Beyond supergravity

(... and generalised complex geometry ?)

with Jim Liu (work in progress)

" ... one small step for man, one giant leap for mankind" (N. Armstrong, Moon, 20/07/69)

R^4 in string theory - the exact opposite (... and very much on Earth)

$\Longleftrightarrow$

From quantum gravity point of view, probing string theory at fundamental level
$\Rightarrow$ study of stringy corrections:

- $\star$ α' expansion $\Rightarrow$ higher derivative corrections to the supergravity
- $\star$ the genus expansion $\Rightarrow$ string quantum corrections in spacetime

$\Longleftrightarrow$

Why more? again? now?!

- things to be learned for 4D $\mathcal{N} = 1$ physics (e.g. Kähler potential)
- test (develop) Generalised complex geometry?
  - ▷ natural appearance of generalised connections, some kinematic structures
  - ▷ the truly **non-linear** tests ahead

$\mathcal{N}=2$ D=4 physics .... Type II strings on CY threefolds at the happy intersection of:

- ⋆ 2D SCFT
- ⋆ special geometry
- ⋆ algebraic geometry

Kähler potential:

◇ $$K = -2\log\left(\mathcal{V}_3 - \frac{\zeta(3)}{32\pi^3 \cdot g_s^{3/2}}\,\chi(X_3)\,\right)$$

▷ $\mathcal{V}_3$ and $\chi(X_3)$ - classical volume of CY and Euler number of X_3 $\Rightarrow$ 10D origin

▷ $\zeta(3)$ - stringy!

... traces back to higher derivative ($\sim \alpha'^3$) R^4 corrections in 10D

- 10D 4pt (linearized) computations are largely sufficient ....
- .... NOT enough for $\mathcal{N}=1$ results (higher-point functions, other fields)

$\mathcal{N} = 2 :$ mostly under control

$\mathcal{N} = 1 :$ gloomy, but ... optimistic

... (!?) non-trivial string-theoretic cancellations **must** hold in $\mathcal{N} = 4$

Geometry of string theory (GCG?) $\Longleftrightarrow$ non-linear completions of gravity

As things stand:

	4pt		5pt		6pt		$\cdots$
	tree	1-loop	tree	1-loop	tree	1-loop	-
strings	$\checkmark$	$\checkmark$	$\odot$	$\checkmark$	$\odot$	$(\sim)\ \checkmark$	-
GCG	$\sim$	$(\sim)\ \checkmark$	?!	?	?!	?	???

- $\diamond$ $(\sim)\ \checkmark$ - work with André Coimbra
- $\diamond$ $\checkmark$, $(\sim)\ \checkmark$, $\odot$ - work with Jim Liu
- $\diamond$?, ?! - need to be explained
- $\diamond$??? - .... GCG making predictions

R^4 corrections in string theory (and GCG)

⋆ Type II theories

- tree-level: $e^{-2\phi}(t_8 t_8 + \frac{1}{4}\epsilon_8\epsilon_8)R^4$
- one loop: $(t_8 t_8 \mp \frac{1}{4}\epsilon_8\epsilon_8)R^4$ (IIA/IIB)
  $X_8(R) \sim (t_8\epsilon_8 + \epsilon_8 t_8)R^4 \sim [\frac{1}{4}p_1^2(TX) - p_2(TX)]$ (coupling $B \wedge X_8(R)$)

⋆ t_8 tensor (.... which we can see in GCG!)

- $t_8 t_8 R^4 = t_{\mu_1\cdots\mu_8} t_{\nu_1\cdots\nu_8} R^{\mu_1\mu_2}{}_{\nu_1\nu_2}\cdots R^{\mu_7\mu_8}{}_{\nu_7\nu_8}$ $\left(t_8 M^4 = 24\left(\text{tr}\, M^4 - \frac{1}{4}(\text{tr}\, M^2)^2\right)\right)$
- $\gamma_{\mu_1\mu_2}\cdots\gamma_{\mu_7\mu_8} \longrightarrow t_{\mu_1\cdots\mu_8} + \gamma_{\mu_1\cdots\mu_8}$
- All 6d (0,2) multiplet anomalies $\sim X_8(R)$ ... (too easy?)

⋆ At linearised level : $\Omega^{\text{LC}} \longrightarrow \Omega_\pm$

∗ $R(\Omega_\pm)_{\mu\nu}{}^{\alpha\beta} = R_{\mu\nu}{}^{\alpha\beta} \pm \nabla_{[\mu} H_{\nu]}{}^{\alpha\beta} + \frac{1}{2} H_{[\mu}{}^{\alpha\gamma} H_{\nu]\gamma}{}^{\beta}$ ($R(\Omega_+)_{\mu\nu\alpha\beta} = R(\Omega_-)_{\alpha\beta\mu\nu}$)

⋆ At non-linear level : $R(\Omega^{\text{LC}}) \longrightarrow R(\Omega_\pm)$ (✓ GCG!) ... but not enough

∗ **Non-linear** objects... e.g. tree-level $T_{\mu\nu\alpha\beta} \sim H_{\mu\nu}{}^{\gamma} H_{\gamma\alpha\beta} + H_{[\mu}{}^{\alpha\beta}\partial_{\nu]}\phi + \cdots$

∗ Can GCG explain T ... or $t_8 t_8 T R^3$?

∗ Dilaton!?!?!

Summary of type IIA $(\alpha')^3$ one-loop couplings (10D):

	No B (linearised)	With B (non-linear)
e-o	$\frac{1}{8}(t_8\epsilon_{10} + \epsilon_{10}t_8)BR^4$	$B \wedge X_8(\Omega^{\rm LC}) +$ exact terms
+	$= B \wedge X_8(\Omega^{\rm LC})$	
o-e	$= \frac{1}{192(2\pi)^4} B \wedge \left(\text{tr}\, R^4 - \frac{1}{4}(\text{tr}\, R^2)^2\right)$	?
e-e	$t_8 t_8 R^4$	?
o-o	$\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4$	??

◇ $t_8 t_8 R^4 = t_{\mu_1\cdots\mu_8} t_{\nu_1\cdots\nu_8} R^{\mu_1\mu_2}{}_{\nu_1\nu_2} R^{\mu_3\mu_4}{}_{\nu_3\nu_4} R^{\mu_5\mu_6}{}_{\nu_5\nu_6} R^{\mu_7\mu_8}{}_{\nu_7\nu_8}$

◇ $\epsilon_{10}\epsilon_{10} R^4 = \epsilon_{\alpha\beta\mu_1\cdots\mu_8} \epsilon^{\alpha\beta\nu_1\cdots\nu_8} R^{\mu_1\mu_2}{}_{\nu_1\nu_2} R^{\mu_3\mu_4}{}_{\nu_3\nu_4} R^{\mu_5\mu_6}{}_{\nu_5\nu_6} R^{\mu_7\mu_8}{}_{\nu_7\nu_8}$

◇ At linearised level (5 & 4-pt functions at one-loop) : $\Omega^{\rm LC} \longrightarrow \Omega_\pm$

◇ Curvature: $R(\Omega_\pm)_{\mu\nu}{}^{\alpha\beta} = R_{\mu\nu}{}^{\alpha\beta} \pm \nabla_{[\mu} H_{\nu]}{}^{\alpha\beta} + \frac{1}{2} H_{[\mu}{}^{\alpha\gamma} H_{\nu]\gamma}{}^{\beta}$

◇ $\mathcal{N} = 1$ superinvariants: $J_0(\Omega) = \left(t_8 t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4$

$$J_1(\Omega) = t_8 t_8 R^4 - \frac{1}{4}\epsilon_{10} t_8 B R^4$$

$\mathcal{N} = 2$: M-theory/IIA on Calabi-Yau threefolds

- 4D quantum corrected effective action:

$$S = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{g^\sigma} \Big[\Big((1 + \tfrac{\chi_T}{v_6})\, e^{-2\phi_4} - \chi_1\Big) \mathcal{R}_{(4)} + \Big((1 - \tfrac{\chi_T}{v_6})\, e^{-2\phi_4} - \chi_1\Big) G_{vv} (\partial v)^2 + \Big((1 + \tfrac{\chi_T}{v_6})\, e^{-2\phi_4} + \chi_1\Big) G_{hh} (\partial h)^2\Big]$$

  - ▷ $v_6 = \mathcal{V}_3 (2\pi l_s)^{-6}$
  - ▷ G_{vv} - the metric of the $h_{(1,1)} - 1$ vector-multiplets
  - ▷ G_{hh} - the metric of the $h_{(1,2)}$ non-universal hypermultiplets
  - ▷ $\chi_T = 2\zeta(3)\chi/(2\pi)^3$
  - ▷ $\chi_1 = 4\zeta(2)\chi/(2\pi)^3$

- Weyl rescaling:
  - ▷ Quantum corrections to vector and hyper moduli space metrics
  - ▷ Corrections to the Kähler potential $\quad (\mathcal{V}_3 \to \mathcal{V}_3 - \frac{\zeta(3)}{32\pi^3\, g_s^{3/2}} \chi(X_3))$

B-field couplings

⋆ Appearance of (at linearized level)

$$\hat{R}_{\mu\nu}{}^{\lambda\sigma}(\omega + \tfrac{1}{2}\mathcal{H}) = R_{\mu\nu}{}^{\lambda\sigma} + \tfrac{1}{2}\nabla_{[\mu}H_{\nu]}{}^{\lambda\sigma}$$

⋆ Inclusion of higher orders in B_2 required by

- supersymmetry
- T-duality (similarly for RR couplings to D-branes $C \wedge \sqrt{\hat{A}(X)}\mathrm{ch}(x)$)
- generalized geometry ??? hope for systematic geometric calculation?
- Heterotic/Type II duality ((refined) map between tree-level and one-loop terms)

⋆ In fact, 10d one-loop term ${\alpha'}^3\, R^3 H^2$ (in the string frame):

$$\int d^{10}x\sqrt{G}\,\delta^{r_1\cdots r_9}_{s_1\cdots s_9}\,R^{r_1r_2}_{s_1s_2}R^{r_3r_4}_{s_3s_4}R^{r_5r_6}_{s_5s_6}\left(H_{r_7r_8s_9}H_{s_7s_8r_9} - \tfrac{1}{9}H_{r_7r_8r_9}H_{s_7s_8s_9}\right)$$

$$\rightarrow \quad \chi \int d^4x\sqrt{g^\sigma}\,H_{r_1r_2r_3}H^{r_1r_2r_3}$$

is at the origin of ....

Quantum corrections to $\mathcal{N}=2$ moduli spaces:

- vector moduli (IIA): $G_{v\bar{v}} \to (1 - \chi \frac{4}{(2\pi)^3} \frac{\zeta(3)}{v_6}) G_{v\bar{v}}$ ($e^{-2\phi_4} = v_6\, e^{-2\phi_{10}}$)
- (non-“universal”) hyper moduli (IIA): $G_{h\bar{h}} \to (1 + e^{2\phi_4} \frac{1}{6\pi} \chi) G_{h\bar{h}}$

2 “loop counting” parameters:

- for the corrections to the metric of vector multiplets(σ-model) :
$e^{-2\tilde{\phi}_4} \simeq e^{-2\phi_4} \left(1 + \mu_T \frac{\chi_T}{v_6} + \cdots\right)$ $(\chi_T = 2\zeta(3)\chi/(2\pi)^3)$
- for the corrections to the metric of hypermultiplets :
$\tilde{v}_6 \simeq v_6 \left(1 - \frac{3\mu_1}{2} \chi_1\, e^{2\phi_4} + \mathcal{O}(e^{4\phi_4})\right)$ ($\mu_1^2 = 4$ and $\chi_1 = 4\zeta(2)\chi/(2\pi)^3$)

classical “universal” hypermultiplet (ϕ_4, B_2, C_0 : $C_3 \to C_0 \Omega_3 + \bar{C}_0 \wedge \bar{\Omega}_3 + \cdots$)

- classically - $SU(2,1)/U(2)$ coset (3 isometries)
- Loop corrections - self dual Enstein metric defined by a single function:
$e^{-2\tilde{\phi}_4} - 4\zeta(2)\chi/(2\pi)^3$

Summary of type II $(\alpha')^3$ one-loop couplings (10D):

	No B (linearised)	With B (non-linear)
e-o **+** **o-e**	$\frac{1}{8}(t_8\epsilon_{10} + \epsilon_{10}t_8)BR^4$ $= B \wedge X_8(\Omega^{\mathrm{LC}})$ $= \frac{1}{192(2\pi)^4} B \wedge \left(\mathrm{tr}\, R^4 - \frac{1}{4}(\mathrm{tr}\, R^2)^2\right)$	$\frac{1}{8}(t_8\epsilon_{10} + \epsilon_{10}t_8)BR^4(\Omega_+)$ $= \frac{1}{8}t_8\epsilon_{10}B(R^4(\Omega_+) + R^4(\Omega_-))$ $= \frac{1}{2}B \wedge [X_8(\Omega_+) + X_8(\Omega_-)]$ $= \frac{1}{192\cdot(2\pi)^4} B \wedge \left(\mathrm{tr}\, R^4 - \frac{1}{4}(\mathrm{tr}\, R^2)^2 + \text{exact terms}\right)$
e-e	$t_8t_8R^4$	$t_8t_8R^4(\Omega_+) = t_8t_8R^4(\Omega_-)$
o-o	$\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4$	$\frac{1}{8}\epsilon_{10}\epsilon_{10}\left(R(\Omega_+)^4 + \frac{8}{3}H^2R(\Omega_+)^3 + \cdots\right)$ $= \frac{1}{8}\epsilon_{10}\epsilon_{10}\left(R(\Omega_-)^4 + \frac{8}{3}H^2R(\Omega_-)^3 + \cdots\right)$

- new kinematic structures in o-o sector
- lifting to 11d: $H \mapsto G_4$ (with lifting ambiguities)
- get RR couplings via reduction

Lift from D=10 to D=11

... is heavy and the ambiguities are not fully resolved.

Lift from D=6 to D=7 has the main features and can be done explicitly.

CP-odd part:

$$\frac{1}{4}B_2 \wedge \overline{X}_4 \quad \longrightarrow \quad -\frac{1}{32\pi^2} \quad C_3 \wedge \left(\text{tr}\, R^2 - \frac{1}{12} d\left(\mathcal{G}^{abc} \wedge (\nabla\mathcal{G})^{abc}\right)\right).$$

◇ $\mathcal{G}_1^{abc} = 4G_{\mu\rho\lambda} d\hat{x}^\mu \hat{e}^{a\nu} \hat{e}^{b\rho} \hat{e}^{c\lambda}$

CP-even part:

$$\begin{aligned} e^{-1}\delta\mathcal{L}^{\text{lift}} \quad &= \quad R^2_{\mu\nu\lambda\sigma} - 2R^2_{\mu\nu} + \tfrac{1}{2}R^2 - \tfrac{1}{6}R_{\mu\nu}G^{2\,\mu\nu} + \tfrac{1}{48}RG^2 + \tfrac{1}{6}\nabla_\mu G_{\nu\alpha\beta\gamma}\nabla^\nu G^{\mu\alpha\beta\gamma} \\ &\quad + \tfrac{1}{48}G_{\mu\nu\lambda\rho}G^{\mu\rho\alpha}{}_\beta G^{\nu\sigma\beta}{}_\gamma G^{\lambda\gamma}{}_{\alpha\sigma} + \tfrac{1}{288\cdot 12}(G^2)^2 - \tfrac{1}{216}(G_{\mu\nu})^2 + (\text{eom})^2 \end{aligned}$$

Reducing back to 10D/6D

◇ $\mathcal{G}_1^{abc} \quad \longrightarrow \quad (e^{\phi/2}\mathcal{F}^{abc}; \mathcal{H}^{ab})$

allows to recover RR completion (1-loop only!)

Puzzles (at tree-level)

- One-loop results would suggest

$$J_0(\Omega) = \left(t_8 t_8 + \tfrac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4 \quad \longrightarrow \quad \left(t_8 t_8 + \tfrac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4(\Omega_+) + \tfrac{1}{3}\epsilon_{10}\epsilon_{10} H^2 R^3(\Omega_+) + \dots$$
$$= J_0(\Omega_+) + \Delta J_0(\Omega_+, H)$$
$$J_1(\Omega) = t_8 t_8 R^4 - \tfrac{1}{4}\epsilon_{10} t_8 B R^4 \quad \longrightarrow \quad t_8 t_8 R^4(\Omega_+) - \tfrac{1}{8}\epsilon_{10} t_8 B \left(R^4(\Omega_+) + R(\Omega_-)\right) = J_1(\Omega_+)$$

$\Delta J_0(\Omega_+, H)$ needs susy completion. $\mathcal{N} = 2$ tests:

⇒ problems at tree-level - 4D $\mathcal{N} = 2$

⇒ ${c_2}^I (t_I \mathrm{tr}\, R^2 + u_I R \wedge R)$ with ${c_2}^I = \int_X \omega^I \wedge \mathrm{tr}\, R^2, \quad \omega^I \in H^{(1,1)}(X)$

- ◇ $\mathcal{F}_1 W^2|_{\text{F-term}}$ with W - $\mathcal{N} = 2$ chiral Weyl superfield and $\mathcal{F}_1$ - function of chiral vector superfields

⇒ important cancellations

▷ Extra corrections (??): $J_0(\Omega) \longrightarrow J_0(\Omega_+) + \Delta J_0 + \not{2\delta}$ $\quad J_1(\Omega) \longrightarrow J_1(\Omega_+) + \not{\delta}$

- ◇ not supported by string calculations

▷ *Different* invariants at tree-level and 1-loop (!?): $J(\Omega) \longrightarrow J_0(\Omega_+) + \alpha_{(l)} \Delta^{(l)} J_0$

⇒ $\mathcal{N} = 4$ tests to follow

Towards $\mathcal{N}=1$: $SL(2)$, F-theory and exotic kinematics

- IIB strings with varying dialton-axion $\tau = C_0 + ie^{-\phi}$
  - ◇ $S_{IIB} \sim \frac{1}{l_s^8} \int (R - P\bar{P}) *_{10} 1$
    - ▷ $P = \frac{i}{2\mathrm{Im}\tau} \nabla\tau$ ($U(1)_R$ covariant, charge 2)
    - ▷ *formally* obtained from $R^{(12)}$ at fixed volume ν
    - ▷ $\mathbb{T}^2$ metric: $\frac{\nu}{\mathrm{Im}\tau} \begin{pmatrix} 1 & \mathrm{Re}\tau \\ \mathrm{Re}\tau & |\tau|^2 \end{pmatrix}$
    - ▷ F-theory space - *elliptically fibered CY* ◁
    - ▷ D7/O7 - codim 2 defects (with *deficit angle*) - degenerations of el. fiber
    - ▷ 10D slice integral: $S_0^{12} \sim \frac{1}{l_s^8} \int R^{(12)} *_{10} 1$

- Decompactification limit of M-theory (on X_e)
  - ◇ $S^{11} \sim \frac{1}{l_M^9} \int R *_{11} 1 \quad \Rightarrow \quad S^9 \sim \frac{\nu}{l_M^7} \int (R - P\bar{P}) *_9 1 \quad \Rightarrow \quad S_{IIB}$
    - ▷ IIB limit $\nu \to 0$:

$\sim \alpha'^3$ in IIB

- kinematics:
  - ▷ $(t_8 t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10})R^4$
  - ▷ no CP-odd (GS-like) couplings
  - ▷ tree-level, 1 loop and non-perturbative contributions
- M-theory (11D) perspective
  - ▷ For M-th/$\mathbb{T}^2$, $\quad \nu \to 0 \quad \Rightarrow \quad S_3^9 \to 0$
  - ▷ need to account for KK modes on $\mathbb{T}^2$
- For constant τ: $\Delta S^{11} \sim \frac{1}{l_M^3} \int t_8 t_8 R^4 *_{11} 1 \quad \Rightarrow \quad \Delta S_{IIB} \sim \frac{1}{l_s^2} \int f_0(\tau,\bar\tau)\, t_8 t_8 R^4 *_{10} 1$
  - ▷ $f_0(\tau,\bar\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^{3/2}}{|m+n\tau|^3} \to \frac{2\zeta(3)}{g_3^{3/2}} + \frac{2\pi}{3} g_s^{1/2} + \mathcal{O}(e^{-1/g_s})$
- Varying dilaton-axion:
  - ▷ $\Delta S_{IIB}|_{4-\mathrm{pt}} \sim \frac{1}{l_s^2} \int \left\{ t_8 t_8 [R^4 + 24 R^2 |DP|^2] + \hat{\mathcal{O}}_1 \left[(|DP|^2)^2 \right] \right\} *_{10} 1$
  - ▷ Complete agreement with 0-mode reduction of

$$S_3^{12}(t_8) \quad \sim \quad \frac{1}{l_s^2} \int \hat t_8 \hat t_8 {R^{12}}^4 *_{10} 1$$

$\sim \alpha'^3$ in F-theory...

- Odd-odd sector:
  - ▷ 0-mode reduction of

$$S_3^{12}(\epsilon_8) \quad \sim \quad \frac{1}{l_s^2} \int \hat{\epsilon}_8 \hat{\epsilon}_8 R^{12^4} *_{10} 1$$

  - ▷ $S_3^{12}(\epsilon_8)$ restricted to 4pt *vanishes*
- Missing τ dynamics - all g_s corrections
- (Conjectured) complete coupling:

$$S_3^{12} \quad = \quad \frac{1}{(4\pi)^9 \cdot 3 \cdot l_s^4} \int f_0(\tau, \bar{\tau}) \left[\hat{t}_8 \hat{t}_8 + \frac{1}{96} \hat{\epsilon}_{12} \hat{\epsilon}_{12} \right] (R^{(12)})^4 *_{10} 1$$

  - ▷ $SL(2, \mathbb{Z})$ and SUSY compatibility
  - ▷ perturbatively *tree* + 1-loop terms
  - ▷ No "cusp forms" - non-perturbative part captured by $f_0(\tau, \bar{\tau})$
  - ∗ g_s-exact $\sim \mathcal{O}(\alpha'^3)$ Type IIB action without flux
  - ∗ R and τ couplings *beyond* 4pt

4D $\mathcal{N}=1$ compactifications of F-theory

- Smooth four-fold $B_3 \to CY_4$, fibered over B_3 with zero-section
- 2 + 8 derivative reductions ($+\mathcal{O}(\alpha'^4)$):

$$S^4_{0+3} = \frac{1}{2\pi\alpha'} \int \left(\mathcal{V}_b - \frac{1}{64\pi^3} \int_{B_3} f_0(\tau,\bar{\tau})\, c_3(X_4)|_{B_3} \right) R_{(4\text{d})} \ast_4 1$$

  - ◇ Correction

$$\sim \int R_{(4\text{d})} \ast_4 1 \int_{B_3} f_0 \ast_8 (J \wedge c_3(X_4)) \ast_6 1 + \dots ,$$

  - ◇ Use $\ast_8(J \wedge \ast_6 1) = 1 + \mathcal{O}(\alpha')$

▷ Verify that the correction is *finite*

▷ *constant* τ ($B_3 = CY_3$) known $\mathcal{N}=2$ results

▷ τ varies over B_3 - the correction is *non-topological*

▷ $\Rightarrow$ Kähler potential via Weyl rescaling (plenty of ifs and buts!)

Weak string coupling limit

Sen limit: a region of the complex structure moduli space of the CY fourfold X_4, where none of the monodromies acting on τ involves the string coupling τ_2^{-1}:

- ▷ τ_2^{-1} kept small in a globally well-defined way
- ▷ Type IIB on orientifolded CY threefold X_3 - branched double cover of B_3
- ▷ O7-plane - branching locus: in cohomolgy $D_{\mathrm{O7}} \equiv c_1(B_3)$
- ∗ Correction (*topological*) to the classical volume $\mathcal{V}_3$ of the CY threefold:

$$\tilde{\mathcal{V}}_3 = \mathcal{V}_3 \; - \; \frac{\zeta(3)}{32\pi^3 \, g_s^{3/2}} \left(\chi(X_3) + 2 \int_{X_3} D_{\mathrm{O7}}^3 \right) \quad + \quad \mathcal{O}(g_s^{-1/2})$$

  - ◇ Note integration over X_3 (not B_3)
  - ◇ Only Weyl rescaling contribution to Kähler potential is computed here
- ▷ New terms from tree-level closed string scattering in this CY orientifold background
- ▷ *Absent* in toroidal models!

$\mathcal{N}=4$ tests : 6D $(1,1)$ and $(2,0)$ theories

Supersymmetry (classification of supervertices):

- $(1,1)$ theory (IIA/$K3$):
  - ◇ tree-level: $\not{0}$
  - ◇ 1-loop: $B_2 \wedge \overline{X}_4 + \cdots$ ✓
- $(2,0)$ theory (IIB/K3)
  - ◇ tree-level: $\not{0}$
  - ◇ 1-loop $\not{0}$ (gravity multiplet)
  - ◇ 4-derivative interaction quartic in tensor multiplets

Cancellations:

- On $K3:$ $J_0 \longrightarrow \left(t_4 t_4 + \frac{1}{8}\epsilon_6\epsilon_6\right) R^2 \longrightarrow$ Ricci $\longrightarrow$ 0 ... Linear
- **Complete** cancellations are very non-trivial
  - ◇ verify ΔJ for one-loop $J_0(\Omega_+) + \Delta J_0(\Omega_+, H)$ (+more!)
  - ◇ three-level 5pt-functions

6D $(2,0)$ theory at one loop (testing $\Delta^{(1)} J_0$)

Linearised CP-odd: $\mathcal{I} \longrightarrow H \wedge \text{tr}\,(\mathcal{H} \wedge R) \longrightarrow R_{\mu\nu}(\epsilon \cdot H \cdot H)_{\mu\nu} \longrightarrow 0$

The non-linear result (Note $SO(5)$ R-symmetry: $5 \longrightarrow 1 + \not{4}$)

- $\mathcal{L}_{\mathrm{CP-even}} = (t_4 t_4 - \epsilon_4 \epsilon_4) R(\Omega_+)^2 - \Delta J$
- For gravity multiplet only - H is **self-dual**:

$$\mathcal{L}_{\mathrm{CP-even}} + 4\mathcal{I} = 4(R_{\mu\nu} - \tfrac{1}{4} H^2_{\mu\nu})^2 - (R + \tfrac{1}{12} H^2)^2$$

- gravity + 1 TM (for the scalars again $5 \longrightarrow 1 + \not{4}$)

$$\mathcal{L}_{\mathrm{CP-even}} + 4\mathcal{I} = \tfrac{4}{3} H^{(-)\,4} - 8 H^{(-)\,2}_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \tfrac{4}{3} H^2 \partial\phi^2 + 16(\partial\phi^2)^2$$

expect only 4pt+ couplings in TM.... not there yet

options:

- ◇ ... pushing to higher point-functions
- ◇ dilaton-dependent 6pt contributions that cancel out in $J_0 - 2J_1$

Nonlinear tree-level cancellations....

...are much harder!

$R(\Omega^{\mathrm{LC}}) \quad \longrightarrow R(\Omega_{\pm})$ needed but not enough

- New couplings $t_8 t_8 T^{\langle 2 \rangle} R^3$
- with **non-linear** objects... $T^{\langle 2 \rangle}_{\mu\nu\alpha\beta} \sim H_{\mu\nu}{}^{\gamma} H_{\gamma\alpha\beta} + H_{[\mu}{}^{\alpha\beta} \partial_{\nu]} \phi + \cdots$
- absence of $T^{\langle 4 \rangle}$ (and $t_8 t_8 T^{\langle 4 \rangle} R^2$)

$*$ ... tree-level 5pt calculations ...

$\Longleftrightarrow$

$$\Longleftrightarrow$$

$\Longleftrightarrow$

LBT - Lichnerowicz-Bismut theorem

- Lichnerowicz theorem:

  $\left(\nabla^a \nabla_a - \nabla\!\!\!/^2\right) \epsilon = \frac{1}{4} R$ tensorial action!

  - ∇_a - Levi-Civita connection (no torsion)
  - $\nabla\!\!\!/ = \gamma^a \nabla_a$ - Dirac operator

- Torsion H ($\mathrm{d}H = 0$)

  $$\begin{cases} D_a \epsilon = \nabla_a \epsilon - \alpha H_{abc} \gamma^{bc} \epsilon \\ D\!\!\!/ \epsilon = \left(\gamma^a \nabla_a - \beta H_{abc} \gamma^{abc}\right) \epsilon \end{cases}$$

  $\leftarrow$ torsionful Dirac operator

  - Note $D\!\!\!/ \neq \gamma^a D_a = \gamma^a \nabla_a - \alpha H_{abc} \gamma^{abc}$

- LBT: $\left(D^a D_a - D\!\!\!/^2\right) \epsilon$ **tensorial**

  $$\left(D^a D_a - D\!\!\!/^2\right) \epsilon = \frac{1}{4}\left[R - \# H^2\right] \epsilon + \gamma^{abcd} \nabla_a H_{bcd} \epsilon + (\alpha - 3\beta) \gamma^{ab} \nabla^c H_{abc} \epsilon + (\alpha - 3\beta) \gamma^{ab} H_{abc} \left(\nabla^c \epsilon\right)$$

- $\alpha = 3\beta$ ($\alpha = \frac{1}{8}$ for normalisation: $\frac{1}{12} H^2$)

gen. Lichnerowicz theorem (gLBT) $\Rightarrow$ effective actions (*Local data*)

- (gen.) Lichnerowicz theorem: $\left(D^A D_A - D^2\right)\epsilon = \left[\frac{1}{4}S + \gamma^{abcd} I_{abcd}\right]\epsilon$ (S tensorial!)

◇ Heterotic effective action: $S = R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12}H^2 - \frac{\alpha'}{4}\mathrm{tr}\,\hat{\mathcal{F}}^2$

◇ $I_{abcd} = \frac{1}{6}\nabla_{[a}H_{bcd]} - \frac{\alpha'}{8}\mathrm{tr}\,\hat{\mathcal{F}}_{[ab}\hat{\mathcal{F}}_{cd]} = 0$

◇ $\left.\begin{aligned} \delta\psi_a &= D_a\epsilon = \nabla_a\epsilon - \tfrac{1}{8}H_{abc}\gamma^{bc}\epsilon \\ \delta\zeta_\alpha &= D_\alpha\epsilon = -\tfrac{1}{8}\sqrt{2\alpha'}\hat{\mathcal{F}}_{ab\alpha}\gamma^{ab}\epsilon \end{aligned}\right\}$ $\leftarrow$ covariant derivative ($A = \{a, \alpha\}$)

◇ $\delta\lambda = D\epsilon = \left(\gamma^a\nabla_a - \frac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon$ $\leftarrow$ Dirac operator

Gravitational terms (obstruction to E) ?

◇ picking a substructure of the GFB ($O(d) \times G \times O(d)$ PB) splits $E = \tilde{C}_+ \oplus \tilde{C}_{\mathfrak{g}} \oplus \tilde{C}_-$

◇ take $G \to G_{\mathrm{gauge}} \times O(d)$....

◇ reduce the structure group of E to $O(d) \times G \times O(d) \subset O(d + \dim(\mathfrak{g})) \times O(d)$

◇ Identify $O(d) \in G$ with $O(d)$ in $\tilde{C}_+$

▷ Works only for $\hat{\mathcal{A}} = \Omega_+ = \omega^{\mathrm{LC}} + \frac{1}{2}\mathcal{H}$!!! (cf susy for Ω_-!!!)

▷ For type II $G \to O(d) \times O(d)$ does **NOT** work

- ▷ Flip of the sign in $\mathcal{O}(\alpha')$ effective action wrt D_a : $\Omega_- \longrightarrow \Omega_+$!!!
  - ◇ $$R_{mnpq}(\Omega^-) - R_{pqmn}(\Omega^+) = -12\mathrm{d}H_{mnpq}$$
  - • leading to corrections all orders in α':
    - ▷ "gaugino" $\psi_{ab} \in \Gamma(\Lambda^2 C_+ \otimes S(C_-))$ for "gauge group" $O(d)_+$
      - ◇ $$\delta\psi_{O(d)ab} = \tfrac{1}{8}\sqrt{\alpha'}R(\Omega^+)_{\bar{a}\bar{b}ab}\gamma^{\bar{a}\bar{b}}\epsilon \ldots\ldots = D_{ab}\epsilon \ \ (?)$$
    - ▷ ψ_{ab} - *composite* "gravitino curvature"
      - ◇ $\delta\psi_{ab} = D_{ab}\epsilon + \tfrac{1}{8}\sqrt{\alpha'}\left(\tfrac{1}{8}\alpha'[\mathrm{tr}\, F\wedge F - \mathrm{tr}\, R(\Omega^+)\wedge R(\Omega^+)]_{ab\bar{a}\bar{b}}\right)\gamma^{\bar{a}\bar{b}}\epsilon \ \rightarrow\ \hat{D}_{ab}\epsilon$
    - ▷ $D_{ab} \ \rightarrow\ \hat{D}_{ab}$ in LBT $\Rightarrow$ $\mathcal{O}(\alpha'^2)$ modifications of susy for
      $\gamma^{\bar{a}}\hat{D}_{\bar{a}}\gamma^{\bar{b}}\hat{D}_{\bar{b}}\epsilon - \hat{D}^a\hat{D}_a\epsilon + \hat{D}^\alpha\hat{D}_\alpha\epsilon + \hat{D}^{ab}\hat{D}_{ab}\epsilon = -\tfrac{1}{4}S^-\epsilon + \gamma^{abcd}I_{abcd}\epsilon$
    - ▷ hierarchy of higher α' corrections (consistent with GCG)
    - ▷ $\mathcal{O}(\alpha'^3)$ agreement with literature
    - ▷ new $\mathcal{O}(\alpha'^4)$ corrections
    - ▷ iterative all order formulae ?

▷ “Tensoriality” without generalised geometry:

$$\not{D}\not{D}\epsilon - D_M D^M \epsilon - \frac{\alpha'}{64}\left(\text{tr } \not{F}\not{F}\epsilon - \text{tr } \not{R}^+\not{R}^+\epsilon\right) + 2\nabla^M\phi D_M\epsilon = -\frac{1}{4}\mathcal{L}_b\epsilon + \mathcal{O}(\alpha'^2)$$

(mod. heterotic BI: $\mathrm{d}H = \frac{\alpha'}{4}\left(\text{tr } F\wedge F - \text{tr } R^+\wedge R^+\right)$)

▷ $\mathcal{L}_b$ - bosonic Lagrangian

▷ Multiply by $e^{-2\phi}\epsilon^\dagger$ and integrate by parts ($\epsilon^\dagger\epsilon = 1$):

$$\frac{1}{4}\mathcal{L}_b = (\not{D}\epsilon)^\dagger\not{D}\epsilon - (D_M\epsilon)^\dagger D^M\epsilon + \frac{\alpha'}{64}\left(\text{tr } \epsilon^\dagger\not{F}\not{F}\epsilon - \text{tr } \epsilon^\dagger\not{R}^+\not{R}^+\epsilon\right) + \mathcal{O}(\alpha'^2)$$

▷ The (bosonic) action

$$S_b = \int_{M_{10}} e^{-2\phi}\mathcal{L}_b = BPS^2$$

▷ Susy + BI $\Rightarrow$ solutions .... alternative to Gen. Ricci computation:

$$\Gamma^M D^-_{[N}D^-_{M]}\epsilon - \frac{1}{2}D^-_N(\mathcal{O}\epsilon) + \frac{1}{2}\mathcal{O}D^-_N\epsilon = -\frac{1}{4}\mathcal{E}_{NM}\Gamma^M\epsilon + \frac{1}{8}\mathcal{B}_{NM}\Gamma^M\epsilon + \frac{1}{48}\mathrm{d}H_{NMPQ}\Gamma^{MPQ}\epsilon$$

$\mathcal{O} = \not{\partial}\phi - \frac{1}{12}\not{H}$ and $\mathcal{E}^0_{NM}, \mathcal{B}^0_{NM}$ - EOMs for metric and B-field

M-theory & GCG

- Fields: $\{g_{mn}, \mathcal{A}_{mnp}, \psi_m\}$
  - ◇ $S_{\mathsf{B}} = \frac{1}{2\kappa^2}\int\left(\sqrt{-g}\,R - \frac{1}{2}\mathcal{F}\wedge *\mathcal{F} - \frac{1}{6}\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}\right)$
  - ◇ $S_{\mathsf{F}} = \frac{1}{\kappa^2}\int\sqrt{-g}\Big(\bar\psi_m\gamma^{mnp}\nabla_n\psi_p + \mathcal{F}_{p_1\dots p_4}\left(\frac{1}{96}\bar\psi_m\gamma^{mp_1\dots p_4 n}\psi_n + \frac{1}{8}\bar\psi^{p_1}\gamma^{p_2p_3}\psi^{p_4}\right)\Big)$
    - ▷ susy $\quad \delta\psi_m = \nabla_m\varepsilon + \frac{1}{288}\left(\gamma_m{}^{n_1\dots n_4} - 8\delta_m{}^{n_1}\gamma^{n_2n_3n_4}\right)\mathcal{F}_{n1\dots n_4}\varepsilon = D_m\varepsilon$
    - ▷ eom $\quad \gamma^{mnp}\nabla_n\psi_p + \frac{1}{96}\left(\gamma^{mnp_1\dots p_4}\mathcal{F}_{p_1\dots p_4} + 12\mathcal{F}^{mn}{}_{p_1p_2}\gamma^{p_1p_2}\right)\psi_n = 0 = L^{mn}\psi_n$

- exact sequence: $\quad S \xrightarrow{D} V\otimes S \xrightarrow{L} V\otimes S \xrightarrow{D^\dagger} S$
  - ▷ $L\circ D = 0$ follows from supersymmetry
  - ▷ reality of $\int\bar\psi_a L^{ab}\psi_b \Rightarrow L = -L^\dagger \Rightarrow D^\dagger\circ L = 0$.

- Lichnerowicz-type relation $\tilde D^a D_a\varepsilon = \gamma^{ab}D_aD_b\varepsilon \propto$ (trace of Einstein $+$ 8-form)
  - ▷ $L^{ab}\psi_b = \gamma^{abc}D_b\psi_c$ and $\tilde D^c = \frac{1}{9}\gamma_a L^{ac} = \gamma^{bc}D_b$
  - ▷ coefs in D_a and $\tilde D^a$ are *uniquely* fixed by tensoriality of rhs!

- LT $\quad\Rightarrow\quad S_{\mathsf{B}} = \int\sqrt{-g}(\mathcal{R} - \frac{1}{3}\frac{1}{48}\mathcal{F}_{b_1\dots b_4}\mathcal{F}^{b_1\dots b_4}) - \frac{1}{3}\mathcal{A}\wedge(\mathrm{d}*\mathcal{F} + \frac{1}{2}\mathcal{F}\wedge\mathcal{F})$

Higher order (type IIA 1-loop) terms

$$D : S \to T^* \otimes S$$

$$(D\varepsilon)_a = \nabla_a \varepsilon + \alpha \left(\nabla^b X_{abcd}\right) \gamma^{cd} \varepsilon + \beta X_{abcd} \gamma^{cd} \nabla^b \varepsilon$$

$$\tilde{D} : T^* \otimes S \to S$$

$$(\tilde{D}\psi) = \gamma^{ab} \left(\nabla_a \psi_b + \tilde{\alpha} \left(\nabla^c X_{acef}\right) \gamma^{ef} \psi_b + \tilde{\beta} X_{acef} \gamma^{ef} \nabla^c \psi_b\right)$$

where $X_{abcd} \in [0, 2, 0, 0, 0]$ and α, β etc. parametrise higher-order corrections

$$\begin{aligned}(\tilde{D}D\varepsilon) &= \gamma^{ab} \nabla_a \nabla_b \varepsilon + (\alpha - \tilde{\alpha} - \beta) \left(\nabla^a X_{abcd}\right) \gamma^{cd} \nabla^b \varepsilon \\ &\quad - \alpha \left(\nabla^a \nabla^b X_{abcd}\right) \gamma^{cd} \varepsilon - \left(\tilde{\beta} + \beta\right) X_{abcd} \gamma^{cd} \nabla^a \nabla^b \varepsilon + \ldots \\ (\text{if } \alpha - \tilde{\alpha} - \beta = 0) &= -\tfrac{1}{4} \mathcal{R} \varepsilon + \tfrac{1}{2} \left(2\alpha - \tilde{\beta} - \beta\right) R^{abe}{}_c X_{abed} \gamma^{cd} \varepsilon \\ &\quad + \tfrac{1}{4} \left(\tilde{\beta} + \beta\right) R^{abcd} X_{abcd} \varepsilon - \tfrac{1}{8} \left(\tilde{\beta} + \beta\right) R^{ab}{}_{cd} X_{abef} \gamma^{cdef} \varepsilon + \ldots \end{aligned} \qquad (1)$$

Ambiguities:

- ▷ $\alpha = \tilde{\alpha} = \beta = 0$ consistent: keeping susy classical and only correcting S_{F}
- ▷ $\tilde{\alpha} = 0, \alpha = \beta = \tilde{\beta}$: $\tilde{D}^a \nabla_a \varepsilon$ and $\gamma^{ab} \nabla_a D_b \varepsilon$ are separately tensorial (the fermionic action is in terms of ”supercovariant” objects)

Everything that can modify susy:

Projection of R^3	Rep of $so(10,1)$	Multiplicity	multiplicity of embeddings in $\delta\psi$	of which result in $[\nabla,\nabla]R^3$	projected into form of rank
X^i	[0,2,0,0,0]	8	1	1	2
W^i	[2,0,0,0,0]	3	3	1	2
S^i	[0,0,0,0,0]	2	1	1	2
Y^i	[0,1,0,0,2]	2	1	1	6
V^i	[1,0,0,0,2]	2	4	2	$4, 6$
T^i	[0,0,0,1,0]	3	5	3	$2, 4, 6$
Z^i	[0,1,0,1,0]	3	1	1	4
U^i	[1,0,1,0,0]	3	4	2	$2, 4$
L^i	[2,1,0,0,0]	3	1	0	-
M^i	[2,0,0,1,0]	6	1	0	-

The last two lead to symmetrised ∇ and so are immediately ruled out. The other terms all admit at least one combination which corresponds to $R^3[\nabla,\nabla]\varepsilon$ in the Lichnerowicz, which thus give rise to R^4 terms.

These R^4 will appear as p-forms contracted with gamma-matrices acting on the spinor. The different terms contribute to different forms as follows:

p-form :	0	1	2	3	4	5
R^4 multiplicity:	7	0	1	2	17	0
$X^i \otimes R$	•	-	•	-	•	-
$W^i \otimes R$	-	-	-	-	-	-
$S^i \otimes R$	-	-	-	-	-	-
$Y^i \otimes R$	-	-	-	•	•	-
$V^i \otimes R$	-	-	-	-	•	-
$T^i \otimes R$	-	-	-	-	•	-
$Z^i \otimes R$	-	-	•	-	•	-
$U^i \otimes R$	-	-	•	-	•	-

!!! Nothing is possible at R^2 and R^3 order !!!

- ▷ Tensoriality of $\tilde{D}D$ + cancellation of 2,4,6-forms yields 2 independent invariants
  - ◇ $x\Big(\left(t_8 t_8 - \frac{1}{8}\epsilon\epsilon\right) R^4 + \frac{1}{2}\epsilon t_8 C R^4\Big)$
  - ◇ $y \left(t_8 t_8 + \frac{1}{8}\epsilon\epsilon\right) R^4$
- ▷ what fixes $y = 0$?
  - ◇ fermion terms and “actual” supersymmetry
  - ◇ inclusion of $\mathcal{F}$
  - ◇ next order $\sim R^7$ (lift from 2-loop string terms)

$\Longleftrightarrow$