Strings on Celestial Sphere

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based on:

St.St., T.R. Taylor:

Strings on Celestial Sphere

arXiv:1806.05688

to appear in Nucl. Phys. B

+ work to appear

Recap: studying scattering amplitudes:

deep connections between
gravity and gauge interactions

e.g.: KLT, BCJ, EYM (double-copy-construction)

(in momentum or twistor space)

traditional momentum space description:

$$p_k^\mu, \quad k = 1,\ldots,N$$
$$p_k^2 = - m_k^2$$

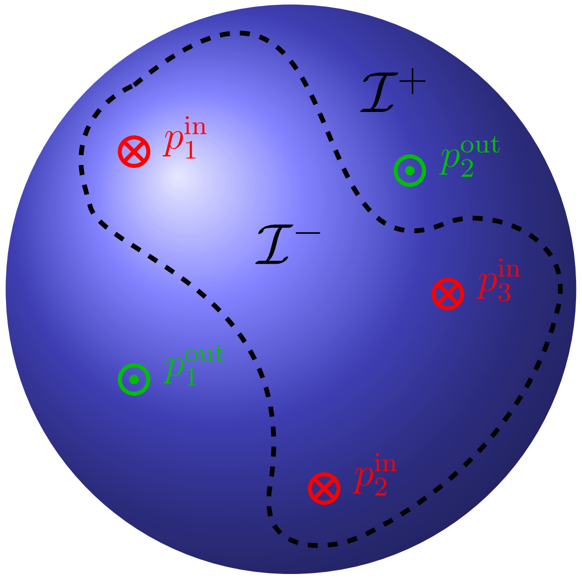
- amplitudes specified by asymptotic wave functions, which transform simply under space-time translations
- with manifest translation symmetry
- traditional amplitudes describe transitions between momentum eigenstates

D=4 Minkowski probably not the right space
to see **all** symmetries
of scattering amplitudes

Lorentz group in $\mathbf{R}^{1,D+1}$ is identical
to Euklidian D-dimensional conformal group SO(1,D+1)

Scattering amplitudes in $\mathbf{R}^{1,D+1}$
interpretation
as Euklidian D-dimensional conformal correlators

D=2: celestial sphere



Can 2D CFT on celestial sphere offer some new insight into gauge-gravity connections ?

N particles on celestial sphere

$p_k \longrightarrow (E_k, z_k, \bar{z}_k)$ represent points z_k on CS^2

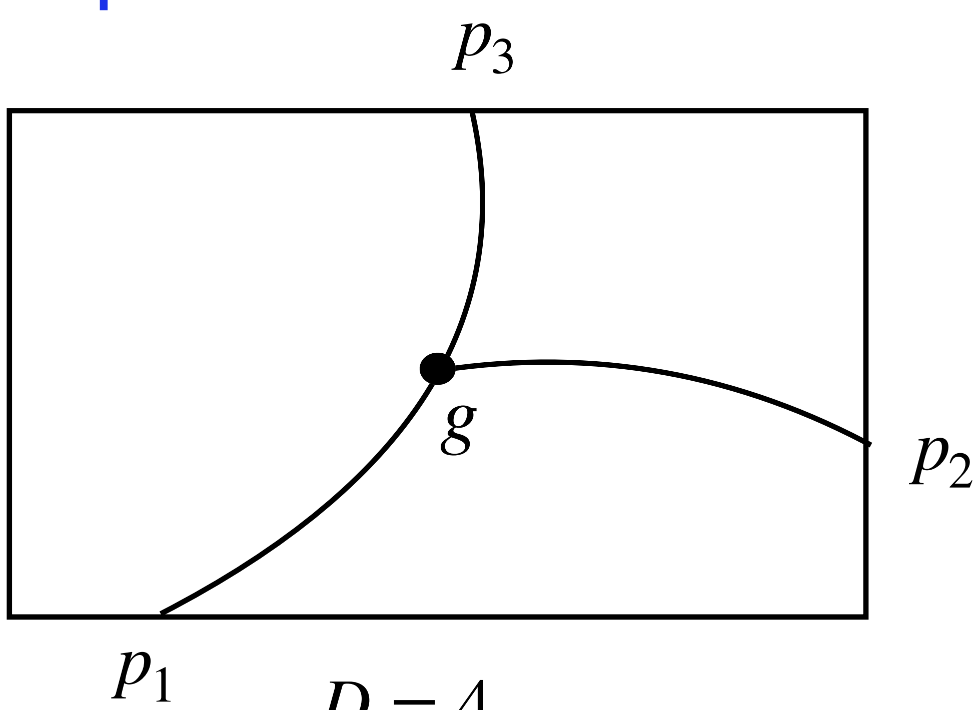
with: $$z_k = \frac{p_k^1 + i p_k^2}{p_k^0 + p_k^3} \quad , \quad E_k = p_k^0 \; , \quad (\overrightarrow{p}_k)^2 = (p_k^0)^2$$

$$p_k^\mu = E_k \left(1 \, , \frac{z_k + \bar{z}_k}{1 + |z_k|^2} \, , \frac{-i(z_k - \bar{z}_k)}{1 + |z_k|^2} \, , \frac{1 - |z_k|^2}{1 + |z_k|^2} \right)$$

$$:= \omega_k \, q_k^\mu \qquad\qquad \omega_k = \frac{2 \, E_k}{(1 + |z_k|^2)}$$

Lorentz symmetry: $$\boxed{z \rightarrow \frac{az + b}{cz + d}}$$ global conformal symmetry on CS^2

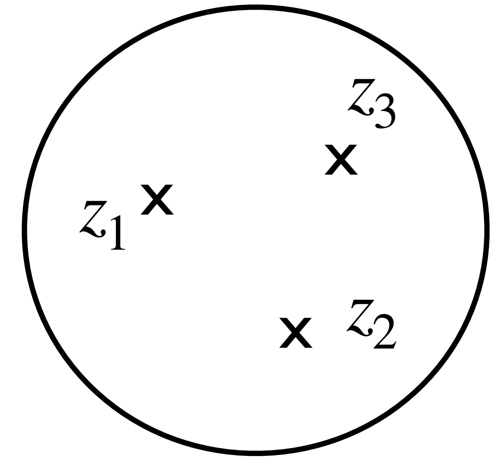
Amplitudes = conformal correlators of primary fields on CS^2



$D = 4$

$$z_k = \frac{p_k^1 + ip_k^2}{p_k^0 + p_k^3}$$

$=$



$D = 2$

$$\sim \frac{g}{|z_1 - z_2|^{h_1+h_2-h_3}\,|z_2 - z_3|^{h_2+h_3-h_1}\,|z_1 - z_3|^{h_1+h_3-h_2}}$$

$\Rightarrow$ D=4 space-time QFT correlators

D=2 Euklidian CFT correlators

D=2 CFT correlators involve conformal wave packets

In practice: in momentum basis: plane waves with momentum p

conformal basis: conformal primary wave functions Δ

→ Mellin transformation

$$\tilde{\phi}(\Delta) = \int_0^\infty d\omega \, \omega^{\Delta-1} \, \phi(\omega)$$

In the massless case, with or without spin, the transition from momentum space to conformal primary wavefunctions with Δ_j is implemented by Mellin transform

$$\mathcal{A}(\{p_i, \xi_j\}) = i(2\pi)^4 \, \delta^{(4)}\left(p_1 + p_2 - \sum_{k=3}^{N} p_k\right) \, \mathcal{M}(\{p_i, \xi_j\})$$

Mellin transform, with: $\Delta_j = 1 + i\lambda_j$

$$\tilde{\mathcal{A}}_{\{\lambda_n\}}(z_n, \bar{z}_n) = \left(\prod_{n=1}^{N} \int_0^\infty \omega_n^{i\lambda_n} \, d\omega_n\right) \, \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^{N} \omega_k q_k)$$

$$\times \, \mathcal{M}(\omega_n, z_n, \bar{z}_n)$$

construct <u>complete set</u> of on-shell wave functions in D=4:
solves D=4 wave equations
transforms as SL(2,Z) conformal primaries

	momentum basis	conformal basis
bases	plane waves	conformal primary wavefunctions
notations	$\exp(\pm ip\cdot x)$	$\varphi^{\pm}_{\Delta}(x^\mu; z, \bar{z}) = [-q(z,\bar{z})\cdot x \mp i\epsilon]^{-\Delta}$
labels	$p^\mu \ (p^2=0,\ p^0>0)$	$\Delta \in 1+i\lambda, \quad \lambda \in \mathbf{R},\ z \in CS^2$

in the massless case the change of basis is furnished
by **Mellin transform** of plane wave (or plus a shadow transform):

$$\varphi^{\pm}_{\Delta}(x^\mu; z, \bar{z}) = \int_0^\infty \omega^{\Delta-1} e^{\pm i\omega q\cdot x - \epsilon\omega} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{[-x\cdot q(z,\bar{z}) \mp i\epsilon]^\Delta}$$

D=4 scalar wave function (solution to Klein-Gordon equation),

specified by x and conformal dimension $\quad \Delta = 1 + i\lambda, \quad \lambda \in \mathbf{R}$

no dependence on D=4 momentum $\quad p^\mu$

similarly for higher spin partners, e.g. spin 1:

$$A^{\Delta,\pm}_{\mu a}(x^\mu; z, \bar{z}) = \frac{\partial_a q_\mu}{(-q \cdot x \mp i\epsilon)^\Delta} + \frac{\partial_a q \cdot x}{(-q \cdot x \mp i\epsilon)^{\Delta+1}}\; q_\mu$$

convenient gauge representative:

$$A^{\Delta,\pm}_{\mu a}(x^\mu; z, \bar{z}) = (\mp i)^\Delta\; \Gamma(\Delta)\, \frac{\partial_a q_\mu}{(-q \cdot x \mp i\epsilon)^\Delta}$$

from Mellin transform:

$$A^{\Delta,\pm}_{\mu a}(x^\mu; z, \bar{z}) = \int_0^\infty d\omega\; \omega^{\Delta-1}\; \partial_a q_\mu\; e^{\pm i\omega q\cdot x - \epsilon\omega}$$

Pasterski, Shao arXiv:1705.01027

Three-point Amplitudes

(i) Mostly-plus three-gluon amplitude

$$\mathscr{M}(-,-,+)=\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}=\frac{\omega_1\omega_2}{\omega_3}\frac{z_{12}^3}{z_{13}z_{23}}$$

$$\tilde{\mathscr{A}}(-,-,+)=4\,z_{21}^{1-i(\lambda_1+\lambda_2)}z_{23}^{i\lambda_1-1}z_{31}^{i\lambda_2-1}\,\delta(\bar{z}_{13})\delta(\bar{z}_{23})\underbrace{\int_0^\infty \omega_3^{i(\lambda_1+\lambda_2+\lambda_3)-1}d\omega_3}_{=2\pi\ \delta(\lambda_1+\lambda_2+\lambda_3)}$$

logarithmically divergent in the infra-red and in ultra-violet
any cutoff would violate SL(2,C) symmetry

conformal transformation properties, read off:

$$\left.\begin{aligned} h_1 &= \frac{i}{2}\lambda_1, & \bar{h}_1 &= 1+\frac{i}{2}\lambda_1,\\ h_2 &= \frac{i}{2}\lambda_2, & \bar{h}_2 &= 1+\frac{i}{2}\lambda_2,\\ h_3 &= 1+\frac{i}{2}\lambda_3, & \bar{h}_3 &= \frac{i}{2}\lambda_3, \end{aligned}\right\}$$

$$\Delta_n = 1+i\lambda_n$$

$$J_1=J_2=-1,\quad J_3=+1$$

Pasterski, Shao, Strominger, 2017

(ii) Mostly-plus three-graviton amplitude

$$\mathcal{M}(--,--,++) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2} = \frac{\omega_1^2\omega_2^2}{\omega_3^2}\frac{z_{12}^6}{z_{13}^2 z_{23}^2}$$

$$\tilde{\mathcal{A}}(--,--,++) = 4\, z_{21}^{2-i(\lambda_1+\lambda_2)} z_{23}^{i\lambda_1-1} z_{31}^{i\lambda_2-1}\,\delta(\bar{z}_{13})\;\delta(\bar{z}_{23}) \underbrace{\int_0^\infty \omega_3^{i(\lambda_1+\lambda_2+\lambda_3)} d\omega_3}_{=\ \text{UV divergent}}$$

The degree of this divergence will grow with the number of external gravitons, reflecting the violation of unitarity bounds at each order of perturbative Einstein's gravity

conformal transformation properties, read off:

$$h_1 = -\frac{1}{2} + \frac{i}{2}\lambda_1, \qquad \bar{h}_1 = \frac{3}{2} + \frac{i}{2}\lambda_1,$$

$$h_2 = -\frac{1}{2} + \frac{i}{2}\lambda_2, \qquad \bar{h}_2 = \frac{3}{2} + \frac{i}{2}\lambda_2,$$

$$h_3 = \frac{3}{2} + \frac{i}{2}\lambda_3, \qquad \bar{h}_3 = -\frac{1}{2} + \frac{i}{2}\lambda_3$$

$$\Delta_n = 1 + i\lambda_n$$

$$J_1 = J_2 = -2, \qquad J_3 = +2$$

(iii) Mostly-plus EYM (one graviton, two gluon) amplitude

$$\mathscr{M}(\,--\,,-\,,+\,) = \frac{\langle 12\rangle^4}{\langle 23\rangle^2} = \frac{\omega_1^2\omega_2}{\omega_3}\frac{z_{12}^4}{z_{23}^2}$$

$$\tilde{\mathscr{A}}(\,--\,,-\,,+\,) = 4\, z_{21}^{1-i(\lambda_1+\lambda_2)}\; z_{23}^{i\lambda_1-1}\; z_{31}^{i\lambda_2}\,\delta(\bar{z}_{13})\;\delta(\bar{z}_{23})\;\underbrace{\int_0^\infty \omega_3^{i(\lambda_1+\lambda_2+\lambda_3)}d\omega_3}_{=\ \mathrm{UV\ \ divergent}}$$

three-point amplitudes in string-theory: same !

Four-point Gauge Amplitudes

$$r = \frac{z_{12}\, z_{34}}{z_{23}\, z_{41}}$$

conformal invariant cross-ratio on CS^2

actually:

$$\frac{s_{23}}{s_{12}} = \frac{1}{r} = -\frac{u}{s} = \sin^2\left(\frac{\theta}{2}\right)$$

$$s = s_{12} = (p_1 + p_2)^2$$

$$u = -s_{23} = (p_2 - p_3)^2$$

θ = scattering angle in center of mass frame

$$\begin{aligned}\tilde{\mathcal{A}}(-,-,+,+) \quad &= \quad 8\pi\, \delta(r-\bar{r})\, \delta\left(\sum_{n=1}^{4} \lambda_n\right) \\ &\times \quad \left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\, \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right) r^{\frac{5}{3}}\, (r-1)^{\frac{2}{3}}\, \theta(r-1)\end{aligned}$$

Pasterski, Shao, Strominger, 2017

type I superstring theory:

$$\tilde{\mathcal{A}}_I(-,-,+,+) = 4\,(\alpha')^{\beta}\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right) \times r^{\frac{5-\beta}{3}}\,(r-1)^{\frac{2-\beta}{3}}\,I(r,\beta)$$

$$\beta := -\frac{i}{2}\sum_{n=1}^{4}\lambda_n$$

$$I(r,\beta) = -\Gamma(1-\beta)\,\frac{r}{2}\int_0^1 \frac{dx}{x}\,\left[r\ln x - \ln(1-x)\right]^{\beta-1}$$

$$I(r,\beta) = 2\pi\,\delta\left(\sum_{n=1}^{4}\lambda_n\right) + \frac{i\pi}{2}(-r)^{\beta-1}\sinh\left(\frac{1}{2}\sum_{n=1}^{4}\lambda_n\right)^{-1}\sum_{k=0}^{\infty}(-r)^{-k}\zeta\left(-\frac{i}{2}\sum_{n=1}^{4}\lambda_n-k,\{1\}^k\right)$$

Remarks:

- no α' - expansion (trivial dependence on α') !
- instead expansion in small scattering angle

$$r^{-1} = \sin^2\left(\frac{\theta}{2}\right)$$

- all heavy string modes participate on same footing
- field-theory is recovered in the limit of forward scattering $\theta = 0$

Question:

celestial CFT_2
string world-sheet CFT_2

} any relation ? →

String world-sheet as celestial sphere

$$\mathcal{M}_I(-,-,+,+) = \mathcal{M}(-,-,+,+)\ F_I(s,u)$$

with string formfactor:

$$F_I(s,u) = -\alpha' s_{12} B(-\alpha s_{12}, 1+\alpha' s_{23}) = -s\ B(-s,1-u) = \frac{\Gamma(1-s)\Gamma(1-u)}{\Gamma(1-s-u)}$$

consider high-energy limit: $B(-s,1-u) = \int_0^1 x^{-1-s}(1-x)^{as}$

saddle-point approximation: $x_0 = \dfrac{1}{1-a} \in CS^2$ $\quad a = r^{-1} < 0$

world-sheet vertex position = point on celestial sphere

= solutions to scattering equations

→ celestial sphere = world-sheet → CFT on celestial sphere related to free world-sheet CFT

Four-point Closed String Amplitudes

heterotic gauge amplitude:

$$\tilde{\mathcal{A}}_H(-,-,+,+) \;=\; 4(\alpha')^{\beta}\;\delta(r-\bar{r})\;\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\;\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$

$$\times\;\; r^{\frac{5-\beta}{3}}\,(r-1)^{\frac{2-\beta}{3}}\;H(r,\beta)$$

$$H(r,\beta) = -\,\Gamma(1-\beta)\,\frac{r}{2\pi}\int_{\mathbf{C}}\frac{d^2z}{|z|^2(1-z)}\;\Big[r\ln|z|^2-\ln|1-z|^2\Big]^{\beta-1}$$

$$H(r,\beta) = 2\pi\,\delta\!\left(\sum_{n=1}^{4}\lambda_n\right)$$

$$+\,\frac{i\pi}{2}(-r)^{\beta-1}\sinh\left(\frac{1}{2}\sum_{n=1}^{4}\lambda_n\right)^{-1}\sum_{k=0}^{\infty}(-r)^{-k}\;S^{\mathbf{c}}\!\left(-\frac{i}{2}\sum_{n=1}^{4}\lambda_n-k-1,k+1\right)$$

Four-point Gravity Amplitudes

heterotic graviton amplitude:

$$\tilde{A}_H(--,--,++,++) = 4\ (\alpha')^{\beta-1}\ \delta(r-\bar{r})\ \theta(r-1)\ \left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\ \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right) \times\ r^{\frac{11-\beta}{3}}\ (r-1)^{\frac{-1-\beta}{3}}\ G(r,\beta)$$

$$G(r,\beta) = H(r,\beta-1)$$

$$H(r,\beta) = -\,\Gamma(1-\beta)\,\frac{r}{2\pi}\int_{\mathbf{C}} \frac{d^2z}{|z|^2(1-z)}\ \Big[r\ln|z|^2 - \ln|1-z|^2\Big]^{\beta-1}$$

$$H(r,\beta) = 2\pi\,\delta\left(\sum_{n=1}^{4}\lambda_n\right) + \frac{i\pi}{2}(-r)^{\beta-1}\sinh\left(\frac{1}{2}\sum_{n=1}^{4}\lambda_n\right)^{-1}\sum_{k=0}^{\infty}(-r)^{-k}\ S^{\mathbf{c}}\left(-\frac{i}{2}\sum_{n=1}^{4}\lambda_n - k - 1, k+1\right)$$

Properties:

- Finite result for any r !

ultra-soft high energy behaviour of string formfactors ensures the convergence of energy integrals

- UV completion provided by string theory
- first calculation of graviton amplitudes in the conformal basis

Alert:

- Divergent for $r \to \infty$ (field-theory limit)

every order in the perturbative expansion of gravity violates the unitarity bounds by growing powers of energy. This uncontrollable growth at large energies poses an obstacle for transforming gravitational amplitudes to celestial sphere

Single-valued Nielsen polylogarithms

Nielsen's polylogarithm functions (real):

$$S_{n,p}(t) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dx}{x} \ \ln^{n-1} x \ \ln^p(1-xt) \quad , \quad t \in \mathbf{C}$$

in particular:

$$\boxed{S_{n,p}(1) = \zeta(n+1, \{1\}^{p-1})}$$

$$\zeta(n+1,\{1\}^{p-1}) = \zeta(n+1, \underbrace{1,\ldots,1}_{p-1}) = \sum_{n_1>n_2>\ldots>n_p} \frac{1}{n_1^{n+1} n_2 \cdots n_p}$$

Single-valued descendants:

$$S^{\mathbf{c}}(n,p) = \pi^{-1} \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_{\mathbf{C}} \frac{d^2 z}{|z|^2} (1-z)^{-1} \ \ln^{n-1}|z|^2 \ \ln^p |1-z|^2$$

$$\boxed{S^{\mathbf{c}}(n,p) = \mathrm{sv}\ S_{n,p}(1) = \mathrm{sv}\ \zeta\left(n+1,\{1\}^{p-1}\right)}$$

Concluding remarks

- explicit and compact expressions
  for string amplitudes on celestial sphere:
  intriguing examples for the study of flat space holography

- string amplitudes on celestial sphere:
  no α'- expansion (trivial dependence on α')

- all heavy string modes participate on same footing

- high-energy limit: string world-sheet = celestial sphere

- first calculation of graviton amplitudes in the conformal basis
  (with gravity UV completed):
  - ultra-soft high energy behaviour of string formfactors ensures the convergence of energy integrals

  - important for the soft graviton theorem in this basis

Can 2D CFT on celestial sphere offer some new insight into gauge-gravity connections ?

for YM scattering amplitudes soft gluon theorem
can be phrased in terms of
tree-level Ward identities of D=2 Kac-Moody symmetry $J(z)$

He, Mitra, Strominger, arXiv:1503.02663

for quantum gravity scattering amplitudes
the global conformal group (Lorentz symmetry) is enhanced
to infinite-dimensional local D=2 conformal symmetry $T(z)$
(full Virasoro symmetry)

Kapec, Lysov, Pasterski, Strominger, arXiv:1406.3312

↓

understanding the nature of 2D CFT on celestial sphere
would enable a **holographic description of flat spacetime**

Thank you