The fluid road to flat holography

Marios Petropoulos

CPHT – Ecole Polytechnique – CNRS

Conference on Geometry and Strings

Schloss Ringberg – Tegernsee – July 2018

with L. Ciambelli, C. Marteau, A. Petkou, K. Siampos

Highlights

Foreword

Carrollian geometries and Carrollian hydrodynamics

Relativistic fluid / AdS gravity

Ricci-flat gravity / Carrollian fluid

Summary

AdS/CFT

Holographic correspondence: *microscopic type IIB string – super-Yang–Mills duality with* $g_{string} \leftrightarrow 1/g_{YM}$ *involving* AdS_5

Anti-de Sitter space: *homogeneous spacetime with* $\Lambda < 0$

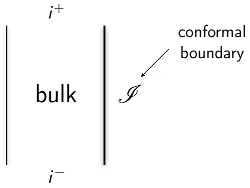


Figure: AdS_4 Penrose–Carter diagram

Thoughts on flat or de Sitter extensions

Minkowski spacetime

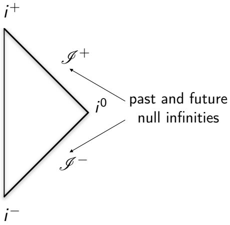


Figure: Minkowski Penrose–Carter diagram

Asymptotically flat holographic correspondence

- *microscopically: hard – perhaps easier in $2+1$ dimensions or in higher-spin theories* [as e.g. in Gaberdiel, Gopakumar '11]
- *macroscopically: better prospects more generally*

Macroscopic approach: fluid/gravity correspondence

Branch of AdS/CFT: *Einstein's and* ***relativistic*** *Euler's equations*

[Bhattacharyya, Haack, Hubeny, Loganayagam, Minwalla, Rangamani, Yarom, ... '07]

$$\text{Einstein asymptotically locally AdS spacetime } \mathscr{E} \text{ with } \Lambda < 0$$
$$\updownarrow$$
$$\text{relativistic fluid on } \mathscr{I} = \partial\mathscr{E} \equiv \text{ conformal boundary}$$

Historically: ***non-relativistic incompressible-fluid*** *equations emerge from perturbations of the black-hole horizon – membrane paradigm* *[Damour '79; Price, Thorne '86; Oz et al '09–16]* *– possibly inaccurate*

General: *in arbitrary bulk dimension* $D+1$

Challenges in flat-spacetime holography

1. *Which spacetime would replace the AdS conf. bry. $\mathscr{I}$ and what is its geometry?*
2. *Which are the macroscopic degrees of freedom hosted by $\mathscr{S}$, what is their dynamics, how are the observables packaged?*
3. *What is the microscopic dynamics (action)?*

Known facts: *mostly in $2+1$ bulk dim* *[Bagchi et al '09–16; Barnich et al '10–12; Duval et al '14; Hartong '15]*

- *null infinity $\mathscr{I}^{\pm}$ plays a privileged role*
- *asymptotic symmetry towards $\mathscr{I}^{\pm}$ is* ***conformal Carrollian – BMS*** *[Bondi, van der Burg, Metzner, Sachs '62]*

Wide dispersion *[Strominger et al '10–17; Compère et al '11–12; Caldarelli et al '13; Klemm et al '14]*

- *"holography" involving Navier–Stokes equations – Galilean incompressible fluids / "holography" involving relativistic fluids at finite r with Brown–York energy–momentum tensor / specific attempts in Rindler – none respects asymptotic symmetry*
- $3+1$*-dim Minkowski* $\leftrightarrow$ 2*-dim CFT on spatial section of* $\mathscr{I}^+$ *again cannot be the final word symmetry-wise*

Our aim: *unravel a clear & generalizable fluid/gravity* $3+1$ *pattern*

The method: *setting* $k \to 0$ *inside the appropriate expansion that reconstructs the bulk from the boundary* ($\Lambda = -3k^2$)

Ricci-flat spacetime $\leftrightarrow$ conformal Carrollian fluid on $\mathscr{S} \times \mathbb{R}$ at $\mathscr{I}^+$

Highlights

Galilean vs. Carrollian contractions of Poincaré group

[Lévy–Leblond '65]

Both non-relativistic limits with decoupling of time

Lorentz boosts: $\begin{cases} t' = \gamma \left(t - \frac{V_i}{c^2} x^i \right) \\ x^{i\prime} = \gamma \left(x^i - V^i t \right) \end{cases}$

with $\gamma = {}^1/\sqrt{1 - V^2/c^2}$

- *Galilean* limit $c \to \infty$: $\begin{cases} t' = t \\ x^{i\prime} = x^i - V^i t \end{cases}$

- *Carrollian* limit $c \to 0$: everything is at rest $\Rightarrow \begin{cases} \frac{V_i}{c} \to 0 \\ \gamma \to 1 \end{cases}$

  but $\lim_{c \to 0} V_i/c^2 = B_i \Rightarrow \begin{cases} t' = t - B_i x^i \\ x^{i\prime} = x^i \end{cases}$

Carrollian covariance in d spatial plus 1 time dimensions

Geom. on $\mathscr{S} \times \mathbb{R}$: $d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j \quad \Omega(t, \mathbf{x}) \quad \boldsymbol{b} = b_i(t, \mathbf{x}) dx^i$

- Covariant under Carrollian diffs.: $t' = t'(t, \mathbf{x}) \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$
  - Jacobian: $J(t, \mathbf{x}) = \frac{\partial t'}{\partial t} \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i} \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}$
  - transfs.: $a'_{ij} = a_{kl} J^{-1k}_{\ i} J^{-1l}_{\ j} \quad \Omega' = \frac{\Omega}{J} \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}_{\ k}$

[Bekaert, Bergshoeff, Duval, Gibbons, Gomis, Hartong, Horvathy, Longhi, Morand, Obers]

- The geometry may have isometries but we do not assume any
  - Carrollian group realized in tangent space
  - Carrollian *invariance* iff $a_{ij} = \delta_{ij}$, $\Omega = 1$, $b_i = \text{const.}$
- The geometry can be equipped with Ehresmann connection / Weyl connection with curvature / Weyl curvature

Relativistic uplift

Riemannian $d+1$-dim fibration à la Randers–Papapetrou

- $\mathrm{d}s^2 = -c^2\left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j$
- reproduces the wanted transformation under $x^{0\prime} = x^{0\prime}(x^0, \mathbf{x})$ $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$ (here $x^0 = ct$) – Carrollian diffeomorphisms

$$J^\mu_\nu(t,\mathbf{x}) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} = \begin{pmatrix} J(t,\mathbf{x}) & cj_j(t,\mathbf{x}) \\ 0 & J^i_j(\mathbf{x}) \end{pmatrix}$$

At $c \to 0$ this provides the Carrollian geometry on $\mathscr{S} \times \mathbb{R}$

Relativistic fluids

Obey $\nabla_\mu T^{\mu\nu} = 0$ *with*

$$T^{\mu\nu} = (\varepsilon + p)\frac{u^\mu u^\nu}{c^2} + pg^{\mu\nu} + \frac{u^\mu q^\nu}{c^2} + \frac{u^\nu q^\mu}{c^2} + \tau^{\mu\nu}$$

- $\|\mathrm{u}\|^2 = -c^2$, $u^0 = \gamma c$, $u^i = \gamma v^i$
- ε, p: energy density and pressure
- q^μ, $\tau^{\mu\nu}$: heat current and viscous stress tensor – transverse

$$u^\mu q_\mu = 0 \quad u^\mu \tau_{\mu\nu} = 0$$

q^i and τ^{ij}

- carry all information on heat exchange and friction processes
- are usually expressed in terms of a u-derivative expansion

Carrollian limit: Carrollian fluid dynamics

Relativistic fluid on Randers–Papapetrou at $c \to 0$

[worth comparing with de Boer, Hartong, Obers, Sybesma, Vandoren '17]

- Kinematics: v^i must vanish faster than c

$$\lim_{c\to 0} \frac{v^i}{c^2} = \Omega\beta^i$$

  avoids blow-ups without trivializing

- Transport: limit inside the fluid data (microscopic justification yet to come)
  - $\varepsilon \to \varepsilon$, $p \to p$
  - $q^i \to Q^i + c^2\pi^i$ and $\tau^{ij} \to -\frac{1}{c^2}\Sigma^{ij} - \Xi^{ij}$

Limit inside the relativistic-fluid equations

$$\begin{cases} 0 = \frac{c}{\Omega}\nabla_\mu T^\mu{}_0 = \frac{1}{c^2}\mathcal{F} + \mathcal{E} + \mathrm{O}\left(c^2\right) \\ 0 = \nabla_\mu T^{\mu i} = \frac{1}{c^2}\mathcal{H}^i + \mathcal{G}^i + \mathrm{O}\left(c^2\right) \end{cases}$$

→ *Carrollian equations*

- scalar equations: $\mathcal{E} = 0 \quad \mathcal{F} = 0$
- vector equations: $\mathcal{G}^j = 0 \quad \mathcal{H}^i = 0$

→ *Covariant under Carrollian diffs.*

- $\mathcal{E}' = \mathcal{E} \quad \mathcal{F}' = \mathcal{F}$
- $\mathcal{G}'^i = J^i_j \mathcal{G}^j \quad \mathcal{H}'^i = J^i_j \mathcal{H}^j$

Highlights

The bulk reconstruction

Given the "initial data"

- *boundary metric* ds^2 *(neither flat nor conformally flat)*
- *conserved energy–momentum tensor* T

Two options exist to get perturbatively the asymptotically AdS bulk ($\Lambda = -3k^2$):

1. Fefferman–Graham expansion: mathematically robust, not resummable, does not discriminate asympt. locally vs. globally AdS bulks, with singular $k \to 0$ limit
2. Derivative expansion: designed for fluid/gravity correspondence requires an extra piece of bry. data – time-like hydrodynamic congruence u (possibly constrained)

Fefferman–Graham: expansion of any $3+1$-dim Einstein metric for large r in a specific gauge (no lapse/shift) [Fefferman, Graham '85]

$$\mathrm{d}s^2_{\text{bulk}} = \frac{\mathrm{d}r^2}{k^2 r^2} + r^2 \mathrm{d}s^2_{\text{bry.}} + \cdots + \frac{16\pi G}{3(kr)} T_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \cdots$$

- boundary metric: leading term
- boundary energy–momentum: subleading term

The derivative expansion – fluid/gravity [Bhattacharyya et al '07; Haack et al '07]

- Guideline: Weyl covariance – the bulk metric must be invariant under boundary Weyl transformations
- Output: $\mathrm{d}s^2_{\mathrm{bulk}} =$ complicated expression based on the boundary data & their derivatives – order by order

Advantages [Leigh et al '10; Caldarelli et al '12; Mukhopadhyay et al '13; Gath et al '15]

- potentially resummable
- controls locally vs. globally AdS bulks
- the limit $k \to 0$ is regular: flat holography

A parenthesis for lunch

- *The velocity u^μ looks redundant – is it arbitrary?*
- *Does every relativistic fluid have a dual Einstein spacetime?*

Naively yes but ...

...the derivative expansion is not manifestly δu-invariant & the boundary data $g_{\mu\nu}$, $T_{\mu\nu}$ and u^μ are subject to remarkable 3rd-order differential constraints [Ciambelli et al '17; Campoleoni et al '18]

The resummation in 4 dimensions [Caldarelli et al '12; Mukhopadhyay et al '13; Gath et al '15]

Assuming u shear-free a resummation is performed ($\Lambda = -3k^2$):

$$\mathrm{d}s^2_{\text{res. Einstein}} = 2\frac{\mathrm{u}}{k^2}(\mathrm{d}r + r\mathrm{A}) + r^2\mathrm{d}s^2 + \frac{\mathrm{S}}{k^4} + \frac{\mathrm{u}^2}{k^4\rho^2}\left(8\pi G\varepsilon r + c\gamma\right)$$

- boundary metric: $\mathrm{d}s^2 = -k^2\left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$
- velocity: $\mathrm{u} = \frac{1}{\Omega}\partial_t \Rightarrow \|\mathrm{u}\|^2 = -k^2$ [*cf.* Barnich, Gomberoff, González, '12]
- conformal-fluid energy density: $\varepsilon = 2p = \frac{1}{k^2}T_{\mu\nu}u^\mu u^\nu$
- genuine resummation: $\rho^2 = r^2 + \frac{1}{2k^4}\omega_{\alpha\beta}\omega^{\alpha\beta} = r^2 + \gamma^2$
- S: Weyl-covariant boundary tensor
- c: Cotton (3rd-order der. of the metric) $\nabla^\lambda C_{\lambda\mu} = C_\lambda{}^\lambda = 0$

$$C_{\mu\nu} = \frac{3c}{2}\frac{u_\mu u_\nu}{k} + \frac{ck}{2}g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}$$

Resummability conditions among boundary $g_{\mu\nu}$, $T_{\mu\nu}$ and shearless u^μ

- transverse duality (with $\eta_{\mu\nu} = -\frac{u^\rho}{k}\eta_{\rho\mu\nu}$)

$$q_\mu = \frac{1}{8\pi G}\eta^\nu{}_\mu c_\nu \quad \tau_{\mu\nu} = -\frac{1}{8\pi G k^2}\eta^\rho{}_\mu c_{\rho\nu}$$

  boundary "electric–magnetic gravitational duality" [Bakas '08]

- energy–momentum conservation $\nabla^\lambda T_{\lambda\mu} = 0$: equation for all boundary data ε, a_{ij}, Ω and b_i

Output: algebraically special Einstein spacetimes – Goldberg–Sachs generalizations – asymptotically ***locally*** *AdS*

- Kerr–Taub–NUT (perfect fluids)
- Robinson–Trautman
- Plebański–Demiański
- ...

Highlights

Foreword

Carrollian geometries and Carrollian hydrodynamics

Relativistic fluid / AdS gravity

Ricci-flat gravity / Carrollian fluid

Summary

The Ricci-flat limit $\Lambda = -3k^2 \to 0$

In the boundary data: $k \equiv$ *speed of light*

$$
\begin{aligned}
\mathrm{d}s^2 &= -k^2\left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j \\
&\underset{k\to 0}{\longrightarrow} 0 \times \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + \mathrm{d}\ell^2
\end{aligned}
$$

$\Rightarrow$ Carrollian limit: $\mathscr{I} \to \mathscr{S} \times \mathbb{R}$ reached at $\mathscr{I}^+$ (or $\mathscr{I}^-$)

The relativistic conformal fluid $\underset{k\to 0}{\longrightarrow}$ *Carrollian conformal fluid*

The Ricci-flat derivative expansion

The limit of the Einstein derivative expansion is well-defined and boundary Weyl- and Carrollian-diffeomorphism-covariant

$$
\begin{aligned}
\lim_{k\to 0} \mathrm{d}s^2_{\text{res. Einstein}} &= \mathrm{d}s^2_{\text{res. flat}} \\
&= -2\left(\Omega \mathrm{d}t - \boldsymbol{b}\right)\left(\mathrm{d}r + r\boldsymbol{\alpha} + \frac{r\theta\Omega}{2}\mathrm{d}t\right) + r^2 \mathrm{d}\ell^2 \\
&\quad + \boldsymbol{s} + \frac{\left(\Omega \mathrm{d}t - \boldsymbol{b}\right)^2}{\rho^2}\left(8\pi G\varepsilon r + c * \varpi\right)
\end{aligned}
$$

- $\mathrm{d}\ell^2$, Ω, $\boldsymbol{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{s}$, θ, $*\varpi$: Carrollian-geometric data
- $\rho^2 = r^2 + *\varpi^2$: genuine resummation
- $\varepsilon = 2p$: conformal Carrollian-fluid energy density
- c: descendent of the Cotton

The boundary Cotton tensor $\underset{k\to 0}{\longrightarrow}$ *conformal Carrollian 3rd-derivative geometric objects*

$$c,\ c_\mu,\ c_{\mu\nu}\ \underset{k\to 0}{\longrightarrow}\ c,\ \chi_i,\ \psi_i,\ X_{ij},\ \Psi_{ij}$$

The Carrollian-boundary resummability conditions (shear-free case)

$$Q_i = \frac{1}{8\pi G}\eta^j{}_i\chi_j \quad \Sigma_{ij} = \frac{1}{8\pi G}\eta^l{}_i X_{lj} \quad \Xi_{ij} = \frac{1}{8\pi G}\eta^l{}_i \Psi_{lj}$$

Output: *algebraically special Ricci-flat spactimes – Goldberg–Sachs asymptotically* ***locally*** *flat*

Example I: stationary

Boundary data: $d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta}$, $\Omega = 1$, $\boldsymbol{b} = b_\zeta d\zeta + b_{\bar{\zeta}} d\bar{\zeta}$

$\xi_{ij} = 0$, $\varphi_i = 0$, $\theta = 0$, $\boldsymbol{\varpi} = \frac{1}{2}\mathrm{d}\boldsymbol{b}$ time-independent

Perfect fluid: ε, $\boldsymbol{\pi} = 0$, $\boldsymbol{Q} = \frac{1}{8\pi G} * \chi = 0$, $\boldsymbol{\Sigma} = \frac{1}{8\pi G} * \boldsymbol{X} = 0$,
$\boldsymbol{\Xi} = \frac{1}{8\pi G} * \boldsymbol{\Psi} = 0$

fluid equations: $\varepsilon = {}^{M}/_{4\pi G}$, $P(\zeta, \bar{\zeta})$, $b_\zeta(\zeta, \bar{\zeta})$, $b_{\bar{\zeta}}(\zeta, \bar{\zeta})$

Bulk: Ricci-flat Kerr–Taub–NUT family plus A-metrics

Example II: time-dependent

Boundary data: $d\ell^2 = \frac{2}{P^2(t,\zeta,\bar\zeta)} d\zeta d\bar\zeta$, $\Omega = 1$, $b_i = 0$

- $\xi_{ij} = 0$, $\varpi_{ij} = 0$, $\varphi_i = 0$, $\theta = -2\partial_t \ln P$
- "Cotton": $c = 0$, $\boldsymbol{\psi} = 0$, $\boldsymbol{\chi} = \frac{i}{2}\left(\partial_\zeta K d\zeta - \partial_{\bar\zeta} K d\bar\zeta\right)$, $\boldsymbol{\Psi} = 0$,
  $\boldsymbol{X} = \frac{i}{P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right) d\zeta^2 - \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta} \ln P\right) d\bar\zeta^2\right)$

$(K = 2P^2\partial_{\bar\zeta}\partial_\zeta \ln P)$

Fluid data: ε, $\boldsymbol{\pi} = 0$, $\boldsymbol{Q} = \frac{1}{8\pi G} * \chi$, $\boldsymbol{\Sigma} = \frac{1}{8\pi G} * \boldsymbol{X}$, $\boldsymbol{\Xi} = 0$

- momentum equation: $\partial_i \varepsilon = 0 \Rightarrow \varepsilon(t) = {}^{M(t)}/_{4\pi G}$
- energy equation: $\Delta\Delta \ln P + 12M\partial_t \ln P - 4\partial_t M = 0$

Bulk: Ricci-flat Robinson–Trautman (algebraically special solution radiating & stabilizing at Schwarzschild)

Remark on example II – Robinson–Trautman

The Carrollian boundary fluid stress tensor $\boldsymbol{\Sigma}$ is related to the bulk **news**

$$\boldsymbol{\Sigma} = \frac{1}{8\pi G} * \boldsymbol{X} = -\frac{1}{8\pi G}\partial_t \boldsymbol{N}$$

with

$$\boldsymbol{N} = \frac{2}{P}\left(\partial_\zeta^2 P \, \mathrm{d}\zeta^2 + \partial_{\bar{\zeta}}^2 P \, \mathrm{d}\bar{\zeta}^2\right)$$

a weight-0 Carrollian rank-2 tensor carrying all news information

More generally

$$*\boldsymbol{X} = -\hat{\mathscr{D}}_t \boldsymbol{N}$$

hence "no Cotton $\leftrightarrow$ no news"

Highlights

Foreword

Carrollian geometries and Carrollian hydrodynamics

Relativistic fluid / AdS gravity

Ricci-flat gravity / Carrollian fluid

Summary

About Carrollian hydrodynamics on a_{ij}, Ω, b_i

- obtained from Randers-Papapetrou relativistic fluids at $c \to 0$
- described in terms of ε, p, β_i, Q_i, π_i, Σ_{ij}, Ξ_{ij}
- obey Carrollian-covariant (possibly conformal) set of equations

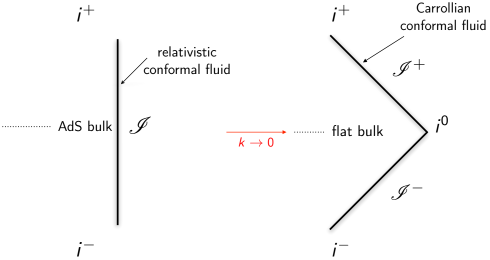
Aside observation: abandon the revered energy–momentum tensor and its equation "$\mathbf{div} T = 0$"!

In example II with $\mathrm{d}\ell^2 = \frac{2}{P^2(t,\zeta,\bar{\zeta})}\mathrm{d}\zeta \mathrm{d}\bar{\zeta}$ the conformal Carrollian fluid equations read (here $\varepsilon = 2p$, $\boldsymbol{\pi} = 0$, $\boldsymbol{\Xi} = 0$):

$$\begin{cases} 3\varepsilon\partial_t \ln P - \partial_t \varepsilon - \mathbf{div}\, \boldsymbol{Q} = 0, \\ \mathbf{grad}\, p = 0 \\ \partial_t \boldsymbol{Q} - 2\boldsymbol{Q}\partial_t \ln P - \mathbf{div}\, \boldsymbol{\Sigma} = 0 \end{cases}$$

About flat holography and Carrollian fluids

1. Which spacetime would replace the AdS conf. bry. $\mathscr{I}$ and what is its geometry? $\mathscr{S} \times \mathbb{R}$ at $\mathscr{I}^+$ equipped with Carrollian geometry
2. Which are the degrees of freedom hosted by $\mathscr{S}$, what is their dynamics, how are the observables packaged? Carrollian conformal fluid with ε, Q_i, π_i, Σ_{ij}, Ξ_{ij} obeying Carrollian fluid equations



Confirms previous results regarding Carroll, null infinity and flat holography

Calls for better microscopic understanding of the $c \to 0$ limit

- First step: Boltzmann equation
- QFT on D-dim null surfaces ($D = d + 1$) with conformal Carrollian symmetries: Hilbert space, Green's functions, unitarity, ...
  - *not* a $D-1$-dim CFT *but* the $\lim_{c\to 0}$ of a D-dim CFT with $\mathrm{BMS}(D+1) \equiv \mathrm{CCarroll}(D) = \mathrm{Conf}(D-1) \ltimes \mathcal{T}(D)$
  - fundamental observables are *not* a conserved $T_{ij}(\mathbf{x})$ (for $d = 2$ $T_{\zeta\zeta}(\zeta)$ and $T_{\bar{\zeta}\bar{\zeta}}(\bar{\zeta})$) [as suggested in He *et al.* '15–17 vs. Fareghbal *et al.* '15; Bagchi *et al.* '16]

Highlights

Galilean covariance

Galilean fluids

Carrollian covariance

Carrollian fluids

Conformal Carrollian geometry and conformal Carrollian fluids

Curvature and Cotton

Galilean covariance in d spatial dimensions

Geometry: $d\ell^2 = a_{ij}(t,\mathbf{x})dx^i dx^j, \quad \Omega = \Omega(t), \quad \mathbf{w} = w^i(t,\mathbf{x})\partial_i$

- Galilean diffs.: $t' = t'(t), \quad \mathbf{x}' = \mathbf{x}'(t,\mathbf{x})$
- Jacobian: $J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t,\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial t}, \quad J^i_j(t,\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}$
- transfs.: $a'_{ij} = a_{kl} J^{-1k}{}_i J^{-1l}{}_j, \; \Omega' = \frac{\Omega}{J}, \; w'^k = \frac{1}{J}\left(J^k_i w^i + j^k\right)$
- absolute Newtonian time (invariant): $\Omega(t)\mathrm{d}t$

[Cartan, Bekaert, Bergshoeff, Duval, Gibbons, Gomis, Hartong, Horvathy, Longhi, Morand, Obers]

Do not confuse with the Galilean group of invariance present when $a_{ij} = \delta_{ij}$, $\Omega = 1$, $w^i =$ *constant*

$$\begin{cases} t' = t + t_0, \\ x'^k = R^k_i x^i + V^k t + x^k_0 \end{cases}$$

Simple realization and relativistic uplift

Particle: $x^i = x^i(t), \quad v^i = {}^{dx^i}/_{dt}, \; \mathbf{v} = v^i \partial_i$

- transfs.: $v'^k = \frac{1}{J}\left(J^k_i v^i + j^k\right), \quad \frac{\mathbf{v}-\mathbf{w}}{\Omega}$ d-dim vector
- free dynamics: $\mathcal{L}(\mathbf{v}, \mathbf{x}, t) = \frac{1}{2\Omega^2} a_{ij}\left(v^i - w^i\right)\left(v^j - w^j\right)$

Relativistic uplift: $d+1$*-dim Zermelo form*

- $\mathrm{d}s^2 = -\Omega^2 c^2 \mathrm{d}t^2 + a_{ij}\left(\mathrm{d}x^i - w^i \mathrm{d}t\right)\left(\mathrm{d}x^j - w^j \mathrm{d}t\right)$
- form-invariant under $J^\mu_\nu(x) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} = \begin{pmatrix} J(t) & 0 \\ \frac{j^i(t,\mathbf{x})}{c} & J^i_j(t,\mathbf{x}) \end{pmatrix}$
- relevant limit: $c \to \infty$

Highlights

Galilean covariance

Galilean fluids

Carrollian covariance

Carrollian fluids

Conformal Carrollian geometry and conformal Carrollian fluids

Curvature and Cotton

Non-relativistic Galilean fluid

Relativistic fluid on Zermelo at $c \to \infty$: Galilean fluid

$$u^0 = \frac{c}{\Omega} + \mathrm{O}\left(1/c\right) \quad u^i = \frac{v^i}{\Omega} + \mathrm{O}\left(1/c^2\right)$$

- v^i
- $e, \; p, \; \varrho$
- $q_i \to Q_i$ and $\tau_{ij} \to -\Sigma_{ij}$

Galilean-covariant equations on $a_{ij}(t, \mathrm{x}), \; \Omega(t), \; w^i(t, \mathrm{x})$

Highlights

Galilean covariance

Galilean fluids

Carrollian covariance

Carrollian fluids

Conformal Carrollian geometry and conformal Carrollian fluids

Curvature and Cotton

Carrollian covariance in d spatial dimensions

Geometry on $\mathscr{S}$: $d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j \quad \Omega(t, \mathbf{x}) \quad \boldsymbol{b} = b_i(t, \mathbf{x}) dx^i$

- Carrollian diffs.: $t' = t'(t, \mathbf{x}) \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$
- Jacobian: $J(t, \mathbf{x}) = \frac{\partial t'}{\partial t} \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i} \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}$
- transfs.: $a'_{ij} = a_{kl} J^{-1k}_{\ \ i} J^{-1l}_{\ \ j} \quad \Omega' = \frac{\Omega}{J} \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}_{\ \ k}$

Do not confuse with the Carrollian group of invariance present when $a_{ij} = \delta_{ij}$, $\Omega = 1$, $b_i =$ constant (here realized in tangent space)

$$\begin{cases} t' = t + B_i x^i + t_0, \\ x'^k = R^k_i x^i + x^k_0 \end{cases}$$

Simple realization and relativistic uplift

Extended object: $t = t(\mathbf{x})$, $\beta_i = \Omega\partial_i t - b_i$, $\boldsymbol{\beta} = \beta_i dx^i$

- transfs.: $\beta'_k = \beta_i J^{-1i}{}_k$ (d-dim form)
- free dynamics: $\mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x}) = \frac{1}{2} a^{ij} \left(\Omega\partial_i t - b_i\right) \left(\Omega\partial_j t - b_j\right)$

Relativistic uplift: $d+1$-dim Randers–Papapetrou form

- relevant limit: $c \to 0$
- $\mathrm{d}s^2 = -c^2 \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j$
- form-invariant under Carrollian diffeomorphisms ($x^0 = ct$)

$$J^\mu_\nu(t, \mathbf{x}) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} = \begin{pmatrix} J(t, \mathbf{x}) & c j_j(t, \mathbf{x}) \\ 0 & J^i_j(\mathbf{x}) \end{pmatrix}$$

More on Carrollian geometries

- *isometries*
- *time and space connections, covariant derivatives, curvatures*
- *time and space Weyl connection, Weyl curvature*

Example: Carrollian space derivative $\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t$

- transfs.: $\hat{\partial}'_i = J^{-1j}_{\ \ i}\hat{\partial}_j$
- connection: $\hat{\gamma}^i_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right)$
- covariant metric-compatible derivative: $\hat{\boldsymbol{\nabla}} = \hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\gamma}}$

Similarly: Weyl-covariant metric-compatible derivatives $\hat{\mathscr{D}}_i$, $\hat{\mathscr{D}}_t$
built on $\varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right)$ and $\theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a}$

Highlights

Galilean covariance

Galilean fluids

Carrollian covariance

Carrollian fluids

Conformal Carrollian geometry and conformal Carrollian fluids

Curvature and Cotton

Carrollian limit: Carrollian fluid

Relativistic fluid on Randers–Papapetrou at $c \to 0$: kinematics

v^i must vanish faster than c:

$$v^i = c^2 \Omega \beta^i + \mathrm{O}\left(c^4\right)$$

avoids blow-ups without trivializing

- kinematic variable $\beta^i = -\frac{\Omega u_i}{c u_0} - b_i = {}^{v_i}/{}_{c^2\Omega\left(1 - \frac{v^j b_j}{\Omega}\right)}$
- $u^0 = \gamma c = {}^{c}/{}_{\Omega} + \mathrm{O}\left(c^3\right) \quad u^i = \gamma v^i = c^2 \beta^i + \mathrm{O}\left(c^4\right)$
- $u_0 = -c\Omega + \mathrm{O}\left(c^3\right) \quad u_i = c^2\left(b_i + \beta_i\right) + \mathrm{O}\left(c^4\right)$

[worth comparing with de Boer, Hartong, Obers, Sybesma, Vandoren '17]

Limit inside the fluid data (microscopic justification yet to come)

- $\varepsilon, \ p$
- $q^i \to Q^i + c^2 \pi^i$ and $\tau^{ij} \to -\frac{1}{c^2}\Sigma^{ij} - \Xi^{ij}$

Inside the perfect-fluid energy–momentum tensor

$$\gamma = \frac{1 + c^2 \boldsymbol{\beta} \cdot \boldsymbol{b}}{\Omega \sqrt{1 - c^2 \boldsymbol{\beta}^2}} = \frac{1}{\Omega} \left(1 + \frac{c^2}{2} \boldsymbol{\beta} \cdot (\boldsymbol{\beta} + 2\boldsymbol{b}) + \mathrm{O}\left(c^4\right) \right)$$

$$\begin{cases} T_{\mathrm{perf}}{}^{0}{}_{0} = -\varepsilon - c^2 (\varepsilon + p) \beta^k \left(b_k + \beta_k\right) + \mathrm{O}\left(c^4\right) \\ c\Omega T_{\mathrm{perf}}{}^{0}{}_{i} = c^2 (\varepsilon + p) \left(b_i + \beta_i\right) + \mathrm{O}\left(c^4\right) \\ \frac{c}{\Omega} T_{\mathrm{perf}}{}^{j}{}_{0} = -c^2 (\varepsilon + p) \beta^j + \mathrm{O}\left(c^4\right) \\ T_{\mathrm{perf}}{}^{j}{}_{i} = p \delta^j_i + c^2 (\varepsilon + p) \beta^j \left(b_i + \beta_i\right) + \mathrm{O}\left(c^4\right) \end{cases}$$

Inside the relativistic-fluid equations

$$\begin{cases} 0 = \frac{c}{\Omega}\nabla_\mu T^{\mu}{}_{0} = \frac{1}{c^2}\mathcal{F} + \mathcal{E} + \mathrm{O}\left(c^2\right) \\ 0 = \nabla_\mu T^{\mu i} = \frac{1}{c^2}\mathcal{H}^i + \mathcal{G}^i + \mathrm{O}\left(c^2\right) \end{cases}$$

→ *Carrollian equations*

- scalar equations: $\mathcal{E} = 0 \quad \mathcal{F} = 0$
- vector equations: $\mathcal{G}^j = 0 \quad \mathcal{H}^i = 0$

→ *Covariant under Carrollian diffs.*

- $\mathcal{E}' = \mathcal{E} \quad \mathcal{F}' = \mathcal{F}$
- $\mathcal{G}'^i = J^i_j \mathcal{G}^j \quad \mathcal{H}'^i = J^i_j \mathcal{H}^j$

Carrollian hydrodynamics with $\boldsymbol{\beta} = 0$

Scalar equations

- $\mathcal{E} = -\frac{1}{\Omega}\partial_t\varepsilon - (\varepsilon + p)\theta - \hat{\nabla}_i Q^i - 2\varphi_i Q^i + \Xi^{ij}\xi_{ij} + \frac{1}{2}\Xi^i{}_i\theta = 0$
- $\mathcal{F} = \Sigma^{ij}\xi_{ij} + \frac{1}{2}\Sigma^i{}_i\theta = 0$

Vector equations

- $\mathcal{G}_j = \hat{\partial}_j p + (\varepsilon + p)\varphi_j + \frac{1}{\Omega}\partial_t\pi_j + \pi_j\theta + 2Q^i\varpi_{ij} - \hat{\nabla}_i\Xi^i{}_j - \varphi_i\Xi^i{}_j = 0$
- $\mathcal{H}^i = \frac{a^{ij}}{\Omega}\partial_t Q_j + Q^i\theta - \hat{\nabla}_j\Sigma^{ji} - \varphi_j\Sigma^{ji} = 0$

Remarks

- *more involved for* $\beta^i \neq 0$
- *more elegant for conformal fluids*
- *not based on an "energy–momentum tensor"*

φ_i, θ, ξ_{ij}, ϖ_{ij}: kinematic observables

Relativistic origin: acceleration, expansion, shear, vorticity

- $\frac{a_i}{c^2} = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right) + \mathrm{O}\left(c^2\right) = \varphi_i + \mathrm{O}\left(c^2\right)$
- $\Theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a} + \mathrm{O}\left(c^2\right) = \theta + \mathrm{O}\left(c^2\right)$
- $\sigma_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij} - \frac{1}{d} a_{ij}\partial_t \ln\sqrt{a}\right) + \mathrm{O}\left(c^2\right) = \xi_{ij} + \mathrm{O}\left(c^2\right)$
- $\frac{\omega_{ij}}{c^2} = \partial_{[i} b_{j]} + \frac{1}{\Omega} b_{[i}\partial_{j]}\Omega + \frac{1}{\Omega} b_{[i}\partial_t b_{j]} + \mathrm{O}\left(c^2\right) = \varpi_{ij} + \mathrm{O}\left(c^2\right)$

Remarks

- all Carrollian-covariant
- purely geometric
- more terms if $\beta^i \neq 0$

Highlights

Galilean covariance

Galilean fluids

Carrollian covariance

Carrollian fluids

Conformal Carrollian geometry and conformal Carrollian fluids

Curvature and Cotton

Conformal Carrollian geometry

Weyl transformation on Carrollian geometry

$$a_{ij} \to \frac{a_{ij}}{\mathcal{B}^2} \quad b_i \to \frac{b_i}{\mathcal{B}} \quad \Omega \to \frac{\Omega}{\mathcal{B}}$$

Spatial Weyl derivative for a weight-w vector V^l

$$\hat{\mathscr{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i \varphi_i$$

Temporal Weyl derivative for a weight-w vector V^l

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l = \frac{1}{\Omega}\partial_t V^l + \frac{w}{2}\theta V^l + \xi^l{}_i V^i$$

Conformal Carrollian fluids $\boldsymbol{\beta} = 0$

From relativistic to Carrollian conformal properties

- $\varepsilon = dp$ and $\tau^{\mu}{}_{\mu} = 0 \underset{c\to 0}{\longrightarrow} \Xi^{i}{}_{i} = \Sigma^{i}{}_{i} = 0$
- $\varepsilon \to \mathcal{B}^{d+1}\varepsilon,\ \pi_i \to \mathcal{B}^{d}\pi_i,\ Q_i \to \mathcal{B}^{d}Q_i,$
  $\Xi_{ij} \to \mathcal{B}^{d-1}\Xi_{ij},\ \Sigma_{ij} \to \mathcal{B}^{d-1}\Sigma_{ij}$

Scalar equations

- $\mathcal{E} = -\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon - \hat{\mathscr{D}}_i Q^i + \Xi^{ij}\xi_{ij} = 0$
- $\mathcal{F} = \Sigma^{ij}\xi_{ij} = 0$

Vector equations

- $\mathcal{G}_j = \frac{1}{d}\hat{\mathscr{D}}_j\varepsilon + \frac{1}{\Omega}\hat{\mathscr{D}}_t\pi_j + \pi_i\xi^{i}{}_{j} + 2Q^i\varpi_{ij} - \hat{\mathscr{D}}_i\Xi^{i}{}_{j} = 0$
- $\mathcal{H}_j = \frac{1}{\Omega}\hat{\mathscr{D}}_t Q_j + Q_i\xi^{i}{}_{j} - \hat{\mathscr{D}}_i\Sigma^{i}{}_{j} = 0$

Highlights

Galilean covariance

Galilean fluids

Carrollian covariance

Carrollian fluids

Conformal Carrollian geometry and conformal Carrollian fluids

Curvature and Cotton

The Conformal Carrollian curvature tensors in 2 dim

The Ricci tensor (space)

$$\hat{\mathscr{R}}_{ij} = \hat{\mathscr{R}}^k{}_{ikj} = \hat{r}_{ij} + a_{ij}\hat{\nabla}_k\varphi^k = \hat{s}_{ij} + \hat{\mathscr{K}} a_{ij} + \hat{\mathscr{A}}\eta_{ij}$$

$$\hat{\mathscr{K}} = \frac{1}{2}a^{ij}\hat{\mathscr{R}}_{ij} = \hat{K} + \hat{\nabla}_k\varphi^k, \quad \hat{\mathscr{A}} = \frac{1}{2}\eta^{ij}\hat{\mathscr{R}}_{ij}$$

$\hat{r}_{ij}$ is the Ricci of $\hat{\nabla}_k$ and $2\hat{K} = \hat{r} = a^{ij}\hat{r}_{ij}$

The vector (time)

$$\hat{\mathscr{R}}_i = \frac{1}{\Omega}\partial_t\varphi_i - \frac{1}{2}\left(\hat{\partial}_i + \varphi_i\right)\theta$$

Reminder: the Cotton tensor

In 3 dim the Weyl tensor vanishes – conformal properties are captured by the Cotton tensor

$$C_{\mu\nu} = \eta_\mu{}^{\rho\sigma} \nabla_\rho \left(R_{\nu\sigma} - \frac{R}{4} g_{\nu\sigma} \right)$$

$(\eta_{\mu\nu\sigma} = \sqrt{-g}\epsilon_{\mu\nu\sigma})$

- symmetric and traceless
- conformally covariant of weight 1
- identically conserved: $\nabla_\mu C^{\mu\nu} = 0$

Decomposition wrt u

$$C_{\mu\nu} = \frac{3c}{2}\frac{u_\mu u_\nu}{k} + \frac{ck}{2} g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}$$

At large k *with* $u = \frac{1}{\Omega}\partial_t$

- $c_i = \chi_i + k^2 \psi_i$
- $c_{ij} = X_{ij} + k^2 \Psi_{ij}$
- weight 2 and 1 respectively

The "Cotton" in two-dimensional Carrollian geometry

$$\begin{cases} c = \left(\hat{\mathscr{D}}_l \hat{\mathscr{D}}^l + 2\hat{\mathscr{K}}\right) * \varpi \\ \chi_j = \frac{1}{2}\eta^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{K}} + \frac{1}{2}\hat{\mathscr{D}}_j \hat{\mathscr{A}} - 2 * \varpi \hat{\mathscr{R}}_j \\ \psi_j = 3\eta^l{}_j \hat{\mathscr{D}}_l * \varpi^2 \\ X_{ij} = \frac{1}{2}\eta^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{R}}_i + \frac{1}{2}\eta^l{}_i \hat{\mathscr{D}}_j \hat{\mathscr{R}}_l \\ \Psi_{ij} = \hat{\mathscr{D}}_i \hat{\mathscr{D}}_j * \varpi - \frac{1}{2} a_{ij} \hat{\mathscr{D}}_l \hat{\mathscr{D}}^l * \varpi - \eta_{ij} \frac{1}{\Omega} \hat{\mathscr{D}}_t * \varpi^2 \end{cases}$$

Conservation identities

$$\begin{cases} \frac{1}{\Omega}\hat{\mathscr{D}}_t c + \hat{\mathscr{D}}_i \chi^i = 0 \\ \frac{1}{2}\hat{\mathscr{D}}_j c + 2\chi^i \varpi_{ij} + \frac{1}{\Omega}\hat{\mathscr{D}}_t \psi_j - \hat{\mathscr{D}}_i \Psi^i{}_j = 0 \\ \frac{1}{\Omega}\hat{\mathscr{D}}_t \chi_j - \hat{\mathscr{D}}_i X^i{}_j = 0 \end{cases}$$