{c}} \theta_c g^{c\bar{b}} \right)$$

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- ▶ DFT projection of doubled Poisson Courant sigma-model with generalized complex structure $\Pi^{IJ} = (\pi^{ij}, J^j_i, -J^i_j, 0)$ and 3-vector \mathcal{R}^{IJK} gives unified description in generalized geometry, and allows for coupling A/B-models to geometric and non-geometric fluxes

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- ▶ Exchanges Poisson and complex structure Courant algebroids within Courant algebroid for generalized complex geometry