L_∞-bootsrap approach to non-commutative gauge theories.

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The aim of the talk:

- Explain the concept of L_∞ algebra: motivation, definition, exemples and physical application.
- Explain the idea of L_∞ bootstrap programe as a generalization of a Gauge Principle: starting from a certain initial data to construct the L_∞ algebra which governs both the kinematics, i.e., gauge transformations of fields, and the dynamics, providing EOM invariant under these gauge transformations.
- Exemplify the proposed ideas constructing the gauge theories on the general NC space, with non-constant NC parameter Θ.
- Construction of $L_\infty^{\mathrm{gauge}}$ algebra + recurrence relations.
- Explicit expressions for the non-commutative $su(2)$-like and non-associative octonionic-like deformations of the abelian gauge transformation in slowly varying field approximation.
- Construction of L_∞^{full} algebra for NC Chern-Simons theory.

Introduction

- L_∞ algebras appeared in higher spin gauge theories with field dependent gauge parameters [Berends, Burgers, van Dam' 85]:

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]\,\Phi = \delta_{C(\lambda_1,\lambda_2,\Phi)}\Phi\,.$$

- As "generalized" gauge symmetries of closed string field theory [Zwiebach' 93]. Higher products from

$$\delta_\lambda \Phi = \sum_n \ell_n(\lambda, \Phi^{n-1})\,, \qquad \mathcal{F}(\Phi) = \sum_n \ell_n(\Phi^n)\,.$$

- The "standard" gauge theories (e.g., Yang-Mills) are realized in terms of L_∞ [Hohm, Zwiebach' 2017].
- In mathematics are known as strong homotopy algebras [Lada, Stasheff' 93].
- In particular, the proof of the Formality Theorem by Kontsevich is based on the notion of L_∞ quasi-isomorphysm (QISO).

Definition of L_∞ in ℓ-picture

- is a graded vector space: $X = \bigoplus_n X_n$,
- endowed with multi-linear maps: $\ell_n(x_1, \dots, x_n)$, of degree

$$\deg\big(\,\ell_n(x_1, \dots, x_n)\,\big) = n - 2 + \sum_{i=1}^{n} \deg(x_i)\,,$$

- which are graded anti-symmetric,

$$\ell_n(\dots, x_1, x_2, \dots\,) = (-1)^{1+\deg(x_1)\deg(x_2)}\,\ell_n(\dots, x_2, x_1, \dots\,)\,,$$

- and satisfy the relations (generalized Jacobi identities):

$$\mathcal{J}_n(x_1, \dots, x_n) := \sum_{i+j=n+1} (-1)^{i(j-1)} \sum_\sigma (-1)^\sigma \,\chi(\sigma; x)\; \ell_j\big(\,\ell_i(x_{\sigma(1)}\,, \dots, x_{\sigma(i)})\,, x_{\sigma(i+1)}, \dots, x_{\sigma(n)}\,\big) = 0\,,$$

where the permutations are restricted to the ones with: $\sigma(1) < \cdots < \sigma(i)$, $\sigma(i+1) < \cdots < \sigma(n)$, and the sign $\chi(\sigma; x) = \pm 1$ can be determined from graded anti-symmetry.

Definition of L_∞

The first L_∞ relations read

$$\ell_1(\,\ell_1(x)\,) = 0\,,$$
$$\ell_1(\,\ell_2(x_1,x_2)\,) = \ell_2(\,\ell_1(x_1),x_2\,) + (-1)^{x_1}\ell_2(\,x_1,\ell_1(x_2)\,)\,,$$

meaning that ℓ_1 is a nilpotent derivation with respect to ℓ_2, i.e., the Leibniz rule is satisfied.

$$\begin{aligned} 0 &= \ell_1(\ell_3(x_1,x_2,x_3)\,) + \ell_3(\ell_1(x_1),x_2,x_3\,) \\ &+(-1)^{x_1}\ell_3(x_1,\ell_1(x_2),x_3\,) + (-1)^{x_1+x_2}\ell_3(x_1,x_2,\ell_1(x_3)\,) \\ &+\ell_2(\ell_2(x_1,x_2),x_3\,) + (-1)^{(x_2+x_3)x_1}\ell_2(\ell_2(x_2,x_3),x_1\,) \\ &+(-1)^{(x_1+x_2)x_3}\ell_2(\ell_2(x_3,x_1),x_2\,)\,, \end{aligned}$$

the Jacobi identity for ℓ_2 is violated up to ℓ_1 exact terms.

- Any Lie algebra g can be represented as L_∞, setting $X_0 = g$, and all other X_n empty. Then $\ell_1 = 0$, and $\ell_2(x_1,x_2) = [x_1,x_2]$.

Consider $X = X_0 \oplus X_{-1}$, with X_0 being the space of gauge parameters f, and X_{-1} the space of gauge fields A_a.
The graded anti-symmetry in this case means:

$$\ell_n(\dots,f,g,\dots) = (-1)^{1+|f|\cdot|g|}\ell_n(\dots,g,f,\dots) = -\ell_n(\dots,g,f,\dots)$$
$$\ell_n(\dots,f,A,\dots) = -\ell_n(\dots,A,f,\dots)\,,$$
$$\ell_n(\dots,A,B,\dots) = \ell_n(\dots,B,A,\dots)\,.$$

Since, $\deg(\ell_n) = n-2$, the only non-vanishing brackets can be

$$\ell_{n+1}(f,A^n) \in X_{-1} \qquad \text{and} \qquad \ell_{n+2}(f,g,A^n) \in X_0\,,$$

satisfying the relations

$$\mathcal{J}_{n+2}(f,g,A^n) = 0 \qquad \text{and} \qquad \mathcal{J}_{n+3}(f,g,h,A^n) = 0\,,$$

with $\mathcal{J}_{n+2}(f,g,A^n) \in X_{-1}$, and $\mathcal{J}_{n+3}(f,g,h,A^n) \in X_0$.

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with $\mathcal{J}_{n+2}(f, g, A^n) \in X_{-1}$, and $\mathcal{J}_{n+3}(f, g, h, A^n) \in X_0$.

Gauge transformations

Gauge variations are:

$$\delta_f A \;= \sum_{n\geq 0} \tfrac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\, \ell_{n+1}(f, \underbrace{A, \ldots, A}_{n \text{ times}}) = \ell_1(f) + \ell_2(f, A) + \ldots\,.$$

Off-shell closure of the gauge symmetry variations,

$$[\delta_f, \delta_g]A = \delta_{-C(f,g,A)} A\,,$$
$$C(f,g,A) = \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\, \ell_{n+2}(f, g, \underbrace{A, \ldots, A}_{n \text{ times}})\,.$$

The Jacobi identity

$$\sum_{\text{cycl}} \left[\delta_f, [\delta_g, \delta_h]\right] \equiv 0\,,$$

is equivalent to the L_∞ relations with three gauge parameters.

Gauge field theory and $L_\infty^{\rm full}$ algebra

It is remarkable that L_∞ defines not only kinematics but also dynamics. Extend the vector space by X_{-2}, containing the eom $\mathcal{F}_a$, and so obtain $L_\infty^{\rm full}$ with

$$X = X_0 \oplus X_{-1} \oplus X_{-2}\,.$$

Non-empty X_{-2}, implies additional non-trivial brackets

$$\ell_n(A^n) \qquad \text{and} \qquad \ell_{n+2}(f, E, A^{n+1})\,,$$

as well as (infinitely) many non-trivial identities

$$\mathcal{J}_{n+1}(f, A^n) = 0 \qquad \text{and} \qquad \mathcal{J}_{n+2}(f, E, A^n) = 0\,.$$

The equations of motion can be written as

$$\mathcal{F} = \sum_{n\geq 1} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\,\ell_n(A^n)\,,$$

$$\delta_f \mathcal{F} = \ell_2(f, \mathcal{F}) + \ell_3(f, \mathcal{F}, A) - \frac{1}{2}\ell_4(f, \mathcal{F}, A^2) + \ldots$$

L_∞ bootstrap, arXiv:1803.00732

We have an infinite number of brackets ℓ_n, which are not arbitrary, since they should satisfy an infinite tower of L_∞ relations.

- **Proposal:** Promote the existence of L_∞ to a guiding principle for bootsrapping unknown gauge theories or consistent deformation of well defined theories.
- We start with $L_\infty^{\mathrm{gauge}}$ algebra. Bootstrap input: $\ell_1(f) \in X_{-1}$, and $\ell_2(f,g) \in X_0$.
- Then from $\mathcal{J}_2(f,g) = 0$ one finds $\ell_2(f,A)$.
- After that $\mathcal{J}_3(f,g,h) = 0$, defines $\ell_3(f,g,A)$, etc.
- Once $L_\infty^{\mathrm{gauge}}$ is constructed we specify the undeformed gauge theory by setting $\ell_1(A) \in X_{-2}$, with $\ell_1^2 = 0$.
- Then solving the corresponding L_∞ relations we construct L_∞^{full} algebra.
- Exemplify this idea for general NC gauge theories.

Problem: non-constant Θ

Given undeformed gauge theory, e.g., abelian Chern-Simons. The problem is to construct the consistent gauge theory on the NC space defined by $[x^i, x^j] = i\Theta^{ij}(x)$, which in the commutative limit, $\Theta \to 0$, reproduces the undeformed one.

One cannot simply substitute all point-wise products with a star products in the action, since the Leibniz rule is violated,

$$\partial_a(f \star g) = \partial_a f \star g + f \star \partial_a g + \frac{i}{2}(\partial_a \Theta^{ij})\partial_i f \partial_j g + \mathcal{O}(\Theta^2)\,,$$

and the standard gauge principle is no longer applicable.

- Old: Hopf-algebra approach, generalized Leibniz rule (deformed co-product).
- One may use the inner derivatives, $D_i = c[x_i, \cdot]_\star$, leading to the problems with the commutative limit.
- New: Consider this problem in the framework of L_∞.

Construction of $L_\infty^{\rm gauge}$ algebra

Let: $\ell_1(f) = \partial_a f$; $\ell_1(A) = 0$, and

$$\ell_2(f,g) = i[f,g]_\star = -\{f,g\} + \mathcal{O}(\Theta^3) \in X_0\,.$$

$\ell_2(f,A)$ can be non-zero and should be found from $\mathcal{J}_2(f,g) = 0$,

$$\begin{aligned}\ell_1(\ell_2(f,g)) &= -\{\overbrace{\ell_1(f)}^{\in X_{-1}}, g\} - \{f, \overbrace{\ell_1(g)}^{\in X_{-1}}\} - (\partial_a \Theta^{ij})\,\partial_i f \partial_j g + \mathcal{O}(\Theta^3)\,,\\ &= \ell_2(\ell_1(f),g) + \ell_2(f,\ell_1(g))\,.\end{aligned}$$

which implies that

$$\ell_2(f,A) = i[f,A_a]_\star - \frac{1}{2}(\partial_a\Theta^{ij})\,\partial_i f\, A_j + \mathcal{O}(\Theta^3)\,.$$

Note that the solution is not unique, one may also set, e.g.,

$$\ell_2'(f,A) = \ell_2(f,A) + s_a^{ij}(x)\,\partial_i f\, A_j\,, \qquad s_a^{ij}(x) = s_a^{ji}(x)\,.$$

However, the symmetric part $s_a^{ij}(x)\,\partial_i f\, A_j$ can be always "gauged away" by L_∞-QISO, physically equivalent to SW map, see arXiv:1806.10314 for more details.

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$$\begin{aligned}\ell_1(\ell_2(f,g)) &= -\{\overbrace{\ell_1(f)}^{\in X_{-1}},g\} - \{f,\overbrace{\ell_1(g)}^{\in X_{-1}}\} - (\partial_a\Theta^{ij})\,\partial_i f\,\partial_j g + \mathcal{O}(\Theta^3)\,,\\ &= \ell_2(\ell_1(f),g) + \ell_2(f,\ell_1(g))\,.\end{aligned}$$

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$L_\infty^{\rm gauge}$ algebra

Next step is to check, $\mathcal{J}_3(f,g,h)=0$, and define $\ell_3(f,g,A)$,

$$\begin{aligned} 0 &= \ell_2(\ell_2(f,g),h)+\ell_2(\ell_2(g,h),f)+\ell_2(\ell_2(h,f),g)+ \\ &\quad \ell_3(\ell_1(f),g,h)+\ell_3(f,\ell_1(g),h)+\ell_3(f,g,\ell_1(h))\ . \end{aligned}$$

The first line is a Jacobiator, which in the leading order reads,

$$\ell_2(\ell_2(f,g),h)+\ell_2(\ell_2(g,h),f)+\ell_2(\ell_2(h,f),g)=-\Pi^{ijk}\partial_i f\partial_j g\partial_k h\,.$$

For associative NC deformations we may just set, $\ell_3(A,f,g)=0$, while for non-associative one needs non-vanishing $\ell_3(A,f,g)$ to satisfy it,

$$\ell_3(A,f,g)=\frac{1}{3}\Pi^{ijk}A_i\partial_j f\partial_k g+\mathcal{O}(\Theta^3)\ .$$

Then, we have to analyze $\mathcal{J}_3(f,g,A)=0$, given by

$$\begin{aligned}0 &= \ell_2(\ell_2(A,f),g)+\ell_2(\ell_2(f,g),A)+\ell_2(\ell_2(g,A),f)+\\ &\quad \ell_1(\ell_3(A,f,g))-\ell_3(A,\ell_1(f),g)-\ell_3(A,f,\ell_1(g)).\end{aligned}$$

We replace it with $\mathcal{J}_3(g,h,\ell_1(f))=0$, written in the form

$$\begin{aligned}&\ell_3(\ell_1(f),\ell_1(g),h)-\ell_3(\ell_1(f),\ell_1(h),g)=G(f,g,h)\,,\\ &G(f,g,h):=\ell_1(\ell_3(\ell_1(f),g,h))\\ &+\ell_2(\ell_2(\ell_1(f),g),h)+\ell_2(\ell_2(g,h),\ell_1(f))+\ell_2(\ell_2(h,\ell_1(f)),g)\,.\end{aligned}$$

By construction, $G(f,g,h)=-G(g,f,h)$. The graded symmetry of $\ell_3(\ell_1(f),\ell_1(g),h)$ implies the graded cyclicity (consistency condition) of $G(f,g,h)$:

$$G(f,g,h)+G(h,f,g)+G(g,h,f)=0\,.$$

Below we show that it holds true as a consequence of the previous "Jacobi identities", $\mathcal{J}_2(f,g)=0$ and $\mathcal{J}_3(f,g,h)=0$.

$L_\infty^{\mathrm{gauge}}$ algebra

$$G(f,g,h)+G(h,f,g)+G(g,h,f)=$$
$$\ell_2(\ell_2(\ell_1(h),f),g)+\ell_2(\ell_2(f,g),\ell_1(h))+\ell_2(\ell_2(g,\ell_1(h)),f)+$$
$$\ell_2(\ell_2(\ell_1(g),h),f)+\ell_2(\ell_2(h,f),\ell_1(g))+\ell_2(\ell_2(f,\ell_1(g)),h)+$$
$$\ell_2(\ell_2(\ell_1(f),g),h)+\ell_2(\ell_2(g,h),\ell_1(f))+\ell_2(\ell_2(h,\ell_1(f)),g)$$
$$\ell_1(\ell_3(\ell_1(f),g,h))+\ell_1(\ell_3(f,\ell_1(g),h))+\ell_1(\ell_3(f,g,\ell_1(h)))\,.$$

Using $\mathcal{J}_2(f,g)=0$, we rewrite it as

$$\ell_1\big[\ell_2(\ell_2(f,g),h)+\ell_2(\ell_2(g,h),f)+\ell_2(\ell_2(h,f),g)+$$
$$\ell_3(\ell_1(f),g,h)+\ell_3(f,\ell_1(g),h)+\ell_3(f,g,\ell_1(h))\big]=$$
$$\ell_1\left[\mathcal{J}_3(f,g,h)\right]=0\,.$$

Thus, the combination (symmetrization in f and g):

$$\ell_3(\ell_1(f),\ell_1(g),h)=-\frac{1}{6}\Big(G(f,g,h)+G(g,f,h)\Big)\,,$$

has required graded symmetry and solves $\mathcal{J}_3(g,h,\ell_1(f))=0$.

Setting

$$\ell_3(A,B,h) = \ell_3(\ell_1(f),\ell_1(g),h)\big|_{\ell_1(f)=A;\,\ell_1(g)=B}\ ,$$

one gets in the leading order,

$$\ell_3(A,B,f) = -\frac{1}{6}\left({G_a}^{ijk} + {G_a}^{jik}\right)A_iB_j\partial_k f$$
$$+\frac{1}{6}\Pi^{ijk}(\partial_a A_i B_j\partial_k f - A_i\partial_a B_j\partial_k f) - \frac{1}{2}\Pi^{ijk}(\partial_i A_a B_j\partial_k f - A_i\partial_j B_a\partial_k f)$$
$$+\mathcal{O}(\Theta^3)\,.$$

with

$${G_a}^{ijk} = \frac{1}{3}\partial_a\Pi^{ijk} - \Theta^{im}\partial_m\partial_a\Theta^{jk} - \frac{1}{2}\partial_a\Theta^{jm}\partial_m\Theta^{ki} - \frac{1}{2}\partial_a\Theta^{km}\partial_m\Theta^{ij}\,.$$

- Even in the associative case one needs higher brackets to compensate the violation of the Leibnitz rule.
- The consistency condition (cyclicity) holds true as a consequence of L_∞ construction.

Non-associative case: $\mathcal{J}_4(f,g,h,A)=0$, we substitute with $\mathcal{J}_4(f,g,h,\ell_1(k))=0$, written as

$$\ell_4(\ell_1(f),g,h,\ell_1(k))+\ell_4(f,\ell_1(g),h,\ell_1(k))+\ell_4(f,g,\ell_1(h),\ell_1(k))$$
$$=F(f,g,h,k)\,,$$

with

$$F(f,g,h,k)=\ell_2(\ell_3(f,g,\ell_1(k)),h)+\ell_2(g,\ell_3(f,h,\ell_1(k)))$$
$$-\ell_2(f,\ell_3(g,h,\ell_1(k)))+\ell_3(\ell_2(f,g),h,\ell_1(k))-\ell_3(\ell_2(f,h),g,\ell_1(k))$$
$$+\ell_3(\ell_2(f,\ell_1(k)),g,h)-\ell_3(f,\ell_2(g,h),\ell_1(k))+\ell_3(f,\ell_2(g,\ell_1(k)),h)$$
$$+\ell_3(f,g,\ell_2(h,\ell_1(k)))\,.$$

By the construction $F(f,g,h,k)$ is antisymmetric in first three arguments and the graded symmetry of $\ell_4(\ell_1(f),g,h,\ell_1(k))$ implies the graded cyclicity (consistency condition):

$$F(f,g,h,k)-F(k,f,g,h)+F(h,k,f,g)-F(g,h,k,f)=0\,.$$

Higher relations

Again, the consistency condition holds true as a consequence of the previous Jacobi identities, graded symmetry and multi-linearity of the products ℓ_n. One finds,

$$\ell_4(\ell_1(f),g,h,\ell_1(k)) = \frac{1}{8}\left(F(f,g,h,k)+F(k,g,h,f)\right) .$$

Then,

$$\ell_4(A,g,h,B) = \ell_4(\ell_1(f),g,h,\ell_1(k))|_{\ell_1(f)=A;\,\ell_1(k)=B} \ .$$

The explicit form in the leading oder,

$$\ell_4(A,g,h,B) = \left[\frac{1}{16}\Pi^{jlm}\partial_m\Theta^{ki}+\frac{1}{16}\Pi^{jkm}\partial_m\Theta^{li}-\frac{1}{16}\Pi^{ilm}\partial_m\Theta^{kj}\right.$$
$$\left.-\frac{1}{16}\Pi^{ikm}\partial_m\Theta^{lj}-\frac{1}{24}\Theta^{km}\partial_m\Pi^{ijl}-\frac{1}{24}\Theta^{lm}\partial_m\Pi^{ijk}\right]\partial_i g\partial_j f A_k B_l \,.$$

For, $\mathcal{J}_{n+2}(g,h,A^n)=0$, $n>1$ we proceed in the similar way. First we substitute them by $\mathcal{J}_{n+2}(g,h,\ell_1(f)^n)=0$,

$$\ell_{n+2}(\ell_1(f)^n,\ell_1(g),h)-\ell_{n+2}(\ell_1(f)^n,\ell_1(h),g)=G(f_1,\dots,f_n,g,h)\,,$$

The graded symmetry of $\ell_{n+2}(\ell_1(f)^n,\ell_1(g),h)$ implies the consistency condition,

$$G(f_1,\dots,f_n,g,h)+G(f_1,\dots,f_{n-1},g,h,f_n)+G(f_1,\dots,f_{n-1},h,f_n,g)=0\,,$$

which follows from the previous L_∞ relations and can be proved by the induction.

The solution is constructed by taking the symmetrization of the r.h.s. in the first $n+1$ arguments, i.e.,

$$\ell_{n+2}(\ell_1(f)^n,\ell_1(g),h)=-\frac{1}{(n+1)(n+2)}\Big(G(f_1,\dots,f_n,g,h)$$
$$+\,G(f_2,\dots,f_n,g,f_1,h)+\dots+G(f_n,\dots,f_{n-1},h)\Big)\,.$$

Recurrense relations for L_∞^{gauge} algebra

Identities: $\mathcal{J}_{n+3}(f,g,h,A^n)=0$, $n>1$, are substituted by, $\mathcal{J}_{n+3}(f,g,h,\ell_1(k)^n)=0$, written as:

$$\ell_{n+3}(\ell_1(f),g,h,\ell_1(k)^n)+\ell_{n+3}(f,\ell_1(g),h,\ell_1(k)^n)$$
$$+\ell_{n+3}(f,g,\ell_1(h),\ell_1(k)^n)=F(f,g,h,k_1,...,k_n)\,.$$

The r.h.s. should satisfy the graded cyclicity which follows from the previous Jacobi identities, graded symmetry and multi-linearity of the products ℓ_n.

The solution is constructed by taking the corresponding symmetrization of r.h.s.:

$$\ell_{n+3}\big(f,g,\ell_1(h),\ell_1(k)^n\big)=-\frac{1}{n(n+2)}\Big(F(f,g,h,k_1,...,k_n)$$
$$+F(f,g,k_1,...,k_n,h)+\cdots+F(f,g,k_n,h,k_1,...,k_{n-1})\Big)\,,$$

see arXiv:1805.12040 for details.

Slowly varying field approximation

The main aim here is to do some explicit calculations to illustrate the proposed ideas. Consider the limit of slowly varying, but not necessarily small gauge fields. We discard the higher derivatives terms and take, $\ell_2(f,g) = -\{f,g\}$, as a Poisson bracket. Then

$$\ell_2(f,A) = -\{f,A_a\} - \frac{1}{2}(\partial_a \Theta^{ij})\,\partial_i f A_j\,.$$

For some particular choices of Θ we may do the all orders calculation. Taking, e.g., $\Theta^{ij}(x) = 2\varepsilon^{ijk}x^k$, we may see that

$$\delta_f A_a = \partial_a f + \{A_a, f\}_\varepsilon + \varepsilon^{abc} A_b \partial_c f + \left(\partial_a f A^2 - \partial_b f A^b A_a\right)\chi(A^2)\,.$$

From the gauge closure condition, $[\delta_f,\delta_g]A = \delta_{\{f,g\}_\varepsilon}A$, one finds,

$$\chi(t) = \frac{1}{t}\left(\sqrt{t}\,\cot\sqrt{t} - 1\right), \qquad \chi(0) = -\frac{1}{3}\,.$$

- NC $su(2)$-like deformation of the abelian gauge transformations in the slowly varying field approximation.

Slowly varying field approximation

One can do the same with the quasi-Poisson structure isomorphic to the algebra of the imaginary octonions,

$$\{f,g\}_\eta = 2\,\eta_{ABC}\,\xi_C\,\partial_A f\,\partial_B g\ .$$

In this case, $\ell_{n+2}(f,g,\Phi^n)\neq 0$, implying the modification of the closure condition, $[\delta_f,\delta_g]\Phi=\delta_{-C(f,g,\Phi)}\Phi$, with

$$C(f,g,\Phi) = -\{f,g\}_\eta \\ -2\,\eta_{ABCD}\,\partial_A f\,\partial_B g\,\Phi_C\left(\frac{\sin 2\sqrt{\Phi^2}}{\sqrt{\Phi^2}}\,\xi_D + 2\,\frac{\sin^2\sqrt{\Phi^2}}{\Phi^2}\,\eta_{DEF}\Phi_E\,\xi_F\right)\ .$$

The expression for the gauge variation reads,

$$\delta_f\Phi_A = \partial_A f + \{\Phi_A,f\}_\eta + \eta_{ABC}\,\Phi_B\,\partial_C f + \left(\partial_A f\Phi^2 - \partial_B f\Phi^B\Phi_A\right)\chi(\Phi^2)\,.$$

- Non-associative octonionic-like deformation of the abelian gauge transformations for slowly varying fields.

Defining the coordinates and momenta in terms of the original coordinates ξ_A as

$$x^i = \frac{\sqrt{\lambda\,\ell_s^3\,R}}{2\hbar}\,\xi_{3+i}\ ,\quad p_i = -\frac{\lambda}{2}\,\xi_i\ ,\quad x^4 = \frac{\sqrt{\lambda^3\,\ell_s^3\,R}}{2\hbar}\,\xi_7\ ,$$

we obtain from $\{\xi_A,\xi_B\}_\eta = 2\,\eta_{ABC}\,\xi_C$:

$$\begin{aligned}
\{x^i,x^j\}_\lambda &= \frac{\ell_s^3}{\hbar^2}\,R^{4,ijk4}\,p_k \quad \text{and} \quad \{x^4,x^i\}_\lambda = \frac{\lambda\,\ell_s^3}{\hbar^2}\,R^{4,1234}\,p^i,\\
\{x^i,p_j\}_\lambda &= \delta^i_j\,x^4 + \lambda\,\varepsilon^i{}_{jk}\,x^k \quad \text{and} \quad \{x^4,p_i\}_\lambda = \lambda^2\,x_i\ ,\\
\{p_i,p_j\}_\lambda &= -\lambda\,\varepsilon_{ijk}\,p^k\ .
\end{aligned}$$

with λ being the M-theory radius.
Sending $\lambda \to 0$ one recover the R-flux algebra.

NC Chern-Simons theory, L_∞^{full} algebra

The lower brackets (derivatives) are

$$\ell_1(f) = \partial_a f\,, \qquad \ell_2(f,g) = \Theta\{f,g\}_\varepsilon \qquad \ell_1(A) = \varepsilon_c{}^{ab}\,\partial_a A_b\,.$$

Corresponding EOM are

$$\begin{aligned}\mathcal{F}_a &= \varepsilon^{abc}\partial_b A_c + \Theta\left(\varepsilon^{abc}\{A_b,A_c\} + 2A_b\partial_a A_b - A_a\partial_b A_b - A_b\partial_b A_a\right) \\ &\quad +\Theta^2\left(\frac{1}{4}\varepsilon^{abc}A_c\partial_b(A^2) - \frac{8}{3}\varepsilon^{abc}A^2\partial_b A_c - 2\varepsilon^{abc}A_cA_i\partial_i A_b \right.\\ &\quad \left. - 2\varepsilon^{ijb}A_aA_j\partial_i A_b - \{A^2,A_a\}\right) + O(A^4)\,.\end{aligned}$$

In the limit $\Theta \to 0$, reproduce undeformed CS eom and transform covariantly,

$$\delta_f\mathcal{F} = \ell_2(f,\mathcal{F}) = \{f,\mathcal{F}\}_\varepsilon\,.$$

We don't have yet all order expression for $\mathcal{F}$.

Discussion

- Given undeformed gauge theory and anti-symmetric bi-vector field $\Theta^{ij}(x)$ describing the non-commutativity of the space, we have iterative procedure of the construction of NC gauge theory, which reproduce in the limit $\Theta \to 0$ the undeformed one.
- NC-YM is constructed taking $\ell_1(A) = \Box A_a - \partial_a(\partial \cdot A)$.
- The relation with the previous approaches needs to be better understood.
- Our construction is based on the principle that gauge symmetry should be realized by L_∞ and works for any given Θ.
- Physical consequences: interaction with the meter fields, etc.